20/8/09/20

FOURIER SERIES

This is a mathematical tool that converts periodic non sinusoidal function in sin and cosine functions.

Types:

1. Trignonometric fourier series
2. Exponential fourier series

1. Trignometric fourier series

If a function $f(t)$ is a periodic function, then its F.S. is given by,

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where, a_0, a_n and b_n are known as Fourier coefficients.

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

T = Time period of the function

$\omega_0 = \frac{2\pi}{T}$ (fundamental frequency)

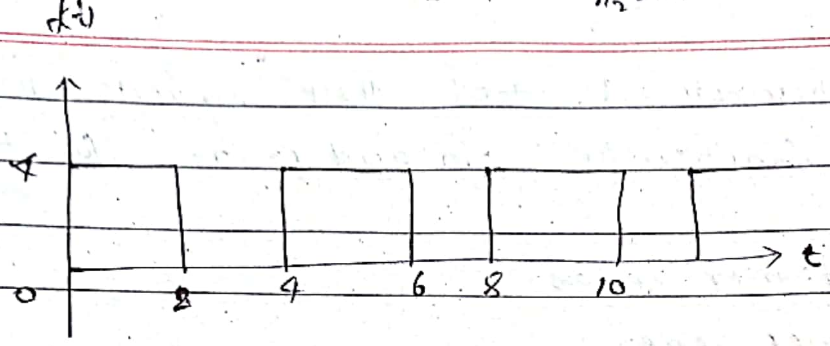
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt.$$

2. $$\begin{cases} \sin\omega_0 t = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \\ \cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \end{cases}$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

eg.



obtain Trigonometric fourier series.

sol^n. Time period $T = 4$

$$f(t) = 4 \qquad 0 \le t \le 2$$
$$= 0 \qquad 2 \le t \le 4$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$
$$= \frac{1}{4}\left(\int_0^2 4\,dt + \int_2^4 0\,dt\right)$$
$$= \frac{1}{4}\cdot 4\,[t]_0^2$$
$$= 2$$

$$a_n = \frac{2}{T}\int_0^T \cos \omega_0 t n f(t)\,dt$$
$$= \frac{2}{4}\left[\int_0^2 4\cos\left(\frac{n \cdot 2\pi}{T}\right)t\,dt + \cancel{\int_2^4 4\cos\left(\frac{2n\pi}{4}\right)t\,dt}^{\,0}\right]$$
$$= 2\int_0^2 \cos\left(\frac{2n\pi}{4}\right)t\,dt$$
$$= 2\int_0^2 \cos\left(\frac{n\pi}{2}\right)t\,dt$$
$$= 2\left[\frac{\sin\left(\frac{n\pi}{2}\right)t}{\frac{n\pi}{2}}\right]_0^2$$

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$$= 2\left[\frac{\sin\frac{n\pi}{2}\times 2}{\frac{n\pi}{2}} - \frac{\sin\frac{n\pi}{2}\cdot 0}{\frac{n\pi}{2}}\right]$$

$$= 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\,\sin n\omega_0 t\,dt$$

$$= \frac{2}{4}\int_0^2 \cancel{4}\sin\left(\frac{2n\pi}{4}\right)t\,dt$$

$$= 2\int_0^2 \sin\frac{n\pi}{2}t\,dt$$

$$= 2\left[\frac{-\cos\frac{n\pi}{2}t}{\frac{n\pi}{2}}\right]_0^2$$

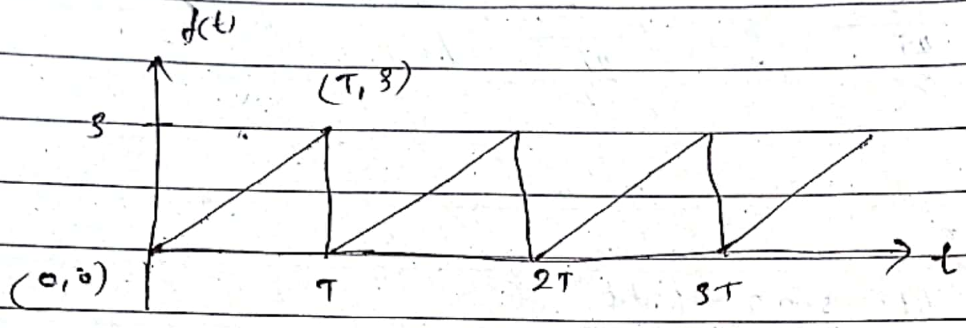
$$= -\frac{2\times 2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\times 2\right) - \cos\left(\frac{n\pi}{2}\times 0\right)\right]$$

