Lecture 10 - Linear Algebra

Sep 24, 2021

Revision.

- Let V be a vector space of dimension n.

  $\mathcal{B}_V = (v_1, .., v_n)$ basis of V.

- F^n : Space of n-dimensional column vector

Theorem. $V \cong F^n$

$$\varphi : F^n \longrightarrow V$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \varphi(X) = \mathcal{B}_V \cdot X = (v_1, .., v_n) \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We verified. φ is well-defined, one-one, onto, and

$$\varphi(X+Y) = \varphi(X) + \varphi(Y)$$

$$\varphi(cX) = c\,\varphi(X)$$

φ is an isomorphism

Corollary. Every vector space V of dimension n is isomorphic to F^n. $V \cong F^n$

$(v_1, \ldots, v_n)$ ordered set of vectors

$$\underset{\mathcal{B}}{\overset{\|}{(v_1, \ldots, v_n)}} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underset{\mathcal{B}X}{\overset{\|}{v_1 x_1 + \cdots + v_n x_m}}$$

Proposition. Let $S = (v_1, \ldots, v_m)$, $\mathcal{V} = (w_1, \ldots, w_n)$ ordered set of vectors in V.

Then

$$\underset{\forall i}{w_i \in \operatorname{Span}(S)} \iff \exists\ A : m \times n \text{ matrix}$$

s.t. $(v_1, \ldots, v_m) \cdot A = (w_1, \ldots, w_n)$

Proof.

$\Longrightarrow$ Assume that $w_i \in \operatorname{Span}(S)$ for all i.

$$w_1 = a_{11} v_1 + \cdots + a_{\cancel{1m}\,1n} v_m$$

$$\vdots$$

$$w_n = a_{1n} v_1 + \cdots + a_{\cancel{1n}\,mn} v_m$$

then choose $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$

then $(v_1, \ldots, v_m) = A \cdot (w_1, \ldots, w_n)$

$\Longleftarrow$ (Easy)

$$\left.\begin{aligned} \mathcal{B}_V &= (v_1, \dots, v_n) \\ \mathcal{B}_V' &= (v_1', \dots, v_n') \end{aligned}\right\}$$

Basis for V

Bases of V.

(How are they related?)

Then

$$v_i \in \operatorname{Span}(\mathcal{B}_V')$$

$\iff$ $\exists$ an $n \times n$ matrix, say P, s.t.

$$\underbrace{(v_1', \dots, v_n')}_{\mathcal{B}_V'} \cdot P = (v_1, \dots, v_n) \quad \text{~~~(i)}$$

$$= \mathcal{B}_V$$

(Matrix of change of basis)

Claim. P is invertible matrix.

Interchanging the role of $\mathcal{B}_V$ and $\mathcal{B}_V'$, we get

$$(v_1, \dots, v_n) \cdot P' = (v_1', \dots, v_n') \quad \sim \text{(ii)}$$

$$\mathcal{B}_V \cdot P' = \mathcal{B}_V'$$

Using (i) and (ii), we get

$$\boxed{\mathcal{B}_V P' P = \mathcal{B}_V}$$

$$(v_1, \dots, v_n) \cdot [\;?\;] = (v_1, \dots, v_n)$$

Matrix

Since $\mathcal{B}_V = (v_1, \dots, v_n)$ is L.I, there is only one way to write v_i as a linear combination of $(v_1, \dots, v_n)$,

namely $v_i = v_i$, or

$$\mathcal{B}_V \cdot I_n = \mathcal{B}_V$$

$$\Rightarrow \quad P' P = I_n$$

P is invertible matrix.

Question. How to compute the matrix of change of basis ?

