Lecture 20 (MA 4020) LINEAR ALGEBRA

Nov 12, 2021

CHAPTER 6 INNER PRODUCT SPACES.

[Textbook: Linear Algebra Done Right by Sheldon Axler]

Recall, dot product:

For $x, y \in \mathbb{R}^n$, the dot product of $\underset{(x_1, \ldots, x_n)}{\underset{\shortparallel}{x}}$ and $y = (y_1, \ldots, y_n)$

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Note. Please see the difference in use of notation.

$x \cdot y$ $\qquad$ $(X \cdot Y)$

Norm of a vector. The norm of a vector is the length of the vector, e.g. for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Note. $x \cdot x = \|x\|^2 \quad \forall\, x \in \mathbb{R}^n$

Observe that

i) $x \cdot x \geq 0 \quad \forall\, x \in \mathbb{R}^n$

(ii) $x \cdot x = 0 \in \mathbb{R} \iff \underline{x = 0}$ in $\mathbb{R}^n$

(iii) If $y \in \mathbb{R}^n$ is fixed, then

$$\underline{\cdot} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \underset{\cdot(x) =}{x \cdot y}$$

is linear map

Linear map.

$$\cdot(c_1 x_1 + c_2 x_2) = (c_1 x_1 + c_2 x_2) \cdot y$$
$$= c_1 x_1 \cdot y + c_2 x_2 \cdot y$$
$$= c_1\, x \cdot y + c_2\, x_2 \cdot y$$
$$= c_1 \cdot(x) + c_2 \cdot(x_2)$$

(iv) $x, y \in \mathbb{R}^n, \quad x \cdot y = y \cdot x$

Symmetry

True / False.

$\boxed{2+3i \geq 0}$

$z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$

$z \cdot w$ and

?

$\underline{z_1 w_1 + \cdots + z_n w_n} \in \mathbb{C}$

want

$z \cdot z \geq 0$

complex numbers.

$z \cdot z = \underline{z_1^2 + \cdots + z_n^2} \in \mathbb{C}$

$|z|^2 \geq 0$

True / False

$\mathbb{C}$ field (un-ordered)	$\mathbb{R}$ field. (ordered)
z_1, z_2	ordering
	α, β
≥ 0	$\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$

$(z_1, \dots, z_n)$ $\quad z \in \mathbb{C}^n$

$z \cdot z \in \mathbb{R}$

$(z_1, \dots, z_n)$

$z \cdot z =$ real number

$z_1 \cdot \bar{z}_1 + z_2 \cdot \bar{z}_2 + \dots + z_n \cdot \bar{z}_n$

$= |z_1|^2 + |z_2|^2 + \dots + |z_n|^2 \geq 0$

$$z \cdot w = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \dots + z_n \bar{w}_n$$

$$w \cdot z = w_1 \cdot \bar{z}_1 + w_2 \cdot \bar{z}_2 + \dots + w_n \cdot \bar{z}_n$$

$$\boxed{z \cdot w = \overline{w \cdot z}}$$

Definition. An inner product on V is a function

$\langle -, - \rangle$

$$\langle -, - \rangle : V \times V \longrightarrow F \quad \text{Field}$$

$$(u, v) \longmapsto \langle u, v \rangle$$

and has the following properties:

$v \cdot v$

Positivity $\quad \langle v, v \rangle \geq 0$ for all $v \in V$

Definiteness $\quad \langle v, v \rangle = 0 \iff v = 0$

Additivity in the first slot.

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ for all } u, v, w \in V.$$

Homogeneity in the first slot

$$\langle av, w \rangle = a \langle v, w \rangle \text{ for all } a \in F, \text{ and } v \in V$$

Conjugate symmetry

conjugate

$$\langle v, w \rangle = \overline{\langle w, v \rangle} \text{ for all } v, w \in V.$$

Definition. An inner-product space is a vector space V along with an inner product on V.

