WORLDQUANT UNIVERSITY

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Submission 2 M5

Part A

1. ***Lookback options:***

Lookback options are a type of exotic option where the holder using historical data can determine when to exercise their option. This type of option is also called a hindsight option for the same reason.

The main idea behind lookback options is to consider the price of the underlying asset before the time of maturity and maximize gains by choosing a point where the difference between the strike price and underlying asset price was the highest.

Given the complicated nature of these contracts, they are not traded in exchanges but traded over the counter.

Lookback options come with major advantages:

- The biggest risk faced by traders - market timing - is practically solved. Since traders can choose a price at any time before the maturity date in hindsight to exercise the option, the risk of market timing is mitigated.

- Profits are guaranteed to be maximised due to the nature of lookback options. It is difficult to know and estimate how the price of the underlying asset would move but knowing the path of the price of the asset allows you to choose a point of maximum profit.

- The risk of the contract becoming worthless on expiry is also very low, much lower compared to other option contracts

  Given all these advantages, it's only right to price these contracts with a higher premium, which they are. Broadly, there are two types of Lookback options - Fixed Strike and Floating Strike:

- Like most other option contracts, Fixed Strike Lookback options have a fixed strike price. The difference being, the price of the underlying asset is not necessarily taken at maturity date but the holder can choose to exercise at any time during the contract where the price of the underlying asset was the highest or lowest depending on whether it was a call or put contract. Fixed Strike Lookbacks are usually cashed settled.

- The Floating Strike lookback option works with the same concept but only a little differently. Instead of fixing the strike price at the time of the contract, at maturity , the price is fixed to either the lowest or highest underlying asset price during the contract period depending on if it is a call or put contract.

- To illustrate the difference between the two, let us look at an example to make it clear. Suppose the strike of a lookback option is $100. During the contract period, the stock had a highest point of $110 and lowest point of $65 but closed at $80 at expiration.

  - For the fixed strike lookback , the highest price was $110 and the strike price was $100. Hence, the profit per option is 10$

  - For the floating strike lookback, the lowest price reached was $65 but the strike price is $80 which is set at maturity, hence the profit per option would be $15

In a sense, these two kinds of options are looking at two different aspects of the contract. By fixing a strike price, you are concerned only about the movement from the strike price. With the floating strike price, you have to estimate how much the contract will move from the price at maturity. That's why floating strike price lookbacks are harder to price.

- ***Pricing Lookback Options***

  For pricing we can use below mentioned formulas of black and Scholes:

  - For Fixed lookback call option

$$c = e^{-rT}(S_{max} - X) + Se^{(b-r)T}N(a_1) - S_{max}e^{-rT}N(a_2) + Se^{-rT}\frac{\sigma^2}{2b}\left[-\left(\frac{S}{S_{max}}\right)^{-\frac{2b}{\sigma^2}} * N\left(a_1 - \frac{2b}{\sigma}\sqrt{T}\right) + e^{bT}N(a1)\right]$$

$$a_1 = \frac{\ln\left(\frac{S}{S_{max}}\right) + \left(b + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T}$$

  - For Floating lookback call option

$$c = Se^{(b-r)T}N(a_1) - S_{min}e^{-rT}N(a_2)$$

$$+Se^{-rT}\frac{\sigma^2}{2b}\left[\left(\frac{S}{S_{min}}\right)^{-\frac{2b}{\sigma^2}} * N\left(-a_1+\frac{2b}{\sigma}\sqrt{T}\right) - e^{bT}N(-a1)\right]$$

$$a_1 = \frac{\ln\left(\frac{S}{S_{min}}\right)+\left(b+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T}$$

- ***Strategies***

Given the nature of lookback options, the greater the volatililty in the market, greater would be the profit generated by lookback options. They are effective in hedging against large price movements and reducing the risk of black swan events such as the Financial Crisis of 2008 ,pandemics ,election results and market anamolies.

Since the volatility plays a major role in lookback options, they can be potentially used to reduce and hegde exposure in cryptocurrency markets where extreme volatility is a norm.

However, the considerable advantage of these options also comes with a high premium that decreases the demand of such contracts. They can be susceptible to become illiquid at times.

