Lecture 1. Linear Algebra (MA4020)

Tuesday, August 17, 2021.

Basics.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ denotes the set of integers.

$\mathbb{N} = \{1, 2, 3, \cdots\}$ positive integers.

$\mathbb{N}_{\geq 0} = \{0\} \cup \mathbb{N}$ non-negative integers.

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$ rational numbers.

$\mathbb{R} :=$ set of real numbers.

$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} ; i^2 = -1\}$ set of complex numbers.

$\left.\mathbb{Z}^+ / \mathbb{Q}^+ / \mathbb{R}^+ \right\}$ positive elements of $\mathbb{Z} / \mathbb{Q} / \mathbb{R}$ respectively.

$$\left.\begin{array}{c} f : A \longrightarrow B \\ \text{or} \\ A \xrightarrow{f} B \end{array}\right\}$$ denote a function f from set A to a set B.

f is well-defined if

one-one if

onto if

familiarize yourself

Exercise

Binary Operation $(*)$.

A binary operation $*$ on a set G (non-empty) is a function

$$* : G \times G \longrightarrow G$$
$$(a, b) \longmapsto \underset{\substack{\| \\ *(a,b)}}{a * b} \in G$$

$$*(a,b) := a * b$$

Notation. f or $*$ or $\circ$ etc. are commonly used for binary operation.

~~$f : G$~~ $\quad f : G \times G \longrightarrow G \qquad f(a,b)$

$$(g_1, g_2) \longmapsto f(g_1, g_2) \in G.$$

or

$$\circ : G \times G \longrightarrow G \qquad \circ(a,b)$$

or

$$* : G \times G \longrightarrow G \qquad *(a,b)$$

Examples.

1. $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$(x, y) \longmapsto x \cdot y \in \mathbb{N}$$

$$(x, y) \longmapsto \max\{x, y\}$$

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

$$(x, y) \longmapsto \frac{x}{y} \times, \ \sin(xy) \times$$

2.. Define $f : Q \times Q \longrightarrow Q$

$$(x, y) \longmapsto \frac{x}{y} \ ; \ y \neq 0 \ \checkmark$$

3. $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

4. Let $GL_n(\mathbb{R})$ denote a set of all $n \times n$ invertible matrices over $\mathbb{R}$. Define

$$f : GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

$$(A, B) \longmapsto A \cdot B \ \checkmark$$

$$A - B \ \times$$

5. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$. Define

$$f : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$
$$(A, B) \longmapsto A + B$$

$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

6. Let $V := \mathbb{R}^2$. Define

$$f : \mathbb{R}^2 \times \mathbb{R}^2 \longmapsto \mathbb{R}^2$$
$$(v, w) \longmapsto v + w$$
$$(v, w) \longmapsto \cancel{v \cdot w}$$

X^2

Binary operation $\qquad f : X \times X \longrightarrow X$

Two - array

There can be operations like unary operation

$$f : X \longrightarrow X$$
$$x \longmapsto f(x) \in X$$

1. Define $f : \mathbb{N} \longrightarrow \mathbb{N}$

n-ary operations.

$$f : X \times X \times \cdots \times X \longrightarrow X$$
$$(x_1, x_2, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n) \in X$$

Uninversal algebra (X, F)

Set

consists of n-ary operations for some n.

We shall see in 3-4 lectures

Vector space (V) over some fixed field (K)

$+ : V \times V \longrightarrow V$ and $\cdot : V \longrightarrow V$

$(v, w) \longmapsto v + w$ $\quad v \longmapsto \alpha \cdot v$

vector addition (linear)

scalar multiplication $\alpha \in K$

Discussion.

Let $*$ be a binary operation on a set G.

I. Associative. $* : G \times G \longrightarrow G$ is associative

if for all $a, b, c \in G$, we have

$$a * (b * c) = (a * b) * c.$$

II. Abelian (commutative). $* : G \times G \longrightarrow G$ is abelian

if for all $a, b \in G$, we have $a * b = b * a$.

Examples.

1. $G := \underline{\underline{M_2(\mathbb{R})}}$ [set of all 2×2 matrices over $\mathbb{R}$]

$(G, +)$ Abelian.

$$* : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$$

$$(A, B) \longmapsto A \cdot B$$

Is it true,

$A * B = B * A$ (usual product)

$A * (B * C) = (A * B) * C$ (usual product)

n fixed. $\quad M_n(\mathbb{R})$ $\quad A(BC) = (AB)C.$

CHAPTER 1 MATRIX OPERATIONS

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{N}_{\geq 0} = \{0\} \cup \mathbb{N}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{R}$: set of real numbers

Let $m, n \in \mathbb{N}$. A $m \times n$ matrix is a collection of mn numbers arranged in a rectangular array:

$$\begin{array}{cc} & n \text{ columns} \\ m \text{ rows} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \end{array}.$$