Set $F = \mathbb{C}$ field of complex numbers.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define the norm of z by

$$\|z\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

$$\boxed{\|z\|^2 = z_1 \cdot \bar{z}_1 + \cdots + z_n \cdot \bar{z}_n .}$$

How to define the dot product ?

$w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, and

$z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

$$w \cdot z := w_1 \bar{z}_1 + w_2 \bar{z}_2 + \cdots + w_n \bar{z}_n .$$

Such a dot product is called inner product on $\mathbb{C}^n$.

over $\mathbb{R}^n$	over $\mathbb{C}^n$
$x \cdot y = y \cdot x$ $\parallel$ $y \cdot x = x \cdot y$	$w \cdot z = \overline{z \cdot w}$

Examples.

$\boxed{F = \mathbb{R} \text{ or } \mathbb{C}}$

1. $V = F^n$, Define

(a) $\langle -,- \rangle : F^n \times F^n \longrightarrow F$

$(w,z) \longmapsto \langle w,z \rangle = ?$

$= w_1\bar{z}_1 + \cdots + w_n\bar{z}_n$

$\langle -,- \rangle$

$\langle w,w \rangle \geq 0$

"Euclidean inner product on F^n".

(b) One may also define

$\langle -,- \rangle : F^n \times F^n \longrightarrow F$

$(w,z) \longmapsto \langle w,z \rangle$

$= c_1 w_1 \bar{z}_1 + \cdots + c_n w_n \bar{z}_n$

$\langle w \cdot w \rangle = c_1|w_1|^2 + \cdots + c_n|w_n|^2$

where $c_1, \ldots, c_n$ are positive real numbers.

2. $V = \mathcal{P}_n(F)$

$\langle -,- \rangle : \mathcal{P}_n(F) \times \mathcal{P}_n(F) \longrightarrow F$

$(f,g) \longmapsto \langle f,g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

$\langle f,f \rangle = \int_0^1 \underbrace{f(x)\cdot\overline{f(x)}}_{|f(x)|^2}\,dx$

Setting. V is a finite-dimensional inner product space over $\mathbb{F}$. (either $\mathbb{R}$ or $\mathbb{C}$.)

Discussion.

$$\langle -,- \rangle : V \times V \longrightarrow \mathbb{F}$$
$$(v,w) \longmapsto \langle v,w \rangle .$$

If we fix w, then for each $v \in V$,

$$\langle -,w \rangle : V \longrightarrow \mathbb{F}$$
$$\cancel{(v,w)}\ v \longmapsto \langle v,w \rangle \quad \text{is a linear map.}$$
$$0 \longmapsto \langle 0,w \rangle$$

$$\Downarrow$$

$\langle 0,w \rangle = 0$ for every choice of $w \in V$

$$\Downarrow$$

$\langle w,0 \rangle = 0$ (conjugate symmetry)

$\langle 0,w \rangle = 0$
$\langle w,0 \rangle = 0$

Additive property in the second slot.

$$\begin{aligned}
\langle u, v+w \rangle &= \overline{\langle v+w, u \rangle} \\
&= \overline{\langle v,u \rangle + \langle w,u \rangle} \\
&= \overline{\langle v,u \rangle} + \overline{\langle w,u \rangle} \\
&= \langle u,v \rangle + \langle u,w \rangle
\end{aligned}$$

Note. We have conjugate homogeneity in the second slot in an inner product space.

$$\langle u, \overset{\in F}{a} v\rangle = \overline{\langle av, u\rangle}$$

$$= \overline{a \langle v,u\rangle}$$

$$= \boxed{\bar{a}}\, \overline{\langle v, u\rangle}$$

$$= \bar{a} \langle u,v\rangle \quad ; \quad a \in F.$$

$$\boxed{\langle a_1 v, b_1 w\rangle = a_1 \bar{b}_1 \langle v, w\rangle}$$

NORMS. For $v \in V$, we define the norm of v, denoted by $\|v\|$,

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Examples.

