10.5
A Special Kind of Nonlinear Recurrence Relation (Optional)

Thus far our study of recurrence relations has dealt with linear relations with constant coefficients. The study of nonlinear recurrence relations and of relations with variable coefficients is not a topic we shall pursue except for one special nonlinear relation that lends itself to the method of generating functions.

We shall develop the method in a counting problem on data structures. Before doing so, however, we first observe that if $f(x) = \sum_{i=0}^{\infty} a_i x^i$ is the generating function for $a_0, a_1, a_2, \ldots,$ then $[f(x)]^2$ generates $a_0a_0, a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, \ldots,$

$a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1 + a_na_0, \ldots,$ the convolution of the sequence $a_0, a_1, a_2, \ldots,$ with itself.

EXAMPLE 10.42

In Sections 3.4 and 5.1, we encountered the idea of a tree diagram. In general, a *tree* is an undirected graph that is connected and has no loops or cycles. Here we examine rooted binary trees.

In Fig. 10.17 we see two such trees, where the circled vertex denotes the *root*. These trees are called *binary* because from each vertex there are at most two edges (called *branches*) descending (since a rooted tree is a directed graph) from that vertex.

In particular, these rooted binary trees are *ordered* in the sense that a left branch descending from a vertex is considered different from a right branch descending from that vertex. For the case of three vertices, the five possible ordered rooted binary trees are shown in Fig. 10.18. (If no attention were paid to order, then the last four rooted trees would be the same structure.)

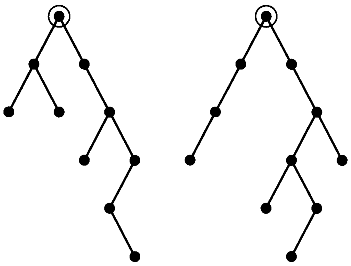

Figure 10.17

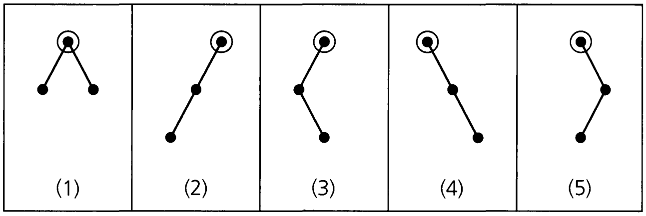

Figure 10.18

Our objective is to count, for $n \geq 0$, the number b_n of rooted ordered binary trees on n vertices. Assuming that we know the values of b_i for $0 \leq i \leq n$, in order to obtain b_{n+1} we select one vertex as the root and note, as in Fig. 10.19, that the substructures descending on the left and right of the root are smaller (rooted ordered binary) trees whose total number of vertices is n. These smaller trees are called *subtrees* of the given tree. Among these possible subtrees is the empty subtree, of which there is only 1 ($= b_0$).

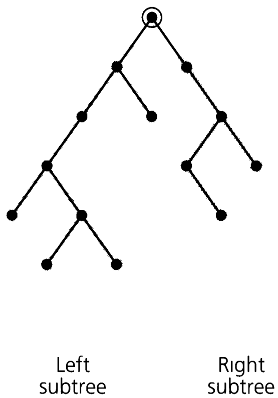

Figure 10.19

Now consider how the n vertices in these two subtrees can be divided up.

(1) 0 vertices on the left, n vertices on the right. This results in b_0b_n overall substructures to be counted in b_{n+1}.

(2) 1 vertex on the left, $n-1$ vertices on the right, yielding b_1b_{n-1} rooted ordered binary trees on $n+1$ vertices.

. . . . . .

($i+1$) i vertices on the left, $n-i$ on the right, for a count of b_ib_{n-i} toward b_{n+1}.

. . . . . .

($n+1$) n vertices on the left and none on the right, contributing b_nb_0 of the trees.

Hence, for all $n \geq 0$,

$$b_{n+1} = b_0b_n + b_1b_{n-1} + b_2b_{n-2} + \cdots + b_{n-1}b_1 + b_nb_0,$$

and

$$\sum_{n=0}^{\infty} b_{n+1}x^{n+1} = \sum_{n=0}^{\infty}(b_0b_n + b_1b_{n-1} + \cdots + b_{n-1}b_1 + b_nb_0)x^{n+1}. \tag{1}$$

Now let $f(x) = \sum_{n=0}^{\infty} b_nx^n$ be the generating function for $b_0, b_1, b_2, \ldots$. We rewrite Eq. (1) as

$$(f(x) - b_0) = x\sum_{n=0}^{\infty}(b_0b_n + b_1b_{n-1} + \cdots + b_nb_0)x^n = x[f(x)]^2.$$

This brings us to the quadratic [in $f(x)$]

$$x[f(x)]^2 - f(x) + 1 = 0, \quad \text{so} \quad f(x) = [1 \pm \sqrt{1-4x}]/(2x).$$

But $\sqrt{1-4x} = (1-4x)^{1/2} = \binom{1/2}{0} + \binom{1/2}{1}(-4x) + \binom{1/2}{2}(-4x)^2 + \cdots$, where the coefficient of x^n, $n \geq 1$, is

$$\begin{aligned}
\binom{1/2}{n}(-4)^n &= \frac{(1/2)((1/2)-1)((1/2)-2)\cdots((1/2)-n+1)}{n!}(-4)^n \\
&= (-1)^{n-1}\frac{(1/2)(1/2)(3/2)\cdots((2n-3)/2)}{n!}(-4)^n \\
&= \frac{(-1)2^n(1)(3)\cdots(2n-3)}{n!} \\
&= \frac{(-1)2^n(n!)(1)(3)\cdots(2n-3)(2n-1)}{(n!)(n!)(2n-1)} \\
&= \frac{(-1)(2)(4)\cdots(2n)(1)(3)\cdots(2n-1)}{(2n-1)(n!)(n!)} = \frac{(-1)}{(2n-1)}\binom{2n}{n}.
\end{aligned}$$

In $f(x)$ we select the negative radical; otherwise, we would have negative values for the b_n's. Then

$$f(x) = \frac{1}{2x}\left[1 - \left[1 - \sum_{n=1}^{\infty}\frac{1}{(2n-1)}\binom{2n}{n}x^n\right]\right],$$

and b_n, the coefficient of x^n in $f(x)$, is half the coefficient of x^{n+1} in

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\binom{2n}{n}x^n.$$

So

$$b_n = \frac{1}{2}\left[\frac{1}{2(n+1)-1}\right]\binom{2(n+1)}{n+1} = \frac{(2n)!}{(n+1)!(n!)} = \frac{1}{(n+1)}\binom{2n}{n}.$$

The numbers b_n are called the *Catalan numbers* — the same sequence of numbers we encountered in Section 1.5. As we mentioned earlier (following Example 1.42), these numbers are named after the Belgian mathematician Eugène Charles Catalan (1814–1894), who used them in determining the number of ways to parenthesize the expression $x_1x_2x_3\cdots x_n$. The first nine Catalan numbers are $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 14$, $b_5 = 42$, $b_6 = 132$, $b_7 = 429$, and $b_8 = 1430$.