Master Theorem Worksheet

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Problem 1-1: $T(n) = 3T(n/2) + n^2$

Identify parameters:

- $a = 3,\ b = 2,\ f(n) = n^2$
- $n^{\log_b a} = n^{\log_2 3} \approx n^{1.585}$

Compare: $f(n) = n^2$ versus $n^{1.585}$
Since $n^2 = \Omega(n^{1.585+\epsilon})$ for $\epsilon = 0.415$, and the regularity condition holds:

$$3 \cdot (n/2)^2 = \frac{3n^2}{4} \leq cn^2 \text{ for } c = 3/4 < 1$$

Answer: Case 3 $\Rightarrow T(n) = \Theta(n^2)$

Problem 1-2: $T(n) = 7T(n/2) + n^2$

Identify parameters:

- $a = 7,\ b = 2,\ f(n) = n^2$
- $n^{\log_b a} = n^{\log_2 7} \approx n^{2.807}$

Compare: $f(n) = n^2$ versus $n^{2.807}$
Since $n^2 = O(n^{2.807-\epsilon})$ for $\epsilon = 0.807$
Answer: Case 1 $\Rightarrow T(n) = \Theta(n^{\log_2 7})$

Problem 1-3: $T(n) = 4T(n/2) + n^2$

Identify parameters:

- $a = 4,\ b = 2,\ f(n) = n^2$
- $n^{\log_b a} = n^{\log_2 4} = n^2$

Compare: $f(n) = n^2 = \Theta(n^2 \cdot \log^0 n)$
Exactly matches the critical exponent with $k = 0$.
Answer: Case 2 $\Rightarrow T(n) = \Theta(n^2 \log n)$

Problem 1-4: $T(n) = 3T(n/4) + n \log n$

Identify parameters:

- $a = 3,\ b = 4,\ f(n) = n \log n$
- $n^{\log_b a} = n^{\log_4 3} = n^{\frac{\log 3}{\log 4}} \approx n^{0.793}$

Compare: $f(n) = n \log n$ versus $n^{0.793}$
Since $n \log n = \Omega(n^{0.793+\epsilon})$ for small ϵ (as $n > n^{0.793}$), and the regularity condition holds.
Answer: Case 3 $\Rightarrow T(n) = \Theta(n \log n)$

Problem 1-5: $T(n) = 4T(n/2) + \log n$

Identify parameters:

- $a = 4,\ b = 2,\ f(n) = \log n$
- $n^{\log_b a} = n^{\log_2 4} = n^2$

Compare: $f(n) = \log n$ versus n^2
Since $\log n = O(n^{2-\epsilon})$ for any $\epsilon < 2$
Answer: Case 1 $\Rightarrow T(n) = \Theta(n^2)$

Problem 1-6: $T(n) = T(n-1) + n$

Analysis: This recurrence has the form $T(n) = T(n-1) + f(n)$, not $T(n) = aT(n/b) + f(n)$.

The Master Theorem requires division of the problem size by a constant factor $b > 1$, not subtraction.

Answer: Master Theorem does not apply
(Note: This solves to $T(n) = \Theta(n^2)$ by other methods)

Problem 1-7: $T(n) = 4T(n/2) + n^2 \log n$

Identify parameters:

- $a = 4,\ b = 2,\ f(n) = n^2 \log n$
- $n^{\log_b a} = n^{\log_2 4} = n^2$

Compare: $f(n) = n^2 \log n = \Theta(n^2 \cdot \log^1 n)$
This matches Case 2 with $k = 1$.
Answer: Case 2 $\Rightarrow T(n) = \Theta(n^2 \log^2 n)$

Problem 1-8: $T(n) = 5T(n/2) + n^2 \log n$

Identify parameters:

- $a = 5,\ b = 2,\ f(n) = n^2 \log n$
- $n^{\log_b a} = n^{\log_2 5} \approx n^{2.322}$

Compare: $f(n) = n^2 \log n$ versus $n^{2.322}$
Since $n^2 \log n = O(n^{2.322-\epsilon})$ for $\epsilon \approx 0.322$ (the logarithmic factor is dominated)
Answer: Case 1 $\Rightarrow T(n) = \Theta(n^{\log_2 5})$

Problem 1-9: $T(n) = 3T(n/3) + n/\log n$

Identify parameters:

- $a = 3,\ b = 3,\ f(n) = n/\log n$
- $n^{\log_b a} = n^{\log_3 3} = n$

Compare: $f(n) = n/\log n$ versus n

The function $f(n) = n/\log n$ differs from n by a logarithmic factor, but this is not a *polynomial* difference. None of the three cases apply.