Given $\mathcal{B}_V$ and $\mathcal{B}_V'$, we have

$$\mathcal{B}_V = \mathcal{B}_V' \cdot P \qquad \left(\text{Here } \mathcal{B}_V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \text{ \& } \mathcal{B}_V' = \begin{bmatrix} | & & | \\ v_1' & \cdots & v_n' \\ | & & | \end{bmatrix} \right)$$

$$\Rightarrow \quad P = \left[\mathcal{B}_V' \right]^{-1} \cdot \mathcal{B}_V$$

In particular, choose $\mathcal{B}_V = E = (e_1, \dots, e_n)$ standard basis

then $$I_n = \mathcal{B}_V' \cdot P$$

$$\Rightarrow \quad P = \left[\mathcal{B}_V' \right]^{-1}.$$

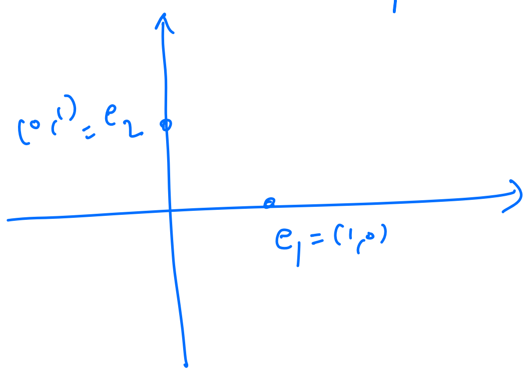
Problems. Textbook.

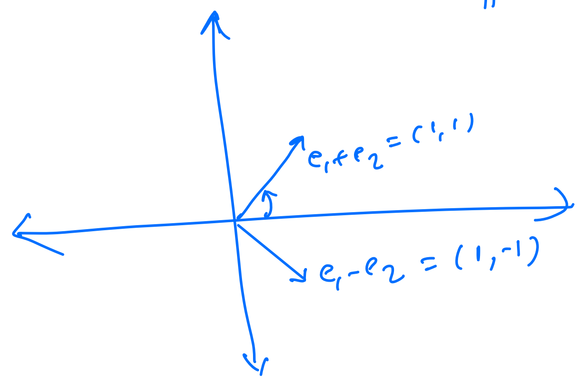
$\mathbb{R}^2$

$\mathcal{B}_1 = (e_1, e_2)$

$\mathcal{B}_2 = (e_1+e_2, e_1-e_2)$

$\mathbb{R}^2$

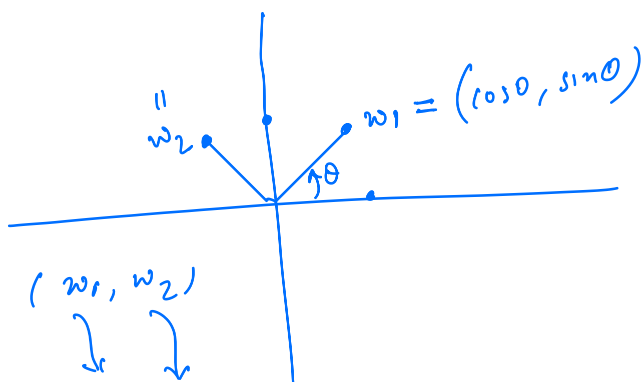




$$\mathcal{B}_1 = \mathcal{B}_2 \cdot P$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot P$$

$$\mathcal{B}_2^{-1} = P$$



(e_1, e_2) $\qquad$ (w_1, w_2)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \underline{\underline{P}}$$

Discussion.

$$\varphi : F^n \longrightarrow V$$
$$X \longmapsto \mathcal{B} \cdot X = v$$

Let X be the co-ordinate vector of v, computed w.r.t. old basis $\mathcal{B}_V$, that is, $\boxed{\mathcal{B} \cdot X = v}$

On the other hand, $\mathcal{B}_V$ and $\mathcal{B}_V'$ are related by

$$\boxed{\mathcal{B}_V' \cdot P = \mathcal{B}_V.}$$

$$\varphi : F^n \longrightarrow V$$
$$Y \longmapsto \mathcal{B}_V' \cdot Y = v$$

How are X and Y related ?

