WORLDQUANT UNIVERSITY

Submission Number: 3 M7

Group Number: 5-D

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Submission 3 M7

Two-Factor Hull-White Model – Part a

1. Two-Factor Hull-White Model in mathematical terms:

From a general Hull-White one factor model, we know that a risk neutral process for short rate, r, is given by:

$$df(r) = [\theta(t) + u - af(r)]dt + \sigma_1 dz_1 \tag{1}$$

Where, stochastic variable u is a component of reversion level of r and has initial value of 0 and follows the process

$$du = -budt + \sigma_2 dz_2 \tag{2}$$

Where, dz_1 and dz_2 are Wiener processes with instantaneous correlations ρ.

When $f(r) = r$, the price at time *t,* of a zero-coupon bond that provides a payoff of $1 at time *T,* is given by:

$$P(t,T) = A(t,T)e^{-B(t,t)r-C(t,T)u} \tag{3}$$

Where,

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$and,$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

The $A(t,T)$ function and associated variables are explained as follows:

$$lnA(t,T) = ln\frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \eta \qquad (4)$$

Where,

$$\eta = \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2 - \rho\sigma_1\sigma_2[B(0,t)C(0,t)B(t,T) + \gamma_4 - \gamma_2] \\ - \frac{1}{2}\sigma_2^2[C(0,t)^2B(t,T) + \gamma_6 - \gamma_5]$$

$$\gamma_1 = \frac{e^{-(a+b)T}(e^{(a+b)t} - 1)}{(a+b)(a-b)} - \frac{e^{-2aT}(e^{2at} - 1)}{2a(a-b)}$$

$$\gamma_2 = \frac{1}{ab}(\gamma_1 + C(t,T) - C(0,T) + \frac{1}{2}B(t,T)^2 - \frac{1}{2}B(0,T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2})$$

$$\gamma_3 = -\frac{e^{-(a+b)t} - 1}{(a+b)(a-b)} + \frac{e^{-2aT} - 1}{2a(a-b)}$$

$$\gamma_4 = \frac{1}{ab}(\gamma_3 - C(0,t) - \frac{1}{2}B(0,t)^2 + \frac{t}{a} + \frac{e^{-at} - 1}{a^2}$$

$$\gamma_5 = \frac{1}{b}[\frac{1}{2}C(t,T)^2 - \frac{1}{2}C(0,T)^2 + \gamma_2]$$

$$\gamma_6 = \frac{1}{b}[\gamma_4 - \frac{1}{2}C(0,t)^2]$$

Where, $F(t,T)$ is the instantaneous forward rate at time *t* for maturity *T*.

The volatility function, σ_P, is

$$\sigma_P^2 = \int_0^t \{\sigma_1^2[B(\tau,T) - B(\tau,t)]^2 + \sigma_2^2[C(\tau,T) - C(\tau,t)]^2 \\ + 2\rho\sigma_1\sigma_2[B(\tau,T) - B(\tau,t)][C(\tau,T) - C(\tau,t)]\}d\tau \qquad (5)$$

Finally, the θ(t) is,

$$\theta(t) = F_t(0,t) + aF(0,t) + \phi_t(0,t) + a\phi(0,t)\phi \qquad (6)$$

Where the subscript denotes a partial derivate and,

$$\phi(t,T) = \frac{1}{2}\sigma_1^2 B(t,T)^2 + \frac{1}{2}\sigma_2^2 C(t,T)^2 + \rho\sigma_1\sigma_2 B(t,T)C(t,T)$$

Using the 2-factor Hull-White model, the prices of a European call and put on a zero-coupon bond are given by

$$c = XP(0,s)N(h) - KP(0,T)N(h - \sigma_P)$$

$$p = KP(0,T)N(-h - \sigma_P) - XP(0,s)N(-h)$$

Where, T is maturity of option, s is maturity of Bond, K is strike price, X is Bond's principal

$$h = \frac{1}{\sigma_P} Ln \frac{XP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2}$$

Model Comparison – Part b

2. Model comparison with Hull-white 2 factor model:

- ***THE HO-LEE MODEL***

  The Ho-Lee Model was the first no arbitrage model for term structure that was proposed in 1986. The model was constructed in the form a binomial tree of bond prices with two related parameters - Short rate Standard Deviation and Market Price of Risk of this short rate.

  The interest rate is given by the Stochastic Equation is given by:

  dr = ϴ (t) dt + σ dz, where s is the instantaneous standard deviation of the short rate and is constant. *ϴ(t)* is chosen to ensure that the model fits the initial term structure and is a function of time. This variable *ϴ(t)* defines the average direction that the rate r moves at time t and is independent of the value of r . The market price of risk considered in this model is irrelevant while pricing interest rate derivatives.

