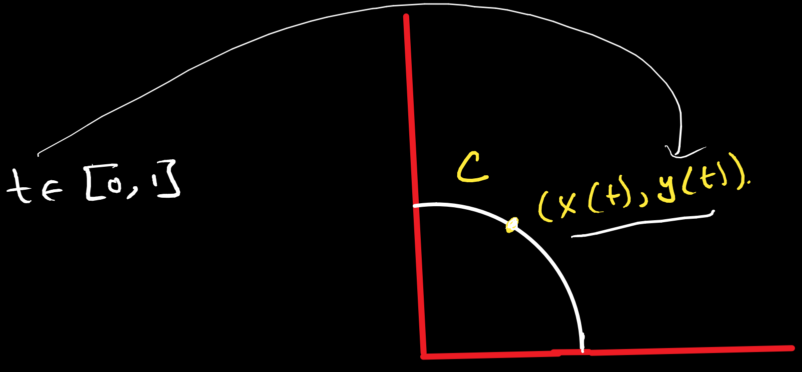
Arc length

A **parametrized curve** $C \subseteq \mathbb{R}^2$ is given by $(x(t), y(t))$ where $x, y : [\alpha, \beta] \to \mathbb{R}$, x, y are continuous.

Examples:

① $\left.\begin{array}{l} x : [0,1] \to \mathbb{R} \\ y : [0,1] \to \mathbb{R} \end{array}\right\} \rightsquigarrow (x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$



Goal: Determine the length of C?

② $(x(t), y(t)) = (t^2, 2t)$ $\quad [0,1]$

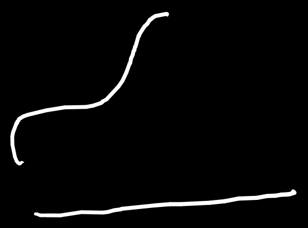
$x, y : [0,1] \to \mathbb{R}$. $\quad [\alpha,\beta] \quad (x(t), y(t))$

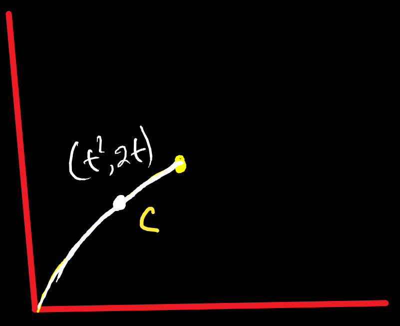
$x(t) = t^2$

$y(t) = 2t$.

$\boxed{y^2 = 4x}$

Parabola:





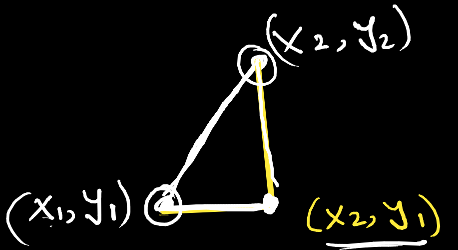
$f : [\alpha, \beta] \to \mathbb{R}^2$.

$f(t) = (x(t), y(t))$

Basic assumptions:

① Length of the line joining the two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$



② **C is smooth:** The functions x, y are continuously differentiable.

$\to$ $x, y : [\alpha, \beta] \to \mathbb{R}$ differentiable.

$\to$ $x', y' : [\alpha, \beta] \to \mathbb{R}$ continuous.

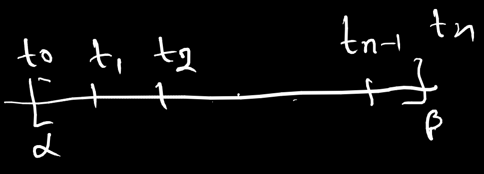
Definition: (Arc length). The length of C as

$$\boxed{\ell(C) := \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt}$$

Rough idea:

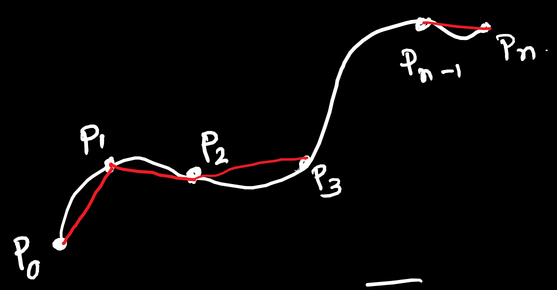
$\to$ Partition $[\alpha, \beta]$ into n parts $\{\alpha = t_0, t_1, \dots, t_n = \beta\}$.

$\to$ $P_i := (x(t_i), y(t_i))$ for $i = 1, \dots, n$.



$(x(t_{i-1}), y(t_{i-1}))$

$(x(t_i), y(t_i))$



$\ell\left(\overline{P_{i-1} P_i}\right)$

$=$

$\to$ Draw line segments $\overline{P_0P_1}, \overline{P_1P_2}, \dots, \overline{P_{n-1}P_n}$.

$[t_{i-1}, t_i]$

$\to$ $\ell(C) \approx \overline{P_0P_1} + \overline{P_1P_2} + \dots + \overline{P_{n-1}P_n}$.

$P_i = (x(t_i), y(t_i))$.

$$= \sum_{i=1}^n \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

$x : [t_{i-1}, t_i] \to \mathbb{R}$ continuous. diff.

