Lecture 5 - MA4020 Linear Algebra

September 03, 2021

Definition. A bijective map $\rho : S \longrightarrow S$ is called a *permutation of the set* S ($S \neq \phi$).

Let $S = \{1, 2, \dots, n\}$.

Remark. Every permutation on n element corresponds to a bijection on n element set.

Cases

$n = 2$, $\{1, 2\}$ → 12, 21

12 → $\rho_1 : \{1,2\} \to \{1,2\}$

$\rho_1(1) = 1$
$\rho_1(2) = 2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

21 → $\rho_2 : \{1,2\} \to \{1,2\}$

$\rho_2(1) = 2$
$\rho_2(2) = 1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$n = 3$: $\{1, 2, 3\}$ →

123 — ρ_1
132 — ρ_2
213 — ρ_3
231 — ρ_4
312 — ρ_5
321 — ρ_6

$p_1 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$123 = id$

$x \longmapsto x$

$1 \longrightarrow 2$

$2 \longrightarrow 3$

$3 \longrightarrow 3$

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Id.$$

$p_2 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$= 132$

$1 \longrightarrow 1$

$2 \longrightarrow 3$

$3 \longrightarrow 2$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2^{-1} = P_2$$

$p_3 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$= 213$

$1 \longrightarrow 2$

$2 \longrightarrow 1$

$3 \longrightarrow 3$

$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3^{-1} = P_3$$

Similarly for

$p_4 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$= 231$

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$p_5 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$= 312$

$$P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$p_6 : \{1,2,3\} \longrightarrow \{1,2,3\}$

$= 321$

$$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Left multiplication by permutation matrix.

Example.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{132} \overset{X}{\overset{\parallel}{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}_{3\times 4}}}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

"Rows of X are permuted w.r.t. permutation"

Discussion. The permutation matrix can be written in terms of the matrix units, e_{ij}, or in terms of certain column vectors denoted by e_i.

I. In terms of column vectors.

$$e_i = \begin{array}{r} \\ \\ \\ i^{th}\text{ row} \to \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \longleftarrow \text{Given permutation}$$

$$p : \{1,2,3\} \longrightarrow \{1,2,3\}$$

$$1 \longmapsto 1$$
$$2 \longmapsto 3$$
$$3 \longmapsto 2$$

$$P = \begin{bmatrix} e_{p^{-1}(1)} & e_{p^{-1}(2)} & e_{p^{-1}(3)} \end{bmatrix}$$

$$e_{p^{-1}(1)} = e_1, \quad e_{p^{-1}(2)} = e_3, \quad e_{p^{-1}(3)} = e_2$$

Similarly,

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longleftrightarrow q : \{1,2,3\} \longrightarrow \{1,2,3\}$$

$$1 \longrightarrow 3$$
$$2 \longrightarrow 1$$
$$3 \longrightarrow 2$$

$$Q = \begin{bmatrix} e_{q^{-1}(1)} & e_{q^{-1}(2)} & e_{q^{-1}(3)} \end{bmatrix}$$

$$e_{q^{-1}(1)} = e_2, \quad e_{q^{-1}(2)} = e_3, \quad e_{q^{-1}(3)} = e_1$$

$$= \begin{bmatrix} e & e & e \end{bmatrix}$$

II. In terms of matrix units.

$$\underline{\underline{e_{ij}}} \qquad \begin{array}{c c} & j^{th} \\ i & \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \end{array}$$

Example.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\updownarrow$

$$P = e_{p(1)1} + e_{p(2)2} + e_{p(3)3}$$

p

$p : \{1,2,3\} \to \{1,2,3\}$

$1 \mapsto 1$

$2 \mapsto 3$

$3 \mapsto 2$

1^{st} column is $e_{p(1)}$

2^{nd} column is $e_{p(2)}$

3^{rd} column is $e_{p(3)}$

Proposition. Let P be the permutation matrix associated to a permutation p. Then $p : \{1, \dots n\} \to \{1, \dots n\}$

(a). The j^{th} column of P is the column vector $e_{p(j)}$

(b). P is a sum of n matrix units:

$$P = e_{p(1)1} + \dots + e_{p(n)n}$$

$$= \sum_j e_{p(j)j}$$

(c). P is invertible and $P^{-1} = P^t$.