Thus
$$\begin{aligned} v &= \mathcal{B}_V \cdot X \\ &= (\mathcal{B}_V' \cdot P) X \\ &= (\mathcal{B}_V')(PX) \end{aligned}$$

Conclusion. If X is the co-ordinate vector of v w.r.t. basis $\mathcal{B}_V$, then PX is the co-ordinate ~~of~~ vector of v w.r.t. basis $\mathcal{B}_V'$.

Generating new basis :

Let V be a vector space of dimension n. Let

$\mathcal{B}_V = (v_1, \dots, v_n)$ be a basis for V.

Pick an invertible matrix, say P, and

Define: $\boxed{\mathcal{B}' = \mathcal{B}_V \cdot P^{-1}}$ claim: $\mathcal{B}'$ is a basis.

Question. Why $\mathcal{B}'$ is a basis ?

Re-writing the expression, $\boxed{\mathcal{B}' \cdot P = \mathcal{B}_V}$

$$\underset{(w_1, \dots, w_n)}{\mathcal{B}'} \cdot P = \underset{(v_1, \dots, v_n)}{\mathcal{B}_V}$$

some w_i's.

note. $v_i \in \mathrm{Span}(\mathcal{B}')$ for all $i = 1, \dots, n$.

and $|\mathcal{B}'| = n$

$\Rightarrow$ $\mathcal{B}'$ is a basis.

Corollary. Let $\mathcal{B}$ be a basis of V. The other bases are the sets of the form $\mathcal{B} \cdot P^{-1}$, where $P \in GL_n(F)$.

We may also write it as

$$\mathcal{B}_V' = \mathcal{B} \cdot Q, \quad \text{where } Q \in GL_n(F).$$

Section 5. Infinite-dimensional spaces.

Let V be a vector space over F. If there does not exist any finite set which can span V, then V is called infinite-dimensional.

Example.

(1). $\left(\underset{\parallel}{\mathcal{C}([0,1])}, +, \cdot \right)$ space of real valued continuous functions on interval $[0,1]$.

$\{ f : [0,1] \to \mathbb{R} \text{ s.t. } f \text{ is continuous} \}$

$$\left\{ 1, x, x^2, x^3, \ldots, x^n, \ldots \right\}$$

$\{ (\underbrace{a_1, a_2, a_3}_{\text{integer}}) \mid a_i \in \mathbb{R} \}$

(2). $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots \longrightarrow, \mathbb{R}^\infty$

finite. dim v.s. over $\mathbb{R}$ — co-ordinate wise

$$\mathbb{R}^\infty := \{ (a) \text{ s.t. } a_i \in \mathbb{R} \}$$

$(a) = (a_1, a_2, \ldots)$

sequence: $\mathbb{N} \to \mathbb{R}$, $n \mapsto a_n$

Think of "Space of sequence of real numbers".

Subspaces of infinite-dimensional vector spaces

Important examples.

(1). $C = \{ (a) \in \mathbb{R}^\infty \text{ s.t. } \lim_{n\to\infty} a_n \text{ exists} \}$

"CONVERGENT SEQUENCES".

(i) $(a), (b) \in C$ $\quad v, w \in W$

$v + w \in W$

Want: $(a+b)$ in C.

$$\underbrace{(a)}_{v} + \underbrace{(b)}_{w} := (a_1+b_1, \dots, a_n+b_n, \dots)$$

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists, then

$$\lim_{n\to\infty} (a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \text{ also exists}$$

(ii) Let $\alpha \in \mathbb{R}$, and $(a) \in C$, then

$$\alpha \cdot (a) = (\alpha a_1, \dots, \alpha a_n, \dots)$$

If $\lim_{n\to\infty} a_n$ exists, then so is $\lim_{n\to\infty} \alpha a_n = \alpha \cdot \lim_{n\to\infty} a_n$.

(iii) $(0) \in \mathbb{R}^\infty$

(2). $\ell^\infty = \{ (a) \in \mathbb{R}^\infty \text{ s.t. } \{a_n\} \text{ is bounded} \}$

" BOUNDED SEQUENCES ".