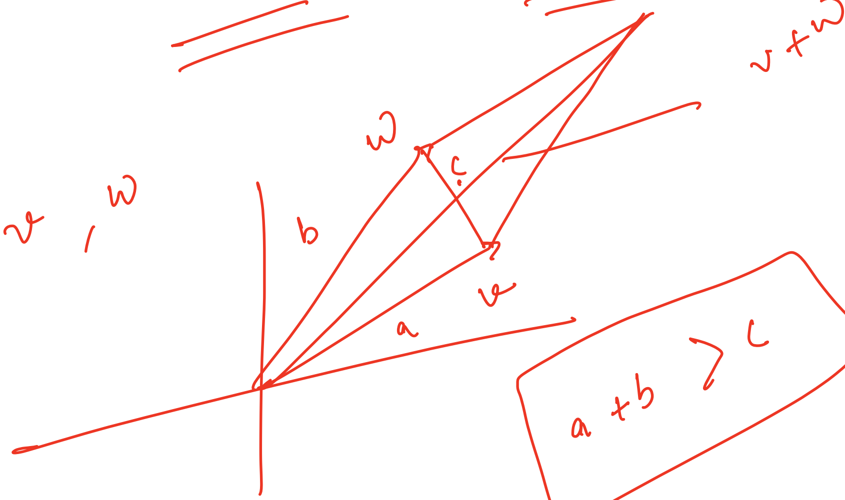
① $V = F^n$

$$\|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

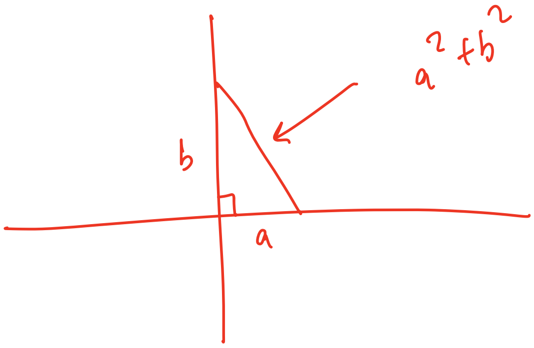
$(F = \mathbb{R}$ or $\mathbb{C})$

$\mathbb{R}^2$ $\mathbb{R}^3$

v, w



$\boxed{a+b > c}$



✓ f.d. vector space.

(2) $V = \mathcal{P}_n(F)$,

$\langle -,- \rangle : \mathcal{P}_n(F) \times \mathcal{P}_n(F) \to F$

$(f,g) \to \int_0^1 f \cdot \bar{g}\, dx$

$$\| f \| = \sqrt{\int_0^1 |f(x)|^2 dx}.$$

Note. we have

$$\| a v \| = |a| \, \| v \| \quad \text{for all } a \in F \text{ and } v \in V.$$

Proof. $\| a v \|^2 = \langle av, av \rangle$

$= a \langle v, av \rangle$

$= a \bar{a} \langle v, v \rangle$

$= |a|^2 \| v \|^2$

Thus taking square root, we get

$$\| a v \| = |a| \, \| v \|.$$

Definition. Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

$\Updownarrow$ same as

$\langle v, u \rangle = 0$

$\langle 0, v \rangle = 0$

0 zero vector

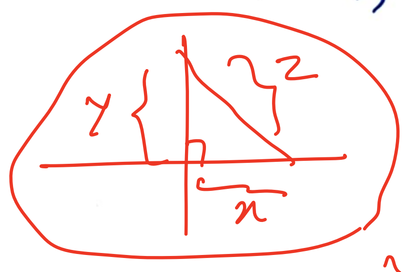
Question

(a) Does there exists a vector which is orthogonal to every vector in V ?

(b) A vector will be orthogonal to itself provided

$$\langle v, v\rangle = \|v\|^2 = 0$$
$$\Updownarrow$$
$$v = 0$$

Theorem. If u, v are orthogonal vectors in V, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$



$z^2 = x^2 + y^2$

Proof.

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \underline{\langle v,v\rangle} \\ &= \|u\|^2 + \underset{0}{\underset{\|}{\langle u,v\rangle}} + \underset{0}{\underset{\|}{\langle v,u\rangle}} + \|v\|^2\end{aligned}$$

" Pythagorean Theorem ".

