World Quant Univeristy

MSc Financial Engineering

Discrete-time Stochastic Processes 2020

Submission 3: American Options

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American Call Option

Unlike the European Option, the American counterpart can be exercised at any time up to and including the expiration date. Early exercise , however, for an American Call Option is never optimal. The exercise requires the payment of the Strike Price,but by holding on to it till the expiration date, the option holder saves the interest on this strike price. The price of an American Call hence is the same as the price of an European Call Option.

Here is a small proof for the same[1]; Consider two portfolios E and F

E : one American call c, $Xe^{-r(T-t)}$ cash

F : one share S.

- If the exercise time is $\tau < T$, the value of E is

  $E = (S - X) + Xe^{-r(T-\tau)}$

  $E < S$, which means $E < F$

- If the exercise is at $\tau = T$, then

  $E = \max(S - X, 0) + X$

  $E = \max(S, X)$

  $E \geq S$, which means $E \geq F$.

By this we can see that for exercise time $\tau = T$, E is always greater than or equal to F and we should never exercise an American call before the expiration date. We will take a closer look at this proof by considering interest rates in the next section

Consider a discrete-time market $\left(\left(\Omega, F, F, P\right), S\right)_n^T = 0$with one stock S and a fixed (continuously compounded) interest rate of $r > 0$ per period.
The proof here carried out in two parts. The first part without including r and the next including r.

Proof 1

Consider an American call option with pay of $H_t = \left(x_t - k\right)^{+¿¿}$, where k is the strike price. Since the function x $\left(x_t - k\right)^{+¿¿}$ is a convex, H is a submartingale with respect to $p^{\square}$. (Since x is a martingale). This is using the Jansen's inequality. Thus $V_T = H_T$ and $U_{t-1} = max$. Continuing in this manner we can see that U=H. As a result the following equation holds true

$$\pi(H) = U_0 = E^{\square}(H_t)$$

Which is equal to the corresponding European call option price. Also $t = \tau$ is the minimal optimal exercise time, implying that for a call option, there is no benefit of early exercise.

Proof 2

Let price of American call option with a stock price S be given by C(S) at maturity, then $C(S) = (S-k)^{+¿¿}$, where K is the strike price. Clearly C(S) is a convex function with respect to S with the property as follows.

C *Equation 1*

Let $S_1 = S \wedge S_2 = 0$, implies

$C(\lambda S) \leq \lambda C(S) + (1-\lambda) C(0) = \lambda C(S)$ *Equation 2*

This is because $C(0) = 0$

Let us consider i situation

Situation (i) if the option is exercised at time t and $payoff = C(S_t) = (S_t - k)^{+¿¿}$

Situation (ii) if the option is not exercised till maturity, then the discounted expected payoff till t is $E^{\square}$ where $\tau = T - t$ and $E^{\square}$ is the risky neutral measure. Under risky neutral probabilities, we also have

$E^{\square}$ *Equation 3*

Where $D_t = e^{-rt}$ is the continuously compounded discounted factor D_t over $\tau = T - t$.

Now in equation 1 , let $S = e^{r\tau} S_t$ and $\lambda = e^{-rt}$, then using equation 3 , we have

$C(\lambda S) = C(S_t) \leq e^{-rt} C(e^{r\tau} S_t) = e^{-rt} C(E^{\square}(S_T))$ Equation 4

Since C is a convex function, by Jensen's inequality $E^{\square}$, using this in equation 4, we get

$C(S_t) \leq e^{-rt} C(E^{\square}(S_T)) = ¿$

That is

Where $C(S_t)$ is a payoff early exercise?

is the expected discounted payoff in the case of no early exercise.

Thus the expected discounted payoff an American option before maturity is always higher than early exercise. Hence, it is never optimal to exercise an American call option before expiration. Since any person would never choose an American call option before maturity, its price will always be equal to the price of European call option.

American Put Option

The above doesn't hold true for American put option. This is because the argument breaks down as equation one above is not valid.

$p(\lambda S) \leq \lambda p(S) + (1-\lambda) p(0) = \lambda C(S)$

But p(0)=k which is different from zero , which is why equation one doesn't hold $p(\lambda S)$ *is not less than* $\vee$ *equal* $\lambda p(S)$. As a result the whole argument breaks.

In fact it is never optimal to exercise an American put option at expiry. This is shown as follows

Let $V_E(t, S_t)$ be the price of a European put with stock price S_t at time t and $V_A(t, S_t)$ be the price of American put with stock price S_t at time t. Then consider an exercise policy which always reads $V_A(t, S_t) \geq V_E(t, S_t)$.

The optimal exercise strategy for American put at min(τ, T)

where $\tau = min\{t \geq 0 : S_t \leq K - K e^{-r(t-T)}\}$

on the event $\tau < T$, the put will be exercised at time τ when $S_t \leq K - K e^{-r(\tau - t)}$ and thus the payoff would be at least $K e^{-r(t-T)}$. This payoff invested over $\tau \leq t \leq T$ would lead to the final payoff K at T. This is greater than any possible payoff from the European put with strike K , since $S_T > 0$ with probability 1.