Exercise : ℓ^∞ is a subspace of $\mathbb{R}^\infty$

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(3). $\ell^1 = \{ (a) \in \mathbb{R}^\infty \text{ s.t. } \sum_{n=1}^{\infty} |a_n| \underbrace{< \infty}_{\text{means finite.}} \}$

" Absolutely convergent series ".

Exercise : ℓ^1 is a subspace of $\mathbb{R}^\infty$.

(4). $Z = \{ (a) \in \mathbb{R}^\infty \text{ s.t. } a_n = 0 \text{ for all but finitely many } n \}$

" Sequences with finitely many nonzero terms ".

Exercise : Z is a subspace of $\mathbb{R}^\infty$

Exercise. $c, \ell^\infty, \ell^1, Z$ are infinite-dimensional subspaces of $\mathbb{R}^\infty$.

Notion of span of an infinite set; say S, of vector space V.

$$\underline{\underline{\text{Span}\,(S)}} \quad ??$$

$S = (v_1, v_2, \dots\dots)$

We may try to write as

$$c_1 v_1 + c_2 v_2 + \dots\dots + c_n v_{n} + \dots\dots$$

Suppose, $S = (v_1, v_2, \dots)$, and $V = \mathbb{R}^\infty$, $S \subset V$. Here $v_1 \in \mathbb{R}$ and $v_2 \in \mathbb{R}$.

then $\boxed{c_1 v_1 + \dots + c_n v_n + \dots}$, where $c_1 \in \mathbb{R}$ and $c_n \in \mathbb{R}$.

" may not make sense "

$1+1+1+1+\dots$ is not a real number

[Expression does not make sense]

Definition. The span of an infinite set S is the set of those vectors v which are linear combination of finitely many elements of S.

$v = c_1 v_1 + \dots + c_r v_r$, where $v_1, \dots, v_r \in S$.

some vector

r depends upon v.

$$\text{Span}\,(S) = \{\text{finite linear combinations of elements of } S\}.$$

Example.

(1). Set $e_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i^{th} \text{ position}}}{1}, 0, \ldots) \in \mathbb{R}^{\infty}$.

Let $S = (e_1, e_2, e_3, \cdots)$ be the infinite set of vectors.

Question. Is it true that $Span(S) = \mathbb{R}^{\infty}$?

NO

Observe that

$$(1, 1, \ldots) \neq c_{i_1} \cdot e_{i_1} + \cdots + c_{i_r} e_{i_r}$$

$(\cdot\ 0\ 0\ 1, 0, 1, 0, 0, 1, 1, 0\ 0 \cdots 0)$ for any finite collection $\{e_{i_1}, \ldots, e_{i_r}\}$ in S.

$$Span(S) \neq \mathbb{R}^{\infty}$$

Remark. $\boxed{Span(S) = Z.}$

Definition.

A set S, infinite or not, is called linearly independent if there is no finite relation

$$c_1 v_1 + \cdots + c_r v_r = 0, \qquad v_1, \ldots, v_r \in S$$

except for the trivial relation, i.e. $c_1 = \cdots = c_r = 0$.

Definition. A basis S of vector space V is linearly independent set which spans V.

Theorem. Every vector space (over a field) has a basis. "1st non-trivial statement"

[Part of E_2]. One needs to apply "Axiom of Choice."

Proposition. Let V be finite-dimensional, and let S be any set which spans V. Then S contains a finite subset which spans V.

Proof. Since $\dim V < \infty \Rightarrow \exists$ finite set, say $(v_1, \dots, v_n)$ s.t. $\mathrm{Span}(v_1, \dots, v_n) = V$.

Now, S also spans V.

$v_1 = \mathrm{Span}($ finite linear combination of elements of $S)$

$\vdots$

$v_n = \mathrm{span}($ finite linear combination of elements of $S)$

Collecting elements of S needed to describe $v_1, \dots, v_n$, say $S' \subseteq S$.

then $\underbrace{(v_1, \dots, v_n)}_{= V} \subset \mathrm{Span}(S')$

$\Rightarrow$ S' also spans V

$\uparrow$ finite set.