2. Barrier options

Barrier options are exotic options which are similar vanilla options with an additional feature of “barrier”. The payoff of such options is based on the contingency event of price reaching or exceeding the barrier level. Barriers can be either Knock-in or Knock-out in nature. In Knock-In options, payoff activates when price of the underlying reaches the barrier level (otherwise remain dormant) while in Knock-out options are active till the price reaches the barrier level, post which they get terminated. Barrier options can be further classified based on the direction of barrier into Up and Down barrier options, indicating the barrier level with respect to the current market price. Another type of barrier option is double knockout options, which are a combination of up & out and down & out options. This means that the option is active between a range of prices otherwise gets knocked out.

Similar to the Put-Call parity to vanilla options, barrier options have In-Out parity; implying the combination of a Knock-In and Knock-Out option would be equal to a plain vanilla option.

Compared to plain vanilla options, barrier options are cheaper; this allows the investor to leverage on the market volatility and again exposure. Due to the customizable nature of barrier options, they are most commonly used in Equity Structured Products; they allow specific yield and stop-loss strategies customized to the need of the investor. Barrier options are also used to hedge FX risks at lower costs compared to participation forwards or futures; and also allow exposure to underlying FX pair with an effective stop-loss using a combination of vanilla and barrier options. Example, if an investor's outlook towards EUR/JPY pair is positive and expects to increase; investor can buy a 128 call with knock-outs at 135 and 125; these limits both upside and down side. This strategy is cheaper than buying a vanilla bull call spread on FX pair. Instead of short straddle a neutral option strategy, trader can buy a double knockout option which will have lower premium compared to neutral option strategies using vanilla options.

The monitoring of barrier can also be customized to continuous, or on expiration. Types of options based on barrier type and direction are below:

- Up and In: Barrier level is higher than market price, option payoff kicks in when price increase above the barrier level. The price Up & In call option is given by below equation:

$$e^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T > B\}} | \mathcal{F}_t\right]$$

- Up and out: Option gets knocked out if the price moves above the barrier level. The option can be worthless or a pre-defined rebate payment can be made if the option gets knocked-out. The price of Up & out call option is given by below equation:

$$e^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{M_0^T < B\}} | \mathcal{F}_t\right]$$

- Down and In: Price of the underlying needs to move lower than the barrier for the option payoff to knock-in. Price of Down and In put option is given by below equation:

$$e^{-(T-t)r}\,\mathbb{E}^*\left[(K - S_T)^+ \mathbb{1}_{\{m_t^T < B\}} | \mathcal{F}_t\right]$$

- Down and Out: If the price of the underlying moves lower than the barrier level, the trades gets knocked-out worthless or a rebate payment is done based on the terms. Price of a down and out put option is given by below equation:

$$e^{-(T-t)r}\,\mathbb{E}^*\left[(K - S_T)^+ \mathbb{1}_{\{m_0^T > B\}} | \mathcal{F}_t\right]$$

Prices of other variations can be obtained similarly using the In-Out and Put-Call parities. Barrier options have been further engineered to get required exposure and payoffs, like Reverse barrier rand Edokko options; for barrier monitoring like Partial Time Barrier and Parisian options.

3. *Compound options*

Compound options are exotic options which are option on an option. These options give the holder the right but not obligation to buy or sell the underlying call or put option. Compared to the plain vanilla options, these options are highly sensitive to market changes. This can be further engineered by adding barrier feature, example a knock-out barrier on compound option; where trade would get knock-out based on the price of the underlying option.

Example for hedging or use of compound options
Compound option can be classified mainly into 4 types based on the combination of call or put nature:

- Call on Call: the price of CoC or Call on Call can be derived from below equation:

  $$Call_{call} = Se^{-DT_2}M(a_1, b_1; \rho) - X_2e^{-rT_2}M(a_2, b_2; \rho) - e^{-rT_1}X_1N(a_2)$$

- Call on put: the price of CoP or Call on Put can be derived from below equation:

  $$Call_{put} = Se^{-DT_2}M(a_1, -b_1; -\rho) - X_2e^{-rT_2}M(a_2, -b_2; -\rho) + e^{-rT_1}X_1N(a_2)$$

- Put on Call: the price of PoC or Put on Call can be derived from below equation:

  $$Put_{call} = X_2e^{-rT_2}M(-a_2, b_2; -\rho) - Se^{-DT_2}M(-a_1, b_1; -\rho) + e^{-rT_1}X_1N(-a_2)$$

