Midsem Exam (Online Mode) - Linear Algebra (MA 4020)

Date: October 12, 2021 — Maximum Marks 15 or 18*

Time: 45 minutes, 4:10 pm - 4:55 pm — Extra uploading time 7 minutes

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Instructions.

1. There are two sections, **Part A**, and **Part B**. Depending upon your roll number ending with even or odd integer, answer the respective parts. All questions are compulsory.
2. Write your name and roll number on each answered pages.
3. Scan the document in the pdf file format.
4. Upload the **pdf file** on the Google classroom. (No .jpeg or .jpg file please)

Note. Write answers carefully. Anyone found copying, even for a single question, will be awarded zero marks. Answering wrong section will result into zero mark. Upload your answers in time, with as many questions as you have done

Part A (Roll numbers ending with 0,2,4,6,8)

1. Let $\phi : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ be the linear transformation defined by $\phi(A) = 2A + 3A^T$. Write down the matrix of this transformation with respect to the basis E_i for $i = 1, 2, 3, 4$, where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Find the condition on the real numbers a, b and c such that the following system of equations has a solution:

$$\begin{aligned} 2x + y + 3z &= a \\ x + z &= b \\ y + z &= c. \end{aligned}$$

3. Let $V = \mathcal{P}_5(\mathbb{R})$ be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 5. Let W be the subspace defined by

$$W = \{f \in V \mid f(1) = f'(2) = 0\}.$$

What is the dimension of W?

4. Answer the following:

(i) Let $A = (a_{i,j}) \in M_n(\mathbb{R})$, where

$$a_{i,j} = \begin{cases} 1 \text{ if } i + j = n + 1, \\ 0 \text{ otherwise} \end{cases}$$

What is the value of $\det(A)$ when (a) $n = 10$ and (b) $n = 100$?

(ii) Write down the inverse of the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

5. Let A,B be any two $n \times n$ matrices. Then prove that

$$\det(AB) = (\det A)\,(\det B).$$

6*. Let V be a finite dimensional vector space and let W_1, W_2 and W_3 be subspaces of V . Which of the following statements are true? (Justify your answer)

(i) $W_1 \cap (W_2 + W_3) = W_1 \cap W_2 + W_1 \cap W_3$;

(ii) $W_1 \cap (W_2 + W_3) \supset W_1 \cap W_2 + W_1 \cap W_3$.

Part B (Roll numbers ending with 1,3,5,7,9)

1. Let $V = \mathcal{P}_3(\mathbb{R})$ be the vector space of all polynomials in a single variable x with real coefficients and of degree less than, or equal to, 3. Assume that $(1, x, x^2, x^3)$ be the ordered basis for V. If

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$

then define $T : V \longrightarrow V$ by

$$T(f)(x) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3.$$

Write down the matrix representing the linear transformation T with respect to the standard basis

2. Let W_1 be a subspace of V. Prove that there is no subspace W_2 such that $W_1 \cap W_2 = 0$ and that $\dim W + \dim W_2 > \dim V$.

3. Using row-reduced echelon form approach, find a solution (if it exists) for the system of equation

$$\begin{aligned} -3x_1 + x_2 + 4x_3 &= \quad 1 \\ x_1 + x_2 + x_3 &= \quad 0 \\ -2x_1 + x_3 &= -1 \\ x_1 + x_2 - 2x_3 &= \quad 0. \end{aligned}$$

4. Let $A = (a_{i,j}) \in M_3(\mathbb{R})$, with

$$W = \left\{ A \in M_3(\mathbb{R}) \mid A^t = -A \text{ and } \sum_{j=1}^{3} a_{1,j} = 0 \right\}.$$

Write down a basis for W.

5. Let A be a square matrix. Prove that if the system of homogeneous equation $AX = 0$ has only the trivial solution, then A can be reduced to the identity matrix by sequence of elementary row operations.

6*. Is the set $\mathbb{R}$ of real numbers a finite-dimensional vector space over the field $\mathbb{Q}$ of all rational numbers? (Justify your answer) (Hint: Use counting argument in the proof)