Fundamental Theorem of Calculus
(Proofs and Remarks)

Domain additivity: $(a \le b)$
If $f:[a,b]\to\mathbb{R}$ is integrable and $c\in[a,b]$, then f is integrable of $[a,c]$ and $[c,b]$.
In this case, we have $\int_a^b f = \int_a^c f + \int_c^b f$.

Convention: If $a<b$, then $\int_b^a f := -\int_a^b f$.
$[b,a]$

Theorem: (FTC 1)
If $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$, then the function
$$F:[a,b]\to\mathbb{R}.$$
$$F(x) = \int_a^x f(t)\,dt.$$
is continuous. Also, if f is continuous at $c\in[a,b]$, then F is differ. at c and $F'(c)=f(c)$.

Pf: F is continuous.
We know that f is integrable $\Rightarrow$ f is bounded
$\Rightarrow \exists \alpha\ge 0$ s.t. $|f(x)|\le\alpha\ \forall x\in[a,b]$

Now, $F(x)-F(c)$
$$= \int_a^x f(t)\,dt - \int_a^c f(t)\,dt. \qquad c<x \text{ or } c>x$$
$$= \int_c^x f(t)\,dt$$
$$|F(x)-F(c)| = \left|\int_c^x f(t)\,dt\right| \le \int_c^x |f(t)|\,dt \le \alpha|x-c|$$
$$x_n\to c \Rightarrow F(x_n)\to F(c)$$

TST, If f is continuous at c, then F is differentiable.
We define a function $F_1:[a,b]\to\mathbb{R}$.
$$F_1(x) = \begin{cases} \dfrac{F(x)-F(c)}{x-c} & \text{if } x\ne c. \\ f(c) & \text{if } x=c. \end{cases}$$

Note that to prove that F is differentiable at c it is enough to show that F_1 is continuous at c.
(Carathéodory's Lemma)

To show that $F_1(x)$ is continuous.

$$f(c) = \frac{1}{x-c}\int_c^x f(c)\,dt$$

$F_1(x)-F_1(c)$
$$= \frac{1}{x-c}\left[\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right] - \frac{1}{x-c}\int_c^x f(c)\,dt$$
$$= \frac{1}{x-c}\int_c^x f(t)\,dt - \frac{1}{x-c}\int_c^x f(c)\,dt.$$
$$= \frac{1}{x-c}\int_c^x \big(f(t)-f(c)\big)\,dt.$$

$|F_1(x)-F_1(c)|$
$$= \frac{1}{|x-c|}\left|\int_c^x \big(f(t)-f(c)\big)\,dt\right|$$
$$\le \frac{1}{|x-c|}\int_c^x |f(t)-f(c)|\,dt. \qquad t\in[c,x]$$

(*) Since f is continuous at c,
Given $\varepsilon>0$, $\exists\delta>0$
$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

Fix $\varepsilon>0$, by (*) $\exists\delta>0$ st. whenever $|x-c|<\delta$
$\Rightarrow |f(x)-f(c)|<\varepsilon$.

$|x-c|<\delta \Rightarrow |t-c|<\delta \Rightarrow |f(t)-f(c)|<\varepsilon$

$\le\varepsilon$. whenever $|x-c|<\delta$.

This shows that $F_1(x)$ is continuous at c
$\overset{\text{C's Lemma}}{\Longrightarrow}$ F is differentiable at c.
$\Rightarrow F'(c)=f(c)$ ∎

FTC (2)
If $f:[a,b]\to\mathbb{R}$ is differentiable and f' is integrable, then
$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Pf: Let $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$.
f is differentiable on $[a,b]$ $\Rightarrow$ f is diff. on $[x_{i-1},x_i]$ $\forall i$.

(Mean value Theorem) (MVT) $\Rightarrow$ for each i, $\exists c_i\in(x_{i-1},x_i)$ with.
$$f(x_i)-f(x_{i-1}) = f'(c_i)(x_i-x_{i-1}). \quad \forall i=1,\dots,n.$$
$$\Rightarrow \underbrace{\sum_{i=1}^n \big(f(x_i)-f(x_{i-1})\big)}_{=f(b)-f(a)} = \sum_{i=1}^n f'(c_i)(x_i-x_{i-1}).$$

$[x_0,x_1],\dots,[x_{i-1},x_i],\dots,[x_{n-1},x_n]$, $x_0=a$, $x_n=b$

$$m_i(f')\le f'(c_i)\le M_i(f') \quad \forall i.$$
$$\Rightarrow \sum_{i=1}^n m_i(f')(x_i-x_{i-1}) \le \sum_{i=1}^n f'(c_i)(x_i-x_{i-1}) \le \sum_{i=1}^n M_i(f')(x_i-x_{i-1}).$$
$$\Rightarrow L(P,f')\le f(b)-f(a)\le U(P,f').$$
But this inequality holds for every partition P of $[a,b]$
$$\Rightarrow \sup_P L(P,f') \le f(b)-f(a) \le \inf_P U(P,f')$$
$$\Rightarrow L(f')\le f(b)-f(a)\le U(f')$$

f' is integrable so $L(f')=U(f')=\int_a^b f'$
$$\Rightarrow \int_a^b f' \le f(b)-f(a) \le \int_a^b f'$$
$$\Rightarrow \int_a^b f' = f(b)-f(a) \quad ∎$$

Remarks:

1. The proof of FTC only relies on MVT and Domain additivity of Riemann integrals.

2. Antiderivatives:
Suppose that I is an interval containing more than one point. ~~$[a,a]$~~ A function $f:I\to\mathbb{R}$ is said to have an antiderivative if $\exists$ a (differentiable) function $F:I\to\mathbb{R}$ such that $F'=f$.
The function F is called an antiderivative/a primitive/an indefinite integral of f.

Lemma: If I is an interval and $f:I\to\mathbb{R}$ has an antiderivative F, then F is unique upto addition of a constant.

Pf: Suppose $F'=G'=f$.
$$\Rightarrow (F-G)'=0$$
$$\overset{\text{MVT}}{\Longrightarrow} F-G=\text{constant} \quad ∎$$

What does FTC say?
1. If f is continuous on $[a,b]$, $\dfrac{d}{dx}\left(\int_a^x f\right)=f(x)\ \forall x\in[a,b]$
2. If f' exists and it is integrable, then $\int_a^b \left(\frac{d}{dx}f\right) = f(b)-f(a)$. ($\frac{d}{dx}f = f'$)
3. An Antiderivative of a continuous function $f:[a,b]\to\mathbb{R}$ is given by $\int_a^x f(x)\,dx$.