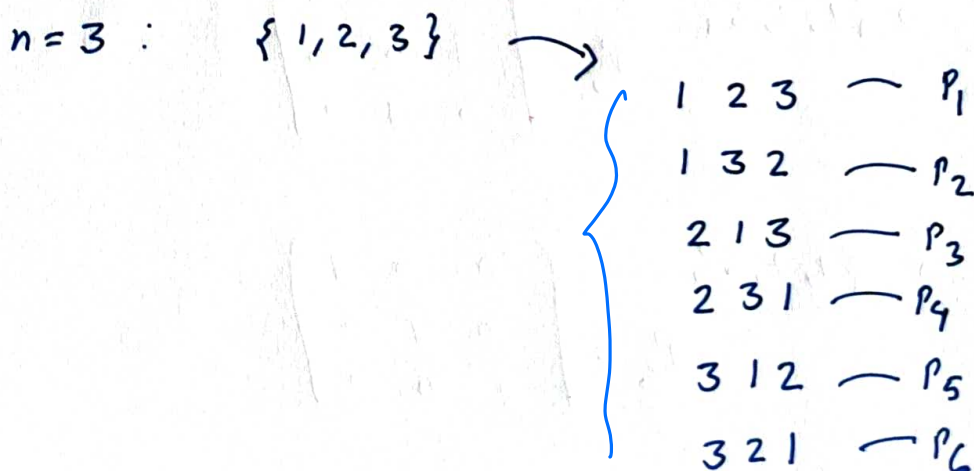
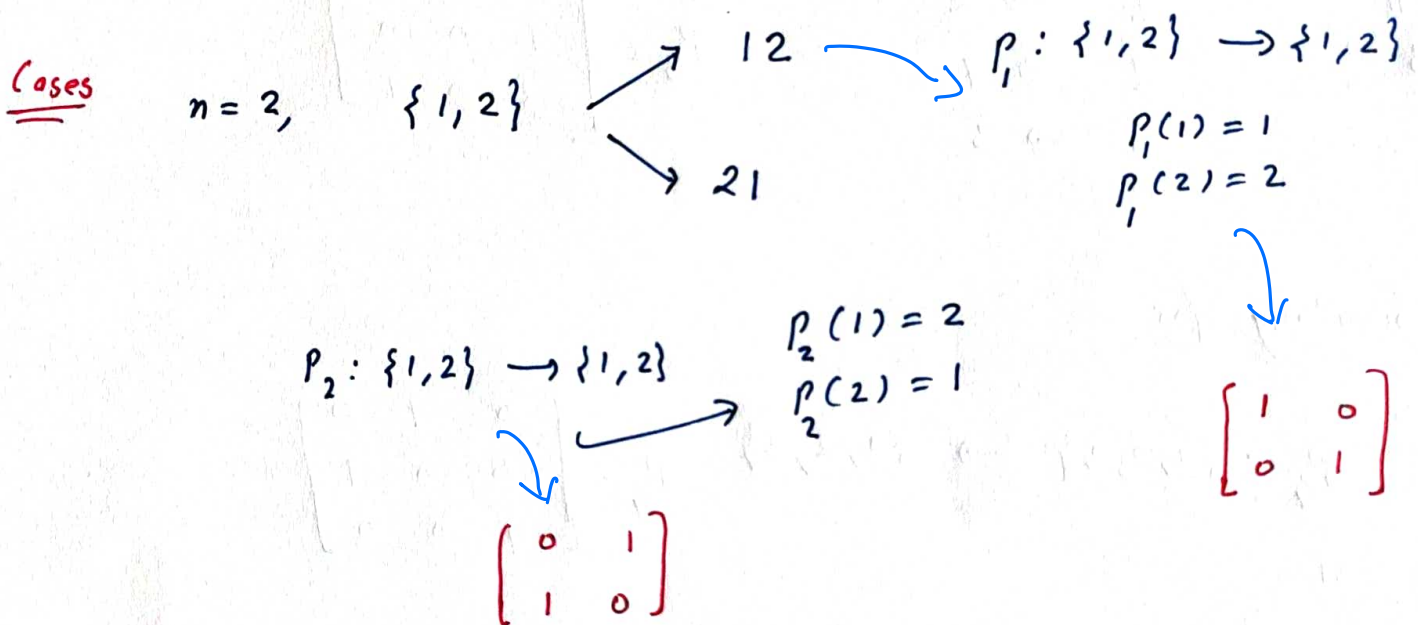


Definition. A bijective map $p : S \rightarrow S$ is called a **permutation** of the set, S ($S \neq \emptyset$).