$$= \frac{4}{n\pi}\left[(-1)^n - 1\right]$$

then F.T.S. is

$$= 2 + \sum_{n=1}^{\infty} 0 + \frac{4}{n\pi}\left[(-1)^n - 1\right]\sin n\omega_0 t$$

Q. Find F-S.



Time period = T

$$f(t) = \frac{3}{T}t \qquad 0 \le t \le T$$

~~y=0~~

now,

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{3}{T}t\,dt$$

$$= \frac{3}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{3}{T^2}\cdot\frac{T^2}{2}$$

$$= \frac{3}{2}$$

$$F.S. = \frac{3}{2} + \sum_{n=1}^{\infty}\left(-\frac{3}{n\pi}\right)\sin n\omega_0 t$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^T \frac{3}{T}t\cdot\cos\left(n\cdot\frac{2\pi}{T}\right)t\,dt$$

$$= \frac{6}{T^2}\int_0^T t\cos\left(\frac{2n\pi}{T}\right)t\,dt$$

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$$a_n = \frac{6}{T^2}\left[t \cdot \frac{\sin\left(\frac{2n\pi}{T}\right)t}{\frac{2n\pi}{T}} + \frac{\cos\left(\frac{2n\pi}{T}\right)t}{\left(\frac{2n\pi}{T}\right)^2} \right]_0^T$$

$$= \frac{6}{T^2}\left[T \cdot \cancel{\frac{\sin(2n\pi)}{\frac{2n\pi}{T}}}^{0} + \cancel{\frac{\cos(2n\pi)}{\left(\frac{2n\pi}{T}\right)^2}} - 0 - \frac{1}{\left(\frac{2n\pi}{T}\right)^2} \right]$$

$$= \frac{6}{T^2}\left[\frac{1}{\left(\frac{2n\pi}{T}\right)^2} \left\{ \cos 2n\pi - 1 \right\} \right]$$

$$= \frac{6}{4n^2\pi^2} * (1-1)$$

$$= 0$$

$$b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega_0 t \, dt$$

$$= \frac{2}{T}\int_0^T \frac{3}{T} t \cdot \sin\left(\frac{2n\pi}{T}\right)t \, dt$$

$$= \frac{6}{T^2}\left[t \, \frac{-\cos\left(\frac{2n\pi}{T}\right)t}{\frac{2n\pi}{T}} + \frac{\sin\left(\frac{2n\pi}{T}\right)t}{\left(\frac{2n\pi}{T}\right)^2} \right]_0^T$$

$$= \frac{6}{T^2}\left[-T \frac{\cos 2n\pi}{\frac{2n\pi}{T}} + 0 + 0 - 0 \right]$$

$$= \frac{6}{\cancel{T^2}}\left[-\cancel{T^2} \frac{\cos 2n\pi}{2n\pi} \right]$$

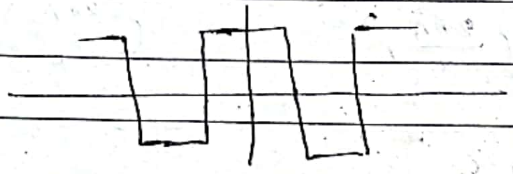
$$= \frac{-6 \cdot 1}{2n\pi}$$

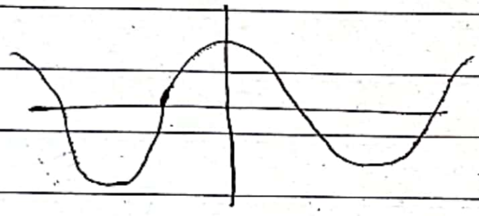
$$= \frac{-3}{n\pi}$$

Even symmetry

$f(t) = f(-t)$

eg. cos function





Calculate:

