Lecture 21 (MA 4020) LINEAR ALGEBRA

Nov 16, 2021

Revision.

Inner Product. An inner product on V is a function

$$\langle -,- \rangle : V \times V \longrightarrow F$$
$$(u,v) \longmapsto \langle u,v \rangle$$

and has the following properties:

Positive $\langle v,v \rangle \geq 0$ for all $v \in V$

Definiteness $\langle v,v \rangle = 0 \iff v = 0$

Additivity in the first slot:

$$\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle \quad \text{for all } u,v,w \in V$$

Homogeneity in the first slot

$$\langle av, w \rangle = a \langle v,w \rangle \quad \text{for all } a \in F,\ v \in F$$

Conjugate symmetry: $\langle u,v \rangle = \overline{\langle v,u \rangle}$ for all $u,v \in V$.

Definition. An INNER PRODUCT SPACE is a vector space V along with an inner product on V.

Exercise.

Define
$$\langle -,- \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\Big(\underset{\substack{\| \\ (x_1,x_2)}}{\underline{x}}\, , \underset{\substack{\| \\ (y_1,y_2)}}{\underline{y}}\Big) \longmapsto (x_1+x_2)(y_1+y_2) + (2x_1+x_2)(2y_1+y_2).$$

Is this $\langle -,- \rangle$ on inner product on $\mathbb{R}^2$?

Solution. See Lecture Recording.

NORM For $v \in V$, the norm of v is defined as
$$\|v\| = \sqrt{\langle v,v \rangle}.$$

- We will get different norms for different inner product that we define.
- Since many choices for inner product, and hence many possibilities for $\|v\|$.

"See discussion on this, in recording"

$V = \mathcal{P}_n(\mathbb{R})$

$v \in$

$\uparrow$

polynomial

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

$V = \mathcal{C}([a,b])$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

$V = \mathcal{C}([-\pi, \pi])$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$\{ \sin x, \sin(2x), \ldots, \sin(nx), \ldots \cos x, \cos(2x), \ldots, \cos(mx), \ldots \}$

Compute

$\langle \sin(mx), \cos(nx) \rangle =$

$\langle \sin(mx), \sin(nx) \rangle =$

$$\langle \sin(mx), \sin(nx) \rangle \overset{\text{defn}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$$

$$= \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\{\sin x, \sin(2x), \dots, \sin(mx), \dots\}$$

Orthogonal / Orthonormal vectors

Definition. Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

$$\{\sin x, \sin 2x, \dots, \sin 3x, \dots, \cos x, \cos(2x), \dots, \cos(mx), \dots\}$$

$$\in C([-\pi, \pi])$$

Orthogonal functions.

Theorem. (Cauchy – Schwarz Inequality).

If $u, v \in V$, then

$$|\langle u, v \rangle| \leqslant \|u\| \, \|v\|.$$

equality $\Leftrightarrow$ one of u, v is a scalar multiple of the other.

Exercise. Let $a_1, a_2, \dots, a_n \in \mathbb{R}_{\geqslant 0}$. Prove that

$$(a_1 + a_2 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right) \geqslant n^2$$

Solution.

Set $V = \mathbb{R}^n$

$$u = (\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n})$$

$$v = \left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \dots, \frac{1}{\sqrt{a_n}}\right)$$

then apply Cauchy – Schwarz inequality.

Similar problems:

$$V = C([a,b])$$

$$\left(\int_a^b f(x)\, g(x)\, dx \right)^2 \leq \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx$$

Orthonormal vectors.

$(v_1, \dots, v_n) \in V$ is orthonormal if

$\langle v_i, v_j \rangle = 0$ when $i \neq j$, and

$\langle v_i, v_i \rangle = 1$ for all i.