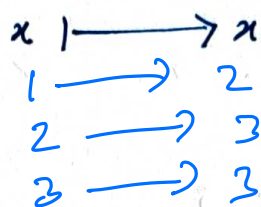
Let $S = \{1, 2, \dots, n\}$.

Remark. Every permutation on n element corresponds to a bijection on n element set.



$$p_1 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

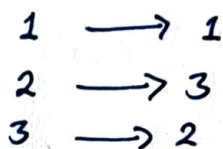
123
||
id



$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Id.}$$

$$p_2 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

||
132

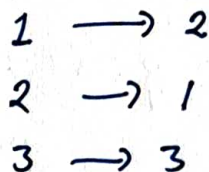


$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2^{-1} = P_2$$

$$p_3 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

||
213



$$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3^{-1} = P_3$$

Similarly for

$$p_4 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

||
231

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$p_5 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

||
312

$$P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_6 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$$

||
321

$$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Left multiplication by permutation matrix.

Example.

$$\begin{matrix}
 & & X \\
 & & \parallel \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}
 \end{matrix}$$

3×4

132

$$= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

"Rows of X are permuted w.r.t. permutation"

Discussion. The permutation matrix can be written in terms of the matrix units, e_{ij} , or in terms of certain column vectors denoted by e_i .

I. In terms of column vectors.

$$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

i^{th} row \rightarrow

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

← Given permutation
 ~~p~~ $p: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

$$1 \mapsto 1$$

$$2 \mapsto 3$$

$$3 \mapsto 2$$

$$P = \begin{bmatrix} e_{\vec{p}(1)} \\ e_{\vec{p}(2)} \\ e_{\vec{p}(3)} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_3 \\ e_2 \end{bmatrix}$$

Similarly,

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$q: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

$$1 \rightarrow 3$$

$$2 \rightarrow 1$$

$$3 \rightarrow 2$$

$$Q = \begin{bmatrix} e_{\vec{q}(1)} \\ e_{\vec{q}(2)} \\ e_{\vec{q}(3)} \end{bmatrix} = \begin{bmatrix} e_3 \\ e_1 \\ e_2 \end{bmatrix}$$

II. In terms of matrix units.

Example.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sum_i \sum_j e_{ij} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

↕

$$P = e_{p(1)1} + e_{p(2)2} + e_{p(3)3}$$

\wp
 $\wp: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

$1 \mapsto 1$

$2 \mapsto 3$

$3 \mapsto 2$

1st column is $e_{p(1)}$

2nd column is $e_{p(2)}$

3rd column is $e_{p(3)}$

Proposition. Let P be the permutation matrix associated to a permutation \wp . Then $\wp: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$

(a). The j th column of P is the column vector $e_{p(j)}$

(b). P is a sum of n matrix units:

$$P = e_{p(1)1} + \dots + e_{p(n)n}$$

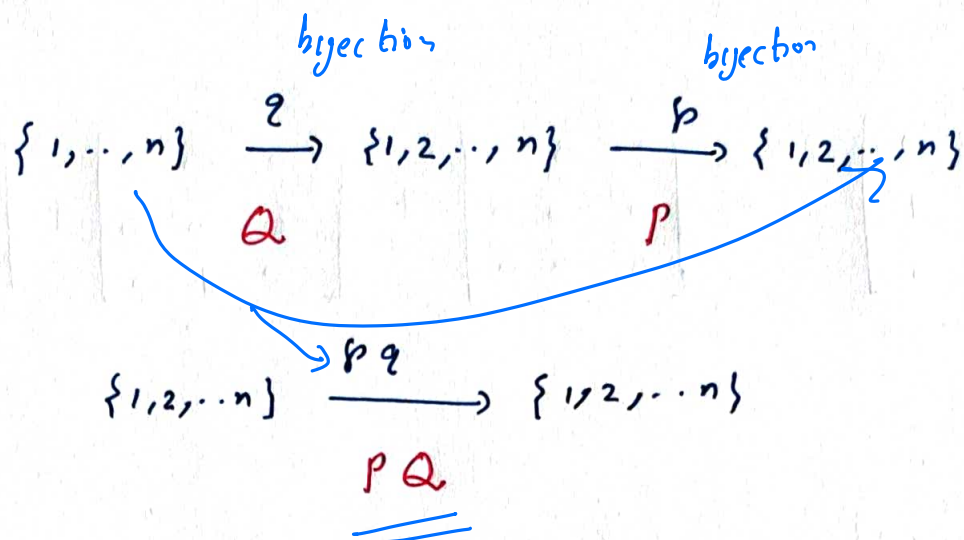
$$= \sum_j e_{p(j)j}$$

(c). P is invertible and $P^{-1} = P^t$.

~~Let~~

\wp

(d)



Notation / Definition. $p: \{1, \dots, n\} \xrightarrow{P} \{1, \dots, n\}$

The sign of a permutation, p , is denoted by $\text{sign } p$

and is defined to be

$$\text{sign } p := \det P \quad (= \pm 1)$$

p is odd permutation if $\text{sign } p = -1$

p is even permutation if $\text{sign } p = 1$

Discussion. Let us try to write expression for determinants.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$123 \longrightarrow$
 $3! = 6$
 $\begin{matrix} 123 \leftarrow + \\ 132 \leftarrow - \\ 213 \\ 231 \\ 312 \\ 321 \end{matrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= \sum (\text{sign } p) a_{1p_1} a_{2p_2} a_{3p_3}$$

$p: \text{permutation}$
 $p: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

Con we describe

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= (\text{sign } p) a_{1p_1} a_{2p_2} a_{3p_3} a_{4p_4}$$

$\{1, 2, 3, 4\}$
 $4! = 24 \text{ elements.}$

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

Goal is to describe $\det(A) = \dots$

Using linearity of the determinant, we expand $\det(A)$

$$\det A = \det \begin{bmatrix} \boxed{a_{11}} & \overbrace{0 \dots 0}^0 \\ - & R_2 \\ \vdots & \\ - & R_n \end{bmatrix} + \det \begin{bmatrix} 0 & \boxed{a_{12}} & 0 \dots 0 \\ - & R_2 \\ \vdots & \\ - & R_n \end{bmatrix} + \dots + \det \begin{bmatrix} 0 \dots 0 & \boxed{a_{1n}} \\ - & R_2 \\ \vdots & \\ - & R_n \end{bmatrix}$$

Now, continue expanding each of these determinants on the 2nd row, 3rd row and so on.

Then $\det A$ is an expression as a sum of n^n ~~n^2~~ determinants, each of which have only one non-zero entry in each row.

Let M be one among the n^2 determinants.

$$M = \begin{bmatrix} \boxed{a_{1*}} \\ a_{2*} \\ \vdots \\ a_{n*} \end{bmatrix}$$

Example.

$$\det \begin{bmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + \det \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

$$= \underset{\parallel}{0} + \det \begin{bmatrix} \underline{a_{11}} & 0 \\ 0 & \underline{a_{22}} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \underset{\parallel}{0}$$

$$= \det \left(\sum_{j=1}^2 a_{p(j)j} e_{p(j)j} \right) + \det \left(\sum_{j=1}^2 a_{p(j)j} e_{p(j)j} \right)$$

$$p: \{1, 2\} \rightarrow \{1, 2\}$$

$$\begin{cases} 1 \rightarrow 1 \\ 2 \rightarrow 2 \end{cases}$$

$$p: \{1, 2\} \rightarrow \{1, 2\}$$

$$\begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 1 \end{cases}$$

$$= a_{p(1)1} a_{p(2)2} (\det P) + a_{p(1)1} a_{p(2)2} (\det P)$$

$$= \sum (\text{sign } p) a_{p(1)1} a_{p(2)2}$$

p : permutation
on $\{1, 2\}$

Observe that every square matrix

[illegible]

is like a permutation matrix, except 1's are replaced by the matrix entries A .

