Lecture 12 – MA4020 (LINEAR ALGEBRA)

Oct. 01, 2021.

RECALL

Isomorphism. An isomorphism φ from a vector space V to a vector space V', both over the same field F, is a bijective map

$$\varphi : (V, +, \cdot) \longrightarrow (V', +', \cdot')$$

compatible with the addition and scalar multiplication map.

$$\varphi(v + v') = \varphi(v) +' \varphi(v') \quad \text{for all } v, v' \in V;$$

$$\varphi(c \cdot v) = c \cdot' \varphi(v) \quad \text{for all } v \in V, c \in F.$$

Examples of vector space to isomorphism.

1. F^n : n-dimensional row vector

(n-dimensional column vector)

$$\varphi : (F^n, +, \cdot) \longrightarrow (F^n, +, \cdot)$$

$$X = [x_1, \cdots, x_n] \longmapsto \varphi(X) = X^t = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(same map. denote by

φ as $T : F^n \longrightarrow F^n$

$$X \longmapsto T(X) = X^t$$

$$\left.\begin{aligned} T(X+Y) &= (X+Y)^t \\ &= X^t + Y^t \\ &= T(X) + T(Y) \end{aligned}\right\} \text{Addition}$$

$$\left.\begin{aligned} T(cX) = (cX)^t &= c \cdot X^t \\ &= c \cdot T(X) \end{aligned}\right\} \text{Scalar multiplication}$$

φ is well-defined

φ is one-one

φ is onto

} Done before

2. $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic vector spaces over $\mathbb{R}$.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{C}$$
$$(a,b) \longmapsto a+ib$$

3. Let $S = (s_1, \dots, s_n)$, a finite set. Define

$$V(S) = \{ a_1 s_1 + \cdots + a_n s_n \mid a_i \in F \}$$

Then $V(S)$ is a vector space with $+$ and $\cdot$

(Lecture 9, Sep 17).

Moreover, $V(S) \cong F^n$

$$T : F^n \longrightarrow V(S)$$

$$X = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto a_1 s_1 + \cdots + a_n s_n$$

$$T(X+Y) = T(X) + T(Y)$$

$$T(cX) = c\,T(X).$$

4. V : vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$.

$$T : F^n \longrightarrow V$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \mathcal{B} \cdot X$$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= v_1 x_1 + \cdots + v_n x_n \in V$$

T is well-defined.

If $X = Y$, then $T(X) = \mathcal{B} X$
$= \mathcal{B} Y$
$= T(Y)$

$$\begin{aligned} T(X+Y) &= \mathcal{B} \cdot (X+Y) \\ &= \mathcal{B} \cdot X + \mathcal{B} \cdot Y \\ &= T(X) + T(Y) \end{aligned}$$ for all $X, Y \in F^n$

$$\begin{aligned} T(cX) &= \mathcal{B}(cX) \\ &= c \cdot \mathcal{B} \cdot X \\ &= c\, T(X) \end{aligned}$$ for all $X \in F^n$, and $c \in F$.

Definition.

Let V and V' be vector space over the field F.

A map $T : V \longrightarrow V'$ compatible with addition and scalar multiplication:

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad \text{for all } v_1, v_2 \in V$$

$$T(cv) = c\,T(v) \quad \text{for all } v \in V \text{ and } c \in F$$

is called <u>linear transformation</u> (or <u>linear map</u>).

Examples.

1. $\varphi : V \longrightarrow V'$

$\uparrow$

Id. ?