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

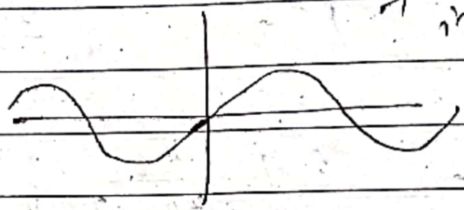
$b_n = 0$ (always)

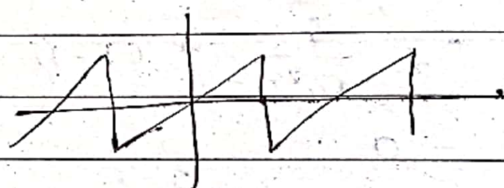
Odd symmetry.

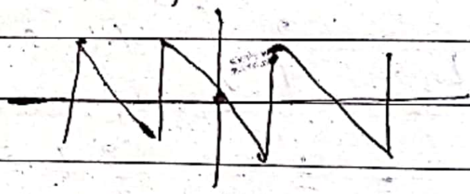
$-f(t) = f(-t)$

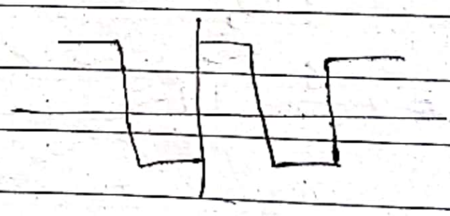
eg. sin function

same inverted by y-axis









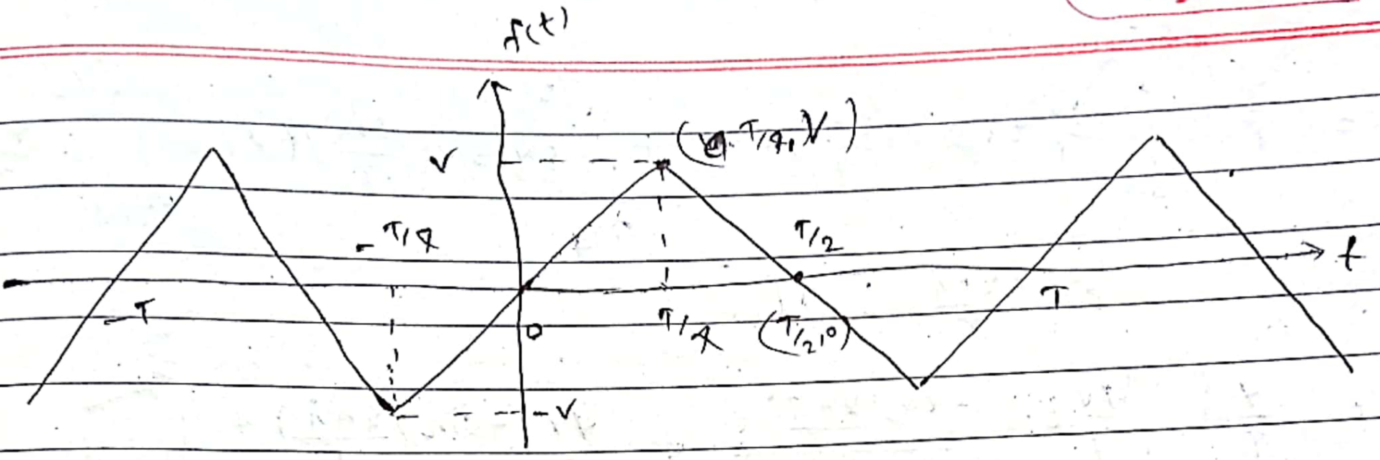
Calculate:

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Q.



Find the Trigonometric F.S.

