Improper integrals

$f : [a, b] \to \mathbb{R}$ $\quad$ f is bounded. $\quad$ $\int_a^b f(x)\,dx$

What if we wanted to consider integrals of the form

① $\int_a^{\infty} f(x)\,dx$, $\quad$ $\int_{-\infty}^{b} f(x)\,dx$, $\quad$ $\int_{-\infty}^{\infty} f(x)\,dx$.

② $\int_a^b f(x)\,dx$ $\quad$ f is unbounded?

Improper integrals of first kind: interval is unbounded.

(i) $\int_a^\infty f(t)\,dt$ where f is integrable on $[a,x]$ $\forall x \geq a$

(ii) $\int_{-\infty}^b f(t)\,dt$ where f is integrable on $[x,b]$ $\forall x \leq b$.

(iii) $\int_{-\infty}^\infty f(t)\,dt$ where f is integrable on $[a,b]$ $\forall a,b \in \mathbb{R}$ with $a \leq b$

(i) $\int_a^\infty f(t)\,dt$. f is integrable on $[a,x]$ $\forall x \geq a$.

We say that $\int_a^\infty f(t)\,dt$ is convergent if $\lim_{x\to\infty} \int_a^x f(t)\,dt$ exists.

If $\int_a^\infty f(t)\,dt$ is convergent, then define

$$\int_a^\infty f(t)\,dt := \lim_{x\to\infty} \int_a^x f(t)\,dt.$$

Example:

$\sum \frac{1}{n}$ diverges

(1) $\int_1^\infty \frac{1}{t}\,dt$.

$$\lim_{x\to\infty} \int_1^x \frac{1}{t}\,dt$$
$$= \lim_{x\to\infty} (\ln x - \ln 1)$$
$$= \lim_{x\to\infty} \ln x = \infty$$

diverges.

$\sum \frac{1}{n^2}$ converges

(2) $\int_1^\infty \frac{1}{t^2}\,dt$.

$$\lim_{x\to\infty} \int_1^x \frac{1}{t^2}\,dt$$
$$= \lim_{x\to\infty} \left(-\frac{1}{t}\right]_1^x$$
$$= \lim_{x\to\infty} \left[\left(-\frac{1}{x}\right) - (-1)\right]$$
$$= \lim_{x\to\infty} \left(1 - \underbrace{\frac{1}{x}}_{\to 0}\right) = 1.$$

Converges!!.

(ii) $\int_{-\infty}^b f(t)\,dt$. where f is integrable on $[x,b]$ $\forall x \leq b$.

We say that $\int_{-\infty}^b f(t)\,dt$ is convergent if $\lim_{x\to-\infty} \int_x^b f(t)\,dt$ exists

If $\int_{-\infty}^b f(t)\,dt$ is convergent, define

$$\int_{-\infty}^b f(t)\,dt := \lim_{x\to-\infty} \int_x^b f(t)\,dt$$

Example:

$\int_{-\infty}^0 te^t\,dt$ is convergent!!.

$x<0$

$\int_x^0 te^t\,dt$. $[x,0]$

$f(t)=t$
$g(t)=e^t$

$$= \cancel{f(0)G(0)} - f(x)G(x) - \int_x^0 e^t\,dt$$
$$= -xe^x - e^t\Big]_x^0$$
$$= -xe^x - 1 + e^x$$

$$\lim_{x\to-\infty} \int_x^0 te^t\,dt$$
$$= \lim_{x\to-\infty} \left(\underbrace{-xe^x}_{\to 0} + \underbrace{e^x}_{\to 0} - 1\right)$$
$$= -1.$$

Integration by parts:

$f, g : [a,b] \to \mathbb{R}$

→ f differentiable, f' integrable.

→ g is integrable with an antiderivative G on $[a,b]$

$$\Rightarrow \int_a^b f(x)g(x)\,dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)\,dx$$

(iii) $\int_{-\infty}^\infty f(t)\,dt$. f is integrable on $[a,b]$ for all $a,b \in \mathbb{R}$ with $a \leq b$.

$\int_{-\infty}^\infty f(t)$ is said to be convergent if both $\boxed{\int_0^\infty f(t)\,dt}$ $\lim_{x\to\infty}\int_0^x f(t)\,dt$ and $\boxed{\int_{-\infty}^0 f(t)\,dt}$ exist. $\lim_{x\to-\infty}\int_x^0 f(t)\,dt$.

In such a case, we define

$$\int_{-\infty}^\infty f(t)\,dt = \int_{-\infty}^0 f(t)\,dt + \int_0^\infty f(t)\,dt.$$

$[-x,x]$

> $\lim_{x\to\infty} \int_{-x}^x f(t)\,dt$ exists, then it is called the Cauchy principal value of the integral $\int_{-\infty}^\infty f(t)\,dt$.

If $\int_{-\infty}^\infty f(t)\,dt$ is convergent, then

Domain additivity

$$\int_{-x}^x f(t)\,dt = \int_{-x}^0 f(t)\,dt + \int_0^x f(t)\,dt.$$

implies that the Cauchy principal value exists.

The converse is not true!!

Try $f(t) = \sin t$.

Example: $\int_{-\infty}^\infty \frac{1}{1+t^2}\,dt$.

$\int_{-\infty}^\infty \frac{1}{1+t^2}\,dt$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

$\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1}(x) - \underbrace{\tan^{-1}(0)}_{=0}$

$\lim_{x\to\infty} \int_0^x \frac{1}{1+t^2}\,dt = \frac{\pi}{2}$.

$\lim_{x\to-\infty} \int_x^0 \frac{1}{1+t^2}\,dt = \frac{\pi}{2}$.

$$\int_1^\infty \frac{1}{t^p}\,dt.$$ (when is this integral convergent)

$(p=1)$. $\int_1^\infty \frac{1}{t}\,dt$. diverges.

$(p\neq 1)$.

$$\lim_{a\to\infty}\int_1^a \frac{1}{t^p}\,dt = \frac{t^{1-p}}{1-p}\Big]_1^a = \frac{a^{1-p}-1}{1-p} = \boxed{\frac{1}{1-p}\left(\frac{1}{a^{p-1}}-1\right)}.$$

$$\Rightarrow \lim_{a\to\infty}\int_1^a \frac{1}{t^p}\,dt = \lim_{a\to\infty}\frac{1}{1-p}\left(\frac{1}{a^{p-1}}-1\right) = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases}$$

Convergent if $p>1$.