(i) identity map

$$\begin{aligned} id : V &\longrightarrow V' \\ v &\longmapsto v \end{aligned}$$

(ii) zero map

$$\begin{aligned} 0 : V &\longrightarrow V' \\ v &\longmapsto 0 \end{aligned}$$

(iii)

$$\begin{aligned} T : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (x, 0) \end{aligned}$$

(iv)

$$\begin{aligned} T : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x \end{aligned}$$

(v)

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{C} \\ (a,b) &\longmapsto a+ib \end{aligned}$$

2. All vector space isomorphism are example of linear transformations.

3. $\mathcal{P}_n(\mathbb{R})$: vector space of real polynomial functions of degree $\leq n$.

vector $f \in \mathcal{P}_n(\mathbb{R})$

$$f = a_n x^n + \cdots + a_1 x + a_0 \; ; \quad a_i \in \mathbb{R}$$

(a)
$$\frac{d}{dx} : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_{n-1}(\mathbb{R})$$
$$f \longmapsto f'$$

Derivative is a linear map

$\left\{ \begin{array}{l} \text{Onto map ?} \\ \text{YES} \end{array} \right.$

$$\frac{d}{dx}(f+g) = (f+g)' = f' + g' = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

$$\frac{d}{dx}(\alpha \cdot f) = (\alpha f)' = \alpha \cdot f' = \alpha \cdot \frac{d}{dx}(f)$$

(b)
$$\int_0^1 : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$f \longmapsto \int_0^1 f(x) \cdot dx$$

$\left\{ \begin{array}{l} \text{Onto map ?} \\ \text{NO} \end{array} \right.$

4. Let $V = \mathbb{R}^m$ and $V' = \mathbb{R}^n$ with $m \leq n$.

Define the map

$$T : V \longrightarrow V' \text{ by}$$

$$T(x_1, \dots, x_m) = (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{n-m \text{ zero's}}).$$

Note that T is linear map

(one - one)

" Natural inclusion of $\mathbb{R}^m$ into $\mathbb{R}^n$."

5. In a similar way, assuming $m \geq n$

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$$

$$\uparrow$$

$$(x_1, \dots, x_n, \underbrace{x_{n+1}, \dots x_m})$$

" Natural projection of $\mathbb{R}^m$ onto $\mathbb{R}^n$."

6. $T : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\underline{x} \longmapsto T(\underline{x}) := \alpha_1 x_1 + \cdots + \alpha_n x_n$

defined as

$\underline{x} = (x_1, \cdots, x_n)$

$\alpha_1, \cdots, \alpha_n$ are fix scalars in $\mathbb{R}$

Is this T a linear transformation? (YES)

$$T(\underline{x} + \underline{y}) = \alpha_1 (x_1 + y_1) + \cdots + \alpha_n (x_n + y_n)$$

$$= \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_1 y_1 + \cdots + \alpha_n y_n$$