Time period (T) = T

$$f(t) = \frac{V}{T/4}t = \frac{4V}{T}t \quad \text{for } 0 \le t \le T/4.$$

$$y - 0 = \frac{V-0}{\frac{T}{4} - \frac{T}{2}}\left(t - \frac{T}{2}\right)$$

$$= \frac{V}{\frac{T-2T}{4}}\left(\frac{2t-T}{2}\right) = \frac{\cancel{4}^{2}V}{T-2T} \cdot \frac{2t-T}{\cancel{2}}$$

$$= \frac{2V(2t-T)}{-T}$$

$$f(t) = 2V - \left(\frac{4V}{T}\right)t \quad \text{for } \frac{T}{4} \le t \le \frac{T}{2}.$$

$$= -\frac{4Vt}{T} + \frac{2V\cancel{T}}{\cancel{T}}$$

$$= 2V - \left(\frac{4V}{T}\right)t$$

given function is odd.

Hence,

$$a_0 = 0$$

$$a_n = 0$$

and
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{4}{T}\left\{\int_0^{T/4} \frac{4V}{T}t \sin\left(\frac{2n\pi}{T}\right)t\, dt + \int_{T/4}^{T/2}\left[2V - \left(\frac{4V}{T}\right)t\right] \cdot \sin\left(\frac{2n\pi}{T}\right)t\, dt\right.$$

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$$= \frac{4}{T}\left\{ \int_0^{T/4} \frac{4V}{T} t \cdot \sin\left(\frac{2n\pi}{T}\right) t \, dt + \int_{T/4}^{T/2} 2V \sin\left(\frac{2n\pi}{T}\right) t \, dt - \int_{T/4}^{T/2} \frac{4V}{T} t \cdot \sin\left(\frac{2n\pi}{T}\right) t \, dt \right\}$$

$$= \frac{4}{T}\left\{ \left[\frac{4V}{T} t \cdot \frac{-\cos\left(\frac{2n\pi}{T}\right) t}{\frac{2n\pi}{T}} - \frac{4V}{T} \cdot \frac{-\sin\left(\frac{2n\pi}{T}\right) t}{\frac{4n^2\pi^2}{T^2}} \right]_0^{T/4} + \left[2V \frac{-\cos\left(\frac{2n\pi}{T}\right) t}{\frac{2n\pi}{T}} \right]_{T/4}^{T/2} - \left[\frac{4V}{T} t \cdot \frac{-\cos\left(\frac{2n\pi}{T}\right) t}{\left(\frac{2n\pi}{T}\right)} - \frac{4V}{T} \frac{-\sin\left(\frac{2n\pi}{T}\right) t}{\frac{4n^2\pi^2}{T^2}} \right]_{T/4}^{T/2} \right\}$$

$$= \frac{4}{T}\left\{ \frac{\cancel{4}V}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{4}} \cdot \frac{-\cos \frac{2n\pi}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{4}_2}}{\frac{2n\pi}{T}} - \frac{4V}{T} \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{4n^2\pi^2}{T^2}} \cancel{+0+0-0} + 2V \cdot \frac{-\cos \frac{2n\pi}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{2}}}{\frac{2n\pi}{T}} \right.$$

$$- 2V \frac{-\cos \frac{2n\pi}{\cancel{T}} \times \frac{\cancel{T}}{4}}{\frac{2n\pi}{T}} - \frac{\cancel{4}^2 V}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{2}} \frac{-\cos \frac{2n\pi}{\cancel{T}} \times \frac{\cancel{T}}{\cancel{2}}}{\frac{2n\pi}{T}} - 0 +$$

$$\left. \frac{4V}{T} \times \frac{T}{4} \cdot \frac{-\cos \frac{2n\pi}{T} \times \frac{T}{4}}{\frac{2n\pi}{T}} + 0 \right\}$$

$$= \frac{4}{T}\left\{ \frac{-2V \cos\left(\frac{n\pi}{2}\right)}{\frac{2n\pi}{T}} - \frac{2V \cos(n\pi)}{\frac{2n\pi}{T}} + \frac{2V \cos\left(\frac{n\pi}{2}\right)}{\frac{2n\pi}{T}} + 2V \frac{\cos(n\pi)}{\frac{2n\pi}{T}} \right\}$$

