Fundamental theorem of Calculus

Proofs and other things.

Domain additivity of Riemann integrals.

Let $f:[a,b]\to\mathbb{R}$ be a bounded function, and $c\in[a,b]$.
Then f is integrable on $[a,b] \iff f$ is integrable on $[a,c]$ and $[c,b]$.
Furthermore, in such a case we have $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
(If $c=a$ or $c=b$, then there is nothing to prove)

Pf: ($\Rightarrow$) Let $f:[a,b]\to\mathbb{R}$ is Riemann integrable.
We show that f is integrable on $[a,c]$ (similar argument works for $[c,b]$).

We assume that $c\in(a,b)$.

Let $\varepsilon>0$. We must produce a partition $P_\varepsilon^{[a,c]}$ of $[a,c]$ such that $U(P_\varepsilon^{[a,c]}, f) - L(P_\varepsilon^{[a,c]}, f) < \varepsilon$.
By Riemann's condition, applied on the integrability f in $[a,b]$ we see that, there exists a partition P_ε of $[a,b]$ suc that
$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$
We distinguish the proof into two cases:

Case ①
$c\in P_\varepsilon$.
Write $P_\varepsilon = \{x_0,\dots,x_k,\dots,x_n\}$ such that $c=x_k$, by our assumption that $c\in(a,b)$, we have $k<n$.
Now $U(P_\varepsilon,f) - L(P_\varepsilon,f) < \varepsilon$.

Note that $P_\varepsilon^{[a,c]} = \{x_0,\dots,x_k\}$ is a partition of $[a,c]$

$$\iff \sum_{i=1}^n M_i(f)(x_i-x_{i-1}) - \sum_{i=1}^n m_i(f)(x_i-x_{i-1}) < \varepsilon.$$
$$\iff \sum_{i=1}^n (M_i(f)-m_i(f))(x_i-x_{i-1}) < \varepsilon$$
$$\iff \sum_{i=1}^k (M_i(f)-m_i(f))(x_i-x_{i-1}) + \sum_{i=k+1}^n (M_i(f)-m_i(f))(x_i-x_{i-1}) < \varepsilon$$
$$\iff U(P_\varepsilon^{[a,c]}, f) - L(P_\varepsilon^{[a,c]}, f) + I < \varepsilon.$$

Now, $M_i(f)-m_i(f)\geq 0 \ \forall i$, $x_i - x_{i-1} \geq 0 \ \forall i$. $\Rightarrow U(P_\varepsilon^{[a,c]}, f) - L(P_\varepsilon^{[a,c]}, f) < \varepsilon$.

Consequently, $I\geq 0$.

Case ②
$c\notin P_\varepsilon$.
Let $P_\varepsilon^* = P_\varepsilon \cup \{c\}$.
But P_ε^* is a refinement of P_ε.
We know that
$$U(P_\varepsilon^*, f) \leq U(P_\varepsilon, f)$$
and $L(P_\varepsilon^*, f) \geq L(P_\varepsilon, f)$.
Combining,
$$U(P_\varepsilon^*, f) - L(P_\varepsilon^*, f) \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$$
Now we have a partition P_ε^* of $[a,b]$ satisfying the Riemann condition. Moreover $c\in P_\varepsilon^*$.
By case ① we are done.

($\Leftarrow$) Fix $\varepsilon>0$.
f is integrable on $[a,c] \xRightarrow{R.C.} \exists$ a partition $P_\varepsilon^{[a,c]}$ s.t. $U(P_\varepsilon^{[a,c]}, f) - L(P_\varepsilon^{[a,c]}, f) < \varepsilon/2$.
f is integrable on $[c,b] \Longrightarrow \exists$ " " $P_\varepsilon^{[c,b]}$ s.t. $U(P_\varepsilon^{[c,b]}, f) - L(P_\varepsilon^{[c,b]}, f) < \frac{\varepsilon}{2}$.
Take $P_\varepsilon = P_\varepsilon^{[a,c]} \cup P_\varepsilon^{[c,b]}$. Then P_ε is a partition of $[a,b]$.
We see that $U(P_\varepsilon,f) - L(P_\varepsilon,f)$.
$$= \left(U(P_\varepsilon^{[a,c]}, f) - L(P_\varepsilon^{[a,c]}, f)\right) + \left(P_\varepsilon^{[c,b]}, f) - L(P_\varepsilon^{[c,b]}, f)\right)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

To show that $\int_a^c f + \int_c^b f = \int_a^b f$.

$$L(P_\varepsilon, f) = L(P_\varepsilon^{[a,c]}, f) + L(P_\varepsilon^{[c,b]}, f) \leq \int_a^c f + \int_c^b f \leq U(P_\varepsilon^{[a,c]}, f) + U(P_\varepsilon^{[c,b]}, f) = U(P_\varepsilon, f).$$
$$\Rightarrow \quad L(P_\varepsilon, f) \leq \int_a^c f + \int_c^b f \leq U(P_\varepsilon, f) \rightarrow ①$$
We also have, $L(P_\varepsilon, f) \leq \int_a^b f \leq U(P_\varepsilon, f) \rightarrow ②$.

① − ②: $\int_a^b f - \left(\int_a^c f + \int_c^b f\right) \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$

② − ①: $\left(\int_a^c f + \int_c^b f\right) - \int_a^b f \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$

$$\Rightarrow \left|\int_a^b f - \left(\int_a^c f + \int_c^b f\right)\right| < \varepsilon$$
But, this is true for all $\varepsilon>0$
$$\Rightarrow \int_a^b f = \int_a^c f + \int_c^b f.$$

Convention: If $b>a$, then $\int_b^a f := -\int_a^b f$.