$A = [a_{ij}]$ where i, j are indices (integers) with $1 \leq i \leq m$ and $1 \leq j \leq n$.

a_{ij} row index column index

a_{ij} entry

$$\begin{array}{cc} & j^{th}\text{ column} \\ i^{th}\text{ row} & \begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{bmatrix} \end{array}$$

- A $\underline{\underline{1 \times n}}$ matrix is called an n-dimensional row vector.

$$[\,-, -, \cdots, -\,]$$

- An $m \times 1$ matrix is called an m-dimensional column vector.

$$\begin{bmatrix} - \\ - \\ \vdots \\ - \end{bmatrix}$$

$$A = [a_1 \cdots a_n]$$

or

$$A = [a_1, \ldots, a_n]$$

or

$$A = (a_1, \ldots, a_n)$$

n-dimensional row vector

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

m-dimensional column vector.

Addition of matrices.

(i) $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

Let $C = A + B$, then if $C = [c_{ij}]_{m \times n}$, then

$$c_{ij} = a_{ij} + b_{ij} \quad \forall \, ij .$$

(ii) Scalar multiplication of a matrix by a number,

$A = [a_{ij}]_{m \times n}$, Let $\lambda \in \mathbb{R}$

$\underset{\substack{\uparrow \\ \text{scalar.}}}{\lambda} \cdot A = [\alpha_{ij}]$, then $\alpha_{ij} = \lambda \cdot a_{ij} \quad \forall \, ij.$

(iii) Matrix multiplication.

$A = [a_{ij}]_{l \times m}$ and $B = [b_{ij}]_{m \times n}$, then

AB is a $l \times n$ matrix.

$AB = [p_{ij}]$, then $p_{ij} =$

$$\begin{array}{cc} & \\ \begin{array}{l} i^{th} \text{ row} \\ \text{of } A \end{array} & \begin{bmatrix} a_{i1} & \cdots & a_{im} \\ & & \end{bmatrix} \end{array} \cdot \begin{array}{c} j^{th} \text{ column of } B \\ \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix} \end{array} = \begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & p_{ij} & \cdots \\ & \vdots & \end{bmatrix} \end{array}$$

$$A \cdot B = [p_{ij}],$$

$$p_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}$$

$$= \sum_{k=1}^{m} a_{ik} b_{kj}$$

$M_n(\mathbb{R})$ set of all $n \times n$ matrices over $\mathbb{R}$.

$$+ : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$

$$(A, B) \longmapsto A + B$$

$$\cdot : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$

$$(A, B) \longmapsto A \cdot B$$

YES

(i) Is it true that if $B = C$, then $AB = AC$?
($M_n(\mathbb{R})$) & $BA = CA$?

(ii) Given, $B = C$, in $M_n(\mathbb{R})$, then for any $A \in M_n(\mathbb{R})$

$A \cdot B = C \cdot A$?

NO

" Proof by counter example. "

Diagonal matrix. A matrix A is called a diagonal matrix if its only nonzero entries are diagonal entries.

$$\begin{bmatrix} \lambda_{11} & & & \\ & \lambda_{22} & & 0 \\ & 0 & \ddots & \\ & & & \lambda_{nn} \end{bmatrix}$$

Identity matrix.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

$n \times n$ square matrix

Upper triangular matrix

$$\begin{bmatrix} * & & \cdots & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix}$$

$*$: some entries.

all zero below the diagonal.

A^{-1} exist.

Inverse of a square matrix.

Let A be a $n \times n$ matrix. If there is a matrix B such that $A \cdot B = I_n$ and $B \cdot A = I_n$, then B is called an inverse of A and is denoted by A^{-1}:

$$\boxed{A^{-1} A = I_n = A \cdot A^{-1}}$$

When A has an inverse, it is said to be invertible matrix.

~~Proposition.~~

$A = \begin{bmatrix} \text{—} \\ \text{—} \end{bmatrix}_{n \times n}$ invertible.

Lemma. Let A be a square matrix. An inverse of A is unique if it exists.

In other words, there can be only one inverse.

Proof. Suppose A^{-1} exists and let B_1 and B_2 be two inverses to A.

$$
\begin{aligned}
B_1 &= I_n \cdot B_1 \\
&= (B_2 \cdot A) \cdot B_1 \\
&= B_2 \cdot (A \cdot B_1) \qquad [\text{Associative law}] \\
&= B_2 \cdot (I_n) \\
&= B_2
\end{aligned}
$$

Proposition. Let A, B be $n \times n$ matrices. If both are invertible, so is their product AB, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

$(AB) \cdot (B^{-1}A^{-1}) = I_n$

$(B^{-1}A^{-1})(AB) = I_n$

$A_1, A_2, \ldots, A_m \quad \forall \; m \geqslant 2$

$$(A_1 A_2 \cdots A_m)^{-1} = A_m^{-1} A_{m-1}^{-1} \cdots A_2^{-1} A_1^{-1}$$

Induction $\quad n = 2$