Assignment 4

Suraj - CS20BTECH11050

Download all python codes from

https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment%204/Python%20codes

Download Latex-tikz codes from

https://github.com/Suraj11050/Assignments-AI1103/blob/main/Assignment%204/Assignment%204.tex

1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0,2]$ is

$$f(x) = \begin{cases} \frac{7}{32}\, x^6\ (2-x)\,, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

then k equals ______________

2 SOLUTION

Method 1:
Let $X \in [0,2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x)\ dx = 1 \tag{2.0.1}$$

$$\Pr(x) = \frac{1}{2}\ (\because \text{Uniform order}) \tag{2.0.2}$$

The PDF for X is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.0.3}$$

The CDF for X is

$$F(x) = \begin{cases} 0, & x \le 0, \\ \frac{x}{2}, & 0 < x < 2, \\ 1, & x > 2 \end{cases} \tag{2.0.4}$$

Theorem 2.1. Let $\{X_1, X_2, \cdots X_n\}$ be n i.i.d random variables with common CDF = $F(x)$ and common PDF = $f(x)$, then the marginal probability density of k^{th} order statistic (PDF) is denoted by $f_{(k,n)}(x)$ and it is given by

$$f_{(k,n)}(x) = n\ {}^{n-1}\mathrm{C}_{k-1}\ f(x)\,(F(x))^{k-1}\ (1-F(x))^{n-k} \tag{2.0.5}$$

PDF of k^{th} order statistic of given sample from equation (2.0.5)

$$f_{(k,n)}(x) = n\ {}^{n-1}\mathrm{C}_{k-1}\,\frac{1}{2}\left(\frac{x}{2}\right)^{k-1}\left(1-\frac{x}{2}\right)^{n-k} \tag{2.0.6}$$

$$f_{(k,8)}(x) = \frac{8}{2^{(1+(k-1)+(8-k))}} \times {}^{7}\mathrm{C}_{k-1}\ x^{k-1}\ (2-x)^{8-k} \tag{2.0.7}$$

$$f_{(k,8)}(x) = {}^{7}\mathrm{C}_{k-1}\,\frac{1}{32}\ x^{k-1}\ (2-x)^{8-k} \tag{2.0.8}$$

Comparing the PDF obtained in equation (2.0.8) with the equation given in question

$$\frac{1}{32}\ {}^{7}\mathrm{C}_{k-1}\ (2-x)^{8-k}\ x^{k-1} = \frac{7}{32}\ (2-x)\ x^6 \tag{2.0.9}$$

$$\therefore k = 7 \tag{2.0.10}$$

Hence the marginal probability density given is 7^{th} order statistic and **the value of k is 7**

Method 2:
we know that, PDF of k^{th} order statistic of a uniform distribution on $[0,1]$ follows beta distribution

$$\int_0^2 f(x)\,dx = \int_0^2 \frac{7}{32}\,x^6\ (2-x)\ dx \tag{2.0.11}$$

$$\int_0^2 f(x)\,dx = \int_0^2 56\left(\frac{x}{2}\right)^6\left(1-\frac{x}{2}\right)d\left(\frac{x}{2}\right) \tag{2.0.12}$$

Let new random variable be t such that $t = x/2$, New sample be $\{T_1, \cdots T_8\}$ such that $T_i = X_i/2$.

$$f(t) = 56\,t^6\,(1-t) \tag{2.0.13}$$

$$\int_0^2 f(x)\,dx = \int_0^1 f(t)\,dt = 1 \tag{2.0.14}$$

The Uniform distribution of new random sample is on $[0, 1]$ such that PDF = 1 and CDF = t
Given k^{th} order statistic (after conversion)

$$f_{(k,8)}(t) = \begin{cases} 56\,t^6\,(1-t)\,, & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.0.15}$$

Since equation (2.0.15) is a Beta distribution with $r = k,\ s = n - k + 1$

$$r - 1 = k - 1 = 6 \tag{2.0.16}$$

$$\therefore k = 7 \tag{2.0.17}$$

Hence the **value of k is 7**

Presentation link :
https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment4presentation

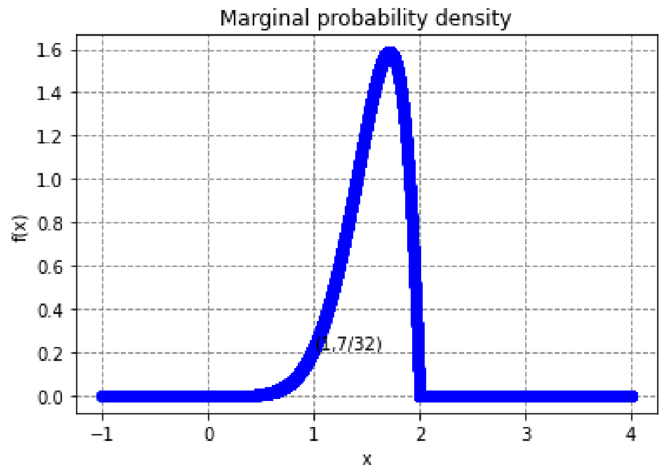

Fig. 1: PDF of $f_{(7,8)}(x)$

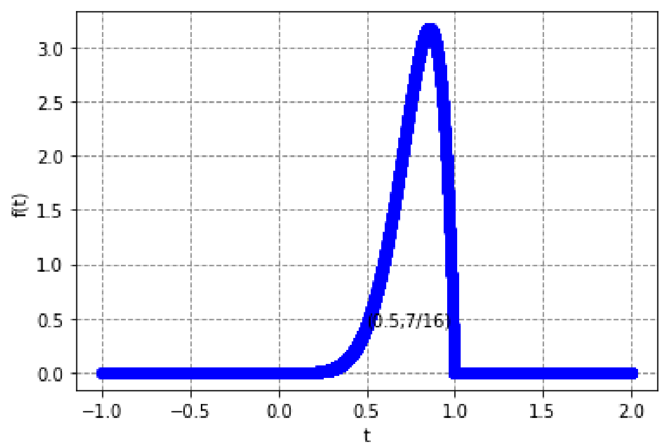

Fig. 2: PDF of $f_{(7,8)}(t)$