In the event that $T \geq \tau$, the American put will be exercised at T and it pays off will be the same as that of European put.

Thus, in both cases ($\tau < T$, $T \geq \tau$, the payoff of American input is at least the payoff of the European put. Hence exercising an American put at maturity is no the optimal strategy. This implies that $V_A(t, S_t) \geq V_E(t, S_t)$ at all time

Example

Here is numerical example with Snell envelope, optimal exercise strategy, Doob's decomposition, and hedging portfolio. Note all calculations are carried out in two decimal places. Consider a binomial tree with T = 2 and r = 10% and the following parameters: u=1.25, d=1/u, S_0=400.

Solution

The solution was carried out both for American call and American put options in comparison with the corresponding European call and put options. But first let us summarize general properties of the given example.

Let S_0= 400 , $S_n = S_{n-1} u^{z_n} d^{1-z_n}$

Given d=1/u , this implies that $S_n = S_{n-1} u^{2z_n - 1}$

On continuously compounded interest rate or r>0 $X_n = e^{-rn} S_n = e^{-rn} S_{n-1} u^{2z_n - 1}$

An EMM ,P* should make the discounted assets $X_n = e^{-rn} S_n$ to be martingale. Hence the following holds true

e^{-r}

This implies that

$$p^{\square} = \frac{e^r - d}{u - d}$$

Which in our case

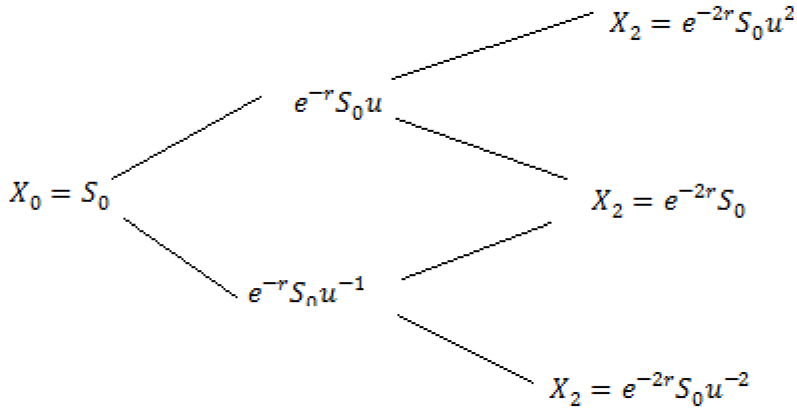
$$p^{\square} = \frac{1.11 - 0.8}{1.25 - 0.8} = 0.69$$

This holds true if 0 < d < e^r < u so in our case 0 < 0.8 < 1.11 < 1.25

$X_0 = S_0$

$X_n = e^{-rn} S_{n-1} u^{2z_n - 1}$

This can be represented on binomial tree diagram as follows



It can be summarized in the following table

X0	X1	X2
400	452.42	511.71
400	452.42	327.49
400	289.55	327.49
400	289.55	209.60

Call option

American call option , let American call option be $H_t = (X_t - K)^{+¿¿}$, where K is striking price which in our case it is K=319.

X0	X1	X2	H_0 $(X_0 - K)^{+¿¿}$	H_1 $(X_1¿ - K)^{+¿¿}$	H_2 $(X_2 - K)^{+¿¿}$
400	452.42	511.71	81	133.42	192.71
400	452.42	327.49	81	133.42	8.49
400	289.55	327.49	81	0	8.49
400	289.55	209.60	81	0	0

Now let us find snell’s envelop

At time 2, $U_2 = H_2$

At time 1, $U_1 = max$

Let us find

It has two values (192.71*0.69 + 8.49*0.31)/1 = 135.6018 ($I_{\{a,b\}}$) and (8.49*0.69) / 1 = 5.86 ($I_{\{c,d\}}$)

Then maximum of (133.42, 135.6018)=135.6018 and maximum of (0,5.86) is 5.86

At time 0, the Snell’s value can be calculated

$U_0 = 135.6018\, I_{\{a,b\}} + 5.86\, I_{\{c,d\}}$

$U_0 = 135.6018 * 0.69 * 0.69 + 8.49 * 0.69 * 0.31 = 95.38$

The corresponding European call

$\pi(H) = 192.71 * 0.69^2 + 2 * 8.49 * 0.69 * 0.31 = 95.34$

Which the American call option is equal to the European option.

