Circuit Analysis using Laplace Transform

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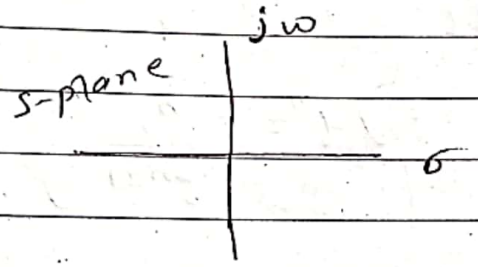
Def^n → If a function in time domain is $f(t)$, then its laplace Tx is

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

time domain $\xrightarrow{L.Tx}$ frequency domain

$(t) \longrightarrow (s)$

$s \longrightarrow$ complex frequency $= \sigma + j\omega$

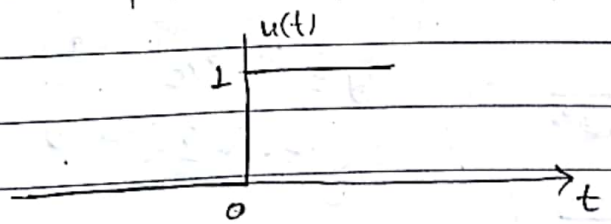


* Advantages of L.T. over classical method.

- The differential equations are greatly simplified by Laplace transform method.
- In Laplace transform, the equation incorporates the provision for initial values.
- No need to calculate initial values seperately.

* Laplace transform of some basic forcing functions

(1) Unit step function.



$u(t) = 1 \quad t \geq 0$

$\quad\;\; = 0 \quad t < 0$

L.T. of unit step function $= \int_0^\infty f(t)\, e^{-st} dt$

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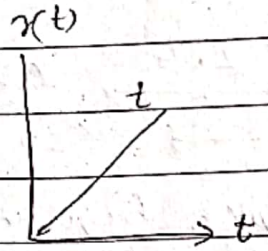
$$= \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \frac{1}{s}$$

② Ramp function



$$r(t) = t$$

$$= \int_0^\infty t\, e^{-st}\, dt$$

$$= \frac{1}{s^2}$$

$$\left[\mathcal{L}\, t^n = \frac{n!}{s^{n+1}} \right]$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3}$$

③ Exponential function

$$e^{at} \xrightarrow{L.T} \frac{1}{s-a}$$

$$e^{-at} \xrightarrow{L.T} \frac{1}{s+a}$$

$y = mx$

$y = 3/2\, x$

$$y - 0 = \frac{3-0}{0-2}(t-2)$$

$$y = -3/2\,(t-2)$$

$$y = -\frac{3}{2}t + 3$$

④ Sinusoidal function.

$$\sin \omega t \xrightarrow{L.T.} \frac{\omega}{s^2+\omega^2}$$

$$\cos \omega t \xrightarrow{L.T.} \frac{s}{s^2+\omega^2}$$

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(5)

shifting property.

$$e^{-at}\sin\omega t \xrightarrow{L.T.} \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t \xrightarrow{L.T.} \frac{(s+a)}{(s+a)^2+\omega^2}$$

$$t\sin\omega t \xrightarrow{L.T.} \frac{\omega}{(s^2+\omega^2)^2}$$

$$te^{-at} \xrightarrow{L.T.} \frac{1}{(s+a)^2}$$

$$t^2e^{-at} \xrightarrow{L.T.} \frac{2!}{(s+a)^3}$$

* Derivative property.

$$\frac{dy}{dx} \xrightarrow{L.T.} sY(s) - y(0^-)$$

$$\frac{di(t)}{dt} \xrightarrow{L.T.} sI(s) - i(0^-)$$

$$\frac{d^2 i(t)}{dt^2} \xrightarrow{L.T.} s^2 I(s) - s\,i(0^-) - i'(0^-)$$

* Integration property.

Initial charge.

$$\frac{1}{C}\int i(t)\,dt = \frac{1}{C}\left[\frac{I(s)}{s} + \frac{Q(0^-)}{s}\right]$$

initial current.

$$\frac{1}{L}\int v(t)\,dt = \frac{1}{L}\left[\frac{V(s)}{s} + \frac{i(0^-)}{s}\right]$$