Proposition. Let V be a finite-dimensional vector space.

(a) Every set S which spans V contains a finite basis.

(b) Every linearly independent set L is finite, and therefore extends to a finite basis.

(c) Every basis is finite.

(Exercise.)

Section 6. DIRECT SUMS

Let V be a vector space, and let $W_1, \dots, W_n$ be subspaces of V.

Consider vectors $v \in V$ which can be written as a sum

$$v = w_1 + \dots + w_n \; ; \quad \text{where } w_i \in W_i .$$

The set of all such vectors is called the sum of the subspaces, and is denoted by

$$W_1 + \dots + W_n = \{ v \in V \text{ s.t. } v = w_1 + \dots + w_n , \text{ with } w_i \in W_i \} .$$

Observation. (i) $W_1 + \dots + W_n$ is a subspace of V

End of lecture 10

Definition. The subspaces $W_1, \dots, W_n$ are called **independent** if no sum $w_1 + \dots + w_n$ with $w_i \in W_i$ is zero, except for the trivial sum, i.e.

$$w_1 + \dots + w_n = 0 \text{ and } w_i \in W_i \Rightarrow w_i = 0 \ \forall i.$$

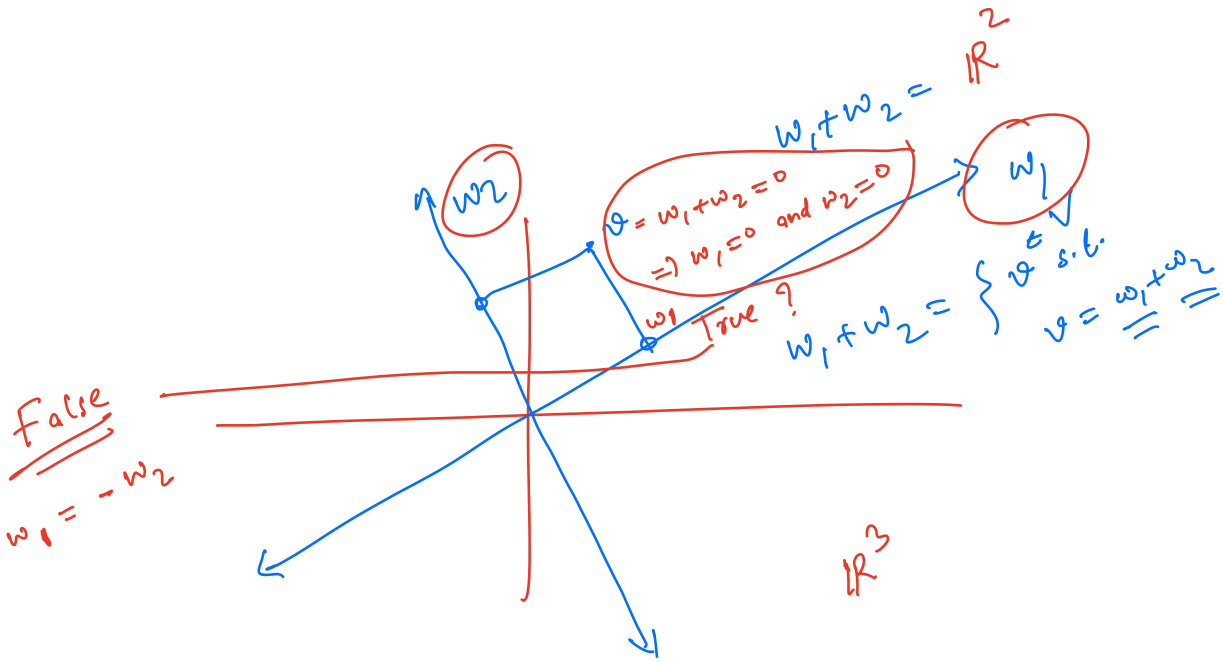
Definition. If subspaces $W_1, \dots, W_n$ are independent and their span is the whole space V, then we say that V is the **direct sum** of $W_1, \dots, W_n$.