~~Pf.~~ p

(d)

$$\begin{tikzcd} \{1,\dots,n\} \arrow[r, "\overset{\text{bijection}}{q}", "Q"'] \arrow[rr, bend right, "pq"'] & \{1,2,\dots,n\} \arrow[r, "\overset{\text{bijection}}{p}", "P"'] & \{1,2,\dots,n\} \end{tikzcd}$$

$$\{1,2,\dots n\} \xrightarrow[PQ]{pq} \{1,2,\dots n\}$$

Notation / Definition. $\quad p : \{1,\dots,n\} \xrightarrow{P} \{1,\dots,n\}$

The sign of a permutation, p, is denoted by $\operatorname{sign} p$ and is defined to be

$$\operatorname{sign} p := \det P \quad (= \pm 1)$$

p is odd permutation if $\operatorname{sign} p = -1$

p is even permutation if $\operatorname{sign} p = 1$

Discussion. Let us try to write expression for determinants.

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21} .$$

$$\det \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$123 \longrightarrow$

$123 \leftarrow +$
$1\boxed{32}$ $-$
213
231
312
321

$3! = 6$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

$$= \sum (\operatorname{sign} p)\, a_{1*}\, a_{2*}\, a_{3*}$$

p: permutation

$p : \{1,2,3\} \to \{1,2,3\}$

Can we describe

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = (\operatorname{sign} p)\, a_{1-}\, a_{2-}\, a_{3-}\, a_{4-}$$

$\{1,2,3,4\}$

$4! = 24$ elements.

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

Goal is to describe $\det(A) = \cdots$

Using linearity of the determinant, we expand $\det(A)$

$$\det A = \det \begin{bmatrix} \boxed{a_{11}} & \overbrace{0 \cdots\cdots 0}^{0} \\ - & R_2 & - \\ & \vdots & \\ - & R_n & - \end{bmatrix} + \det \begin{bmatrix} 0 & \boxed{a_{11}} & 0 \cdots 0 \\ - & R_2 & - \\ & \vdots & \\ - & R_n & - \end{bmatrix} +$$

$$\cdots + \det \begin{bmatrix} 0 \cdots & 0 & \boxed{a_{1n}} \\ - & R_2 & - \\ & \vdots & \\ - & R_n & - \end{bmatrix}$$

Now, continue expanding each of these determinants on the 2^{nd} row, 3^{rd} row and so on.

Then $\det A$ is an expression as a sum of n^n $\cancel{n^2}$ determinants, each of which have only one non-zero entry in each row.

Let M be one among the $\cancel{n^2}\ n^n$ determinants.

$$M = \begin{bmatrix} — & \boxed{a_{1*}} & — \\ a_{2*} & — & \\ — & — & \\ & & \\ — & a_{n*} & — \end{bmatrix}$$

Example.

$$\det\begin{bmatrix} \boxed{a_{11} \quad a_{12}} \\ a_{21} \quad a_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \cancel{\det\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}} + \det\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \cancel{\det\begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}}$$

$$= \underset{\| \atop }{0} + \det\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + 0$$

$$= \det\left(\sum_{j=1}^{2} a_{p(j)j}\, e_{p(j)j}\right) + \det\left(\sum_{j=1}^{2} a_{p(j)j}\cdot e_{p(j)j}\right)$$

$p : \{1,2\} \to \{1,2\}$

$\begin{cases} 1 \to 1 \\ 2 \to 2 \end{cases}$

$p : \{1,2\} \to \{1,2\}$

$1 \to 2$

$2 \to 1$

$$= a_{p(1)1}\, a_{p(2)2}\,(\det P) + a_{p(1)1}\, a_{p(2)2}\,(\det P)$$

$$= \sum_{\wp:\ \text{permutation on } \{1,2\}} (\operatorname{sign} \wp)\, a_{p(1)1}\, a_{p(2)2}.$$

Observe that every square matrix

$$M = \begin{bmatrix} & a_{1*} & \\ a_{2*} & & \\ & & a_{n*} \end{bmatrix}$$

is like a permutation matrix, except 1's are replaced by the matrix entries A.