$C([-\pi, \pi])$

$(v_1, v_2, \dots \quad)$

$\{ \sin x, \sin(2x), \dots, \sin(mx), \dots, \cos x, \cos(2x), \dots \}$

$$\langle -,- \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot g(x)\, dx.$$

Theorem. Suppose $(v_1, \ldots, v_n)$ is an orthonormal basis of V. Then $v \in V$ can be written as

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n$$

and

$$\|v\|^2 = |\langle v, v_1 \rangle|^2 + \cdots + |\langle v, v_n \rangle|^2.$$

for every $v \in V$.

Gram-Schmidt process:

Theorem. If $(w_1, \ldots, w_n)$ is a linearly independent list of vectors in V, then there exists an orthonormal list $(v_1, \ldots, v_n)$ of V, such that

$$\mathrm{Span}(w_1, \ldots, w_m) = \mathrm{Span}(v_1, \ldots, v_m)$$

for all $m = 1, 2, \ldots, n$.

Theorem [Gram-Schmidt]

If $(w_1, \dots, w_n)$ is a linearly independent list of vectors in V, then there exists an orthonormal list $(v_1, \dots, v_n)$ of V such that

$$\operatorname{span}(w_1, \dots, w_j) = \operatorname{span}(v_1, \dots, v_j) \quad \text{— (A)}$$

for $j = 1, 2, \dots, n$.

Proof.

Suppose $(w_1, \dots, w_n)$ is a linearly independent list of vectors in V.

$\operatorname{Span}(v_1) = \operatorname{Span}(w_1)$

Set $v_1 = \dfrac{w_1}{\|w_1\|}$, then (A) holds.

$$v_2 = \frac{w_2 - \langle w_2, v_1 \rangle v_1}{\| \quad \|}$$

Proof by induction:

$j = 1$, $\operatorname{Span}(w_1) = \operatorname{Span}(v_1)$

Induction hypothesis. Assume that $(v_1, \dots, v_{j-1})$ orthonormal list of vectors have been chosen with

$$\left[\operatorname{Span}(w_1, \dots, w_{j-1}) = \operatorname{Span}(v_1, \dots, v_{j-1}). \right]$$

$v_j \in \mathrm{Span}(w_1, \dots, w_j)$

non-zero
some vector

Let

$$v_j = \frac{w_j - \langle w_j, v_1\rangle v_1 - \cdots - \langle w_j, v_{j-1}\rangle v_{j-1}}{\| w_j - \langle w_j, v_1\rangle v_1 - \cdots - \langle w_j, v_{j-1}\rangle v_{j-1} \|}.$$

$\|v_j\| = 1$

$w_j = * v_j + (\ )v_1 + (\ )v_2 + \cdots + (\ )v_{j-1}$

Notice that $w_j \notin \mathrm{Span}(v_1, \dots, v_{j-1})$

$\|$ by induction hypothesis.

$w_j \notin \mathrm{Span}(w_1, \dots, w_{j-1})$

Hence v_j is a non-zero vector with $\|v_j\| = 1$.

Now, observe that

$\langle v_j, v_k \rangle = 0$ for all $1 \leq k < j$.

$$\|$$

$$\left\langle \frac{w_j - \langle w_j, v_1\rangle v_1 - \cdots - \langle w_j, v_{j-1}\rangle v_{j-1}}{*}, v_k \right\rangle$$

$k = 1, 2, \dots, j-1$

$$\|$$

$$\left\langle \frac{\langle w_j, v_k\rangle - \langle w_j, v_k\rangle}{*} \right\rangle$$

$$\|$$

$$0$$

$v_1, \dots, v_{j-1}$

$\langle v_i, v_j \rangle = 0$ $i \neq j$

$\langle v_i, v_j \rangle = 1$ if $i = j$

Thus,

$$(v_1, \dots, v_j) \text{ is an orthonormal list.}$$

Note that

$$w_j \in \operatorname{Span}(v_1, \dots, v_j)$$

$$\Downarrow$$

$$\operatorname{Span}(w_1, \dots, w_{j-1}, w_j) \subset \operatorname{Span}(v_1, \dots, v_j).$$

Linearly independent set, and hence subspaces have same dimension j.