$$= \frac{4}{T}\left\{ \frac{4V}{T}\times\frac{T}{4}\cdot\frac{-\cos\frac{2n\pi}{T}\times\frac{T}{4}}{\frac{2n\pi}{T}} + \frac{4V}{T}\,\frac{\sin\frac{2n\pi}{T}\times\frac{T}{4}}{\frac{4n^2\pi^2}{T^2}} + 0 - 0 \right.$$

$$+ 2V\cdot\frac{-\cos\frac{2n\pi}{T}\times\frac{T}{2}}{\frac{2n\pi}{T}} - 2V\cdot\frac{-\cos\frac{2n\pi}{T}\times\frac{T}{4}}{\frac{2n\pi}{T}} -$$

$$\frac{4V}{T}\times\frac{T}{2}\,\frac{-\cos\frac{2n\pi}{T}\times\frac{T}{2}}{\frac{2n\pi}{T}} + \frac{4V}{T}\,\frac{-\sin\frac{2n\pi}{T}\times\frac{T}{2}}{\frac{4\pi^2n^2}{T^2}} + \cancel{\frac{4V}{T}\times\frac{T}{4}\,\frac{-\cos 2n\pi}{T}}$$

$$\left. \frac{4V}{T}\times\frac{T}{4}\,\frac{-\cos\frac{2n\pi}{T}\times\frac{T}{4}}{\frac{2n\pi}{T}} - \frac{4V}{T}\,\frac{-\sin\frac{2n\pi}{T}\times\frac{T}{4}}{\frac{4n^2\pi^2}{T^2}} \right\}$$

$$= \frac{4}{T}\left\{ -2V\,\frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{2n\pi}{T}} + \frac{4V}{T}\,\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{4n^2\pi^2}{T^2}} - \frac{2V\cos(n\pi)}{\frac{2n\pi}{T}} + \right.$$

$$\left. \frac{2V\cos\left(\frac{n\pi}{2}\right)}{\frac{2n\pi}{T}} + 2V\,\frac{\cos(n\pi)}{\frac{2n\pi}{T}} + \frac{4V}{T}\,\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{4n^2\pi^2}{T^2}} \right\}$$

$$= \frac{4}{\cancel{T}}\; 2\cdot\frac{4V}{\cancel{T}}\,\frac{\sin\left(\frac{n\pi}{2}\right)}{\cancel{4}n^2\pi^2}\times \cancel{T^2}$$

$$= \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

Hence, F.S is

$$= \sum_{n=1}^{\infty} \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\sin n\omega_0 t$$

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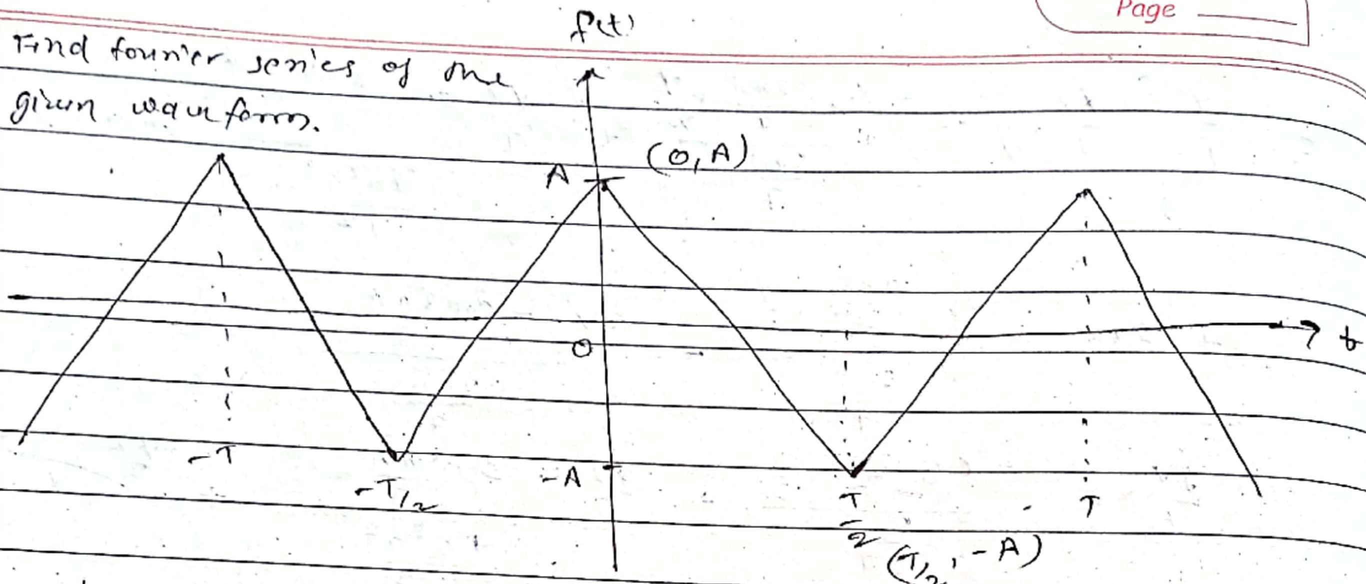
Q. Find fourier series of the given waveform.