Answer: Master Theorem does not apply

Problem 1-10: $T(n) = 2T(n/4) + c$

Identify parameters:

- $a = 2,\ b = 4,\ f(n) = c$ (constant)
- $n^{\log_b a} = n^{\log_4 2} = n^{1/2} = \sqrt{n}$

Compare: $f(n) = c = O(1)$ versus $\sqrt{n}$
Since $c = O(n^{1/2-\epsilon})$ for any $\epsilon < 1/2$
Answer: Case 1 $\Rightarrow T(n) = \Theta(\sqrt{n})$

Problem 1-11: $T(n) = T(n/4) + \log n$

Identify parameters:

- $a = 1,\ b = 4,\ f(n) = \log n$
- $n^{\log_b a} = n^{\log_4 1} = n^0 = 1$

Compare: $f(n) = \log n$ versus 1
Since $\log n = \Omega(1 + \epsilon)$ for any $\epsilon < \log n$, and the regularity condition holds:

$$1 \cdot \log(n/4) = \log n - \log 4 \leq c \log n \text{ for } c < 1$$

Answer: Case 3 $\Rightarrow T(n) = \Theta(\log n)$

Problem 1-12: $T(n) = T(n/2) + T(n/4) + n^2$

Analysis: This recurrence involves subproblems of two different sizes: $n/2$ and $n/4$.

The Master Theorem requires all subproblems to be of the same size (n/b).

Answer: Master Theorem does not apply

Problem 1-13: $T(n) = 2T(n/4) + \log n$

Identify parameters:

- $a = 2,\ b = 4,\ f(n) = \log n$
- $n^{\log_b a} = n^{\log_4 2} = n^{1/2} = \sqrt{n}$

Compare: $f(n) = \log n$ versus $\sqrt{n}$
Since $\log n = O(n^{1/2-\epsilon})$ for any $\epsilon < 1/2$
Answer: Case 1 $\Rightarrow T(n) = \Theta(\sqrt{n})$

Problem 1-14: $T(n) = 3T(n/3) + n \log n$

Identify parameters:

- $a = 3,\ b = 3,\ f(n) = n \log n$
- $n^{\log_b a} = n^{\log_3 3} = n$

Compare: $f(n) = n \log n$ versus n

We have $f(n) = n \log n$ which is n times a logarithmic factor. This doesn't fit Case 2 exactly (which requires $f(n) = \Theta(n \log^k n)$ to give $T(n) = \Theta(n \log^{k+1} n)$), and it's not polynomially larger than n.

Answer: Master Theorem does not apply

(Note: By extended analysis, $T(n) = \Theta(n \log^2 n)$)

Problem 1-15: $T(n) = 8T((n - \sqrt{n})/4) + n^2$

Analysis: The subproblem size is $(n - \sqrt{n})/4$, not exactly $n/4$.

The Master Theorem requires subproblems of size exactly n/b for constant b.

Answer: Master Theorem does not apply

Problem 1-16: $T(n) = 2T(n/4) + \sqrt{n}$

Identify parameters:

- $a = 2,\ b = 4,\ f(n) = \sqrt{n} = n^{1/2}$
- $n^{\log_b a} = n^{\log_4 2} = n^{1/2}$

Compare: $f(n) = n^{1/2} = \Theta(n^{1/2} \cdot \log^0 n)$

Exactly matches the critical exponent with $k = 0$.

Answer: Case 2 $\Rightarrow T(n) = \Theta(\sqrt{n} \log n)$

Problem 1-17: $T(n) = 2T(n/4) + n^{0.51}$

Identify parameters:

- $a = 2,\ b = 4,\ f(n) = n^{0.51}$
- $n^{\log_b a} = n^{\log_4 2} = n^{0.5}$

Compare: $f(n) = n^{0.51}$ versus $n^{0.5}$

Since $n^{0.51} = \Omega(n^{0.5+\epsilon})$ for $\epsilon = 0.01$, and the regularity condition holds:

$$2 \cdot (n/4)^{0.51} = 2 \cdot \frac{n^{0.51}}{4^{0.51}} \approx 0.486 n^{0.51} \leq c n^{0.51}$$

Answer: Case 3 $\Rightarrow T(n) = \Theta(n^{0.51})$

Problem 1-18: $T(n) = 16T(n/4) + n!$

Identify parameters:

- $a = 16,\ b = 4,\ f(n) = n!$
- $n^{\log_b a} = n^{\log_4 16} = n^2$

Compare: $f(n) = n!$ versus n^2

Since $n!$ grows much faster than any polynomial, $n! = \Omega(n^{2+\epsilon})$ for any ϵ, and the regularity condition holds.