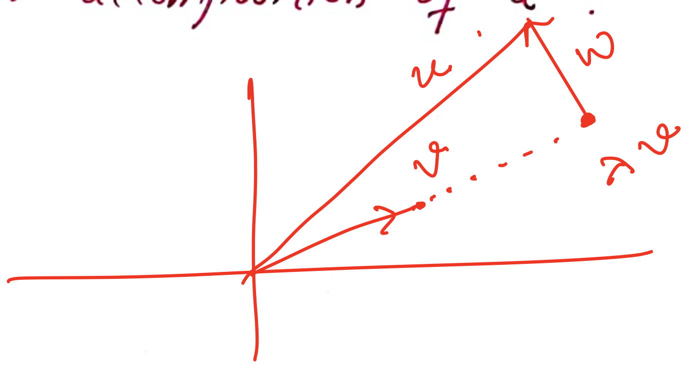
Discussion. Let $u, v \in V$. Let $v \neq 0$.

We want to express

$$u = (\ )\, v + w \quad \text{for some } w \in V$$

$$\text{with } \langle v, w \rangle = 0$$

"Orthogonal decomposition of u".



Start with $u = u$

$$= av + u - av \quad ; \quad a \in F$$

then, we want $\langle u - av, v \rangle = 0$

$$\|$$

$$\langle u, v \rangle - a \langle v, v \rangle = 0$$

$$\Downarrow$$

Thus, we may write, $\boxed{a = \dfrac{\langle u, v \rangle}{\|v\|^2}}$ (non zero scalar)

$v \neq 0$

$$\boxed{u = \frac{\langle u, v \rangle}{\|v\|^2} v + u - \frac{\langle u, v \rangle}{\|v\|^2} v}$$

$$= (\ \lambda\ )\, v + w$$

Theorem [Cauchy-Schwarz Inequalities]:

If $u, v \in V$, then

$$\boxed{|\langle u,v\rangle| \leq \|u\| \, \|v\|.}$$

$V = \mathbb{R}^2$

$u = (a_1, a_2)$

$v = (b_1, b_2)$

$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Proof. If $v = 0$, then L.H.S = R.H.S. = 0.

Assume $v \neq 0$; then write the orthogonal decomposition of u:

$$\boxed{u = \frac{\langle u,v\rangle}{\|v\|^2} v + w}$$; where $\langle u,w\rangle = 0$,

and $w = u - \dfrac{\langle u,v\rangle}{\|v\|^2} v$

By the Pythagorean theorem;

$$\|u\|^2 = \Big\| \underbrace{\frac{\langle u,v\rangle}{\|v\|^2}}_{a} v \Big\|^2 + \underbrace{\|w\|^2}_{\geq 0}$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2$$

$\|w\|^2 = 0$ $[\, \|av\|^2 = |a|^2 \|v\|^2 \,]$

$$\boxed{\|u\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2}}$$

$[\, \|v\| \neq 0 \,]$

Multiplying both sides of this inequality by $\|v\|^2$ and taking square roots, we get

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|.$$

Equality happens $\iff \|w\|^2 = 0$

$$\Updownarrow$$

$$w = 0$$

$$\Downarrow$$

$$u - \frac{\langle u, v\rangle}{\|v\|^2} v = 0$$

$$\Downarrow$$

$$u = \underbrace{\frac{\langle u, v\rangle}{\|v\|^2}}_{\text{scalar}} v$$

Theorem [Triangle Inequality]

If $u, v \in V$, then $\|u+v\| \leq \|u\| + \|v\|$.

This inequality is an equality if and only if one of u, v is a non-negative multiple of the other.