given waveform is even.

Symmetrical about x-axis so $a_0 = 0$. It is average value.

$$f(t) + A = \frac{A+A}{0 - T/_2}(t - T/_2)$$

$$= \frac{2A \times 2}{-T}\left(t - \frac{T}{2}\right)$$

$$= \frac{2 \cancel{4}A}{-T} \cdot \frac{2t - T}{\cancel{2}}$$

$$= \frac{2A}{-T}(2t - T)$$

$$f(t) = 2A - \frac{4A}{T}t - A$$

$$f(t) = A - \frac{4A}{T}t \quad \text{for} \quad 0 \le t \le T/_2$$

F.s is, $\Rightarrow a_0 + \sum_{n=1}^{\infty} a_n \cos n w_0 t + b_n \sin n w_0 t$ ——— (1)

where $b_n = 0$ for even.

$$a_0 = \frac{2}{T}\int_0^{T/_2}\left(A - \frac{4A}{T}t\right)dt$$

average value

$$= \frac{2}{T}\left[At - \frac{4A}{T}\cdot\frac{t^2}{2}\right]_0^{T/_2}$$

$$= \frac{2}{T}\left[\frac{AT}{2} - \frac{2AT^2}{T \cdot 4}\right]$$

$$= \frac{2}{T}\left[\frac{AT}{2} - \frac{AT}{2}\right]$$

$$= 0$$

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$$a_n = \frac{4}{T}\int_0^{T/2} \left(A - \frac{4A}{T}t\right)\cos \omega_0 t \, dt$$

$$= \frac{4}{T}\left[\left(A - \frac{4A}{T}t\right)\cdot\frac{\sin n\omega_0 t}{n\omega_0} - \frac{4A}{T}\cdot\frac{\cos n\omega_0 t}{n^2\omega_0^2}\right]_0^{T/2}$$

$$= \frac{4}{T}\left[\left(A - \frac{4A}{T}\times\frac{T}{2}\right)\frac{\sin \frac{n2\pi}{T}\cdot\frac{T}{2}}{\frac{n2\pi}{T}} - \frac{4A}{T}\frac{\cos \frac{n2\pi}{T}\times\frac{T}{2}}{\frac{4n^2\pi^2}{T^2}} + \frac{4A}{T}\frac{1}{\frac{n^2 4\pi^2}{T^2}}\right]$$

$$= \frac{4}{T}\left[\frac{4A}{T}\cdot\frac{T^2}{4n^2\pi^2} - \frac{4A}{T}\cdot\frac{\cos n\pi \cdot T^2}{4n^2\pi^2}\right]$$