$$= T(\underline{x}) + T(\underline{y}).$$

$$T(c\underline{x}) = \alpha_1 \cdot c x_1 + \cdots + \alpha_n \cdot c x_n$$

$$= c \cdot (\alpha_1 x_1 + \cdots + \alpha_n x_n)$$

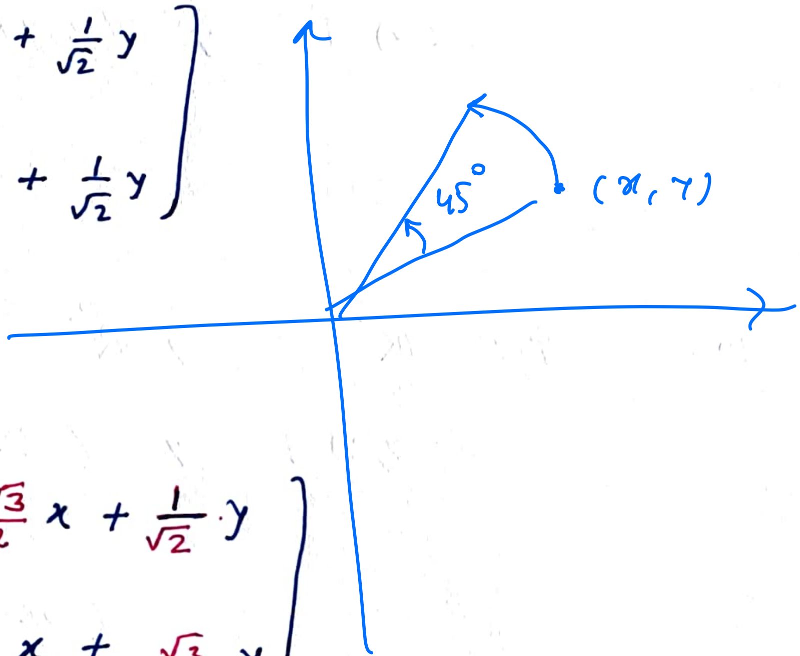
$$= c \cdot T(\underline{x}).$$

7.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{bmatrix}$$

$$T_{45^\circ}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{bmatrix}$$



$$T_{60^\circ}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{2}}\cdot y \\ -\frac{1}{\sqrt{2}}x + \frac{\sqrt{3}}{2}y \end{bmatrix}$$

$$T : R_\theta\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \cdot x + \sin\theta\, y \\ -\sin\theta \cdot x + \cos\theta \cdot y \end{bmatrix}$$

R_θ is a linear map, geometrically rotation of the co-ordinate by angle θ w.r.t. x-axis

8. $T : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto x^2$

Linear transformation ?

NO

9. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (x, y+3)$

Linear transformation ?

NO

10. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x,y,z) \longmapsto (2x - y + 3z,\ 7x + 5y - 6z)$ YES

$(3x - 5y + 2z,\ 5x + 2y - 7z)$ YES

$(x - y + z,\ x + y - z)$ YES

11. $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$X \longmapsto AX$; where A is some fixed matrix $A \in M_n(\mathbb{R})$

- $T(X+Y) = A \cdot (X+Y)$
  $= A \cdot X + A \cdot Y = T(X) + T(Y)$
- $T(cX) = A \cdot (cX) = c \cdot AX = c \cdot T(X)$

T is a linear transformation.

Proposition. Let $T : V \longrightarrow V'$ be a linear map. Then the following are true:

(i) $T(0_V) = 0_{V'}$

(ii) $T(-v) = -T(v)$ for all $v \in V$

(iii) $T(v_1 - v_2) = T(v_1) - T(v_2)$.

$v_1 + (-1)\cdot v_2$

Proof.

$$T(0_V) = T(0_V + 0_V)$$
$$= T(1\cdot 0_V + 1\cdot 0_V)$$
$$= T(2\cdot 0_V)$$
$$= 2\cdot T(0_V)$$

$$\Rightarrow \quad 2\cdot T(0_V) - T(0_V) = 0_{V'}$$

$$\Rightarrow \quad T(0_V) = 0_{V'}$$

$$T(-v) = T((-1)\cdot v)$$
$$= (-1)\, T(v)$$
$$= -T(v)$$

$$T(v_1 - v_2) = T(v_1 + (-1) v_2)$$
$$= T(v_1) + T((-1) v_2)$$
$$= T(v_1) - T(v_2)$$

Let $T : V \longrightarrow W$ be any linear transformation.

Define

$$\text{kernel of } T = \{ v \in V \mid T(v) = 0_W \} \subseteq V$$

$$\underset{\left(\text{range of } T\right)}{\overset{}{\text{Image of } T}} = \{ w \in W \mid w = T(v) \text{ for some } v \in V \} \subseteq W$$

Claim. (i) $\ker T$ is a subspace of V, and

(ii) $\operatorname{im} T$ is a subspace of W.

Let $v_1, v_2 \in V$ s.t. $T(v_1) = 0$ and $T(v_2) = 0$, $\in \ker T$

then $T(v_1+v_2) = T(v_1) + T(v_2)$

$= 0 + 0 = 0$

$\Rightarrow v_1 + v_2 \in \ker T$

Let $v \in V$ s.t. $T(v) = 0$, then

$T(cv) = c\,T(v) = c \cdot 0_W = 0_W$

$= 0$

$\Rightarrow cv \in \ker T$

By definition, $0 \in \ker T$

To show im T is a subspace of W.