- Put on Put: the price of PoP or Put on Put can be derived from the below equation:

  $$Put_{put} = X_2e^{-rT_2}M(-a_2, -b_2; \rho) - Se^{-DT_2}M(-a_1, -b_1; \rho) - e^{-rT_1}X_1N(-a_2)$$

Where;

X_1 and X_2 are exercise prices of underlying option and compound option
T_1 and T_2 are expiry of underlying and compound option

While

$$a_1 = \frac{-Ln\left(\frac{S}{S^*}\right) + (r - D + 0.5\sigma^2)T_1}{\sigma\sqrt{T_1}} \; ; a_2 = a_1 - \sigma\sqrt{T}$$

$$b_1 = \frac{Ln\left(\frac{S-De^{-r(t1-t)}}{S*}\right)+(r+0.5\sigma^2)(t_1-t)}{\sigma\sqrt{(t_1-t)}}; \, b_2 = b_1 - \sigma\sqrt{T_1}$$

M is two-dimensional cumulative normal distribution function and ρ correlation coefficient which is the correlation coefficient of overlapping Brownian increments.

$$\rho = \sqrt{\frac{T2}{T1}}$$

S* is the critical stock price for the chooser option, based on (Rubinstein,1992), which holds below criteria:

$$Call_{European}(S^*, X_1, D, r, V, T_2 - T_1) = X_2$$

This can be replicated by combination of call and put option.

Compound options can also be classified based on creation of the underlying option into simultaneous and Sequential (only when the first option is exercised). This allows the use of compound options in real option scenarios like capital budgeting, R&D investments, etc. For example, investments in clinical trials for a pharmaceutical company; investment requirements are based on contingency of certain phase to go through. This allows the company to maximize net project value based on the success or failure of clinical phases.

- ***Application:***

Compound Options are used commonly in fixed income and currency markets where the option's risk capabilities are uncertain. They allow for larger leverage than normal options as they are initially cheaper. They become costlier only if both the options in the compound option are exercised.

They are also used in the mortgage market to hedge the risk of interest rate changes. The changes between the rates at the time of mortgage commitment and schedule date can be offset useing Call-Put options.

Speculation in the financial markets is a major chunk of the application of compound options but they can also be used by large businesses to finance their supplies before a business venture. If they do not build or go ahead with the project, as an insurance policy, compound options work great. Let us take a closer look at how this would work.

Let's say a total amount of X million dollars is required by the business for the next Y years. This number would have been calculated with the current interest rate values and pitched to the investors. The company's risk would be the possible difference in interest rate between the time of bid and the time of agreement to the project commencement. For one, they could buy a Y year interest cap to begin with, but that would be very expensive if they do not get the project.

So what do they do? They buy a call on the on Y year interest cap. If the project is handed to them then they can exercise the option at a premium. However, if they do not require it, then they can let the option expire and the only expenses would be the premium amount which is much lesser than the amount they would have lost otherwise.

Part B

4. *Derivation of pricing formulas used for hindsight options:*

- ***Fixed Strike call:***

In the fixed strike lookback call case, the holder has the option to exercise at any historical price point (achieved in the holding period). Obviously, for a call option user will prefer, if at all, to exercise at the highest price point. Formally, we can write the price of such a call option (at maturity) as:

$$LC_{fix} = \max(S_{max} - K, 0)$$

Where, S_{max} = highest price point of stock in holding period
K = fixed strike price

Now, if we are operating in a risk-neutral regime, the price process of our asset will follow:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

$$=> \quad dU_t = d(\ln\frac{S_t}{S_0}) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

Let’s define 2 stochastic variables for minimum and maximum movement of underlying asset

$$y_t = \ln\frac{m_t^T}{S_t} = \min\{U_\alpha, \alpha \in [t,T]\}$$

and,

$$Y_t = \ln\frac{M_t^T}{S_t} = \max\{U_\alpha, \alpha \in [t,T]\}$$

For $y <= 0$ and $y <= u$, the joint distribution function of U_T and y_T can be calculated from transition density function of Brownian Motion considering a downstream barrier as

$$P[U_T \geq u, y_T \geq y] = N\left(\frac{-u + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N(\frac{-u + 2y + \mu\tau}{\sigma\sqrt{\tau}})$$

