CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 3

Instructions

- Try these questions before class. Do not submit!

(1) Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Let $r \in \mathbb{Q}$ and i be an irrational number. Then T.S.T. $s = r + i$ is a irrational number.
We need to show by proof by contradiction. So let us assume that s is rational. But then $s + (-r)$ is a rational number. Since if $s = m/n$ and $r = k/l$ where $m, n, k, l \in \mathbb{Z}$ then,

$$s + -r = (ml - nk)/(nl),$$

where $ml - nk \in \mathbb{Z}$ and $nl \in \mathbb{Z}$ and therefore $s + -r$ is a rational number. But $s + -r = i$ and therefore contradicts our assumption that i is irrational. So s is rational is an incorrect assumption and we have our result.

(2) P.T. if x is irrational, then $1/x$ is irrational.

Proof by contraposition. If $1/x$ is rational then x is rational.
If $1/x$ is rational then $1/x = a/b$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$. Since $1/x$ is valid, this implies $x \neq 0$ and therefore $a \neq 0$. So now $x = 1/(a/b) = b/a$ is valid since $a \neq 0$ and therefore is a rational number.

(3) Are these steps for finding solutions of $\sqrt{x+3} = 3 - x$ correct?
(a) $\sqrt{x+3} = 3 - x$ is given
(b) $x + 3 = x^2 - 6x + 9$ obtained by squaring both sides of (a),
(c) $0 = x^2 - 7x + 6$ obtained by subtracting $x + 3$ from both sides of (b),
(d) $0 = (x - 1)(x - 6)$ obtained by factoring the RHS of (c),
(e) $x = 1$ or $x = 6$ which follows from (d) because $ab = 0$ implies that $a = 0$ or $b = 0$.

The steps are valid for obtaining *possible* solutions to the equations. If the given equation is true, then we can conclude that $x = 1$ or $x = 6$.
However it is not an if and only if which means if we go in the opposite direction it may not be true. Especially step (a). Therefore the possible solutions have to be checked by plugging in the original equation. *We know that no other solutions are possible.*

If we plug in $x = 1$ we get the tautology $2 = 2$; but if we plug in $x = 6$ we get the false statement $3 = -3$. Therefore, $x = 1$ is the only solution.

(4) P.T. at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

Proof by contradiction. Suppose instead that all of the numbers $a_1, a_2, \ldots, a_n$ are less than their average, A. In symbols, we have $a_i < A$ for all i. If we add these n inequalities

$$a_1 + a_2 + \cdots + a_n < nA.$$

But by definition,

$A = \frac{a_1+a_2+\cdots+a_n}{n}$. We basically get $nA < nA$, so our assumption was incorrect and at least one of the numbers is greater than or equal to the average.

(5) P.T. $log_4 6$ is irrational.

Suppose that $log_4 6$ is rational, then $log_4 6 = m/n$.

$$\begin{aligned} 4^{m/n} &= 6 \\ 4^m &= 6^n \\ 2^{2m} &= 2^n 3^m \\ 2^{2m-n} &= 3^n \end{aligned}$$

Since $n > 0$, $3|3^n$ but 3 does not divide 2^{2m-n} so it cannot be true and we have a contradiction.

(6) Let the coefficients of the polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.

(a) Explain why the lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.

(b) Carefully prove the lemma.

You may find it helpful to appeal to:

Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer.

k is not an mth power of some integer is equivalent to saying that $x^m = k$ has no integer solutions, or $x^m - k$ has no integer root. If we prove the lemma then we have the root of $x^m - k$ is irrational. By definition, $\sqrt[m]{k}$ is a root of this polynomial so it is irrational.