$$V = W_1 \oplus \dots \oplus W_n \quad \text{if } V = W_1 + \dots + W_n \text{ and}$$
$$\text{if } W_1, \dots, W_n \text{ are independent.}$$

This is equivalent to saying that

every vector $v \in V$ can be written as

$v = w_1 + \dots + w_n$ in exactly one way.


$W_1 + W_2 = \mathbb{R}^2$
W_2
W_1
$\vartheta = w_1 + w_2 = 0$
$\Rightarrow w_1 = 0$ and $w_2 = 0$
w_1
True ?
$W_1 + W_2 = \{ \vartheta$ s.t.
$\vartheta = w_1 + w_2$
False
$w_1 = -w_2$
$\mathbb{R}^3$

Discussion.

$W_1 + \cdots + W_n$ is a subspace of V.

If $W_1, \ldots, W_n$ are subspaces (independent) and

$W_1 + \cdots + W_n \neq V$, then let

$U = W_1 + \cdots + W_n$, subspace of V.

Here, U is the direct sum of $W_1, \ldots, W_n$,

$U = W_1 \oplus \cdots \oplus W_n$.

Proposition.

(a). A single subspace W_1 is independent.

(b). Two subspaces W_1, W_2 are independent if and only if $W_1 \cap W_2 = (0)$.

Proof. (a) "Easy".

(b) W_1 and W_2 are independent $\Longleftrightarrow$ $W_1 \cap W_2 = (0)$.

(A) $\Longleftrightarrow$ (B).

Proof.

$(\Rightarrow)$ We will prove this by $\neg(B) \Rightarrow \neg(A)$.

$$\uparrow$$

$$W_1 \cap W_2 \neq (0)$$

Take $v \in W_1 \cap W_2$.

Note that we can always write

$$\begin{array}{c} 0 = v + (-v) \\ \| \\ (-v) \\ + \\ v \end{array}$$

Thus 0 vector is written in two different ways,

$\Rightarrow$ W_1 and W_2 are not independent.

$(\Leftarrow)$ $(A) \overset{\Leftarrow}{\cancel{\Rightarrow}} (B)$ $W_1 \cap W_2 = (0)$

$$\|$$

W_1 & W_2 are independent

Let $w_1 + w_2 = 0$; $w_1 \in W_1$ and $w_2 \in W_2$.

This implies $\left.\begin{array}{l} w_1 = -w_2 \\ \text{and } w_2 = -w_1 \end{array}\right\} \Rightarrow \begin{array}{l} w_1 \in W_1 \cap W_2 \text{ and} \\ w_2 \in W_1 \cap W_2 \end{array}$

But $W_1 \cap W_2 = (0)$ $\Rightarrow$ $w_1 = 0$ and $w_2 = 0$.

Hence W_1 and W_2 are independent.

Proposition. Let $W_1, \dots, W_n$ be subspaces of a finite-dimensional vector space V, and let $\mathcal{B}_i$ be a basis for W_i.

(a) The ordered set $\mathcal{B}$ obtained by listing the bases $\mathcal{B}_1, \dots, \mathcal{B}_n$ in order is a basis of V if and only if $V = W_1 \oplus \cdots \oplus W_n$.

(b) $$\dim(W_1 + \cdots + W_n) \leq (\dim W_1) + \cdots + (\dim W_n),$$

with equality if and only if the spaces are independent.

Corollary. Let W be a subspace of a finite-dimensional vector space V. There is another subspace W' such that $V = W \oplus W'$.

Proof. Let $(w_1, \dots, w_d)$ be a basis for W.

We extend to a basis $(w_1, \dots, w_d, v_1, \dots, v_{n-d})$ for V.