Proof.
$$
\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \|v\|^2 + \underline{\langle u,v\rangle + \overline{\langle u,v\rangle}} \\
&= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v\rangle \\
&\leq \|u\|^2 + \|v\|^2 + 2\,|\langle u,v\rangle| \\
&\leq \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\| \\
&= (\|u\|+\|v\|)^2
\end{aligned}
$$

Thus
$$\boxed{\|u+v\| \leq \|u\| + \|v\|.}$$

"Exercise: Discuss the case of equality"

Theorem. [By Parallelogram Equality].

If $u, v \in V$, then
$$\boxed{\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).}$$

Proof. "Easy".

Orthonormal Bases.

Orthonormal vectors. A list of vectors is called orthonormal if the vectors in it are pairwise orthogonal and each vector has norm 1.

In other words, $(v_1, \dots, v_n)$ of V is orthonormal if

$$\begin{cases} \langle v_i, v_j \rangle = 0 & \text{when } i \neq j; \text{ and} \\ \langle v_i, v_i \rangle = 1 & \text{for all } i. \end{cases}$$

Example.

(1). Standard basis in F^n are orthonormal.

$$(e_1, e_2, \dots, e_n)$$

Proposition. If $(v_1, \ldots, v_n)$ is an orthonormal list of vectors in V, then

proof

$$\| a_1 v_1 + \cdots + a_n v_n \|^2 \left(= |a_1|^2 \|v_1\|^2 + \cdots + |a_n|^2 \|v_n\|^2 \right)$$

$$= |a_1|^2 + \cdots + |a_n|^2$$

$$\left(\because \|v_i\|^2 = 1 \ \forall i \right)$$

for all $a_1, \ldots, a_n \in F$.

Corollary. Every orthonormal list of vectors is linearly independent.

$(v_1, \ldots, v_n)$

Proof. Suppose $a_1 v_1 + \cdots + a_n v_n = 0$, then

$$|a_1|^2 + \cdots + |a_n|^2 = 0$$

$$\Downarrow$$

$$a_i = 0 \text{ for all } i.$$

Orthonormal basis. An orthonormal basis of V is an orthonormal list of vectors that is also a basis of V.

Discussion.

If $(v_1, \dots, v_n)$ is a basis of V, then any vector $v \in V$ can be written as

$$v = c_1 v_1 + \cdots + c_n v_n.$$

If $(v_1, \dots, v_n)$ is an orthonormal basis, then c_i's can be computed (or expressed) effectively.

Theorem. Suppose $(v_1, \dots, v_n)$ is an orthonormal basis of V. Then $v \in V$ can be written as

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n \qquad \text{—(i)}$$

and

$$\|v\|^2 = |\langle v, v_1 \rangle|^2 + \cdots + |\langle v, v_n \rangle|^2 \qquad \text{—(ii)}$$

for every $v \in V$.

Proof. Let $v \in V$, then

$$v = c_1 v_1 + \cdots + c_n v_n \quad \text{for some } c_1, \dots, c_n \in F.$$

Now, consider the inner product with v_i.

$$\langle v, v_i \rangle = \langle c_1 v_1 + \cdots + c_n v_n, v_i \rangle$$

$$= c_i .$$

Hence

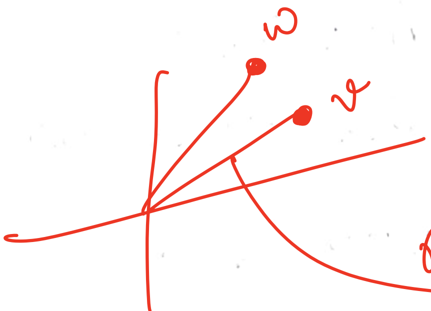
$$v = \underset{c_1}{\underset{\shortparallel}{\langle v, v_1 \rangle}} v_1 + \cdots + \underset{c_n}{\underset{\shortparallel}{\langle v, v_n \rangle}} v_n$$

Also, $\|v\|^2 = |\langle v, v_1 \rangle|^2 + \cdots + |\langle v, v_n \rangle|^2$ is immediate.