Let r be a real root of the polynomial. This implies

$$a_0 + a_1r + a_2r^2 + \cdots + a_{m-1}r^{m-1} + r^m = 0.$$

So r could be an integer, rational or irrational. We need to show that it cannot be a rational number. We use proof by contradiction (again!). So assume $r = a/b$, where a, b are integers and r is in its lowest terms. We substitute for r in the equation and then multiply with b^m on both sides to get,

$$a_0 b^m + a_1 a b^{m-1} + a_2 a^2 b^{m-2} + \cdots + a_{m-1} a^{m-1} b + a^m = 0$$

Taking a^m to the RHS we get,

$$a_0 b^m + a_1 a b^{m-1} + a_2 a^2 b^{m-2} + \cdots + a_{m-1} a^{m-1} b = -a^m.$$

LHS has b as its common factor. b has some prime factor p s.t. it divides the LHS but then that prime factor has to divide the RHS too. Therefore $p \mid (-a^m)$. By using the fact mentioned above we can conclude that p divides a. Therefore a and b have a common factor, a contradiction.

All that we have used is Fundamental Theorem of Arithmetic/ Unique Prime Factorization Theorem, which says that every integer > 1 factors into a product of primes that is unique except for the order in which the primes are multiplied.

(7) Consider a different proof that $\sqrt{2}$ is irrational, taken from American Mathematical Monthly :
Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer $q > 0$ s.t. $(\sqrt{2} - 1)q$ is a nonnegative integer. Let $q^{'} := (\sqrt{2} - 1)q$. Clearly $0 < q^{'} < q$. But an easy computation shows that $(\sqrt{2} - 1)q^{'}$ is a nonnegative integer, contradicting the minimality of q.
(a) This proof was written for an audience of college teachers, and at this point it is a little more concise than desirable. Write out a more complete version which includes an explanation of each step.
(b) Now that you have justified the steps in this proof, do you have a preference for one of these proofs over the other? Why?
(a) Why is there an integer $q > 0$ s.t. $(\sqrt{2} - 1)q$ is a nonnegative integer? Since $\sqrt{2}$ is rational $\sqrt{2} - 1$ is rational which implies it can be written as m/n for integers m, n. Then we let q to be that n and if they are in the lowest terms then q is the least positive integer for which that is possible.
(b) Also the fact that there is one such integer and that it is positive, we will have the least one. This also what is called the Well-ordering principle which we will soon take it up.
(c) Clearly $0 < q^{'} < q$ may not be very clear to all. $\sqrt{2} - 1$ lies between 0 and 1 and therefore the product of $\sqrt{2} - 1$ and a positive real number is always going to be strictly greater than zero and less than that real number.

(d) Easy computation of $(\sqrt{2}-1)q^{'}$ shows it is a nonnegative integer may not be an easy computation after all.

$$\begin{aligned}(\sqrt{2}-1)q^{'} &= (\sqrt{2}-1)^2 q \text{ Substituting for } q^{'}\\ &= 2q - 2q\sqrt{2} + q\\ &= q - 2\cdot[(\sqrt{2}-1)q]\end{aligned}$$

The last term is an integer because q is an integer and $2\cdot[(\sqrt{2}-1)q]$ is also integer.

Which proof is better? Both seem easy to understand. The first proof uses Unique Prime Factorization while this one uses simple algebra.

Exercise: The first proof can be easily generalized from square root of 2 to kth root of 2.

(8) Bogus Proof: It is a fact that arithmetic mean is at least as large as the geometric mean,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

for all nonnegative real numbers a and b. But the following proof has something objectionable about it. Can you identify it? If

$$\begin{aligned}\frac{a+b}{2} &\geq \sqrt{ab},\\ \Rightarrow a+b &\geq 2\sqrt{ab},\\ \Rightarrow a^2+2ab+b^2 &\geq 4ab,\\ \Rightarrow a^2-2ab+b^2 &\geq 0,\\ \Rightarrow (a-b)^2 &\geq 0\end{aligned}$$

which we know is true (since a and b are real numbers, $(a-b)$ is a real number and the square is always positive or zero) so the proof is true.

The problem comes again when you take square root without considering negative roots:$(-4+-4) >= 2*\sqrt{16}$ is not true.

(9) Bogus Proof: $1/8 > 1/4$

$$\begin{aligned}3 &> 2\\ 3log_{10}(1/2) &> 2log_{10}(1/2)\\ log_{10}(1/2)^3 &> log_{10}(1/2)^2\\ (1/2)^3 &> (1/2)^2.\end{aligned}$$

$log_{10}(1/2)$ is a negative value and therefore $3log_{10}(1/2) > 2log_{10}(1/2)$ is not true.