(MVT) $$= \sum_{i=1}^n \sqrt{(x'(s_i))^2 + (y'(u_i))^2}\,(t_i - t_{i-1})$$ for some $s_i, u_i \in (t_{i-1}, t_i)$

$\exists\, s_i \in (t_{i-1}, t_i)$. "

$x'(s_i) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}$ " $\longrightarrow$ "

$$\int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Some special cases:

① $C: y = f(x)$ ← continuously differentiable, $x \in [a,b]$

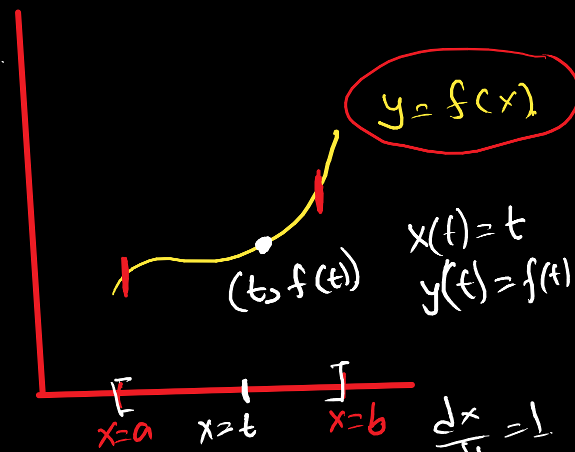
$\alpha := a, \quad \beta = b.$

$$\boxed{x(t) = t, \quad y(t) = f(t)}$$

$$\ell(C) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

$$= \int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$$



$x(t) = t$
$y(t) = f(t).$

$\frac{dx}{dt} = 1$

② $C: x = g(y)$ $\quad y \in [a,b]$

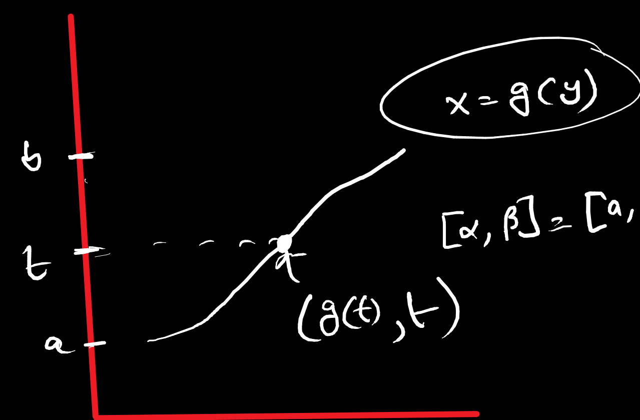
$\alpha = a, \quad \beta = b.$

$$(x(t), y(t)) = (g(t), t).$$

$$\ell(C) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_a^b \sqrt{(g'(t))^2 + 1}\, dt$$

$$= \int_a^b \sqrt{(g'(y))^2 + 1}\, dy.$$



$[\alpha, \beta] = [a,b]$

$x'(t) = g'(t)$
$y'(t) = 1.$

Example:

(1) Perimeter of a circle:

$x(\theta) = r\cos\theta \qquad y(\theta) = r\sin\theta \qquad 0 \le \theta \le 2\pi.$

$x'(\theta) = -r\sin\theta.$

$y'(\theta) = r\cos\theta.$

$$x'(\theta)^2 + y'(\theta)^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2.$$

$$l(c) = \int_0^{2\pi} \sqrt{\underbrace{x'(\theta)^2 + y'(\theta)^2}_{r^2}}\, d\theta = \int_0^{2\pi} r\, d\theta = 2\pi r$$

(2) $x(t) = \cos^3 t \qquad y(t) = \sin^3 t \qquad 0 \le t \le \frac{\pi}{2} \qquad [0, \frac{\pi}{2}]$

$x'(t) = -3\cos^2 t \sin t.$

$y'(t) = 3\sin^2 t \cos t.$

$$\begin{aligned} x'(t)^2 + y'(t)^2 &= 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t \\ &= 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) \\ &= 9\cos^2 t \sin^2 t. \\ &= \frac{9}{4}\sin^2 2t \end{aligned}$$

$$\begin{aligned} l(c) &:= \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt. \\ \text{FTC}(2) &= -\frac{3}{4} \cos 2t \Big]_0^{\pi/2} \\ &= -\frac{3}{4}(\cos\pi - \cos 0) \\ &= \frac{3}{2}. \end{aligned}$$

(3) $y = x^{3/2} \qquad 0 \le x \le 1.$

$y = f(x)$ ← cont. diff.

$$\begin{aligned} l(c) &= \int_0^1 \sqrt{1 + (f'(x))^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{1}{2}\int_0^1 \sqrt{4 + 9x}\, dx \\ \text{FTC}(2) &= \frac{1}{27}\left((13)^{3/2} - 8\right). \end{aligned}$$

$f(x) = x^{3/2}. \qquad f'(x) = \frac{3}{2}x^{1/2}$

$$(4 + 9x)^{1/2} = \left(\frac{2}{3} \cdot \frac{(4+9x)^{3/2}}{9}\right)'$$