MScFE 620: Discrete-time Stochastic Processes

Group Work Project

Submission 2

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Constructing a Concrete Trinomial Tree Model

The trinomial tree is an evolution of the Binomial tree, with the difference between them being that a Binomial tree allows two possible outcomes whereas a Trinomial tree allows three. Binomial models were extremely popular in modeling financial options, but this method became outdated due to it only allowing two outcomes. The three outcomes for a trinomial tree are either a movement up, down, or static. An example of a trinomial tree is:

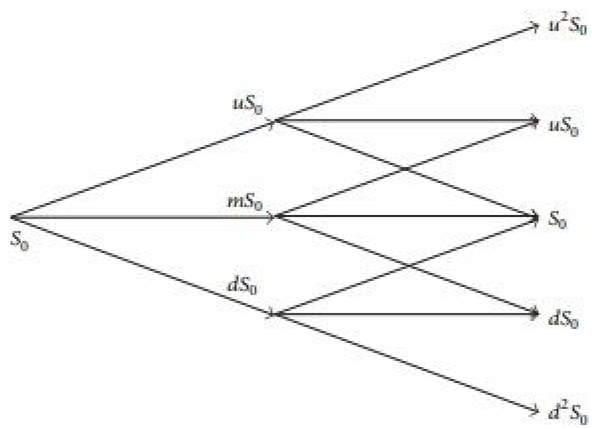

Fig: Trinomial tree [2]

This trinomial tree is slightly simplified as any one of the nodes at t=2 can be achieved by taking one, two, or three different paths. This is due to the up movement being a reciprocal to the down movement, i.e. u = 1/d. As a result, the calculations are simplified.
Let the jump be 'u', 'd', and 'm', with the corresponding probabilities of 'α','β', and 'γ'. The probabilities can also be represented as pu, $1 - pu - pd$ and pd.
So we get;

$$X_{t+1} = \begin{cases} u * X_t & \text{with probability } pu \\ m * X_t & \text{with probability } 1 - pu - pd \\ d * X_t & \text{with probability } pd \end{cases}$$

Now, let's construct a trinomial tree that assumes the evolution of a risk stock price X is,

$$X_{n+1} = X_n Z_{n+1} \ , X_0 = constant$$

Where $(Z_n)^T$ is a sequence of independent random variables each taking three distinct values (u, m and d). Below is how one would construct a concrete trinomial tree model for X with T = 2 with one risky stock. First of all, the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is defined as

$$\Omega = \{ uu, um, ud, mu, mm, md, du, dm, dd \}$$

$$\mathbb{F}^X = \{ \mathcal{F}_t^X : t \in \mathbb{I}\} \ where \ \mathbb{I} = \{0,1,2\}$$

Where 'u' defines the jump up in stock price, 'm' is 1 meaning the stock price is static from one-time period to the next and 'd' is the reduction in the stock price. The three possible paths for this trinomial tree are:

$$X_{t+1} = \left\{ \begin{array}{ll} u * X_t & with\ probability\ \alpha \\ m * X_t & with\ probability\ \beta \\ d * X_t & with\ probability\ \gamma \end{array} \right\}$$

The filtrations are,

$\mathcal{F}_0^X = \sigma(\{X_0\}) = \{\varnothing, \Omega\}$

$\mathcal{F}_1^X = \sigma(\{X_0, X_1\}) = \{\varnothing, \Omega, \{uu, um, ud\}, \{mu, mm, md\}, \{du, dm, dd\}\}$

$\mathcal{F}_2^X = \sigma(\{X_0 , X_1 , X_2 \}) = 2^\Omega$

Arbitrage and Finding Equivalent Martingale Measures

Arbitrage is the ability for a market participant to make a risk-free profit. This can be done by the simultaneous purchase and sale of an asset to profit from an imbalance in the price [3]. When used by academics, an arbitrage transaction is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state [4]. More explicitly an arbitrage strategy φ satisfies the following conditions:

(a) $V_0(\varphi) = 0$

(b) $V_T(\varphi) \geq 0$ $with\ probability$ 1

(c) $\mathbb{P}(V_T(\varphi) > 0) > 0$ [5]

