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1. *Abstract*

This paper presents a historical perspective on the developments of stochastic calculus in the 20th century. In doing so, we have focused primarily on the development of Stochastic Processes and Integration and have, therefore, focused on the works of Louis Bachelier, Paul Samuelson, Kiyosi Itô and Pual-André Meyer.

2. *Louis Bachelier and Paul Samuelson*

If you asked an Economist about the name of Louis Bachelier , you would probably get a blank expression. However, if you were to ask a mathematician about him, the response would be quite the opposite.

Widely regarded as the founder of modern Mathematical Finance as we know it, Louis Bachelier's contribution to mathematics and finance cannot be underscored enough. Every influential paper of the 20th century in the field of mathematical finance has cited Bachelier's work. Even Kolmogorov's most famous paper of 1931 "Analytical Methods of Probability Theory" cites Bachelier's work.

The first concept of Brownian motion was effectively modeled by T.N Thiele, who created the model based on time series in 1880. However, Louis Bachelier improved on this model while studying the dynamic behavior of the Paris stock market.

His thesis, "The Theory of Speculation" aimed to introduce new fundamental concepts while analysing stock markets, mainly the concept of stochastic analysis. In addition to the modern model of Brownian motion, he derived the Markov property and the Chapman - Kolmogorov equation. He also established a relationship between Brownian motion and the heat equation.

Bachelier's work on derivative securities was novel at the time. It was relatively common to model the valuation of stocks with respect to fixed bonds. In his thesis, Bachelier mentions in great detail about the derivative contracts that were available at the time and the working of each one of these. Using a random walk model, he explained that the prices are continuous and homogenous in time and space. He continues this to derive what we now know as the Chapman-Kolmogorov equation. Considering probability of different prices as a function of time, he derived the square root law as well.

For mathematical economists such as Samuelson, the theory of martingales and stochastic integrals, were topics to really ponder upon and subsequent improvements to the same have been accomplished. Samuelson gives credit to Bachelier for "supplying the tools" needed for his research. Already attempted to model the noise of the Paris Bourse, Bachelier also tackled the option pricing problem. His initial thesis, using some ideas of the Central Limit Theorem, stated that the market noise should have no memory, i.e., the movement of the stock prices is independent of the current stock price and is normally distributed. This helped him to connect the Brownian motion with the heat equation. He derived that the fundamental solution to the heat equation is the Gaussian kernel.

Bachelier's attempt to price options came extremely close to the current Noble Prize winning Black-Scholes formula using Stochastic Analysis. Far ahead of his time, he represented the stock prices as stochastic processes and computed the interest by connecting these stochastic processes and partial differential equations. His justification for using the martingale assumption and further martingale properties was based on economic grounds. Samuelson's direction of research became a lot easier due to the tools provided to him by this research.

The influence of Bachelier's work in using mathematical methods like stochastic calculus can be seen in his research concerning the pricing of stocks. His work with Fama leading to the Efficient Market Hypothesis showed this influence as well. An efficient market will always value a share price with all the information that is known and it is impossible to be able to "beat the market”. Samuelson also published a proof stating that if a market is efficient, the stock prices will exhibit a random walk behavior. However, random walk behavior by itself is not enough to prove market efficiency. Using Bachelier's model of stock pricing according to a Brownian motion, Samuelson and McKean derive that a good model for stock price movements is Geometric Brownian Motion. They note that although, Bachelier's model using Brownian motion was inspiring, it failed to ensure the stock prices would be positive.

This is corrected by Samuelson and McKean with the Geometric Brownian Motion model.
Bachelier was the first to investigate the perfomance and behavior of capital markets from a purely mathematicians point of view, giving certain economics reasoning whenever

necessary and is regarding as the father of modern mathematical finance for that very reason. However, Paul Samuelson, being a trained economist, used these mathematical models and improved them based on solid economic reasoning.

3. ***Kiyosi Itō and other Japanese mathematicians***

Kiyosi Itô, a Japanese mathematician, pioneered the theory of stochastic integration and stochastic differential equations. This theory is now named after him – Itô Calculus.