$\operatorname{Span}(v_1, \dots, v_{n-d})$ is the required subspace W'.

Proposition. Let W_1, W_2 be subspaces of a finite-dimensional vector space V. Then

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Proof. Re-write above relation as

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Assume that $\dim W_1 = m$ and $\dim W_2 = n$, for some $m, n \in \mathbb{N}$.

Observe that $W_1 \cap W_2 \subseteq W_1$, and

$W_1 \cap W_2 \subseteq W_2$.

Also, $W_1 \cap W_2$ is a subspace of V, hence finite-dimensional.

Choose $\mathcal{B}_1 = (u_1, \dots, u_r)$, basis for $W_1 \cap W_2$, $r = \dim(W_1 \cap W_2)$.

Extend $\mathcal{B}_1$ to get a basis for W_1:

$\mathcal{B}_1' = (u_1, \dots, u_r; x_1, \dots x_{m-r})$, $m = \dim W_1$

Similarly, extend $\mathcal{B}_1$ to get a basis for W_2

$\mathcal{B}_1'' = (u_1, \dots, u_r; y_1, \dots, y_{n-r})$, $n = \dim W_2$

To prove the proposition, it is enough to show that $\mathcal{B} = (u_1, \ldots, u_r; x_1, \ldots, x_{m-r}; y_1, \ldots, y_{n-r})$ is a basis for $W_1 + W_2$.

We need to show

(i) $\mathcal{B}$ is linearly independent;

(ii) $\text{Span}(\mathcal{B}) = W_1 + W_2$.

Proof of (i)

Suppose $\mathcal{B}$ is linearly dependent, then

A

$$a_1 u_1 + \cdots + a_r \cdot u_r + b_1 x_1 + \cdots + b_{m-r} x_{m-r} + c_1 y_1 + \cdots + c_{n-r} y_{n-r} = 0$$

where some scalars are non-zero.

In short,

$$u + x + y = 0.$$

$$\Rightarrow y = -u - x \in W_1.$$

Also, $y \in W_2 \Rightarrow y \in W_1 \cap W_2$

Then y is a linear combination of $(u_1, \ldots, u_r)$

$y = d_1 u_1 + \cdots + d_r u_r$ for some d_i; $i = 1, \ldots, r$

$$y - (d_1 u_1 + \cdots + d_r u_r) = 0$$

or, $$c_1 y_1 + \cdots + c_{n-r} y_{n-r} + (-d_1) u_1 + (-d_2) u_2 + \cdots + (-d_r) u_r = 0$$

Recall $(y_1, \ldots, y_{n-r}; u_1, \ldots, u_r)$ is a basis for W_2

$$\Rightarrow \quad y = 0$$

Thus our original relation reduces to

$$u + x = 0.$$

Again, since $(u_1, \ldots, u_r; x_1, \ldots, x_{m-r})$ is a basis for W_1

$\Rightarrow$ all scalars are zero

$\Rightarrow$ $u = 0$ and $x = 0$

Thus whole relation in equ. (A) was trivial, and hence $\mathcal{B}$ is a basis.

Proof of (ii).

For any vector v in $W_1 + W_2$ is of the form :

$$v = w_1 + w_2, \quad w_1 \in W_1, \; w_2 \in W_2.$$

$$w_1 = a_1 u_1 + \cdots + a_r u_r + b_1 x_1 + \cdots + b_{m-r} x_{m-r},$$

$$w_2 = a_1' u_1 + \cdots + a_r' u_r + c_1 y_1 + \cdots + c_{n-r} y_{n-r}$$

Then

$$w_1 + w_2 = (a_1 + a_1') u_1 + \cdots + (a_r + a_r') u_r$$
$$+ b_1 x_1 + \cdots + b_{m-r} x_{m-r}$$
$$+ c_1 y_1 + \cdots + c_{n-r} y_{n-r}.$$

Thus any $v \in W_1 + W_2$ is a linear combination of $\mathcal{B}$.