Divide & Conquer / Divide & Combine

1. Merge Sort
2. Quick Sort

— Recursion.

Multiplication of two n digit numbers

Input: $x = x_{n-1} \ldots x_1 x_0 = \sum_{i=0}^{n-1} 10^i \cdot x_i$

$y = y_{n-1} \ldots y_1 y_0 = \Sigma$

Output: $x * y$.

$x = 5472$

$y = 8910$

Primitive (basic operation on one digit numbers

1, 2, 3

5472.

8910

0 000

5 4 72

operation $\leq c \cdot n^2$

$T(n) = \Theta(n^2)$

$x = x_{n-1} x_{n-2} \ldots x_0$

$y = y_{n-1} y_{n-2} \ldots y_0$

$x = \underbrace{10^{n/2}(x_{n-1} \ldots x_{n/2})}_{a} + \underbrace{x_{n/2-1} \ldots x_0}_{b}$

$= 10^{n/2} \cdot a + b$

$y = 10^{n/2}(y_{n-1} y_{n-2} \ldots y_{n/2}) + y_{n/2-1} \ldots y_0$

$= 10^{n/2} \cdot c + d$

Proof: $x \cdot y = (10^{n/2} a + b)(10^{n/2} \cdot c + d)$

$= 10^{n}(ac) + 10^{n/2}(a.d) +$
$10^{n/2}(bc) + (bd)$

Multiplication on $n/2$ digit numbers

F-Multiply (x, y)
If $|x| = 1$ and $|y| = 1$ ✓
• Return $x.y$.

Eg:

$p = $ F-Multiply (a, c) ✓
$q = $ F-Multiply $(a.d)$ ✓
$r = $ F-Multiply (b, c) ✓
$s = $ F-Multiply (b, d) ✓

Combine

$f = 10^{n} p + 10^{n/2}(q+r) + s$.

p: $O(n)$; $(q+r)$: $O(n)$; s: $O(n)$

$\geq (c')n$

$\leq cn$ ✓ where c is constant

$$T(n) \leq 4T(n/2) + 20 \cdot n$$

$$T(1) = 3$$

$$T(n) \in O(n^2)$$

Statement:

$$T(n) \leq 100n^2 - 40n$$

Base-Case:

$$T(1) = 3 \leq 100 \cdot 1^2 - 40 \cdot 1 \quad \checkmark$$

Induction

IHA for all $q \leq n$

(I.H)₁ Assume that $T(q) \leq 100q^2$.

We want to prove that

$$T(n) \leq 100n^2$$

$$T(n) \leq 4T(n/2) + 20 \cdot n$$

$$\leq 4 \cdot \left[100\left(\frac{n}{2}\right)^2 - 40 \cdot \frac{n}{2} \right] + 20n.$$

(By I.H)

$$\leq 100n^2 + -80n + 20n$$

$$\leq 100n^2 - 60n$$

$\leq 100n^2 - 40n$

$T(n) \in O(n^2)$

$T(n) \in \Theta(n^2)$

Divide - Combine

(1) Divide into smaller smaller subproblem.

(2) Solve it using "recursion"

③ Combine (Merge) the solutions to get the answer.

Correct Induction:

$x = \underbrace{\quad}_{a} \underbrace{\quad\quad}_{b}$

$y = \underbrace{\quad}_{c} \underbrace{\quad\quad}_{d}$

a.c, a.d, b.c, b.d

$10^{m+n} a.c + 10^{m} (bc + ad) + bd$

$$bc + ad = ac + bd - (a-b)(c-d)$$

$$= ac + bd - (ac + ad - bc + bd)$$

$a \cdot c$, $b \cdot d$, $(a-b) \cdot (c-d)$.

multiply (a, c)

" (b, d)

" $(a-b, c-d)$

$f = - + - - +$

$\leq cn.$

$T(n) \leq 3T(n/2) + 40n$

$T(0) = 3.$

$T(n) \in O(n^2)$ ✓

$T(n) \in O(n)$ ✗

$T(n) \in O(n \log n)$

① Recurrence Tree method

② Domain and Range Transformations.