Itô made 2 seminal contributions which are now considered the origin of Itô's stochastic calculus. The first one is known as Lévy-Itô Theorem, where he gave a rigorous proof for the structure of the sample functions of Lévy processes. This theorem enabled a complete understanding of Lévy-Khinchin formula for canonical forms of infinitely divisible distributions. In his second work, Itô introduced the notion of stochastic integral and the formula now known as Itô's lemma. This directly led to formation to development of an analogous differential and integral calculus for a class of random functions now known as Itô processes.

During WWII Itô had sent his manuscript (On Stochastic Differential Equations) containing above 2 proofs to J.L. Doob, who got those proofs published in the US (Memoirs of the AMS 1951). After the war ended, Itô visited the Institute for Advanced Study at Princeton from 1954 to 1956 and primarily focused on studying one-dimensional diffusion processes along with Henry P. McKean. Together they developed Itô-Mckean construction of sample paths of one-dimensional diffusions which later played an important role in the stochastic calculus of the random functions called semi-martingales.

In *Itô's calculus*, the classical rule $dX^2 = 2XdX$ is replaced by

$$dX^2 = 2XdX + d < X >, \qquad where$$

$$< X >_t = \sum_{\substack{t_i \in D_n \\ t_i < t}} (X_{t_{i+1}} - X_{t_i})^2$$

Denotes the quadratic variation of path up to time t. Itô proved that solution of stochastic differential equation admits a quadratic variation of form

$$< X >_t = \int_0^t \sigma^2 (X_s) ds$$

He further showed that behaviour of a function $f \in C^2$ is described by the rule

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)d < X > . \quad It\hat{o}'s\ Formula$$

Itô’s works also had impact on martingale theory. Along with his student Watanabe, he, unlike the additive decomposition under Doob-Meyer, studied the multiplicative decomposition of a positive supermartingale into product of a positive martingale and a positive decreasing process. The concept of local martingales was therefore introduced. Watanabe later contributed in development of unified theory of stochastic integrals in a framework of martingale theory.

Later, Itô reformulated stochastic calculus in terms of stochastic differentials and put the formula in a more convenient form. When Stratanovich and Fisk introduced the idea of symmetric stochastic integral, Itô noticed that it can be defined by modifying Itô integrals. If X and Y are continuous semi-martingales, the symmetric stochastic integral of X and Y, is, by definition, a continuous semi-martingale given by

$$Z = \left(\int_0^t X(s)\ o\ dY(s)\right)$$
$$= \int_0^t X(s)dY(s) + \frac{1}{2} < M^X, M^Y > (t)$$

The first term on RHS is Itô’s stochastic integral and the second term is the predictable quadratic co-variation of the martingale parts. Itô represented this in stochastic differential form as

$$X\ o\ dY = X\ dY + \frac{1}{2}dXdY. \qquad It\hat{o}'s\ \ Circle\ Operation$$

Now, under this operation, Itô’s formula has the same form as in ordinary differential calculus. This is, therefore, an indispensable tool for study of random motion on manifolds.
In the Itô-McKean theory, the local time of Brownian motion (i.e., the Brownian local time) plays a fundamental role. H. Tanaka, a Japanese mathematician, gave an alternate proof on its existence and its continuity on the space variable. Tanaka’s proof was basically an extension of Itô’s formula. Itô-Tanaka formula, as it is now known, extends Itô’s formula which deals with continuous semimartingales on $\mathcal{C}^2$-functions to functions that are differences of two convex functions in which the part of process of bounded variation can be expressed by an integral of local times.

M.Fukushima, another Japanese mathematician, further extended the Itô formula in his theory of Dirichlet forms and associated symmetric Markov processes and introduced a class of stochastic processed with zero energy. The semimartingale decomposition is now known as the Fukushima decomposition and is being used in path-theoretic studies in symmetric Markov processes

4. *Paul-André Meyer's 'Strasbourg School' of probability*

Paul-Andre Meyer, was a French mathematician is considered the father of 'Strasbourg' or "French" school of probability. Through his research papers and lectures organized under the name "Seminaire de Probabilities" in Strasbourg, Meyer along with his colleagues developed stochastic geometry, quantum probability most importantly stochastic integration. But Meyer is best known for his seminal work on the Doob- decomposition, where he developed upon continuous – time martingales and super-martingales later known as Doob-Meyer decomposition.

