CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 11

Instructions

- Try these questions before class. Do not submit!

(1) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where $x_i, i = 1, 2, 3, 4, 5$ is a nonnegative integer s.t.
(a) $x_1 \geq 1$?
(b) $x_i \geq 2$, for $i = 1, 2, 3, 4, 5$?
(c) $0 \leq x_1 \leq 10$?
(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$ and $x_3 \geq 15$?

Answer:

(a) Let $x_1 = x_1^{'} + 1$. Now the problem can be thought of as finding non-negative solutions to $x_1^{'} + x_2 + x_3 + x_4 + x_5 = 20$. This we know from combinations with repetitions that it is $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition is allowed.

$C(5 + 20 - 1, 20) = C(24, 20)$ of them.

(b) Substitute $x_i = x_i^{'} + 2$ for all i and working similarly as above we get $x_1^{'} + x_2^{'} + x_3^{'} + x_4^{'} + x_5^{'} = 11$. By same theorem there are $(5 + 11 - 1, 11) = C(15, 11)$ of them.

(c) With no restrictions, there are $C(5 + 21 - 1, 21) = 12650$ solutions. Let $x_1 \geq 11$ then we have from what we did in the first part, $C(5+10-1, 10) = C(14, 10) = 1001$ solutions. Subtracting 1001 from 12650 to get $11,649$ we get the required solution.

(d) Let us impose the easy restrictions first $x_3 \geq 15$ and $x_2 \geq 1$ to get an equivalent problem:
Find number of solutions of

$$x_1 + x_2^{'} + x_3^{'} + x_4 + x_5 = 5,$$

under the conditions, $x_1 \leq 3$, $x_2^{'} \leq 2$ since $x_2 < 4$ implies $x_2 \leq 3$ and $x_2^{'} \leq 2$.
We count the number of solutions of the equation above and subtract the solutions in which $x_1 \geq 4$ and $x_2^{'} \geq 3$. $C(5 + 5 - 1, 5) = 126$ solutions to the unrestricted case. Applying $x_1 \geq 4$ gives us the equation,

$$x_1^{'} + x_2^{'} + x_3^{'} + x_4 + x_5 = 1.$$

This has $C(5 + 1 - 1, 1) = 5$ solutions. Applying $x_2^{'} \leq 2$ we get,

$$x_1 + x_2^{''} + x_3^{'} + x_4 + x_5 = 2,$$

which has $C(5+2-1,2) = 15$ solutions. The final answer is $126-5-15 = 106$ solutions.

(2) How many solutions are there to the inequality $x_1+x_2+x_3 \leq 11$, where x_1, x_2, and x_3 are nonnegative integers?[Hint: Introduce an auxiliary variable x_4 s.t. $x_1 + x_2 + x_3 + x_4 = 11$.]

Answer: Introduce a new variable x_4 to get the equation $x_1+x_2+x_3+x_4 = 11$. This one has $C(4 + 11 - 1, 11) = C(14, 3) = 364$ solutions. Why is this correct? Since if we have a solution for the equation s.t. $a_1+a_2+a_3+a_4 = 11$ then $a_1 + a_2 + a_3 = 11 - a_4 \leq 11$ and therefore a_1, a_2, a_3 is a solution for the inequality. Also if $b_1+b_2+b_3 \leq 11$ then we can consider b_4 as $11-(b_1+b_2+b_3)$ and we get a solution for the equation. Thus a $1-1$ correspondence exists and counting one is the same as counting the other.

(3) How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of 12 0 bits, and must have at least two 0 bits following each 1 bit?

We place 4 1s with space in between and there are four spaces (we do not consider a space before the first 1). Into the spaces we have to place 12 0s. Thus we represent the space as s_i we get, $s_1 + s_2 + s_3 + s_4 = 12$ with $s_i \geq 2$ for each i. Since every 1 should have at least 0s following it, we are actually looking for the number of nonnegative solutions of

$$s_1^{'} + s_2^{'} + s_3^{'} + s_4^{'} = 4.$$

This has $C(4 + 4 - 1, 4 = C(7, 4) = 35$ solutions.

(4) Solve these recurrences with the initial conditions given.

(a) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0, a_1 = 1$

(b) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1, a_1 = 0$

(a) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0, a_1 = 1$

The characteristic equation is $r^2+4r+4 = 0$ and the root is $r = -2$ with multiplicity 2. We have the general solution as

$$a_n = \alpha_1(-2)^n + \alpha_2 n(-2)^n$$

where α_1, α_2 are constants. To solve for these constants, $a_0 = 0$ implies $\alpha_1 = 0$ and $a_1 = 1$ means we have $1 = -2\alpha_1 - 2\alpha_2$. And this gives us $\alpha_2 = -1/2$. So final solution is $a_n = (-1/2)n(-2)^n = n(-2)^{n-1}$.

(b) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1, a_1 = 0$ The characteristic equation is $r^2 - 1/4 = 0$ and the root is $r = 1/2$ and $-1/2$. We have the general solution as

$$a_n = \alpha_1(1/2)^n + \alpha_2(-1/2)^n$$

where α_1, α_2 are constants. To solve for these constants, $a_0 = 1$ implies $\alpha_1 + \alpha_2 = 1$ and $a_1 = 0$ means we have $0 = \alpha_1/2 - \alpha_2/2$. And this gives us $\alpha_1 = \alpha_2 = 1/2$. So final solution is $a_n = (1/2)(1/2)^n + (1/2)(-1/2)^n = (1/2)^{n+1} - (-1/2)^{n+1}$.