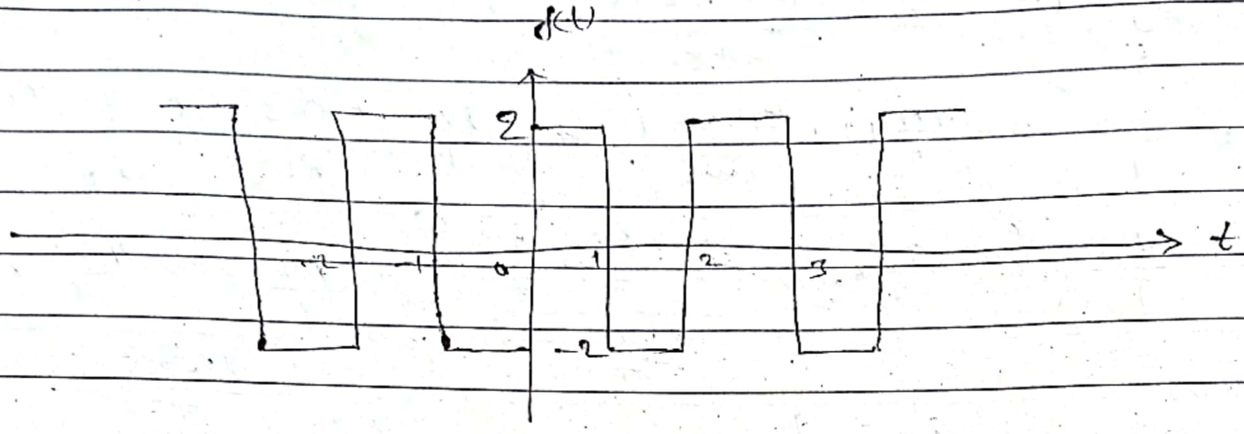
$$= \frac{16A}{4n^2\pi^2}(1-\cos n\pi)$$

$$= \frac{4A}{n^2\pi^2}(1-\cos n\pi)$$

Hence from (1)

$$\text{F.S} \Rightarrow 0 + \sum_{n=1}^{\infty} \frac{4A}{n^2\pi^2}(1-\cos n\pi)\cos n\omega_0 t + 0$$

Q. find F.S.



Time period $T = 2$.

It is odd waveform so clearly,

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$$

$$f(t) = 2 \qquad 0 \le t \le 1$$

Then,

$$b_n = \frac{4}{T}\int_0^{1} 2 \sin n\omega_0 t \, dt$$

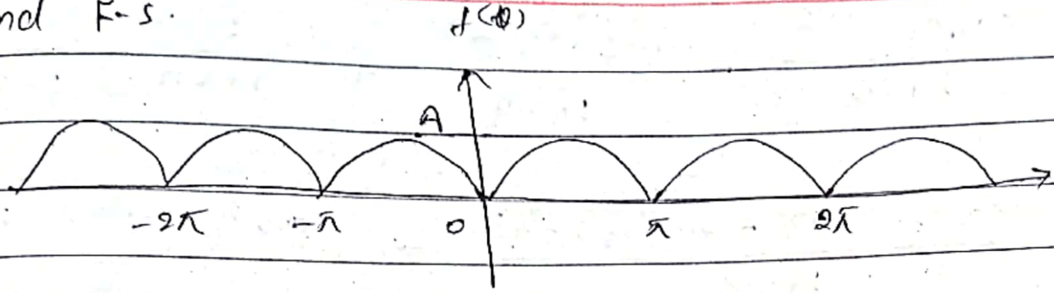
$$= \frac{8}{T}\left[\frac{-\cos n\omega_0 t}{n\omega_0}\right]_0^1$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{T}}\left\{\frac{-\cos n\frac{2\pi}{T}}{\frac{2n\pi}{\cancel{T}}} + \frac{1}{\frac{2n\pi}{\cancel{T}}}\right\}$$

$$= \frac{2}{n\pi}\left(1 - \cos\frac{2n\pi}{T}\right)$$

Then F.S $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{2n\pi}{T}\right) \cdot \sin n\omega_0 t$

Q. Find F.S.



$f(\theta) = A\sin\theta \quad 0 \le \theta \le \pi,$ $\qquad \frac{2\pi}{T} = \frac{2\pi}{\pi}$