Thus, we may write

$$\rho = \sum_j e_{p(j)j} \quad \text{and} \quad M = \sum_j a_{p(j)j} e_{p(j)j}.$$

By linearity of the determinant, we can describe

$$\det M = (a_{p(1)1} \cdots a_{p(n)n}) (\det P)$$

$$= (\text{sign } p) (a_{p(1)1} \cdots a_{p(n)n})$$

for every permutation on $\{1, \dots, n\}$, we get one such term.

Henle

$$\det(A) = \sum_{\text{Permutation } g: \{1, \dots, n\} \rightarrow \{1, \dots, n\}} (\text{sign } g) a_{p(1)1} \cdots a_{p(n)n}$$

or equivalently

$$[\det A = \det A^t]$$

$$\det(A) = \sum_{\sigma \in S_n} (\text{sign } \sigma) a_{1, \sigma(1)} \cdots a_{n, \sigma(n)}$$

S_n : all permutations on $\{1, \dots, n\}$

$$A = [a_{ij}]$$

$$\det A = \sum_{\substack{\text{permutations} \\ p}} (\text{sign } p) a_{1p(1)} \cdots a_{np(n)}.$$

Expanding by minors on the i^{th} row

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots \\ + (-1)^{i+n} a_{in} \det A_{in}$$

or,

$$\det A = (-1)^{j+1} a_{1j} \det A_{1j} + (-1)^{j+2} a_{2j} \det A_{2j} + \cdots \\ + (-1)^{j+n} a_{nj} \det A_{nj}.$$

$$\begin{bmatrix} + & - & + & - \\ - & + & \cdot & \cdot \\ + & - & + & \cdot \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Definition. Let A be an $n \times n$ matrix. The

adjoint of A is the $n \times n$ matrix whose (i, j) entry

$$(adj)_{ij} = (-1)^{i+j} \cdot \det A_{ji}$$

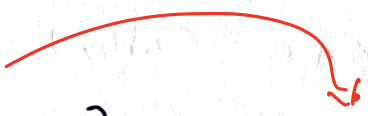
||
 α_{ji}

Then

$$(adj A) = (\alpha_{ij})^t,$$

$$\text{where } \alpha_{ij} = (-1)^{i+j} \det A_{ij}.$$

Example.


$$adj \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$adj \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = [\alpha_{ij}]^t$$

$$= \begin{bmatrix} \\ \\ \end{bmatrix}$$

Theorem. Let $\delta = \det A$. Then

$$(\operatorname{adj} A) \cdot A = \delta I, \text{ and}$$

$$A \cdot (\operatorname{adj} A) = \delta I.$$

Proof.

$$\left[(\operatorname{adj} A) \cdot A \right]_{(i,j)^{\text{th}} \text{ entry}} \begin{cases} \det A & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

\parallel

$$(\operatorname{adj})_{i,1} a_{1j} + \dots + (\operatorname{adj})_{i,n} a_{nj}$$

When $i=j$, we get $\det(A)$ (why), else 0 (why)
(Exercise!)

$$(\operatorname{adj} A) \cdot A = \begin{bmatrix} \det A & & \\ & \ddots & \\ & & \det A \end{bmatrix}$$

Corollary. Let A be a square matrix with $\det A \neq 0$. Then

$$A^{-1} = \frac{1}{\det A} \cdot (\text{adj } A)$$

Proof. "Easy".

Discussion. Consider a system of linear equations

$$AX = B, \text{ where } A \text{ is an } n \times n \text{ matrix with } \det A \neq 0.$$

$$A^{-1} \cdot AX = A^{-1} \cdot B$$

$$\Rightarrow X = A^{-1} \cdot B$$

$$X = \frac{1}{\det A} \cdot (\text{adj } A) B$$

$$n \times n \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

From this, we can write

$$x_j = \frac{1}{\det A} \cdot (b_1 \kappa_{1j} + \dots + b_n \kappa_{nj}),$$

$$\text{where } \kappa_{ij} = \pm \det A_{ij}.$$

Construct a new matrix M_j replacing the j^{th} column of A by the column vector B .

Expansion by minors on the j^{th} column of M_j

is

$$\det M_j = \underline{b_1 a_{1j} + \dots + b_n a_{nj}}.$$

Using this, we can re-write,

$$x_j = \frac{\det M_j}{\det A}$$

known as

"Cramer's Rule".

← x →

FOR DISCUSSION ON PROBLEMS, SEE RECORDING OF
TODAY LECTURE.