Answer: Case 3 $\Rightarrow T(n) = \Theta(n!)$

Problem 1-19: $T(n) = 3T(n/2) + n$

Identify parameters:

- $a = 3,\ b = 2,\ f(n) = n$
- $n^{\log_b a} = n^{\log_2 3} \approx n^{1.585}$

Compare: $f(n) = n$ versus $n^{1.585}$

Since $n = O(n^{1.585-\epsilon})$ for $\epsilon = 0.585$

Answer: Case 1 $\Rightarrow T(n) = \Theta(n^{\log_2 3})$

Problem 1-20: $T(n) = 4T(n/2) + cn$

Identify parameters:

- $a = 4,\ b = 2,\ f(n) = cn$
- $n^{\log_b a} = n^{\log_2 4} = n^2$

Compare: $f(n) = cn$ versus n^2

Since $cn = O(n^{2-\epsilon})$ for any $\epsilon < 1$

Answer: Case 1 $\Rightarrow T(n) = \Theta(n^2)$

Problem 1-21: $T(n) = 3T(n/3) + n/2$

Identify parameters:

- $a = 3,\ b = 3,\ f(n) = n/2$
- $n^{\log_b a} = n^{\log_3 3} = n$

Compare: $f(n) = n/2 = \Theta(n \cdot \log^0 n)$

Exactly matches the critical exponent with $k = 0$ (the constant $1/2$ doesn't matter asymptotically).

Answer: Case 2 $\Rightarrow T(n) = \Theta(n \log n)$

Problem 1-22: $T(n) = 4T(n/2) + n/\log n$

Identify parameters:

- $a = 4,\ b = 2,\ f(n) = n/\log n$
- $n^{\log_b a} = n^{\log_2 4} = n^2$

Compare: $f(n) = n/\log n$ versus n^2
Since $n/\log n = O(n^{2-\epsilon})$ for any $\epsilon < 1$
Answer: Technically Master Theorem does not apply
However, Case 1 pattern gives: $T(n) = \Theta(n^2)$

Problem 1-23: $T(n) = 7T(n/3) + n^2$

Identify parameters:

- $a = 7,\ b = 3,\ f(n) = n^2$
- $n^{\log_b a} = n^{\log_3 7} \approx n^{1.771}$

Compare: $f(n) = n^2$ versus $n^{1.771}$
Since $n^2 = \Omega(n^{1.771+\epsilon})$ for $\epsilon = 0.229$, and the regularity condition holds:

$$7 \cdot (n/3)^2 = \frac{7n^2}{9} \leq cn^2 \text{ for } c = 7/9 < 1$$

Answer: Case 3 $\Rightarrow T(n) = \Theta(n^2)$

Problem 1-24: $T(n) = 8T(n/3) + 2^n$

Identify parameters:

- $a = 8,\ b = 3,\ f(n) = 2^n$
- $n^{\log_b a} = n^{\log_3 8} \approx n^{1.893}$

Compare: $f(n) = 2^n$ versus $n^{1.893}$
Since 2^n grows exponentially while $n^{1.893}$ grows polynomially, $2^n = \Omega(n^{1.893+\epsilon})$ for any ϵ, and the regularity condition holds.
Answer: Case 3 $\Rightarrow T(n) = \Theta(2^n)$

Problem 1-25: $T(n) = 16T(n/4) + n$

Identify parameters:

- $a = 16,\ b = 4,\ f(n) = n$
- $n^{\log_b a} = n^{\log_4 16} = n^2$

Compare: $f(n) = n$ versus n^2
Since $n = O(n^{2-\epsilon})$ for any $\epsilon < 1$
Answer: Case 1 $\Rightarrow T(n) = \Theta(n^2)$