MScFE 620: Discrete-time Stochastic Processes

Group Work Project

Submission 1

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Task : The History of Measure-theoretic Probability and Martingales. Early history of probability theory from a measure theoretic perspective. Your discussion must address the following:

1. Kolmogorov and his axioms of probability

2. Markov and Markov processes

3. J. L. Doob and the development of martingales

1.0 : Brief History of Probability

The concept of probability has always been something famous philosopher's, mathematicians and other great minds have pondered upon. The origins can trace back to Plato and Aristotle, who used to discuss the definition of "chance" philosophically. The concept of probability arose due to the chance of winning or losing a game from a gambling perspective. Gerolamo Cardano, a mathematician,physician and gambler , came up with this concept.

The journey of advancements in the realm of probability can be followed on the path of Jacob Bernoulli ,who used the property of binomial coefficients to come up with the probability of r events from a total of n events, Laplace and Bayes . The Bayes theorem was particularly important in the advancement of the concept of conditional probability.

2.1.1: Kolmogorov's Axioms

Finally, this journey brings us to the most fundamental and influential 20th century theory of probability set by A. N Kologomorov. Kolmogorov created the modern axiomatic foundations in probability theory published in his book *Foundations of Probability* in 1933. A brief overview of the axioms is as follows -

Probability measure is defined on field *F* (subset) of events of the sample space *S* . It is the mapping of the events in *F* to points on the real line between 0 and 1 and satisfy the following *axioms* or *rules* for all events belonging to the set *F* .

1. For any event E that belongs to set F, The probability of an event is a non-negative real number. This follows that P(E) is always finite, in contrast with more general measure theory

2. Assumption of unit measure - The probability that at least one of the elementary events in the entire sample space will occur is 1. This means that P(S) = 1

3. The assumption of sigma additivity - For any countable sequence of disjoint sets , the total probability is the sum of all the probabilities.

2.1.2: Significance

One may wonder why the axioms of Kolmogorov are so important. The work that was done taking these principles as a fundamental truth helped in proving very important and now commonly known probability properties. For example, from these axioms it is easy to derive

1. Monotonicity - If A is a subset of B , then P(A) <= P(B)

2. Probability of an empty set - The probability of an empty set is 0

3. The complement rule - Probability of event A *not occurring* P(A') = 1 - P(A)

These are just a few examples of fundamental derivations. All probability proofs have been based on these axioms or an extension of these axioms. These include the proof numerical bound (between 0 and 1) , probability of two non disjoint events A and B,etc.

2.2.1: Markov Processes

A random process is said to be a Markov process if its future is independent of the past given the present. A stochastic process x(t) is called Markov if for every n and t1<t2...<tn, we have

$$P\left(x\left(t_n\right) \leq x_n \mid x\left(t_{n-1}\right), \ldots, x\left(t_1\right)\right) \\ = P\left(x\left(t_n\right) \leq x_n \mid x\left(t_{n-1}\right)\right). \qquad [1]$$

The Markov process can either be discrete(N) or continuous [0, infinity) in time-space.

When the state space is discrete, Markov processes are known as Markov chains. When the time-space is discrete along with the state space, we call it a discrete-time Markov chain. When the time-space is continuous with state-space being discrete, we call it a continuous-time Markov chain. Both have minimal reliance on measure theory and have reasonably simpler mathematics involved.

One guide to clarify the discrete-time Markov chain is the cost of an asset where the stock price is enlisted uniquely by the day's end. The estimation of the Markov chain in discrete-time called the state relates to the closing price. A continuous-time Markov chain changes at any time. One notable case of a constant time Markov chain is the Poisson procedure.