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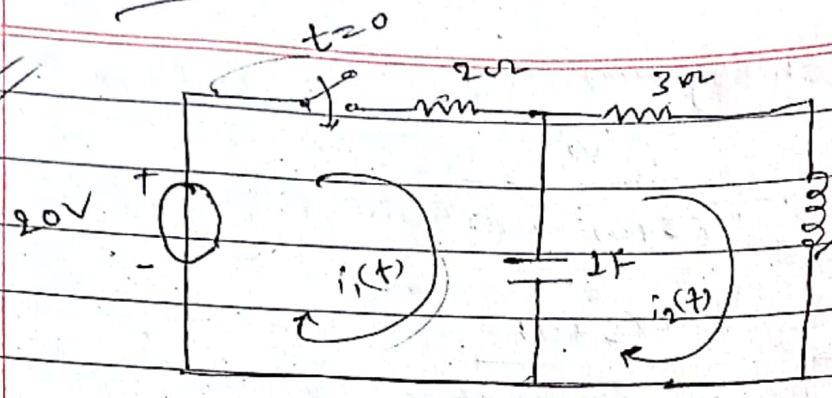
H.W

Pb



$i_1 = 10$

$i_2 = 0$

Switch is closed at $t=0$, find $i_1(t)$, $i_2(t)$, $\frac{di_1(t)}{dt}$, $\frac{di_2(t)}{dt}$, $\frac{d^2 i_1(t)}{dt^2}$ and $\frac{d^2 i_2(t)}{dt^2}$ at $t=0+$

At $t=0$, KVL @ fresh mesh.

$$\boxed{-2\,i_1(t) - \frac{1}{1}\int\left([i_1(t) - i_2(t)]\right)dt + 20\,u(t) = 0}$$

or, $2i_1(t) + \int i_1(t)\,dt - \int i_2(t)\,dt = 20\,u(t)$ —— ①

KVL @ mesh 2,

$$-3i_2(t) - 1\cdot\frac{di_2(t)}{dt} - \frac{1}{1}\int(i_2(t) - i_1(t))\,dt = 0$$

or, $$\boxed{3i_2(t) + \frac{di_2(t)}{dt} + \frac{1}{1}\int\left(\cancel{i_2(t)\,dt} - \cancel{\int i_1(t)\,dt}\right) = 0}$$ —— ②

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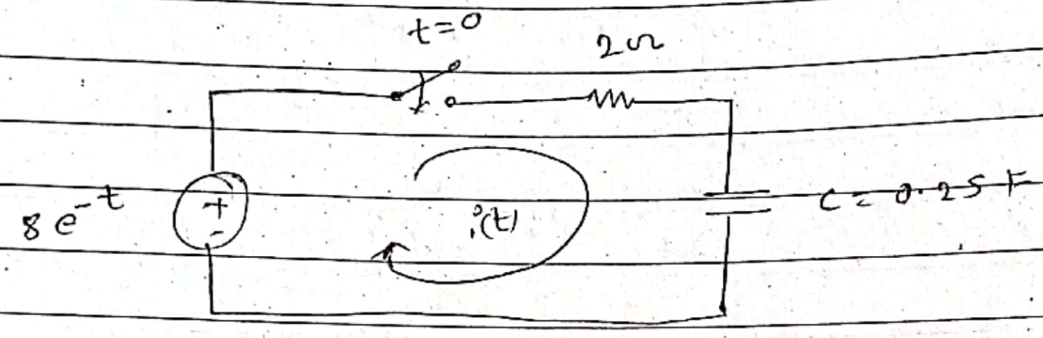
Q. A voltage source of $v = 8e^{-t}$ is supplied to series RC circuit which consists of $R = 2\Omega$, $C = 0.25\,F$ at $t=0$ time. Find the expression for $i(t)$ using Laplace transform method. Assume 0 charge in capacitor initially.



At $t=0$, using KVL

$$-2i(t) - \frac{1}{0.25}\int i(t)\,dt + 8e^{-t} = 0$$

or $$2i(t) + \frac{1}{0.25}\int i(t)\,dt = 8e^{-t}$$

Applying Laplace Tx.

$$2I(s) + 4\left[\frac{I(s)}{s} + \frac{q(0^+)}{s}\right] = 8 \times \frac{1}{s+1}$$

or $$2I(s) + \frac{4I(s)}{s} = \frac{8}{s+1}$$

or $$I(s)\left(2 + \frac{4}{s}\right) = \frac{8}{s+1}$$

or $$I(s)\left(\frac{2s+4}{s}\right) = \frac{8}{s+1}$$

$$I(s) = \frac{8s}{2(s+1)(s+2)}$$

$$= \frac{4s}{(s+2)(s+1)}$$

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Partial fraction,

Residue

$$\frac{4s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = \left.\frac{4s}{s+1}\right|_{s+2=0}$$

$s = -2$

$$A = \frac{-8}{1} = 8$$

$$B = \left.\frac{4s}{s+2}\right|_{s+1=0}$$

$s = -1$

$$= \frac{-4}{1}$$

$$= -4.$$

Now,

$$I(s) = \frac{4s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$I(s) = \frac{8}{s+2} - \frac{4}{s+1}$$

ILT.

$$i(t) = 8e^{-2t} - 4e^{-t}$$ Ans.