$T = \pi$, even waveform.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B.$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(\theta)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} A\sin\theta\,d\theta$$

$$= \frac{2A}{\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{2A}{\pi}(-0+1)$$

$$= \frac{2A}{\pi}.$$

$$a_n = \frac{4}{T}\int_0^{\pi/2} f(\theta)\cos n\omega_0\theta\,d\theta$$

$$= \frac{4}{\pi}\int_0^{\pi/2} A\sin\theta\cos 2n\theta\,d\theta$$

$$= \frac{2A}{\pi}\int_0^{\pi/2} 2\sin\theta\cos 2n\theta\,d\theta$$

$$= \frac{2A}{\pi}\int_0^{\pi/2} \sin(1+2n)\theta + \sin(1-2n)\theta\,d\theta$$

$$= \frac{2A}{\pi}\left[-\frac{\cos(1+2n)\theta}{1+2n} - \frac{\cos(1-2n)\theta}{1-2n}\right]_0^{\pi/2}$$

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$$= \frac{2A}{\pi}\left[-\frac{\cos(\pi/2 + n\pi)}{1+2n} - \frac{\cos(\pi/2 - n\pi)}{1-2n} + \frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= \frac{2A}{\pi}\left[-\frac{\cos\pi/2 \cos n\pi - \sin\pi/2 \sin n\pi}{1+2n} - \frac{\cos\pi/2 \cos n\pi + \sin\pi/2 \sin n\pi}{1-2n}\right.$$

$$\left. + \frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= \frac{2A}{\pi}\left(\frac{1-\cancel{2n}+1+\cancel{2n}}{(1+2n)(1-2n)}\right)$$

$$a_n = \frac{4A}{\pi}\frac{1}{(1-2n)(1+2n)}$$

Hence, fourier series is,

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi}$$

$$\Rightarrow \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi}\frac{1}{(1-2n)(1+2n)} \cos n w_0 \theta + 0$$

$$\Rightarrow \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi(1-2n)(1+2n)} \cdot \cos 2n\theta$$

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Exponential Fourier Series: / complex exponential

Fourier series, $= \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$

where,

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$\sin \omega_0 t \longrightarrow \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\cos \omega_0 t \longrightarrow \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

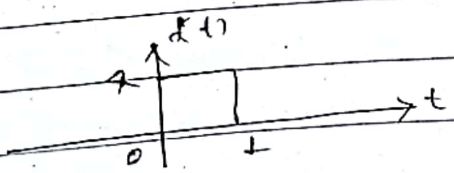
Fourier Transform

This is used for non-periodic function.

$$f(t) \xrightarrow{F.T.} F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Q. Find fourier transform.



$f(t) = 4 \qquad 0 \le t \le 1$

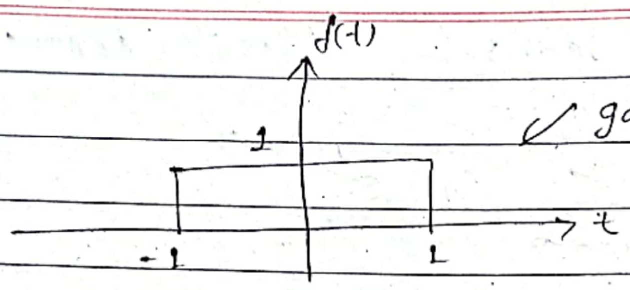
$$F(\omega) = \int_0^1 4 e^{-j\omega t}\, dt$$

$$= 4\left[\frac{e^{-j\omega t}}{j\omega}\right]_0^1$$

$$= -\frac{4}{j\omega}\left(e^{-j\omega} - 1\right)$$

Ans.

Q.



gate function.

$f(t) = 1 \qquad -1 \le t \le 1$

Fourier transform,

$$F(\omega) = \int_{-1}^{1} 1 \cdot e^{-j\omega t} \, dt$$

$$= 1 \cdot \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-1}^{1}$$

$$= \frac{1}{-j\omega} \left[e^{-j\omega} - e^{j\omega} \right]$$

$$= -\frac{2}{\omega} \left(\frac{e^{-j\omega} - e^{j\omega}}{2j} \right)$$

$$= \frac{2}{\omega} \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)$$

$$= \frac{2}{\omega} \cdot \sin \omega .$$