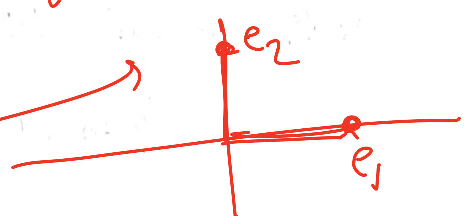
$\{v_1, \dots, v_n\}$ is a L.I | Basis

$\mathcal{B} = (v_1, \dots, v_n)$

$\mathcal{B}' = (w_1, \dots, w_n)$ where $\langle w_i, w_j \rangle = 0$ $i \neq j$ & $\|w_i\|^2 = 1$



Easy to work.



Theorem [Gram - Schmidt]

If $(w_1, \dots, w_n)$ is a linearly independent list of vectors in V, then there exists an orthonormal list $(v_1, \dots, v_n)$ of V such that

$$\text{span}(w_1, \dots, w_j) = \text{Span}(v_1, \dots, v_j) \quad \text{— (A)}$$

for $j = 1, 2, \dots, n$.

Proof. $v_1 = \frac{w_1}{\|w_1\|}$ $j = 1$

Suppose $(w_1, \dots, w_n)$ is a linearly independent list of vectors in V.

Set $v_1 = \frac{w_1}{\|w_1\|}$, then (A) holds.

Proof by induction :

$j = 1$, $\text{Span}(w_1) = \text{Span}(v_1)$

Induction hypothesis. Assume that $(v_1, \dots, v_{j-1})$ orthonormal list of vectors have been chosen with

$$\text{Span}(w_1, \dots, w_{j-1}) = \text{Span}(v_1, \dots, v_{j-1}).$$

Let $v_j \in \operatorname{Span}(w_1, \ldots, w_j)$ some non-zero vector

$$v_j = \frac{w_j - \langle w_j, v_1 \rangle v_1 - \cdots - \langle w_j, v_{j-1} \rangle v_{j-1}}{\| w_j - \langle w_j, v_1 \rangle v_1 - \cdots - \langle w_j, v_{j-1} \rangle v_{j-1} \|}.$$

$\|v_j\| = 1$

$\underline{w_j = * v_j + (\ )v_1 + (\ )v_2 + \cdots + (\ )v_{j-1}}$

Notice that $w_j \notin \operatorname{Span}(v_1, \ldots, v_{j-1})$

$\parallel$ by induction hypothesis.

$w_j \notin \operatorname{Span}(w_1, \ldots, w_{j-1})$

Hence v_j is a non-zero vector with $\|v_j\| = 1$.

Now, observe that

$$\langle v_j, v_k \rangle = 0 \quad \text{for all } 1 \leq k < j.$$

$$\parallel$$

$$\left\langle \underbrace{w_j - \langle w_j, v_1 \rangle v_1 - \cdots - \langle w_j, v_{j-1} \rangle v_{j-1}}_{*},\ v_k \right\rangle$$

$k = 1, 2, \ldots, j-1$

$$\parallel$$

$$\left\langle \underbrace{\langle w_j, v_k \rangle - \langle w_j, v_k \rangle}_{*} \right\rangle$$

$$\parallel$$

$$0$$

$v_1, \ldots, v_{j-1}$

$\langle v_i, v_j \rangle = 0$ $i \neq j$

$\langle v_i, v_j \rangle = 1$ if $i = j$

Thus,

$(v_1, \ldots, v_j)$ is an orthonormal list.

Note that

$$w_j \in \operatorname{Span}(v_1, \ldots, v_j)$$

$$\Downarrow$$

$$\operatorname{Span}(w_1, \ldots, w_{j-1}, w_j) \subset \operatorname{Span}(v_1, \ldots, v_j).$$

Linearly independent set, and hence

subspaces have same dimension j.

Thus $\operatorname{Span}(w_1, \ldots, w_j) = \operatorname{Span}(v_1, \ldots, v_j)$.

L. I. O.N. basis

Remark. The algorithm involved in the proof for constructing an orthonormal set of vectors is known as Gram-Schmidt procedure

or

Gram-Schmidt orthonormalization process.