Thus $$\operatorname{Span}(w_1, \dots, w_j) = \operatorname{Span}(v_1, \dots, v_j).$$

L.I. O.N. basis

Remark. The algorithm involved in the proof for constructing an orthonormal set of vectors is known as Gram-Schmidt procedure

or

Gram-Schmidt orthonormalization process.

$V = P_2(\mathbb{R})$,

define $\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$

$\mathcal{B} = \{1, x, x^2\}$

Is this an orthonormal basis?

NO

Apply Gram-Schmidt process

$w_1 = 1 \longrightarrow v_1 = \frac{w_1}{\|w_1\|} = 1$

$\|w_1\|^2 = \langle w_1, w_1 \rangle = \int_0^1 1 \cdot 1 \cdot dx = 1$

$w_2 = x, \longrightarrow v_2$

$$v_2 = \frac{w_2 - \langle w_2, v_1 \rangle v_1}{\| w_2 - \langle w_2, v_1 \rangle v_1 \|} = \frac{??}{\| \cdots \|}$$

$$= \frac{x - \left\{ \int_0^1 (x \cdot 1)\, dx \right\} 1}{\| \cdots - \cdot \|} = \frac{x - \frac{1}{2}}{\sqrt{}}$$

$$\| \cdot\cdot \|^2 = \int_0^1 \left(x - \tfrac{1}{2}\right)^2 \cdot dx = ?$$

$$v_2 = \sqrt{12} \cdot \left(x - \tfrac{1}{2}\right)$$

$w_3 = x^2$ ✓

$$v_3 = \frac{w_3 - \langle w_3, v_1 \rangle v_1 - \langle w_3, v_2 \rangle v_2}{\| \cdots \cdot \|}$$

$=$ compute this.

$$\| \cdot v \| = \sqrt{\langle v, v \rangle}$$

$$\langle -, - \rangle : V \times V \longrightarrow F$$

$$(f, g) \longmapsto \int_0^1 f(x) g(x)\, dx$$

"Norm of a vector"

Corollary. Every finite-dimensional inner-product space has an orthonormal basis.

Proof.

Choose a basis of V, say $(w_1, \dots, w_n)$

$\downarrow$ Apply Gram-Schmidt procedure to get an orthonormal list

$(v_1, \dots, v_n)$

Since $\operatorname{Span}(v_1, \dots, v_n) = \operatorname{Span}(w_1, \dots, w_n)$

$\Downarrow$

$(v_1, \dots, v_n)$ is an orthonormal basis.

Corollary. Every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Proof.

L.I

Suppose $(w_1, \dots, w_k)$ is an orthonormal list of vectors in V. $\xrightarrow{\text{get to}}$ Basis of V

Extend $(w_1, \dots, w_k)$ to $(w_1, \dots, w_k; x_1, \dots, x_{n-k})$

Basis of V

Apply Gram-Schmidt process to it

$$(v_1, \dots, v_n)$$
$$\|$$
$$(w_1, \dots, w_k; v_{k+1}, \dots, v_n)$$

Orthonormal basis of V

Corollary. Suppose $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix w.r.t. some basis of V, then T has an upper-triangular matrix w.r.t. some orthonormal basis of V.

Proof.

$$T : V \to V \longrightarrow \begin{array}{cccc} T(v_1) & T(v_2) & \cdots & T(v_n) \end{array}$$
$$\begin{bmatrix} * & * & & * \\ 0 & & \ddots & \\ 0 & & 0 & \ddots \\ \vdots & & & \\ 0 & & & * \end{bmatrix}$$

Proof.

Suppose T has an upper-triangular matrix w.r.t. some basis $(v_1, \dots, v_n)$ of V.

$\Rightarrow$ $T(v_k) \in \operatorname{Span}(v_1, \dots, v_k)$ for each $k = 1, \dots, n$.

$\Rightarrow$ $\operatorname{Span}(v_1, \dots, v_k)$ is invariant under T for each $k = 1, \dots, n$.