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Q. Determine the current $i(t)$ from the differential equation

$$i''(t) + 5i'(t) + 6i(t) = 0.$$

Given that, $i(0+) = 0$ and $i'(0+) = 5$

use Laplace transform method.

Sol^n

$$i''(t) + 5i'(t) + 6i(t) = 0$$

Applying Laplace transform, we get

$$[s^2 I(s) - s\,i(0) - i'(0+)] + 5[s\,I(s) - i(0+)] + 6I(s)$$

$$\Rightarrow s^2 I(s) - s\times 0 - 5 + 5s\,I(s) - 5\times 0 + 6I(s) = 0$$

$$\Rightarrow s^2 I(s) + 5s\,I(s) + 6I(s) = 5$$

$$\Rightarrow I(s)(s^2 + 5s + 6) = 5$$

$$\Rightarrow I(s) = \frac{5}{s^2+5s+6} = \frac{5}{(s+2)(s+3)}$$

Partial fraction.

$$\frac{5}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = \left.\frac{5}{s+3}\right|_{s+2=0 \Rightarrow s=-2}$$

$$A = 5$$

$$B = \left.\frac{5}{s+2}\right|_{s+3=0 \Rightarrow s=-3}$$

$$B = -5$$

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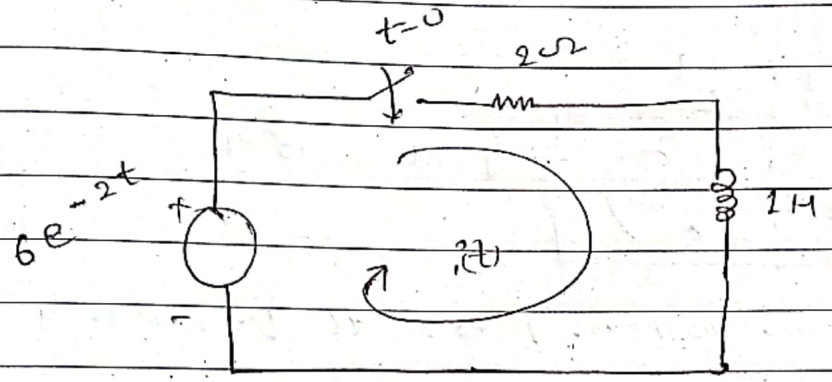
So, $I(s) = \frac{5}{s+2} - \frac{3}{s+3}$

Taking I.L.T.

$$i(t) = 5e^{-2t} - 3e^{-3t}$$ Ans.

Q.



switch was closed at t = 0. Find expression for i(t) using laplace transform.

Sol^n At t = 0, using KVL.

$$2\,i(t) + 1 \cdot \frac{di(t)}{dt} = 6e^{-2t}$$

Applying laplace transform.

$$2\,I(s) + s\,I(s) - i(0+) = 6 \cdot \frac{1}{s+2}$$

Initially, current ~~across~~ ~~that~~ is zero so

$$2\,I(s) + s\,I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6}{(s+2)(s+2)}$$

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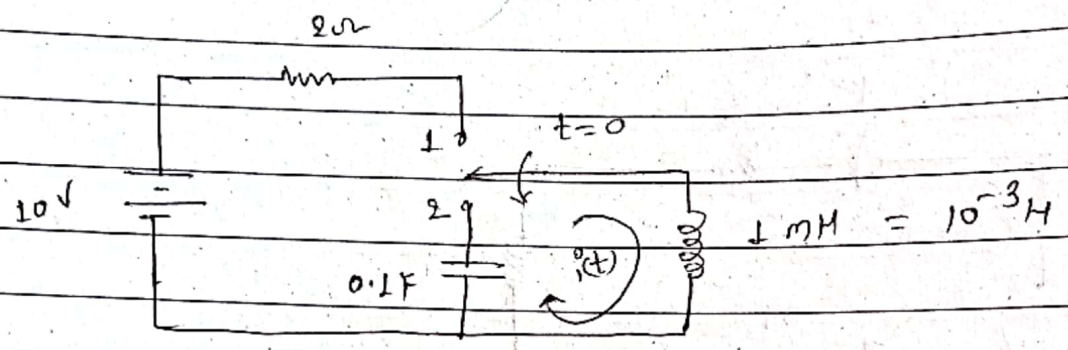
$$I(s) = \frac{6}{(s+2)^2}$$

Taking inverse laplace, we get

$$i(t) = 6 \cdot t \cdot e^{-2t}$$

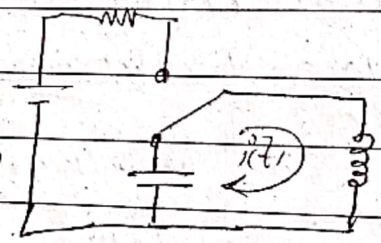
Q.



switch moved from position 1 to 2. at t=0, Find current i(t) using L.T.