Taking into consideration the above definition of arbitrage, there are conditions that need to be imposed on the trinomial model defined in the previous section. They are that the performance of the stock price can't be positive for all outcomes, therefore the down movement must depreciate the price to enforce a negative return. The conditions which can be imposed on the trinomial model to ensure an arbitrage free strategy are:

$$X_{t+1} < d * X_t \Rightarrow d < 1$$

$$X_{t+1} > u * X_t \Rightarrow u > 1$$

Which can be re-written as

$$1 - \alpha - \gamma + \alpha u + \gamma d = 1 \qquad [6]$$

Applying the Fundamental Theorem of Asset Pricing I (FTAP I) one could also prove the trinomial model has no arbitrage strategies. The FTAP I states that the following are equivalent:

- The market has no arbitrage strategies
- The set P, of EMMs for X is non-empty

In order to prove an arbitrage free strategy, one needs to find at least a single element of the set $\mathscr{P}$. An EMM $\mathbb{P}^*$ is characterized by $\mathbb{P}^*(u) = \alpha$, $\mathbb{P}^*(m) = \beta$, $\mathbb{P}^*(d) = \gamma$ as well as:

$$\alpha, \beta, \gamma > 1 \text{ and}$$
$$\alpha * \alpha + \alpha * \beta + \alpha * \gamma + \beta * \alpha + \beta * \beta + \beta * \gamma + \gamma * \alpha + \gamma * \beta + \gamma * \gamma = 1$$
$$\alpha + \beta + \gamma = 1$$

Assuming that $m = 1$, $0 < d < u$.
By using the martingale property on X for time t=1:

$$\alpha u + \beta + \gamma d = 1$$

And substituting $\beta = 1 - \alpha - \gamma$ into the above equation yields:

$$\alpha u + 1 - \alpha - \gamma + \gamma d = 1$$
$$\alpha(u - 1) - \gamma(1 - d) = 0$$
$$\alpha = \gamma * (1 - d)/(u - 1)$$

α and γ are greater than 0 and this can't hold: 1 – d < 0 because d < 1. That means that:

$$(1 - d) > 0 \Rightarrow d < 1$$
$$(u - 1) > 0 \Rightarrow u > 1 \Rightarrow 0 < d < 1 < u$$

Let,

$$\gamma = p \Rightarrow \alpha = p * (1 - d)/ (u - 1)$$

Now substituting α and γ into $\alpha + \beta + \gamma = 1$,

$$p * (1 - d)/ (u - 1) + \beta + p = 1$$
$$\Rightarrow \beta = 1 - p + p * (d - 1)/ (u - 1)$$
$$\Rightarrow \beta = 1 + p*(d - u)/(u - 1)$$

Therefore, the set of all EMMs is,
$\mathscr{P} = \{\mathbb{P}^*(p * (1 - d)/ (u - 1), 1 + p*(d - u)/(u - 1), p): 0 < d < 1 < u, 0 < p < 1\}$

For a market to be complete the set of all EMMs should have only one element. This is inferred from the Fundamental Theorem of Asset Pricing (II) that states:
For a financial market $((\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P}), X)$ with no arbitrage opportunities, the following are equivalent:

1) The market is complete
2) $\mathscr{P}$ has exactly one element
3) X has the predictable representation property with respect to every $\mathbb{P}^* \in \mathscr{P}$: every $(\mathbb{F}, \mathbb{P}^*)$-martingale M with $M_0 = 0$ can be written as a martingale transform with respect to X_t [5]

This implies that if $|\mathscr{P}|=1$ then the market is complete. A possibility of making a market of a trinomial model complete is to consider embedded complete market models inside the incomplete market model.
In our case we could assume values for p, u and d, this would be enough to have only one element in set $\mathscr{P}$.

References

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[5] Worldquant University Discrete-time Stochastic Processes Module 4 Notes

[6] Paul Clifford, Yan Wang, Oleg Zaboronski, Kevin Zhang (November 2010). ‘Pricing Options Using Trinomial Trees’

[7] Takahashi, H. (2000), A Note on Pricing Derivatives in an Incomplete Markets