Put Option

X0	X1	X2	H_0 $(K - X_0)^{+¿¿}$	H_1 $(K - X_2)^{+¿¿}$	H_2 $((\Omega, F, F, P), S)_n^T = 0$
400	452.42	511.71	0	0	0
400	452.42	327.49	0	0	0
400	289.55	327.49	0	29.45	0
400	289.55	209.60	0	29.45	109.4

Now let us find snell’s envelop

At time 2, $H_t = (x_t - k)^{+¿¿}$

At time 1, $(x_t - k)^{+¿¿}$

Let us find

It has two value 0 and $\frac{109.4 * 0.31 * 0.31}{0.31} = 33.91$

Then maximum of (0, 0) =0 and maximum of (33.91, 29.45) is 33.91

At time 0, the snell’s value can be calculated

P

The corresponding European put option

$V_T = H_T$

Which the American put option is equal to its corresponding European put option.

Perpetual American Put Option

Due to early exercise, it is sensible to define American option with infinite maturity. This is called perpetual American option or expiration-less option there are no obstacles produced by the finite horizon (maturity time), the valuation formula is available for these options.

The exercise boundary for the American put with expiration T does not have a closed form, and it is difficult even to establish simple qualitative properties of the boundary, such as smoothness. To illustrate some of the mathematical techniques that are used in studying optimization problems such as the determination of the optimal exercise policy for the American put, we shall analyze a simpler American option, the perpetual put option. The perpetual put works the same way as the American put option, except that there is no expiration date. Thus, one may hold it (or pass it along to one's offspring) until the universe collapses into a final black hole, and even beyond. If at some time τ the owner chooses to exercise the option, the payoff is (K − Sτ)+, where Sτ is the share price of the underlying asset Stock at the instant of exercise.[1]

Metron Explicit Solution

Consider the following

- r is risk free interest rate , expiration time is infinity
- $\delta\,is\,contineously\,divedend\,rate$
- $\sigma\,is\,constant\,volatility$
- The limiting early exercise boundary position be $S_{\square}$,

Then the explicit solution discovered by Metron has the form which is described as follows for the value

$$V(S)=\{(K-S_{\square})\left(\frac{S}{S_{\square}}\right)^{h} for\;S\geq S_{\square}$$

$$V(S)=K-S\;for\;S<S_{\square}$$

Where

$$S_{\square}=\frac{Kh}{1+h}$$

The solution for h is described with it quadratic form has two solutions

$$h=\left(\frac{1}{2}-\left(\frac{r-\delta}{\sigma^2}\right)\right)\pm\sqrt{\left(\left(\frac{r-\delta}{\sigma^2}\right)+\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}$$

[1] Taken from American option lecture

For put option we use the negative part since K > S

$$h=\left(\frac{1}{2}-\left(\frac{r-\delta}{\sigma^2}\right)\right)-\sqrt{\left(\left(\frac{r-\delta}{\sigma^2}\right)+\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}$$

for simplicity let us consider the continuously dividend rate to be zero , then the formula becomes

$$h=\frac{2r}{\sigma^2}=\frac{2*0.1}{0.6^2}=0.56$$

Then this implies that

$$S_{\square}=\frac{Kh}{1+h}=\frac{K\frac{2r}{\sigma^2}}{1+\frac{2r}{\sigma^2}}=\frac{2Kr}{2r+\sigma^2}$$

Therefore $S_{\square}=\frac{2Kr}{2r+\sigma^2}$

$$V(S)=\{(K-S_{\square})\left(\frac{S}{S_{\square}}\right)^h \text{for } S\geq S_{\square}$$

$$V(S)=K-S \text{ for } S<S_{\square}$$

So now let us calculate the price at time zero

$$V(S)=\{(K-S_{\square})\left(\frac{S_0}{S_{\square}}\right)^h \text{for } S_0\geq S_{\square}$$

$$V(S)=K-S \text{ for } S_0<S_{\square}$$

First let us calculate by considering $\sigma=0.6$

$$S_{\square}=\frac{2Kr}{2r+\sigma^2}=\frac{2*319*0.1}{2*0.1+0.6^2}=113.93$$

Since $S_0=400\geq S_{\square}=113.93$ then the price of the perpetual put will be

$$V(S_0)=\{(K-S_{\square})\left(\frac{S_0}{S_{\square}}\right)^h=(319-113.93)\left(\frac{400}{113.93}\right)^{0.56}=414.32$$

References

[1]https://www.google.com/urlsa=t&rct=j&q=&esrc=s&source=web&cd=&ved=2ahUKEwi7mJaIi6nqAhUX4zgGHQiTBA8QFjABegQIChAD&url=https%3A%2F%2Fwww.math.ucla.edu%2F~caflisch%2F181.1.07w%2FLect18.pdf&usg=AOvVaw2031ObSP0X-by__hBCRU2s (Math 181 Lecture 18, UCLA Math)

[2]https://www.investopedia.com/terms/e/earlyexercise.asp

[3] Maria do Rosario Grossinho, Yaser Kord Faghan, Daniel Sevcovic : Pricing Perpetual Put Options by the Black – Scholes Equation with a Nonlinear Volatility Function ; November 9,2017