At t=0, using KVL,

$$10^{-3} \frac{d\,i(t)}{dt} + \frac{1}{0.1}\int i(t)\,dt = 0$$



L.T.

$$10^{-3}\left[s\,I(s) - \underbrace{i(0+)}_{\text{initial current}}\right] + 10\left[\frac{I(s)}{s} + \frac{\overbrace{q(0+)}^{\text{Initial charge}}}{s}\right] = 0$$

or, $1\times10^{-3}[s\,I(s) - 5) + 10\left\{\frac{I(s)}{s} + 0\right\} = 0$

or, $10^{-3}\,s\,I(s) - 5\times10^{-3} + \frac{10\,I(s)}{s} = 0$

or, $1\times10^{-3}\,s^2 I(s) + 10\,I(s) = 5\times10^{-3}\cdot s$

or, $$I(s) = \frac{5\times10^{-3}\,s}{1\times10^{-3}s^2 + 10}$$

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$$\Rightarrow I(s) = \frac{5\times10^{-3}s}{1\times10^{-3}\left(s^2 + \frac{10}{10^{-3}}\right)}$$

$$= \frac{5s}{s^2 + 10^4}$$

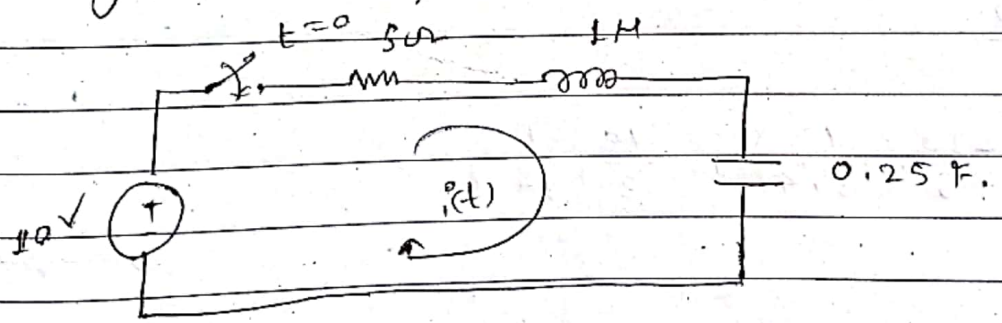
$$\cancel{= 5\cos 10^2 t}$$

I.L.T.

$$i(t) = 5\cos 10^2 t$$

$$= 5\cos 100t$$

Q. A series ~~ser~~ RLC circuit consist of $R = 5\Omega$, $L = 1H$ and $C = 0.25F$ is connected with supply voltage of 10 V at $t = 0$. Find $i(t)$ using Laplace transform. Assume 0 initial current through inductor & 0 initial voltage across capacitor.



At $t=0$, applying KVL.

$$5i(t) + 1\cdot\frac{di(t)}{dt} + \frac{1}{0.25}\int i(t)\,dt = 10$$

L.T.

$$5I(s) + sI(s) - \cancelto{0}{i(0^+)} + 4\left(\frac{I(s)}{s} + \cancelto{0}{\frac{q(0^+)}{s}}\right) = 10\cdot\frac{1}{s}$$

$$\Rightarrow 5I(s) + sI(s) + 4\frac{I(s)}{s} = \frac{10}{s}$$

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or, $5sI(s) + s^2 I(s) + 4\,I(s) = 10$

$$\therefore I(s) = \frac{10}{s^2 + 5s + 4}$$

$$\therefore I(s) = \frac{10}{(s+4)(s+1)}$$

Partial fraction,

$$\frac{10}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$$

$$A = \left.\frac{10}{s+1}\right|_{s+4=0 \Rightarrow s=-4}$$

$$A = \frac{10}{-3}$$

$$B = \left.\frac{10}{s+4}\right|_{s+1=0 \Rightarrow s=-1}$$

$$B = \frac{10}{3}$$

$$\therefore I(s) = \frac{-10}{3}\left(\frac{1}{s+4}\right) + \frac{10}{3}\left(\frac{1}{s+1}\right)$$

I.L.T.

$$i(t) = -\frac{10}{3} e^{-4t} + \frac{10}{3} e^{-t}$$ Ans

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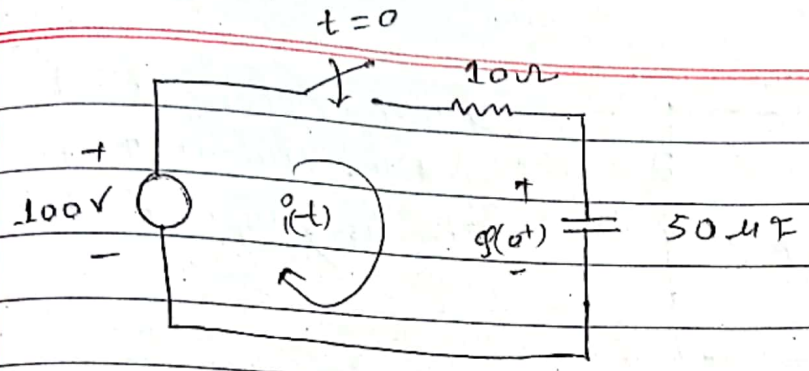
Q.