In these "Seminares" he discussed and extended upon the research done by other mathematicians in the area of probability and potential theories, some of these include below:

- Stochastic integrals and additive-functionals for Markov Processes – based on Kunita-Watanabe (1967)
- Along with C. Dellacherie, a book on general theory of stochastic processes
- Foundational in-equalities by D.L Burkholder in martingale theory
- Follmer's measure presented by Meyer in association to a super-martingale
- "Cours sur les Int´egrales stochastiques" – where he improved upon many aspects of general theory, mainly to develop theory of stochastic-integration in general semi-martingales. Also, this collection of research included the derivation of the Tanaka formula with respect to local times and existence in a general semi-martingale

The French school of Probability had notable contributions to the Potential and martingale theories; some of which are:

- Doob-Meyer decomposition first developed by JL.Doob, states that every cadlag submartingale, can be uniquely decomposed into a sum of a local martingale and an increasing predictable process.

$$X = M + A$$

Where X can be any local sub-martingale, which can be decomposed into M (local sub-martingale) and A (predictable increasing process starting from zero)

This was further developed into the Bichteler-Dellacherie theorem, this shows equivalence of two properties of semimartingales.

It states that if X is cadlag stochastic process, then following would be equivalent:

For every t>0, the set;

- ∫t0 ξdX : |ξ| ≤1 is simple predictable
- Decomposition X = M+V exists M is a local martingale and V is a finite variation process
- Decomposition X = M+V exists M is a local martingale which is uniformly bounded and V is finite variation process

- General theory of Process, further development of Doob's families of increasing sigma algebras by Dellacherie into dual projection of increasing process. These contributions were mainly what constitute stochastic calculus, enlargement of filtrations, martingale inequalities, etc. Generalization of convergence of martingales to amarts or asymptotic martingales both in discrete and continuous time. Further extension by Cairoli and Walsh into multidimensional time processes. F. Knight's prediction theory is linked to the general theory of processes.
- Probabilistic theory of diffusion or Malliavin Calculus: Paul Malliavin worked upon the notion of purely probabilistic method to establish in some stochastic differential equations with coefficient C^{-}, existence of very regular densities. Malliavin also introduced new tool to define flows of C^{-} diffeomorphisms in Ito's stochastic differential equations. Malliavin was key in developing stochastic differential geometry.
- Time reversals: Jacod and Protter worked on the time reversal of semi-martingales and expansion of filtrations to reversed process. They developed theorems related to reversible semi-martingales, reversal of stochastic integrals in case of reversible semi-martingales and additive functionals.
- Exit systems for Markov process: Bernard Maisonneuve developed and worked on Exit systems and excursions of Markov processes from the Borel Set
- Quantum probability: another important contribution of Meyer, was in the field of quantum probability and its relationship with martingales. He provided insight into properties of Azema's martingale, such as projection of real-valued Brownian motion. Quantum probability has wide spread applications in physics, information technology and biology. It was developed as a non-cumulative version of stochastic processes, using elements of classical probability theory and generalizing stochastic process within the framework of quantum probability. Seminal works in this field include Hudson's work on

Quantum Ito's formula in relation to Fock space, Quantum noise modelling by Nurdin etc.

- Marc Yor and Brownian Motion: Marc Yor's research dealt with Brownian motion, particularly in local time. He generalized Trotter's theorem on Brownian local times, worked on Brownian filtrations; published "Local times and Excursion theory of Brownian Motion" with Ju-Yi Yen. He pioneered study in Bessel process and bridges, obtained Levy-Khintchine measures in relation to squared Bessel processes. His work in filtration especially in his proof that martingale stopped at a random, is a semi-martingale with filtration progressively enlarged by this random time. He used many of these results in conjunction with world of finance, like his general model of credit default time, continuous diffusions for stock price and equity option modelling.

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