For a general state space, the theory becomes more complicated and technical. For general state space with discrete time-space, we can consider the example of the partial sum process associated with a sequence of independent, identically distributed real-valued random variables. For continuous time-space, we can consider the example of a diffusion process ie, Brownian motion.[2]

2.2.2: Applications

In economics and finance, they are regularly used to foresee macroeconomic circumstances like market crashes and cycles among downturn and extension. Different zones of utilization incorporate anticipating resource and choice costs and figuring credit dangers. While considering a persistent time budgetary market Markov fastens are utilized to display the randomness.

2.3.1: J. L. Doob and the development of martingales

Joseph Leo "Joe" Doob was an American mathematician and a pioneer of probability theory

Doob's worked extensively on boundary values of analytic functions,eventually writing his thesis about the same topic . He made contributions to the Transactions of the American Mathematical Society by publishing two papers he wrote based on his thesis.

In 1933 after Kolmogorov provided the first axiomatic foundation for the theory of probability, what was once an intuitive idea and generalized, now became a mathematical science with proofs and theorems. Doob recognized this advancement as an opportunity that would make it possible to give extensive proofs for known and existing probability results. The tools that Measure theory brought us could be use to prove and improve probability results.

After working rigorously on multiple papers on the foundations of this new probability theory and stochastic processes that included research in martingales and Markov processes, Doob decided to write a book that would collate all this work into one single book which he named *Stochastic Processes*. To this day, it remains one of the most influential and important books in the field of modern probability theory as we know it

The development Doob has made in the concepts of martingales is revolutionary and has real life applications in the world of mathematical finance. Let us take a look at some examples -

1. Doob's submartingale inequality : This inequality was a result of his study in the area of stochastic processes. For any submartingale X,

$$X_s <= E [X_t \mid F_s] \text{, for all times s and t with } s < t.$$

This theorem holds true for martingales in both continuous and discrete times. This inequality for discrete time also implies the Kolmogorov's inequality.

This theorem is used in one dimensional canonical Brownian motion to prove the required inequality of λ .

2. Doobs' decomposition theorem : For any filtered probability space (Ω,F,F,P) and X={Xn:n∈N} , an adapted stochastic process, the statement theorem states that X can be broken down into a martingale and a predictable process. Here predictable means that A_n is F_{n-1} measurable and this decomposition is almost surely unique.

Applications of this theorem include the largest optimal exercise time of an American Option[4].

3. Optional stopping theorem : This theorem states that, the expected value of a martingale at stopping time is the initial expected value .(under certain conditions) . This theorem elaborates on (a) the stopping time being almost surely bounded. (b) E[X_{t+1} - X_t | F_t] <= c for some constant c and is almost sure on the event of stopping time > t. Stopping time also has a finite expectation .

This theorem is important in the context of the fundamental theorem of asset pricing. More particularly, the first fundamental theorem of asset pricing that lays down the condition for an arbitrage free market in a finite state (discrete) market .

For his exceptional contribution to the field of mathematics and revolutionizing probability theory and measure theory, the award for the most outstanding mathematical book is named after him as the Joseph L Doob Prize.

We have taken a journey through the development of modern probability theory and some of the most revolutionary ideas and people that are responsible for how we understand probability as we do now. Kolmogorov's axioms were the foundations and inspirations for many mathematicians like Doob and Markov to improve on the knowledge and transform probability theory and measure theory into what it is today .

3.0: References:

[1] Wolfram MathWorld.

Available at: https://mathworld.wolfram.com/MarkovProcess.html

[2] randomservices.org.

Available at: https://www.randomservices.org/random/markov/General.html

[3] amberton D. (2009). Optimal stopping and American options, Ljubljana Summer School on Financial Mathematics,

[4] http://www.fmf.uni-lj.si/finmath09/ShortCourseAmericanOptions.pdf

[5] https://en.wikipedia.org/wiki/Doob_decomposition_theorem#Application

[6] https://en.wikipedia.org/wiki/Doob%27s_martingale_inequality

[7] https://www.researchgate.net/publication/271856948_A_short_history_of_probability_theory_and_its_applications

[8] https://en.wikipedia.org/wiki/Probability_axioms