Improper Integrals (of Second kind).

$(a,b]$ $\quad$ (a,b).
$[a,b)$

① Let $a,b \in \mathbb{R}$ with $a<b$.
$f:(a,b] \to \mathbb{R}$ such that f is UNBOUNDED on $(a,b]$
but f is integrable on $[x,b]$ for $x \in (a,b]$.
$a < x \le b$.

$\int_a^b f \leftrightarrow [a,b]$

$\int_{a<t\le b} f \leftrightarrow (a,b]$

$x \in (a,b]$

→ $\int_{a<t\le b} f(t)\,dt$ is convergent if $\lim_{x\to a^+} \boxed{\int_x^b f(t)\,dt}$ exists.

divergent otherwise.

Value: $\int_{a<t\le b} f(t)\,dt := \lim_{x\to a^+} \int_x^b f(t)\,dt$.

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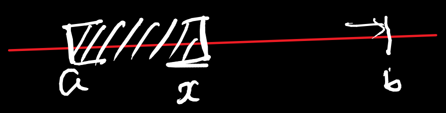
② Let $a,b \in \mathbb{R}$ with $a<b$.
$f:[a,b) \to \mathbb{R}$ such that f is UNBOUNDED on $[a,b)$
but f is integrable on $[a,x]$ for $x \in [a,b)$.

$\int_a^b f \leftrightarrow [a,b]$



$\int_{a\le t<b} f \leftrightarrow [a,b)$

→ $\boxed{\int_{a\le t<b} f(t)\,dt}$ is convergent if $\lim_{x\to b^-} \boxed{\int_a^x f(t)\,dt}$ exists.

divergent otherwise.

Value: $\int_{a\le t<b} f(t)\,dt := \lim_{x\to b^-} \int_a^x f(t)\,dt$.

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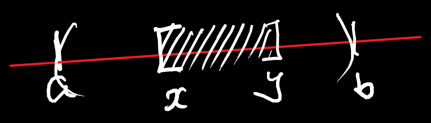
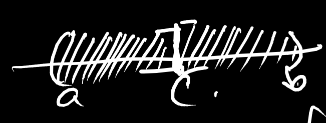
③ Let $a,b \in \mathbb{R}$ with $a<b$.
$f:(a,b) \to \mathbb{R}$ such that f is UNBOUNDED on (a,b)
but f is integrable on $[x,y]$ for $x,y \in (a,b)$

$\int_a^b f \leftrightarrow [a,b]$



$\int_{a<t<b} f \leftrightarrow (a,b)$

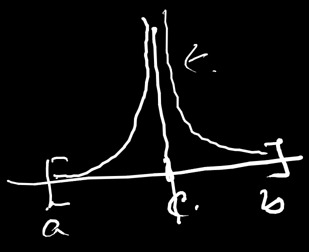


→ $\int_{a<t<b} f(x)\,dx$ is convergent if for all $c \in (a,b)$, $\int_{a<t\le c} f(t)\,dt$ and $\int_{c\le t<b} f(t)\,dt$ are convergent

divergent otherwise.

Value: $\int_{a<t<b} f(t)\,dt = \int_{a<t\le c} f(t)\,dt + \int_{c\le t<b} f(t)\,dt$.

$[a-\varepsilon, b+\varepsilon]$ $\quad$ (a,b) $\quad$ $\boxed{\int_{a<t<b} f(t)\,dt}$



If $\lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b-\varepsilon} f(t)\,dt$ exists, then it is called the

Cauchy principal value of $\int_{a<t<b} f(t)\,dt$.

$\Longrightarrow$ If $\int_{a<t<b} f(t)\,dt$ is convergent, then it has a Cauchy principal value.

the converse is not true.

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④ $a<b$. suppose $c \in (a,b)$. such that

(i) f is integrable on $[a,x]$ $\forall x \in [a,c)$
but unbounded on $[a,c)$

$\forall x \in (c,b]$.

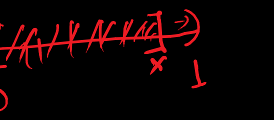
+ (ii) f is integrable on $[x,b]$
but unbounded on $(c,b]$. $\quad$ $[a,c)$ $\quad$ $(c,b]$

Then. $\int_a^b f$ is convergent if $\int_{a\le t<c} f(t)\,dt$ and $\int_{c<t\le b} f(t)\,dt$ are convergent

Value $\int_a^b f = \int_{a\le t<c} f + \int_{c<t\le b} f$.

Examples:

(1) $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$



$f(t) = \frac{1}{1-t}$ continuous on $[0, x]$ for all $x \in [0,1)$

$\Downarrow$

integrable

$$\lim_{x \to 1^-} \int_0^x \frac{1}{1-t}\,dt$$

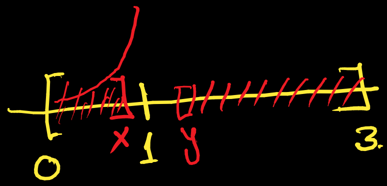
$$= \lim_{x \to 1^-} -\ln|1-t| \Big]_0^x$$

$$= -\lim_{x \to 1^-} \ln|1-x| = \infty$$

Divergent!!

(2) $$\int_0^3 \frac{1}{(t-1)^{2/3}}\,dt.$$

$[0,1)$. $(1,3]$



$$\int_0^1 \frac{1}{(t-1)^{2/3}}\,dt.$$

$[0,x]$ $\quad x\in[0,1)$

$3(t-1)^{1/3}$

$$\lim_{x\to 1^-}\int_0^x \frac{1}{(t-1)^{2/3}}\,dt.$$

convergent

$$= \lim_{x\to 1^-} 3(t-1)^{1/3}\Big]_0^x$$

$$= \lim_{x\to 1^-} 3\left[(x-1)^{1/3} - (-1)^{1/3}\right]$$

$$= 3.$$

$$\int_1^3 \frac{1}{(t-1)^{2/3}}\,dt.$$

convergent

$$\lim_{x\to 1^+}\int_x^3 \frac{1}{(t-1)^{2/3}}\,dt.$$

$$= \lim_{x\to 1^+} 3.(t-1)^{1/3}\Big]_x^3$$

$$= \lim_{x\to 1^+}\left[3(3-1)^{1/3} - 3(x-1)^{1/3}\right]$$

$$= 3\cdot 2^{1/3}$$

$$\int_0^3 \frac{1}{(t-1)^{2/3}}\,dt = 3+3\cdot 2^{1/3}.$$