Find i(t) using L.T. $q_0(0^+) = 2.5\,\mu C$.

At t = 0, applying KVL.

$$10\, i(t) + \frac{1}{50\times10^{-6}} \int i(t)\, dt = 100$$

Using L.T.

$$10\, I(s) + 20000 \left(\frac{I(s)}{s} + \frac{q(0^+)}{s} \right) = \frac{100}{s}$$

or, $$10\, I(s) + 2^4 \left(\frac{I(s)}{s} + \frac{2.5\times10^{-6}}{s} \right) = \frac{100}{s}$$

or, $$10s\, I(s) + 2^4\, I(s) + 4\times10^{-5} = 100$$

or, $$I(s)\,(10s + 2^4) = 100$$

or, $$I(s) = \frac{100}{10s + 2^4}$$

$$= \frac{100}{10(s + 2^3)}$$

$$= \frac{10}{s + 2^3}$$

ILT.

$$i(t) = 10 \cdot e^{-2^3 t}$$

$$i(t) = 10\, e^{-2000t}$$

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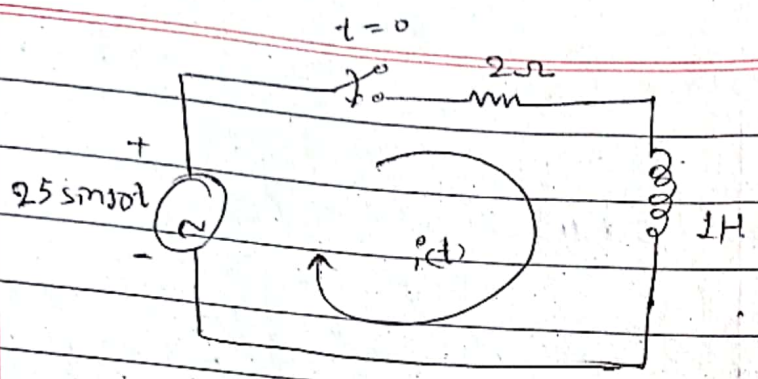
Q.



switch is closed at t = 0
Find the current response i(t) using laplace transform

Sol^n:

At t = 0 using KVL.

$$2\,i(t) + 1 \cdot \frac{d\,i(t)}{dt} = 25 \sin 10t$$

Applying L.T.

$$2\,I(s) + s\,I(s) - i(0+) = 25 \cdot \frac{10}{s^2+100} = \frac{250}{s^2+100}$$

or, $$2\,I(s) + s\,I(s) = \frac{250}{s^2+100}$$

or, $$I(s) = \frac{250}{(s+2)(s^2+100)}$$

partial fraction,

$$\frac{250}{(s+2)(s^2+100)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+100}$$

$$\frac{250}{(s+2)(s^2+100)} = \frac{(s^2+100)A + Bs + C(s+2)}{(s+2)(s^2+100)}$$

or, $$250 = As^2 + 100A + Bs^2 + 2Bs + Cs + 2C$$

or, $$(A+B)s^2 + (2B+C)s + 100A + 2C = 250$$

Equating coefficients,

$$A + B = 0 \quad \text{——— (1)}$$

$$2B + C = 0 \quad \text{——— (2)}$$

$$100A + 2C = 250 \quad \text{——— (3)}$$

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Solving ① ② & ③

$A = 125/52$

$B = -125/52$

$C = 125/26$

$$\therefore I(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+100}$$

$$= \frac{125}{52}\left(\frac{1}{s+2}\right) + \frac{-125/52\, s + 125/26}{s^2+100}$$

$$= \frac{125}{52}\cdot\left(\frac{1}{s+2}\right) - \frac{125}{52}\left(\frac{s}{s^2+10^2}\right) + \frac{125}{26}\left(\frac{1}{s^2+100}\right)$$

$$= \frac{125/52}{s+2} - \frac{125/52\, s}{s^2+10^2} + \frac{12.5/26 \cdot 10}{s^2+10^2}$$

Using I.L.T.

$$i(t) = \frac{125}{52} e^{-2t} - \frac{125}{52}\cos 10t + \frac{12.5}{26}\cdot \sin 10t$$

A

$$\boxed{\sin(\omega t+\phi) \xrightarrow{L.T.} \frac{s\sin\phi + \omega\cos\phi}{s^2+\omega^2}}$$

Q. A RL series ckt is connected to ac supply voltage $v(t) = 80\cos(400t + 20)$, $R = 10\Omega$, & $L = 0.05$ H at $t=0$. The switch K is closed. find $i(t)$ using LT method.