$\Theta(t) = F_t(0,t) + \sigma^2 t$, where $F_t(0,t)$ is the instantaneous forward rate for a maturity t as seen at time zero partially differentiated with respect to t. If approximated, this rate is equal to $F_t(0,t)$.

This means that the slope of the instantaneous forward curve will show the average direction of the movement of the short rate in the future.

- ***THE VASICEK MODEL***

The Vasicek model is given by the following stochastic equation

$dr = (\Theta - ar)\,dt + \sigma\,dz$, where Θ , a and σ are constants and dz is a Wiener process that models the continuous inflow of randomness. It is a process under the risk neutral framework modelling of the random risk factor. The dynamics are characterized by three factors:

- Θ - Long term mean level. All future values of r will revolve around a mean level of Θ as t tends to infinity
- a - Speed of reversion. The rate (velocity) at which these trajectories will regroup around Θ
- σ - Instantaneous volatility. Higher randomness is characterized by a higher value of σ

Since the rate r is normally distributed, the value can be negative as well. The short rate is also mean reverting with a long term mean of Θ / a

- ***THE COX-INGRESOLL-ROSS MODEL***

The Cox - Ingresoll - Ross model, describes the short rate with only one factor - market risk. The standard deviation factor introduced avoids the possibility of negative interest rates which is an improvement on the Vasicek Model. It is characterized by the following stochastic equation

$dr = (\Theta - ar)dt + \sigma \sqrt{r}\, dz$

If 2ϴ is greater than or equal to σ 2, the short rate never reaches zero. The distribution of the short rate in this model is characterized by a non-central chi squared distribution.

- ***THE HULL-WHITE ONE FACTOR MODEL***

The Hull White One factor model was an extended version of the already established Vasicek model that provided an exact fit to the initial term structure. The short rate is given by

$dr = (\Theta(t) - a(t)r(t))dt + \sigma\, dz$, where a and σ are constants.

This basically means that this model is basically a Ho - Lee model with mean reversion at rate a. Also, the Ho - Lee model is a particular case of the Hull White model with a = 0. It is also a Vasicek model with a time dependent reversion level. At time t, the short rate reverts to ϴ(t)/a at rate a.

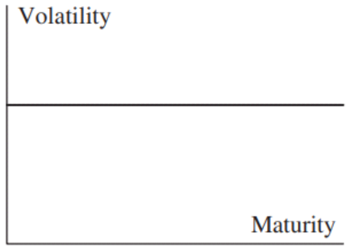
So in a sense, the Hull White model is a generalization of the previously formulated models with small variations.

- ***THE HULL-WHITE TWO FACTOR MODEL***

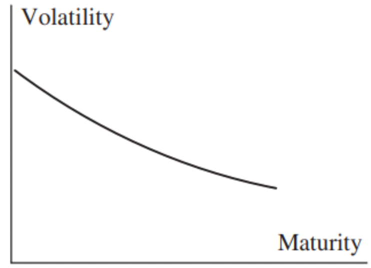
As mentioned previously, the two factor model involves an additional stochastic variable that is a component of the mean reversion level and reverts to zero at a particular rate.

This model provides a richer pattern of term structure movements and a definitive pattern of volatilities than one factor models of short rates.

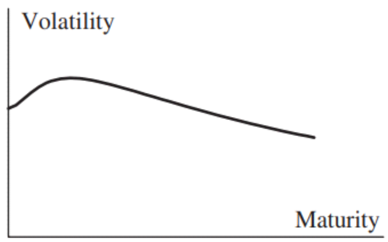
- Volatility Structures -
  a) The Ho - Lee volatility of a 3 month forward rate is the same for all maturities.
  b) The Hull - White model volatility of the 3 month forward rate is a declining function of maturity. This is the effect of the mean reversion.
  c) The Hull - White two factor model volatility for the same forward rate has a humped look when the parameters are chosen appropriately. This is consistent with the empirical evidence and exhibits the actual behavior of the market.



(a)



(b)



(c)

- All these models model the short rate directly and not the forward rate. They are all based on normal distribution of these rates. Since the development of the models happened chronologically, we can pinpoint the developments and additions to each model. To summarize:

  a) The Vasicek model is an improvement on the Ho - Lee model as it adds the mean reversion property, speed of reversion factors.
  b) The CIR model removes the shortcoming of the Vasicek model of possible negative interest rate by adding a standard deviation factor.
  c) The Hull-White model improves upon this to add a time varied mean - reversion factor to better fit the exact initial term structure
  d) The Hull-White two factor model provides a more defined pattern of term structure movements and a more empirical form of volatility measure - the humped look - which is similar to the actual market behavior

- To summarize the various properties of the models explained, the results have been tabulated below

Model	Negative Rates Allowed	Mean Reversion Factor Present	Level of Mean Reversion	Time Varying Volatility
Ho-Lee	Yes	No	N/A	No
Vasicek	Yes	Yes	Static ϴ	No
C I R	No	Yes	Static ϴ	Yes
Hull – White	No (Yes, with calibration)	Yes	Varying	Yes
Hull – White two factor	No (Yes , with calibration)	Yes	Varying	Yes

Derivative Product Payoff – Part c

3. Swaption derivation priced by Hull-white 2 factor model:

Swaption is a derivative which gives the holder the option to enter into interest rate swap or any other type of swap with the counterparty. In other words, a swaption is an option on a swap. Based on type of direction of fixed rate payments, the swaptions are of two types: a) Payer Swaption and b) Receiver Swaption. The payoffs are determined by the level of underlying interest rates.

