CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 9 AND LECTURE 10

Instructions

- Try these questions before class. Do not submit!

(1) Let $a, b \in \mathbb{N}$. Then $ab = gcd(a, b) \cdot lcm(a, b)$.

(2) (Fermat's Little Theorem) If p is prime and a is an integer not divisible by p then

$$a^{p-1} \equiv 1 \text{ mod } p.$$

Furthermore, for every integer a we have

$$a^p \equiv a \text{ mod } p.$$

(3) Find 50^{50} mod 13 using FLT.

(4) Let $\mathbb{Z}_m^*$ be defined as follows:

$$\mathbb{Z}_m^* = \{a \in \mathbb{Z}_m : gcd(a, m) = 1\}.$$

S.T. $\mathbb{Z}_m^*$ is a group with the binary operation $\cdot_m$.

(5) Given any sequence of $mn + 1$ real numbers, some subsequence of $(m + 1)$ numbers is increasing or some subsequence of $(n + 1)$ numbers is decreasing.

(6) How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that a_1 mod $5 = a_2$ mod 5 and b_1 mod $5 = b_2$ mod 5.

(7) Show that a non-empty set **(of finite size!!)** has an equal number of even subsets (that is, subsets with an even number of elements) and odd subsets.
It's false for the empty set (and meaningless for infinite sets).

(8) Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies. The only thing there really is to do here is try things out. In particular, you find that one arrangement that works is as follows: The following people are friends: $(A, B), (B, C), (C, D), (D, E), (E, A)$. The rest are all enemies.

(9) Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

(10) Given the set of natural numbers, we saw that the power set of $\mathbb{N}$ is uncountable. What about the set of all finite subsets of the natural numbers, is it countable?

(11) How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers $k, k+1, k+2$, in the correct order
a) where these consecutive integers can perhaps be separated by other integers in the permutation?
b) where they are in consecutive positions in the permutation?

(12) Let n be a positive integer. S.T.

$$\binom{2n}{n+1} + \binom{2n}{n} = (\binom{2n+1}{n+1})/2.$$

(13) Let n, k be integers with $1 \le k \le n$. S.T.

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}$$