Thus, we may write

$$P = \sum_j e_{p(j)j} \quad \text{and} \quad M = \sum_j a_{p(j)j}\, e_{p(j)j} .$$

By linearity of the determinant, we can describe

$$\det M = (a_{p(1)1} \cdots a_{p(n)n})(\det P)$$

$$= (\operatorname{sign} p)(a_{p(1)1} \cdots a_{p(n)n})$$

For every permutation on $\{1,..,n\}$, we get one such term.

Hence

$$\boxed{\det(A) = \sum_{\substack{\text{Permutation} \\ p : \{1,..n\} \to \{1..n\}}} (\operatorname{sign} p)\, a_{p(1)1} \cdots a_{p(n)n} .}$$

or equivalently

$[\det A = \det A^t]$

$$\det(A) = \sum_{S_n : \text{all permutations on } \{1,..n\}} (\operatorname{sign} p)\, a_{1p(1)} \cdots a_{np(n)} .$$

$$A = [a_{ij}]$$

$$\det A = \sum_{\substack{\text{permutations} \\ p}} (\operatorname{sign} p)\, a_{1p(1)} \cdots a_{np(n)}.$$

Expanding by minors on the i^{th} row

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots$$
$$+ (-1)^{i+n} a_{in} \det A_{in}$$

or,

$$\det A = (-1)^{j+1} a_{1j} \det A_{1j} + (-1)^{j+2} a_{2j} \det A_{2j} + \cdots$$
$$+ (-1)^{j+n} a_{nj} A_{nj}.$$

$$\begin{bmatrix} + & - & + & - \\ - & + & \cdot & \cdot \\ + & - & + & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Definition. Let A be an $n \times n$ matrix. The adjoint of A is the $n \times n$ matrix whose (i,j) entry

$$(\operatorname{adj})_{ij} = (-1)^{i+j} \cdot \det A_{ji}$$

$$\parallel$$

$$\alpha_{ji}$$

Then

$$(\operatorname{adj} A) = (\alpha_{ij})^t,$$

where $\alpha_{ij} = (-1)^{i+j} \det A_{ij}$.

Example.

$$\operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\operatorname{adj} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = [\alpha_{ij}]^t$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Theorem. Let $\delta = \det A$. Then

$$(\operatorname{adj} A)\cdot A = \delta I, \text{ and}$$

$$A\cdot(\operatorname{adj} A) = \delta I.$$

Proof.

$$\left[(\operatorname{adj} A)\cdot A\right]_{(i,j)^{th}\text{ entry}} \begin{cases} \det A & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$$\|$$

$$(\operatorname{adj})_{i1}\, a_{1j} + \cdots + (\operatorname{adj})_{in}\, a_{nj}$$

When $i=j$, we get $\det(A)$ (why), else 0 (why)

(Exercise!)

$$(\operatorname{adj} A)\cdot A = \begin{bmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{bmatrix}$$

Corollary. Let A be a square matrix with $\det A \neq 0$. Then

$$A^{-1} = \frac{1}{\det A} \cdot (\operatorname{adj} A)$$

Proof. "Easy".

Discussion. Consider a system of linear equations

$AX = B$, where A is an $n \times n$ matrix with $\det A \neq 0$.

$$A^{-1} \cdot A X = A^{-1} \cdot B$$

$$\Rightarrow \quad X = A^{-1} \cdot B$$

$n \times n$

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$X = \frac{1}{\det A} \cdot (\operatorname{adj} A) B$$

From this, we can write

$$x_j = \frac{1}{\det A} \cdot (b_1 \alpha_{1j} + \cdots + b_n \alpha_{nj}),$$

where $\alpha_{ij} = \pm \det A_{ij}$.

Construct a new matrix M_j replacing the j^{th} column of A by the column vector B.

Expansion by minors on the j^{th} column of M_j is

$$\det M_j = b_1 \alpha_{1j} + \cdots + b_n \alpha_{nj} .$$

Using this, we can re-write,

$$x_j = \frac{\det M_j}{\det A}$$

Known as "Cramer's Rule".

FOR DISCUSSION ON PROBLEMS, SEE RECORDING OF TODAY LECTURE.