$\cos(\omega t + \phi) \xrightarrow{L.T.} \dfrac{s\cos\phi - \omega\sin\phi}{s^2 + \omega^2}$

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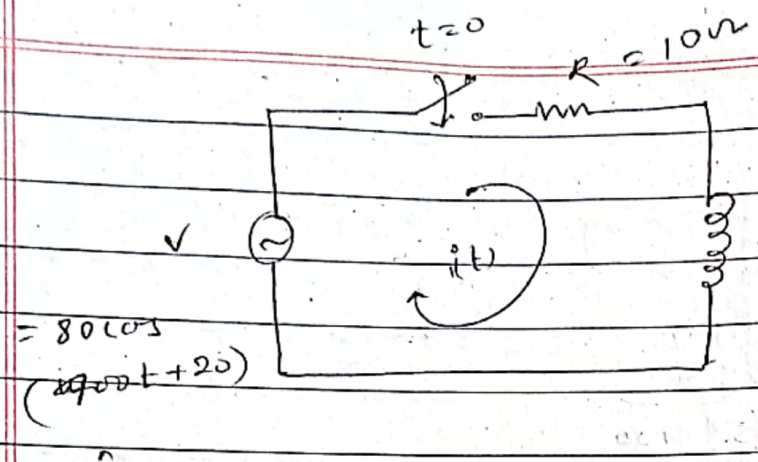

Soln. At $t = 0$ applying KVL.

$$10\, i(t) + 0.05 \frac{d\, i(t)}{dt} = 80 \cos(400t + 20)$$

using L.T.

$$10\, I(s) + 0.05\, [s\, I(s) - i(0^+)] = 80 \cdot \left(\frac{s\cos 20 - 400 \sin}{s^2 + 400^2} \right.$$

or, $$10\, I(s) + 0.05 s\, I(s) = 80 \cdot \frac{s\cos 20 - 400 \sin 20}{s^2 + 400^2}$$

or, $$I(s) = \frac{80 \cdot s\cos 20 - 400 \sin 20}{(s^2 + 400^2) \cdot (10 + 0.05 s)}$$

Now,

$$\frac{80 s \cos 20 - 32000 \sin 20}{(s^2 + 400^2)(10 + 0.05 s)} = \frac{A}{10 + 0.05 s} + \frac{Bs + C}{s^2 + 400^2}$$

or, $$75.17 s - 10944.64 = A s^2 + A \cdot 400^2 + 10 B s + 0.05 + 10 C + 0.05 s C$$

equating coefficient

or,

$$A + 0.05 B = 0 \quad \text{—— (1)}$$

$$10 B + 0.05 C = 75.17 \quad \text{—— (2)}$$

$$160000 A + 10 C = -10944.64 \quad \text{—— (3)}$$

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solving (1) (2) & (3)

$A = -0.13$

$B = 2.6$

$C = 983.8$

Now

$$I(s) = \frac{-0.13}{10 + 0.05s} + \frac{2.6s + 983.8}{s^2 + 400^2}$$

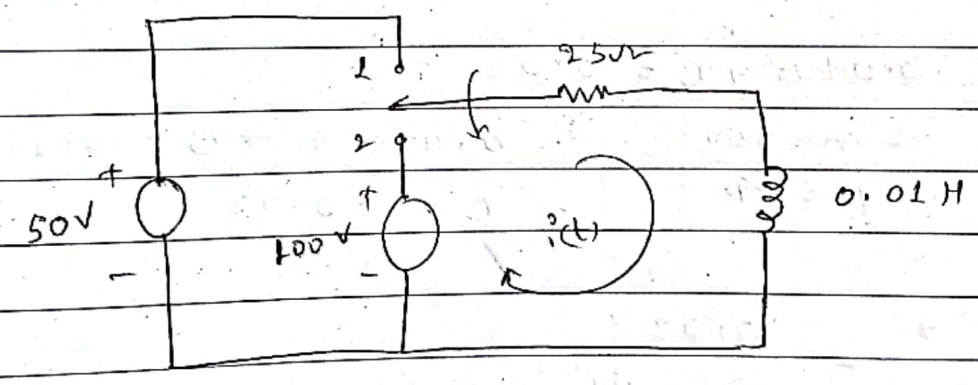
$$I(s) = \frac{-2.6}{s + 200} + \frac{2.6s}{s^2 + 400^2} + \frac{983.8}{400} \cdot \frac{400}{s^2 + 400^2}$$

using I.L.T.

$$i(t) = -2.6e^{-200t} + 2.6\cos 400t + 2.46\sin 400t$$

Ans.