(5) In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces? *A checkerboard is a board of chequered pattern with alternating dark and light color, typically black and white.*

We model this as a recurrence relation. Let T_n be the number of ways to tile a $2 \times n$ rectangular checkerboardusing 1×2 and 2×2 pieces. Consider one end of the board. We can place a 2×2 tile, in which case what remains is a $2 \times (n-2)$ board and this can be done in T_{n-2} ways.

If we consider a 1×2 tile then we can keep it vertically and then what remains is a $2 \times (n-1)$ board which can be done in T_{n-1} ways.

We can also place two 1×2 tiles leaving a $2 \times (n-2)$ board which can be done in T_{n-2} ways.

These are all disjoint cases so we can sum them up to get $T_n = T_{n-1} + 2T_{n-2}$. Initial conditions are $T_0 = T_1 = 1$ since for a 2×0 board there is only one way, i.e no tiles and for a 2×1 board using a 1×2 tile vertically.

The characteristic equation $r^2 - r - 2 = 0$ has roots 2 and -1 so the general solution is $T_n = \alpha_1 2^n + \alpha_2 (-1)^n$. Solving for the constants, we get $1 = T_0 = \alpha_1 + \alpha_2$ and $1 = T_1 = 2\alpha_1 - \alpha_2$ and this gives $\alpha_1 = 2/3, \alpha_2 = 1/3$.

(6) Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7, a_1 = -4$ and $a_2 = 8$

This is a degree 3 recurrence. The characteristic equation is $r^3 - 2r^2 - 5r + 6 = 0$. By rational root test, the possible rational roots are $1, -1, 2, -2, -3, 3, -6, 6$. Trying one out we get $r = 1$ is a root. Dividing by $r - 1$ we get the factorization of $r^3 - 2r^2 - 5r + 6 = (r-1)(r^2 - r - 6) = (r-1)(r-3)(r-2)$ and therefore the roots are $1, 3, -2$.

The general solution is $a_n = \alpha_1 1^n + \alpha_2 3^n + \alpha_3 (-2)^n$.

To find the constants we apply the initial conditions.

$$\begin{aligned} 7 = a_0 &= \alpha_1 + \alpha_2 + \alpha_3 \\ -4 = a_1 &= \alpha_1 + 3\alpha_2 - 2\alpha_3 \\ 8 = a_2 &= \alpha_1 + 9\alpha_2 + 4\alpha_3. \end{aligned}$$

Solving we get $\alpha_1 = 5, \alpha_2 = -1, \alpha_3 = 3$.

So the final solution is $a_n = 5 - 3^n + 3(-2)^n$.

(7) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$. Find the solution with initial condition $a_1 = 4$.

Roots of the characteristic equation: $r - 2 = 0$ is $r = 2$. The solution of the associated homogeneous part is $a_n^{(h)} = \alpha_1 2^n$.

The particular solution : $F(n) = 2n^2 \cdot 1^n$ and 1 is not a root of the characteristic equation and therefore $a_n^{(p)} = p_2n^2 + p_1n + p_0$ is the form of the particular solution.

It should satisfy the recurrence, $a_n = 2a_{n-1} + 2n^2$. Therefore,

$$\begin{aligned} p_2n^2 + p_1n + p_0 &= 2p_2(n-1)^2 + 2p_1(n-1) + 2p_0 + 2n^2 \\ p_2n^2 + p_1n + p_0 &= 2p_2n^2 - 4p_2n + 2p_2 + 2p_1n - 2p_1 + 2p_0 + 2n^2 \\ -p_2n^2 - p_1n - p_0 &= -4p_2n + 2p_2 - 2p_1 + 2n^2 \\ 0 &= (-p_1 - 2)n^2 + (-p_1 + 4p_2)n + (-2p_2 + 2p_1 - p_0) \end{aligned}$$

Solving, we get $p_2 = -2, p_1 = -8, p_0 = -2p_2 + 2p_1 = -12$.

$a_n^{(p)} = -2n^2 - 8n + -12$.

Final solution: $a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 2^n - 2n^2 - 8n + -12$.

For the initial condition $a_1 = 4$ we have at $n = 1$

$$\begin{aligned} 4 = a_1 &= 2\alpha_1 - 2 - 8 - 12 \\ 4 = a_1 &= 2\alpha_1 - 22 \\ 26 &= 2\alpha \\ 13 &= \alpha \end{aligned}$$

So the final solution is $a_n = a_n^{(h)} + a_n^{(p)} = 13 \cdot 2^n - 2n^2 - 8n + -12$.

(8) Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

Associated homogeneous recurrence is $a_n = 4a_{n-1} - 4a_{n-2}$. Characteristic equation: $r^2 - 4r + 4 = 0$, $r = 2$ is a repeated root and therefore we have the following solution,

$$a_n^{(h)} = \alpha_1 2^n + \alpha_2 n \cdot 2^n.$$

To get a particular solution we need to look for a particular solution of the form $a_n = n^2(cn + d)2^n$. Why n^2 because of the repeated root $r = 2$ which has multiplicity 2.

Plugging into the recurrence to get $n^2(cn+d)2^n = 4(n-1)^2(cn-c+d)2^{n-1} - 4(n-2)^2(cn - 2c + d)2^{n-2} + (n+1)2^n$.

Dividing through 2^n and doing some manipulation we get, $0 = (-6c+1)n + (6c - 2d + 1)$.

Equating coefficients of the powers of n we get $c = 1/6$ and $d = 1$. Thus $a_n^{(p)} = n^2(\frac{n}{6} + 1)2^n$. General solution is $(\alpha_1 + \alpha_2 n + n^2 + \frac{n^3}{6}) \cdot 2^n$.