Let $w_1, w_2 \in \text{im } T$

$\Rightarrow \exists\, v_1, v_2 \in V$ s.t. $T(v_1) = w_1$, and
$T(v_2) = w_2$

Now $T(\overset{v_1+v_2}{?}) = \underline{\underline{w_1 + w_2}}$

Let $w \in \text{im } T \Rightarrow \exists\, v \in V$ s.t. $T(v) = w$

$T(cv) = \underline{\underline{cw}} \in \text{im}(T)$

$T(\quad) = 0_w$

$\square$

Notation/(Naming)

($\ker T$ = kernel of T = null space of T = null T)

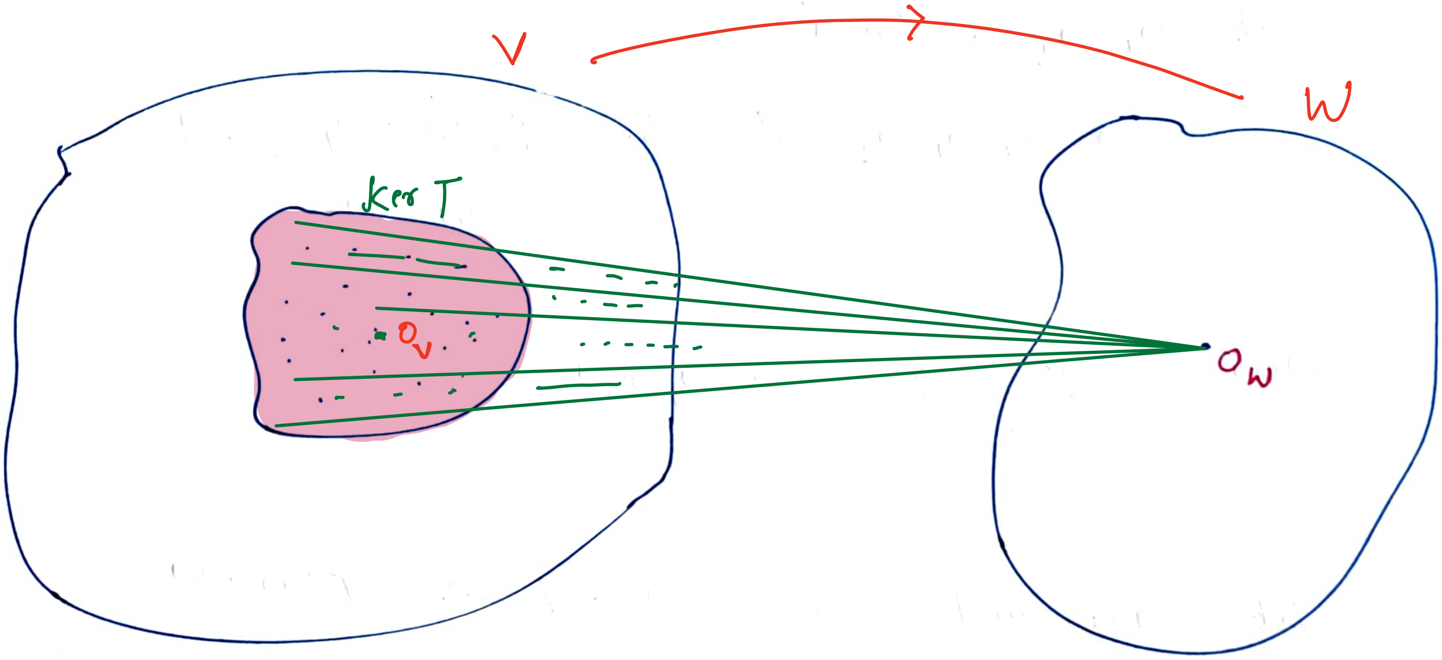
$T : V \longrightarrow W$ linear transformation