Q.



switch move from 1 to 2 at $t=0$. Find $i(t)$ using L.T.

At $t=0$ KVL.

$$25\,i(t) + 0.01\frac{di(t)}{dt} = 100$$

using L.T.

initial current

$$25\,I(s) + 0.01\left[s\,I(s) - i(0+)\right] = \frac{100}{s}$$

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Initially current in inductor $i(0+) = 50/25 = 2\,A$.

So,

$$25\,I(s) + 0.01(s\,I(s) - 2) = \frac{100}{s}$$

or

$$25\,I(s) + 0.01s\,I(s) - 0.02 = \frac{100}{s}$$

or $$25\,I(s) + 0.01s\,I(s) = \frac{100}{s} + 0.02 = \frac{100 + 0.02s}{s}$$

or $$I(s) = \frac{100 + 0.02s}{s(25 + 0.01s)}$$

$$\frac{100 + 0.02s}{s(25 + 0.01s)} = \frac{A}{s} + \frac{B}{25 + 0.01s}$$

$$100 + 0.02s = 25A + 0.01sA + Bs$$

equating.

$$0.01A + B = 0.02$$

$$25A = 100$$

$$A = 4$$

$$0.01 \times 4 + B = 0.02$$

$$B = -0.02$$

$$I(s) = \frac{4}{s} - \frac{0.02}{25 + 0.01s}$$

$$= \frac{4}{s} - \frac{2}{s + 2500}$$

I.L.T.

$$i(t) = 4 - 2e^{-2500t}$$

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Transform Impedence

① For resistor

$$V(s) = I(s)\,R$$

$$\Rightarrow R = \frac{V(s)}{I(s)}$$

② For Inductor

$$v(t) = L\frac{di}{dt}$$

$\Downarrow$ L.T

$$V(s) = L\,[s\,I(s) - i(0+)]$$

$$V(s) = Ls\,I(s) - L\,i(0+)$$

Let the voltage be

$$V_1(s) = Ls\,I(s)$$

$$Z = \frac{V_1(s)}{I(s)} = Ls \quad \Omega$$

↑ impedance.

③ For capacitor

$$v(t) = \frac{1}{C}\int i(t)\,dt$$

$\Downarrow$ L.T.

$$V(s) = \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0+)}{s}\right]$$

$$V(s) = \frac{I(s)}{Cs} + \frac{q(0+)}{Cs}$$

If $V_1(s) = \dfrac{I(s)}{Cs}$

impedance, $Z = \dfrac{V_1(s)}{I(s)} = \dfrac{1}{Cs}\ \Omega$.

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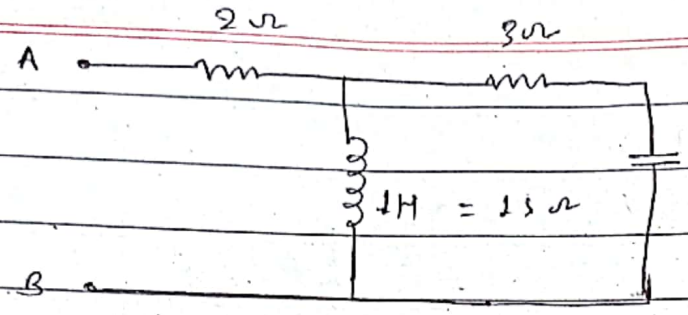
Q.



Find the transform impedance across AB terminals.

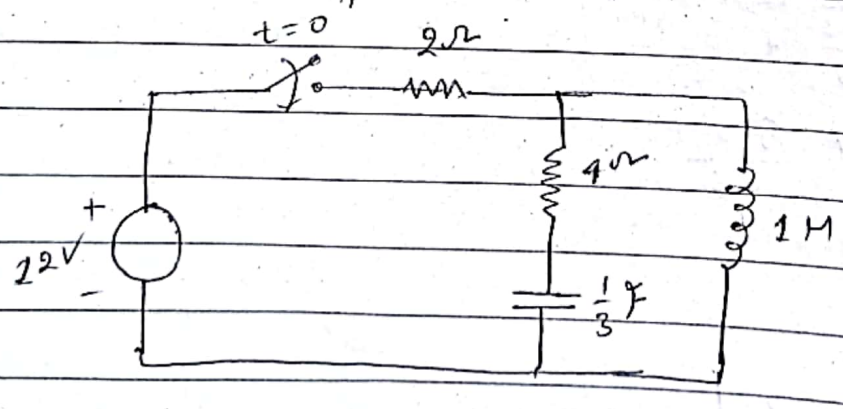
$$3 + \frac{1}{2s} = \frac{6s+1}{2s}$$

$$\frac{\frac{6s+1}{2s} \times 1s}{\frac{6s+1}{2s} + 1s} = \frac{(6s+1)2s}{2\cdot(6s+1+2s^2)} = \frac{6s^2+s}{2s^2+6s+1}$$