Similarly, the joint distribution under an upstream barrier can be written as:

$$P[U_T \leq u, y_T \leq y] = N\left(\frac{u - \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N(\frac{u - 2y - \mu\tau}{\sigma\sqrt{\tau}})$$

Combining the above 2 at $y = u$,

$$P(y_T \geq y) = N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right) \quad , y \leq 0$$

$$P(Y_T \leq y) = N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right) \quad , y \geq 0$$

Returning back to our lookback option, the value of our fixed strike (K) call option at any time 't' is:

$$\begin{aligned} LC_{fix}(S, M, t) &= e^{-r\tau}\mathrm{E}[\max(S_{max} - K, 0)] \\ &= e^{-r\tau}\mathrm{E}[\max(\max(M, M_t^T) - K, 0)] \end{aligned}$$

i. $M <= K$

$$\max(\max(M, M_t^T) - K, 0) = \max(M_t^T, 0)$$

ii. $M > K$

$$\max(\max(M, M_t^T) - K, 0) = (M - X) + \max(M_t^T - M, 0)$$

We know that expectation of a non-negative random variable, X, is given by:

$$E[X] = \int_0^\infty [1 - F_X(t)]dt$$

In our case, $\max(M_t^T - K, 0)$ is a non-negative rv. We can calculate the expected value, which will be our call price, as follows:

$$\begin{aligned} LC_{fix}(S, M, t) &= e^{-r\tau}E[\max(M_t^T - K, 0)] \\ &= e^{-r\tau}\int_0^\infty P[M_t^T - K \geq x]\,dx \\ &= e^{-r\tau}\int_0^\infty P[\ln(\frac{M_t^T}{S}) \geq \ln(\frac{z}{S})]\,dz \qquad \because z = x + K \end{aligned}$$

Let $y = \ln(\frac{z}{S})$,

$$\begin{aligned} &= e^{-r\tau}\int_{ln\frac{K}{S}}^\infty Se^y P[Y_T \geq y]dy \\ &= e^{-r\tau}\int_{ln\frac{K}{S}}^\infty Se^y \left[N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}}N(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}})\right]dy \end{aligned}$$

This reduces to

$$LC_{fix}(S, M, \tau) = SN(d_1) - e^{-r\tau}KN(d_1 - \sigma\sqrt{\tau}) + e^{-r\tau}\frac{\sigma^2}{2r}S\left[e^{r\tau}N(d_1) - \left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N\left(d_1 - \frac{2r}{\sigma}\sqrt{\tau}\right)\right]dy$$

Where,

$$d_1 = \frac{ln\frac{S}{K} + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

The above formula is derived for the case when $M <= K$. For the other case, $M > K$, the formula becomes:

$$LC_{fix}(S,M,\tau) = e^{-r\tau}(M-K) + SN(d_1) - e^{-r\tau}KN\left(d_1 - \sigma\sqrt{\tau}\right) + e^{-r\tau}\frac{\sigma^2}{2r}S\left[e^{r\tau}N(d_1) - \left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N\left(d_1 - \frac{2r}{\sigma}\sqrt{\tau}\right)\right]dy$$

$\therefore proved$

- Floating Strike call:

In the floating strike lookback call case, the holder has the option to exercise at current stock price but the strike price can be chosen as any historical price (achieved in the holding period). Obviously, for a call option user will prefer, if at all, to exercise at the lowest strike price. Formally, we can write the price of such a call option (at maturity) as:

$$LC_{floating} = \max(S_T - \min(m, m_t^T), 0)$$

Where, S_T = current price of stock

Proceeding as in the case of fixed strike lookback call, we can write

$$\begin{aligned} LC_{floating} &= e^{-r\tau}\mathrm{E}[(S_T - \mathrm{m}) + \max(m - m_t^T, 0)] \\ &= S - me^{-r\tau} + LC_{floating}(S, m, \tau) \end{aligned}$$

This reduces to

$$LC_{floating} = SN(d_1) - e^{-r\tau}mN\left(d_1 - \sigma\sqrt{\tau}\right) + e^{-r\tau}\frac{\sigma^2}{2r}S\left[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N\left(-d_1 + \frac{2r}{\sigma}\sqrt{\tau}\right) - e^{r\tau}N(-d_1)\right]$$

Where,

$$d_1 = \frac{ln\frac{S}{m} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

$\therefore proved$

5. References

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