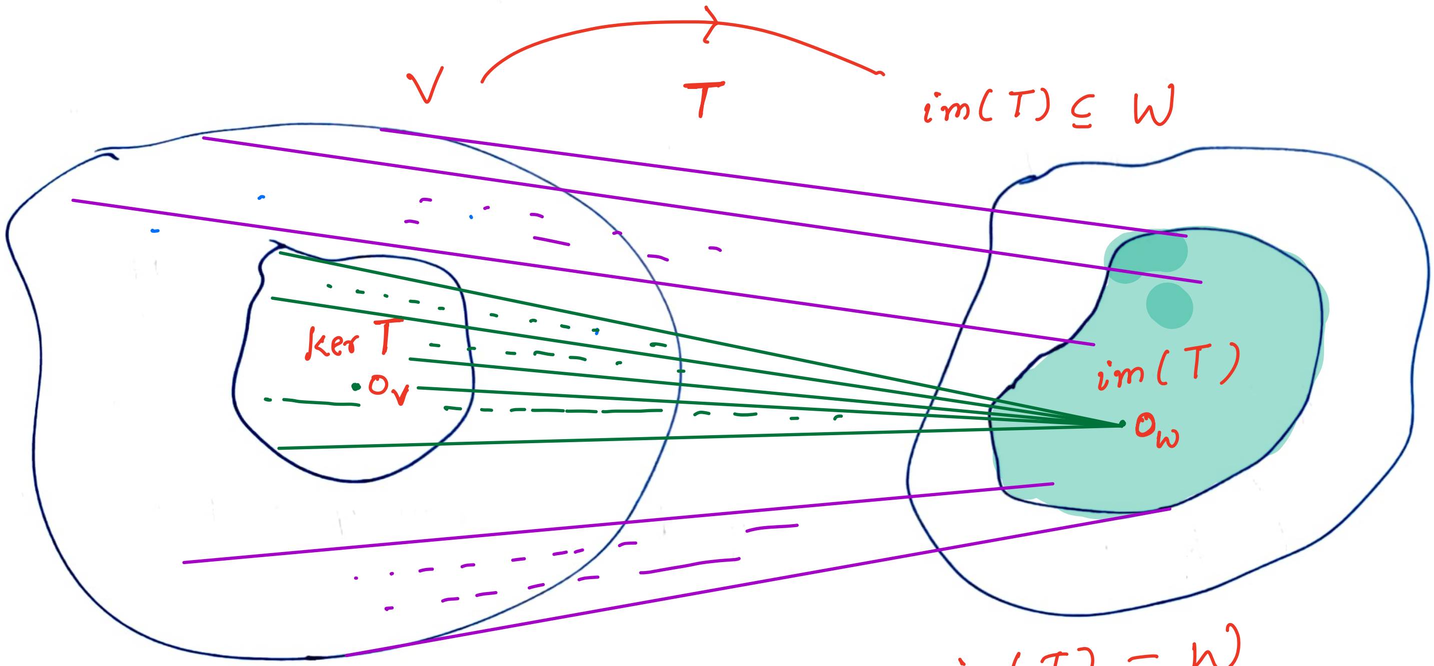
$\quad\ \cup| \qquad\qquad\quad \cup|$

$\ker T$ subspace of V $\qquad$ $\text{im } T$ supspace of W

$\dim(\ker T)$? $\qquad$ $\dim(\text{im } T)$?

$T : V \longrightarrow W$ linear transformation





$im(T) = W$

Onto map

$$\left.\begin{array}{l} \dim(\ker T) \\ \\ \dim(\operatorname{im} T) \end{array}\right\} \quad \text{Discussion !}$$

Examples.