Apply the Gram-Schmidt procedure to $(v_1, \dots, v_n)$, producing an orthonormal basis $(v_1', \dots, v_n')$ of V,

Since $\operatorname{Span}(v_1', \dots, v_m') = \operatorname{Span}(v_1, \dots, v_m)$
$m = 1, 2, \dots, n$.

$\Downarrow$

$\operatorname{Span}(v_1', \dots, v_m')$ is invariant under T

$\Downarrow$

T has an upper-triangular matrix w.r.t. orthonormal basis $(v_1', \dots, v_n')$.

Theorem [Schur's theorem]. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix w.r.t. some orthonormal basis of V.

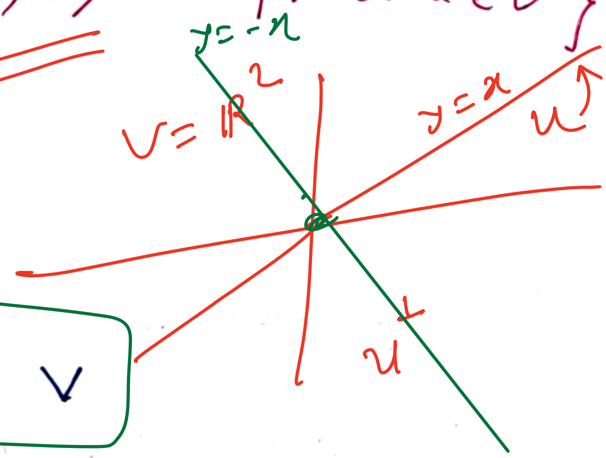
(Think about this)

Definition. [Orthogonal complement].

If $\mathcal{U}$ is a subset of V, then the orthogonal complement of $\mathcal{U}$, denoted $\mathcal{U}^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in $\mathcal{U}$:

in $\mathbb{R}^2$

$$\mathcal{U}^{\perp} = \{ v \in V \ \text{s.t.} \ \langle v, u \rangle = 0 \ \text{for all } u \in \mathcal{U} \}$$



Observations.

1. $\boxed{\mathcal{U}^{\perp} \text{ is a subspace of } V}$

2. $V^{\perp} = \{0\}$ (trivial)

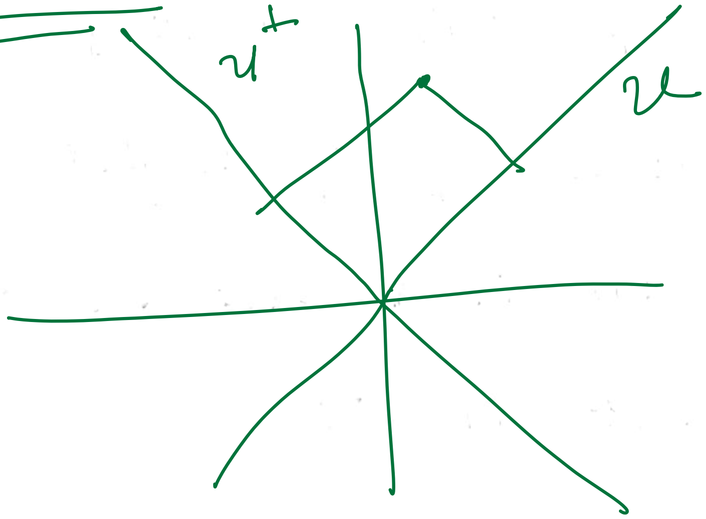
3. $\{0\}^{\perp} = V$

4. If $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then $\mathcal{U}_1^{\perp} \supset \mathcal{U}_2^{\perp}$.

Theorem. If U is a subspace of V, then
$$V = U \oplus U^{\perp}.$$

Discussion.

$$U \cap U^{\perp} = \{0\}$$



Proof of theorem.

1^{st} Claim: $V = U + U^{\perp}$.