- ***Payer Swaption:***

  In Payer Swaption, the holder(buyer) has the right to enter into an Interest rate swap, where the holder pays FIXED rate and receives Floating rate payments from the counterparty. Buyer can benefit from increase in interest rates and can lose maximum of swaption premium.

$$P_{Swpt}(t, T_0, T_n, K) = B(t, T_n)\Phi(d_1) - P(t, T_0)\Phi(d_2)$$

- ***Receiver Swaption:***

In Receiver Swaption, the holder(buyer) has the right to enter into an interest rate swap, where the holder pays FLOATING rate and receives Fixed rate payments from the counterparty. Buyer will benefit when interest rates fall.

$$R_{Swpt}(t, T_0, T_n, K) = P(t, T_0)\Phi(-d_2) - B(t, T_n)\Phi(-d_1)$$

Φ Denotes the cumulative distribution function of the standard Gaussian distribution with

$$d_1 = \frac{ln\left[\frac{B(t, T_n)}{P(t, T_0)}\right] + \frac{1}{2}\Sigma_B^2(t, T_0, T_n)}{\Sigma_B(t, T_0, T_n)} \qquad ----------------(1)$$

$and;$

$$d_2 = \frac{ln\left[\frac{B(t, T_n)}{P(t, T_0)}\right] - \frac{1}{2}\Sigma_B^2(t, T_0, T_n)}{\Sigma_B(t, T_0, T_n)} \qquad ----------------(2)$$

- In equation (1) and (2) above for d_1 , d_2 *ln* of the coupon bond forward price is taken. This price is denoted by $B(t, T_0, T_n) = \frac{B(t,T_n)}{P(t,T_0)}$; the ratio of the coupon bond price and the zero coupon bond price is a martingale. The coupon bond forward price is also a martingale under the T-forward measure.

- The variance of the coupon bond price under the T-forward risk adjusted measure is denoted by $\Sigma_B^2(t, T_0, T_n) = \int_t^{T_0} \sigma_B(u, T_0, T_n)^2 du$

- and the volatility of the coupon bond price under the T-forward risk adjusted measure is denoted by $\Sigma_B(t, T_0, T_n) = \sqrt{\int_t^{T_0} \sigma_B(u, T_0, T_n)^2 du}$

- Interest rates in the above equation for bonds evolve according to the 2-factor hull-white process where:

$$\sigma_B(t,T_0,T_n)^2 = individual\ variance + 2 * correlation * covariance$$
$$= \sigma_x^2 D_{B_x}(t,T_0,T_n)^2 + \sigma_y^2 D_{B_y}(t,T_0,T_n)^2$$
$$+ 2\rho\sigma_x\sigma_y D_{B_x}(t,T_0,T_n)\, D_{B_y}(t,T_0,T_n)$$

- The quantities $D_{B_x}(t,T_0,T_n)\ and\ D_{B_y}(t,T_0,T_n)$ are defined as forward stochastic durations of the coupon bond with respect to the factors x and y respectively.
- $D_{B_x}(t,T_0,T_n)$ is computed by $D_{B_x}(t,T_n) - D_{P_x}(t,T_0)$

$$= \frac{\sum_{i=1}^{n} C_i P(t,T_i)[H_x(t,T_i) - H_x(t,T_0)]}{\sum_{i=1}^{n} C_i P(t,T_i)}$$

- $D_{B_y}(t,T_0,T_n)$ is computed by $D_{B_y}(t,T_n) - D_{P_y}(t,T_0)$

$$= \frac{\sum_{i=1}^{n} C_i P(t,T_i)\left[H_y(t,T_i) - H_y(t,T_0)\right]}{\sum_{i=1}^{n} C_i P(t,T_i)}$$

- here,

$$H_x(t,T) = \frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial x(t)} = D_{P_x}(t,T)\quad and$$

$$H_y(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial y(t)} = D_{P_y}(t,T)$$

The quantities $D_{P_x}(t,T)\ and\ D_{P_y}(t,T)$ are stochastic durations of the zero coupon bond with respect to the factors x and y respectively.

- P^M (0,t) is market price of a zero coupon bond at t=0, at maturity t; P_{XY}(t,T)
- Above method can be used for coupon bond options and swaptions as payoff is similar

4. References

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