1. $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad (m \leqslant n)$

$$(x_1, \cdots, x_m) \longmapsto (x_1, \cdots, x_m, \underbrace{0, 0, \cdots, 0}_{n-m \text{ zeros}})$$

$\ker T$ is a subspace of $\mathbb{R}^m$

$\parallel$

$\{ \underline{x} \in \mathbb{R}^m \mid T(\underline{x}) = 0 \}$ $\qquad \dim(\ker T) = 0.$

$\operatorname{im} T$ is a subspace of $\cancel{\mathbb{R}}$ $\mathbb{R}^n$

$\parallel$

$\{ w \in \mathbb{R}^n \mid w = T(v), v \in \mathbb{R}^m \}$

2. $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad (m \geqslant n)$

$$(x_1, \cdots, x_m) \longmapsto (x_1, \cdots x_n)$$

$$(\underbrace{x_1, \cdots, x_n}, x_{n+1}, \cdots, x_m) \longmapsto (x_1, \cdots, x_n)$$

$\ker T$

$\operatorname{im} T = \mathbb{R}^n$

$\operatorname{im} T$

$\dim(\ker T) =$ $\qquad \dim(\operatorname{im} T) = n$

3. $$\begin{array}{rccc} \underset{id}{\overset{\parallel}{T}} : & V & \longrightarrow & V \\ & v & \longmapsto & v \end{array}$$

identity map

$(\dim V = n)$

$\ker T = \{(0)\}$

$\operatorname{im} T = V$

$\dim(\ker T) = 0$

$\dim(\operatorname{im} T) = n$

Onto $\Rightarrow \operatorname{im}\left(\frac{d}{dx}\right) = \mathcal{P}_{n-1}(\mathbb{R})$

4. $$\begin{array}{rccc} \frac{d}{dx} : & \mathcal{P}_n(\mathbb{R}) & \longrightarrow & \mathcal{P}_{n-1}(\mathbb{R}) \\ & f & \longmapsto & f' \end{array}$$

$\dim(\mathcal{P}_{n-1}(\mathbb{R}))$

$\parallel$

$\#$ Basis

$\# \{1, x, x^2, \dots, x^{n-1}\}$

$= n$

$\ker \frac{d}{dx} = \{ f \mid \frac{d}{dx}(f) = 0 \}$

$\operatorname{im} \frac{d}{dx}$

$\dim\left(\ker \frac{d}{dx}\right) = 1$

$\dim\left(\operatorname{im} \frac{d}{dx}\right) = n$

5. $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$X \longmapsto AX$, where $A \in M_n(\mathbb{R})$

is a fixed matrix.

$\ker T = \{ X \in M_n(\mathbb{R}) \text{ s.t. } AX = 0 \}$

$\operatorname{im} T =$

6. V : vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$.

$T : \mathbb{R}^n \longrightarrow V \cong \mathbb{R}^n$

$X \longmapsto \mathcal{B} \cdot X$

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

$\ker T = \{ X \in \mathbb{R}^n \mid T(X) = 0 \text{, i.e. } [\mathcal{B}] X = 0 \}$

$\operatorname{im} T = \{ w \in V \mid \cancel{\exists}\ w = T(X) \text{ for some } X \in \mathbb{R}^n \}$

$\dim(\ker T) = 0$

$\dim(\operatorname{im} T) = n$

Theorem. Let $T : V \longrightarrow W$ be a linear transformation, and assume that V is finite-dimensional. Then

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

Proof. Let $\dim V = n$.

$\ker T$ is a subspace of V, choose a basis for this, say, $(u_1, \cdots, u_k)$.

Extend $(u_1, \cdots, u_k)$ to a basis for V

$$(\underbrace{u_1, \cdots, u_k}; v_1, \cdots, v_{n-k}) \qquad (n = \dim V)$$

By definition, $T(u_i) = 0$ for $1 \leq i \leq k$, and

$T(v_j) \neq 0$ for $1 \leq j \leq n-k$.
in W

Let $w_i = T(v_i)$; $i = 1, \cdots, n-k$.

