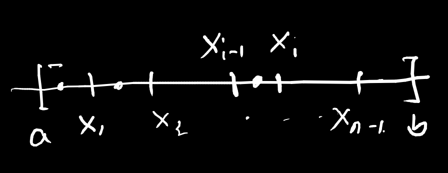
Integration: What we have learnt and the difficulties:

Goal: Given a bounded function $f:[a,b]\to\mathbb{R}$, determine if f is integrable. If yes, then compute $\int_a^b f(x)\,dx$.

Approaches:	Difficulties:
Computing $U(P,f)$ and $L(P,f)$	One needs to compute absolute minima and maxima of f in several subintervals.
Corollary of Riemann condition: f is integrable iff there exists a sequence of partitions (P_n) of $[a,b]$ such that $U(P_n,f)-L(P_n,f)\to 0$ as $n\to\infty$	How do we go about choosing (P_n) so that $U(P_n,f)-L(P_n,f)\to 0$ as $n\to\infty$.
Using FTC: Try to find an antiderivative F with $F'=f$ on $[a,b]$.	f might not have an anti-derivative or even if it has one, it might be very difficult to compute.

Riemann sums

Given a function $f:[a,b]\to\mathbb{R}$ (bounded), and a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$.



Choose $t_1,t_2,\dots,t_n$ in $[a,b]$ such that $t_i\in[x_{i-1},x_i]$ for $i=1,\dots,n$.

Riemann sum for f corresponding to the partition P. $\to$ $S(P,f):=\sum_{i=1}^n f(t_i)(x_i-x_{i-1})$.

Compare: $S(P,f)$, $L(P,f)$, $U(P,f)$.

$$m_i(f)\le f(t_i)\le M_i(f). \quad \forall i=1,\dots,n.$$

$$\Rightarrow \sum_{i=1}^n m_i(f)(x_i-x_{i-1})\le \sum_{i=1}^n f(t_i)(x_i-x_{i-1})\le \sum_{i=1}^n M_i(f)(x_i-x_{i-1})$$

$$\Rightarrow \quad L(P,f)\le S(P,f)\le U(P,f) \longrightarrow (1)$$

The mesh of a partition P is given by

$$\underbrace{\mu(P)}_{\text{mesh}}:=\max\{x_1-x_0,x_2-x_1,\dots,x_n-x_{n-1}\}$$

Theorem: Let f be integrable on $[a,b]$ and let $\varepsilon>0$ be given. Then there exists a $\delta>0$ (depending on ε) such that $U(P,f)-L(P,f)<\varepsilon$ whenever $\mu(P)<\delta$.

Pf: Omitted!!

Corollary: Let f be integrable on $[a,b]$. Suppose that there exists a sequence of partitions (P_n) such that $\mu(P_n)\to 0$ as $n\to\infty$. Then $U(P_n,f)-L(P_n,f)\to 0$ as $n\to\infty$.

Let $S(P_n,f)$ be a Riemann sum corresponding to P_n and f. Then $S(P_n,f)\to\int_a^b f(x)\,dx$ as $n\to\infty$. $\left(\int_a^b f(x)\,dx=\lim_{n\to\infty}S(P_n,f)\right)$

Pf: Let $\varepsilon>0$ be given. Since f is integrable on $[a,b]$, by the previous theorem, there exists $\delta>0$ such that whenever $\mu(P)<\delta$, we have $U(P,f)-L(P,f)<\varepsilon. \longrightarrow (*)$

$\mu(P_n)\to 0$ as $n\to\infty$.

By definition, for the given δ, $\exists\, n_0\in\mathbb{N}$ s.t.

$$\mu(P_n)<\delta \quad \text{whenever } n\ge n_0.$$

$$\Rightarrow U(P_n,f)-L(P_n,f)<\varepsilon. \quad \text{whenever } n\ge n_0.$$

$$\Rightarrow U(P_n,f)-L(P_n,f)\to 0 \text{ as } n\to\infty.$$

$$\Rightarrow \lim_{n\to\infty}U(P_n,f)=\lim_{n\to\infty}L(P_n,f).$$

By (1),

$$L(P_n,f)\le S(P_n,f)\le U(P_n,f). \longrightarrow (1')$$

$$L(P_n,f)\le \underset{\substack{\uparrow\\ \text{Since}\\ f \text{ is integrable}}}{L(f)=}\int_a^b f(x)\,dx=U(f)\le U(P_n,f) \longrightarrow (2)$$

Summarizing,

$$L(P_n,f)\le S(P_n,f)\le U(P_n,f) \longrightarrow (1')$$
$$L(P_n,f)\le \int_a^b f(x)\,dx\le U(P_n,f)\longrightarrow (2)$$

Claim: $\left|S(P_n,f)-\int_a^b f(x)\,dx\right|\le U(P_n,f)-L(P_n,f).$

Aside:
$\lvert x\rvert\le y$ $\Leftrightarrow -y\le x\le y$

Proof of claim:

$$S(P_n,f)\le U(P_n,f)\longrightarrow (3)$$
$$\int_a^b f(x)\,dx\ge L(P_n,f)\longrightarrow (4)$$

(3) − (4) $\quad S(P_n,f)-\int_a^b f(x)\,dx\le U(P_n,f)-L(P_n,f)$

$$S(P_n,f)\ge L(P_n,f)\longrightarrow (5)$$
$$\int_a^b f(x)\,dx\le U(P_n,f)\longrightarrow (6)$$

(5) − (6) $\quad S(P_n,f)-\int_a^b f(x)\,dx\le L(P_n,f)-U(P_n,f).$

$$\Longrightarrow \left|S(P_n,f)-\int_a^b f(x)\,dx\right|\le U(P_n,f)-L(P_n,f)$$

Completes Proof of claim.

$$0\le\left|S(P_n,f)-\int_a^b f\right|\le U(P_n,f)-L(P_n,f)\longrightarrow 0 \text{ as } n\to\infty.$$

By Sandwich theorem,

$$S(P_n,f)-\int_a^b f\to 0 \text{ as } n\to\infty.$$

$$\Rightarrow S(P_n,f)\to\int_a^b f(x)\,dx \quad \square$$

If f is integrable, and (P_n) a sequence of partitions s.t. $\mu(P_n)\to 0$ then

$$S(P_n,f)\longrightarrow\int_a^b f(x)\,dx \text{ as } n\to\infty.$$

$$\lim_{n\to\infty}\sum_{i=1}^n f(t_i)(x_i-x_{i-1})=\int_a^b f(x)\,dx. \quad \square$$

Remarks: (1) $T=\{t_1,\dots,t_n\}$ is called a tag set.

(2) We will mostly choose P_n to be the partition of $[a,b]$ into n equal subintervals.

$$P_n:=\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\dots,b\right\}.$$

$$\Delta x:=\frac{b-a}{n}.$$

Leibnitz

$$\int_a^b f(x)\,dx=\lim_{\substack{(n\to\infty)\\(\mu(P_n)\to 0)}}S(P_n,f). \qquad \Delta x=x_i-x_{i-1}$$

$$=\lim_{n\to\infty}\sum_{i=1}^n f(t_i)\,\Delta x$$

$$=\boxed{\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n f(t_i).}$$