$$Z = 2 + \frac{6s^2+s}{2s^2+6s+1} = \frac{4s^2+12s+2+6s^2+s}{2s^2+6s+1}$$

$$= \frac{10s^2+13s+2}{2s^2+6s+1}$$

Q. Using laplace transform method, find the current through inductor and capacitor for $t > 0$

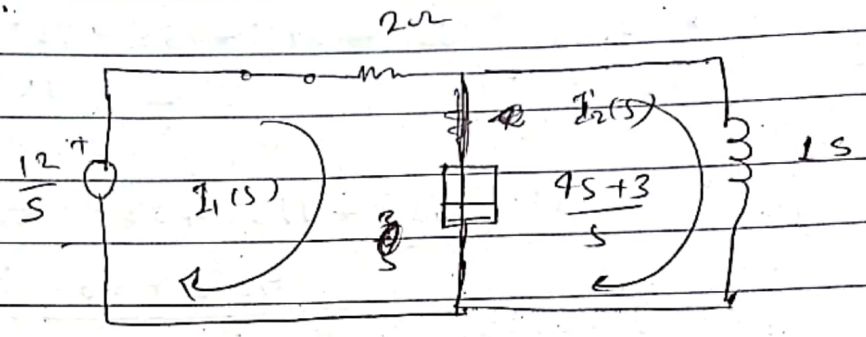


At $t=0$, switch is closed.

$\frac{1}{3}F = \frac{3}{s}\,\Omega$

$2H = 15\,\Omega$

$12V = \frac{12}{s}\,V$



In mesh 1,

$$-2I_1(s) - \frac{4s+3}{s}\left(I_1(s) - I_2(s)\right) + \frac{12}{s} = 0$$

or, $$2I_1(s) + \frac{4s+3}{s} I_1(s) - \frac{4s+3}{s} I_2(s) = \frac{12}{s}$$

or, $$I_1(s)\left(\frac{6s+3}{s}\right) - I_2(s)\left(\frac{4s+3}{s}\right) = \frac{12}{s}$$

or, $$I_1(s)(6s+3) - I_2(s)(4s+3) = 12 \quad \text{——— (1)}$$

In 2nd mesh,

$$-15 I_2(s) - \frac{4s+3}{s}\left(I_2(s) - I_1(s)\right) = 0$$

$$15 I_2(s) + \frac{4s+3}{s} I_2(s) - \frac{4s+3}{s} I_1(s) = 0$$

$$-\frac{4s+3}{s} I_1(s) + I_2(s)\left(\frac{s^2+4s+3}{s}\right) = 0$$

$$-(4s+3) I_1(s) + (s^2+4s+3) I_2(s) = 0 \quad \text{——— (2)}$$

$$I_1(s) = \frac{(s^2+4s+3)}{4s+3} I_2(s)$$

from (1) $$\frac{(s^2+4s+3)(6s+3)}{4s+3} I_2(s) - (4s+3) I_2(s) = 12$$

or, $$I_2(s)\left[(6s+3)(s^2+4s+3) - (4s+3)^2\right] = 12(4s+3)$$

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or, $I_2(s)\left[(6s^3+24s^2+18s+3s^2+12s+9)-16s^2-24s-9\right]$
$= 12(4s+3)$

or, $I_2(s)\,(6s^3+11s^2+6s) = 48s+36$

$$I_2(s) = \frac{48s+36}{6s^3+11s^2+6s} = \frac{12(4s+3)}{6s^3+11s^2+6s}$$

$$I_1(s) = \frac{(s^2+4s+3)}{4s+3} \times \left(\frac{48s+36}{6s^3+11s^2+6s}\right)$$

$$I_1(s) = \frac{12(s^2+4s+3)}{6s^3+11s^2+6s}$$

current through capacitor $= I_1(s) - I_2(s)$

$$= \frac{12(s^2+4s+3)}{6s^3+11s^2+6s} - \frac{12(4s+3)}{6s^3+11s^2+6s}$$

$$= \frac{12\left[\cancel{(4s+3)}+(s^2+\cancel{4s+3})\right]}{6s^3+11s^2+6s}$$

$$= \frac{12\,(\cancel{4s^3+19s^2+24s+9})}{\cancel{6s^3+11s^2+6s}}$$

$$= \frac{12s^2}{6s^3+11s^2+6s}$$