$w_1, \cdots, w_{n-k} \in W$ (in $\operatorname{im} T$)

If we prove that $(w_1, \cdots, w_{n-k}) = S$ is a basis for $\operatorname{im} T$, then ~~im T~~ $\boxed{\dim(\operatorname{im} T) = n-k}$.

This will complete the proof of theorem.

Claim. $S = (w_1, \dots, w_{n-k})$ is a basis for $\operatorname{im} T$.

$\begin{cases} S \text{ is linearly independent set, and} \\ S \text{ spans } \operatorname{im} T \end{cases}$

Let $w \in \operatorname{im} T$ be arbitrary.

Then $w = T(v)$ for some $v \in V$.

$$v = \underbrace{a_1 u_1 + \dots + a_k u_k}_{} + b_1 v_1 + \dots + b_{n-k} v_{n-k}$$

Apply T,

$$T(v) = 0 + \dots + 0 + b_1 w_1 + \dots + b_{n-k} w_{n-k}$$

$$\|$$

$$w = b_1 w_1 + \dots + b_{n-k} w_{n-k}$$

$$w \in \operatorname{Span}(w_1, \dots, w_{n-k})$$

Thus S spans $\operatorname{im} T$.

Suppose $c_1 w_1 + \dots + c_{n-k} w_{n-k} = 0^{\in W}$ (linear relation)

Consider a linear combination $v = c_1 v_1 + \dots + c_{n-k} v_{n-k}$

s.t. $T(v) = c_1 w_1 + \dots + c_{n-k} w_{n-k}$

$$\|$$
$$0$$

$v \in \ker T$

$v = a_1 u_1 + \dots + a_k u_k$ (in terms of basis of $\ker T$)

$$v = c_1 v_1 + \cdots + c_{n-k} v_{n-k}$$
$$\| $$
$$a_1 u_1 + \cdots + a_k u_k$$

We may re-write it as

$$-a_1 u_1 - \cdots - a_k u_k + c_1 v_1 + \cdots + c_{n-k} v_{n-k} = 0.$$

But $(u_1, \ldots, u_k; v_1, \ldots, v_{n-k})$ is a basis for V

$\Rightarrow \quad -a_1 = 0, \ldots, -a_k = 0,$ and $c_1 = \cdots = c_{n-k} = 0.$

$$\Downarrow$$

$$c_1 w_1 + \cdots + c_{n-k} w_{n-k} = 0$$

$$\Downarrow$$

$(w_1, \ldots, w_{n-k})$ is L.I.

$$\boxed{\dim V = \dim(\ker T) + \dim(\operatorname{im} T)}$$

DIMENSION FORMULA

Notations. $\dim(\operatorname{im} T) :=$ rank of T $:= \dim(\text{range } T)$

$\dim(\ker T) :=$ nullity of T $:= \dim(\underline{\text{null}}\, T)$

$$\boxed{\dim V = \text{rank} + \text{nullity}}$$ Rank-Nullity Theorem

Discussion.

Question. ~~Can we hav~~ Is it possible to

Define a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ which is surjective.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad (\text{onto})$$

$$(x, y) \longmapsto (x+y,\ x-y,\ x+2y) \qquad \text{NO}$$

or

$$(x,\ y,\ x+y) \qquad \text{NO}$$

$$\dim(\mathbb{R}^2) = \dim(\ker T) + \dim(\operatorname{im} T)$$

$$2 = (\geq 0) + 3 \qquad (\text{Not possible})$$

Q. Is it possible to Define a linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad (\text{which is one-one})$$

$$\dim(\mathbb{R}^3) = \dim(\ker T) + \dim(\operatorname{im} T)$$

$$3 = 0 + 2$$

(Not possible)

[Mid-sem : October 12, 2021].