Let $v \in V$, and let $(v_1, \dots, v_m)$ be an orthonormal basis of U. Then

~~$v = \langle v, e_1$~~

$$u + v - u$$

$$v = \underbrace{\langle v, v_1\rangle v_1 + \cdots + \langle v, v_m\rangle v_m}_{u} + \underbrace{v - \langle v, v_1\rangle v_1 - \cdots - \langle v, v_m\rangle v_m}_{w}$$

$u \in U$

To show $w \in U^{\perp}$ $\qquad \langle w, v_i \rangle = 0$

2^{nd} claim: $\mathcal{U} \cap \mathcal{U}^\perp = \{0\}$.

Suppose $v \in \mathcal{U} \cap \mathcal{U}^\perp$

$$\Rightarrow \langle v, v \rangle = 0$$

$$\Rightarrow v = 0.$$

Corollary. If $\mathcal{U}$ is a subspace of V, then

$$\mathcal{U} = (\mathcal{U}^\perp)^\perp.$$

Proof. "See Textbook, page 112".

Discussion.

$$V = \mathcal{U} \oplus \mathcal{U}^\perp,$$

$$v = u + w \quad ; \quad \text{where } u \in \mathcal{U},\ w \in \mathcal{U}^\perp.$$

Define,

$$P_{\mathcal{U}} : V \longrightarrow V$$

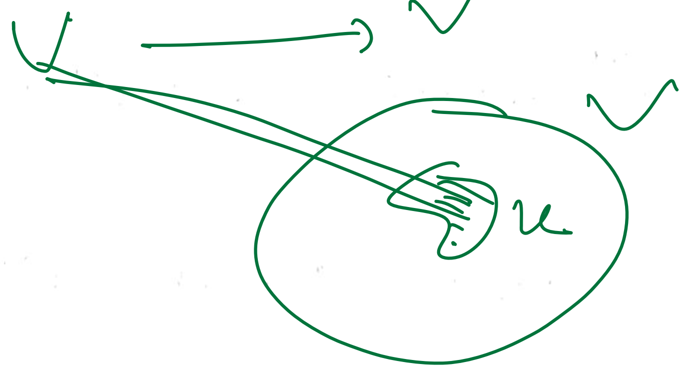
$$v \longmapsto P_{\mathcal{U}}(v) = u$$

Orthogonal projection of V onto $\mathcal{U}$.

Note. $P_{\mathcal{U}}$ is a linear operator.

Remark.

$P_{\mathcal{U}} : V \longrightarrow V$



1) $\text{range } P_{\mathcal{U}} = \mathcal{U}$;

2) $\text{null } P_{\mathcal{U}} = \ker P_{\mathcal{U}} = \mathcal{U}^{\perp}$;

3) $v - P_{\mathcal{U}}(v) \in \mathcal{U}^{\perp}$ for every $v \in V$;

$$\left(\underset{\parallel}{} u + w - u = w \right)$$

4) $P_{\mathcal{U}}^2 = P_{\mathcal{U}}$;

5) $\| P_{\mathcal{U}}(v) \| \leqslant \| v \|$ for every $v \in V$.

$$P_{\mathcal{U}}(v) = u$$

Hence $\| u \| \leqslant \| u + w \| = \| v \|$

$$V = \mathcal{U} \oplus \mathcal{U}^{\perp}$$

$\uparrow$

$(v_1, \dots, v_m)$ Orthonormal basis, then

$$P_{\mathcal{U}}(v) = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_m \rangle v_m$$

for every $v \in V$.

$\mathbb{R}^3$

$w_1 = (1,0,1)$, $w_2 = (2,1,1)$, $w_3 = (0,1,2)$

(w_1, w_2, w_3) Orthogonal ?

$\langle w_i, w_j \rangle = 0$ if $i \neq j$

$\neq 0$ if $i = j$

NO

define an inner product on $\mathbb{R}^2$ s.t.

$\langle e_1, e_1 \rangle = 2$, $\langle e_1, e_2 \rangle = -1$, $\langle e_2, e_2 \rangle = 3$

$\langle -,- \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying

$\{e_1, e_2\}$ std. basis.

i.p $\langle -, - \rangle$ with property

Think about this !!