Question 1.5

Let $0<a<b$ and $f:[a,b]\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}1+x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$$

$y\in\mathbb{Q}\cap[x_{i-1},x_i]$

$f(y)=1+y$

Is f integrable?

Let P be any partition $\{a=x_0,x_1,\dots,x_n=b\}$ of $[a,b]$.

Then P divides $[a,b]$ into n subintervals $\sqrt{2}$

$[x_0,x_1],\dots\dots,[x_{n-1},x_n]$

For all $i=1,\dots,n$.

ⓐ There exists a rational number in $[x_{i-1},x_i]$

This implies that $M_i(f)\geq 1$ (Since $x_{i-1}\geq a>0$).

Consequently, $U(P,f)=\sum_{i=1}^{n}\underset{\overset{\shortparallel}{1}}{M_i(f)}(x_i-x_{i-1})$

$x>y \Rightarrow x\geq y$

$$\geq \sum_{i=1}^{n}1\cdot(x_i-x_{i-1})=b-a>0 \longleftarrow ①$$

ⓑ There exists an irrational number in $\underset{\overset{\psi}{z}}{[x_{i-1},x_i]}$

Thus $m_i(f)=0$.

Consequently, $L(P,f)=0 \longleftarrow ②$

Now, ① and ② holds for all partitions P. This gives

$U(f)\cancel{\geq}\, b-a$. $\quad U(f)\geq b-a$.

$L(f)=0$

Since $a<b$, $U(f)>0$. and thus $U(f)\neq L(f)$.

Thus, f is not integrable.

$a,b>0$ X

$\wp[a,b]=\{P \mid P \text{ is a partition of } [a,b]\}$

$1+x_i\geq 0$

π? $\mathbb{Q}$

$\exists! x_i\in[x_{i-1},x_i]$

$0=\inf\{\frac{1}{n}\mid n\in\mathbb{N}\}$ $\quad \frac{1}{n}>0\ \forall n.$

Exercise 1.7

Let $f:[0,1]\to\mathbb{R}$ be given by $f(x)=\begin{cases}1 & \text{if } x\in[0,1) \\ 0 & \text{if } x=1.\end{cases}$

Use Riemann's condition to show that f is integrable on $[0,1]$

Let $\varepsilon>0$ be given.

If $\varepsilon>1$, we may take $P_\varepsilon=\{0,1\}$, the trivial partition.

$$\left.\begin{aligned} U(P_\varepsilon,f) &= \sup_{x\in[0,1]} f(x)=1 \\ L(P_\varepsilon,f) &= \inf_{x\in[0,1]} f(x)=0 \end{aligned}\right\} \Rightarrow U(P_\varepsilon,f)-L(P_\varepsilon,f)=1<\varepsilon.$$

P $\quad U(P,f)-L(P,f)<\varepsilon$

$n>\frac{1}{\varepsilon}$

If $\varepsilon\le 1$, take $P_\varepsilon=\{0,\alpha,1\}$ such that $1-\varepsilon<\alpha<1$

$1-\frac{\varepsilon}{2}$

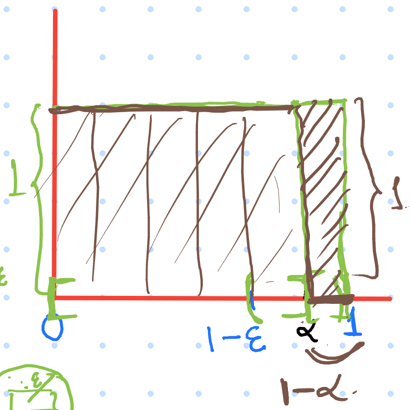
$$U(P_\varepsilon,f)=\sup_{x\in[0,\alpha]} f(x)\,(\alpha-0)+\sup_{x\in[\alpha,1]} f(x)\,(1-\alpha).$$
$$=1\cdot\alpha+1\cdot(1-\alpha)$$
$$=\alpha+1-\alpha$$
$$=1.$$

P

$U(P,f)-L(P,f)<\varepsilon$

$(x_n)\to x$

$\forall\varepsilon>0,\ \exists n_0\in\mathbb{N}$ s.t. $n\ge n_0\Rightarrow |x_n-x|<\varepsilon$



$$L(P_\varepsilon,f)=\inf_{x\in[0,\alpha]} f(x)\,(\alpha-0)+\inf_{x\in[\alpha,1]} f(x)\,(1-\alpha).$$
$$=1\cdot\alpha+0\cdot(1-\alpha)=\alpha.$$

$\alpha>1-\varepsilon$

$\Rightarrow 1-\alpha<\varepsilon.$

$f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}.\end{cases}$

Thus $U(P_\varepsilon,f)-L(P_\varepsilon,f)=1-\alpha<\varepsilon.$

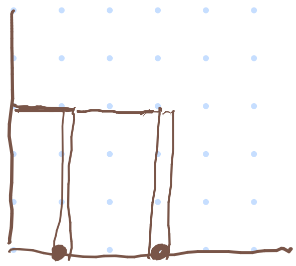
Thus the Riemann's condition is satisfied and f is integrable.

$f,g:[a,b]\to\mathbb{R}$

g is integrable.

$\{x\in[a,b] \mid f(x)\ne g(x)\}$ finite.

$\Rightarrow f$ is integrable. $\int_a^b f=\int_a^b g$



$f-g=\begin{cases}0 & \\ a_1 & P_1\\ \vdots & \\ a_r & P_r\end{cases}$

g

$(f-g)+g$

$=f$

Measure theory

Exercise 4.4

Determine $H'(x)$, where $H(x) = \int_0^{x^2} \frac{\sin \pi t}{1+t^2}\, dt$

$f(t)$. $[a,b]$ x^2 $\int_0^x f(t)\,dt$

Ans Note that $f(t) = \sin(\pi t)$ and $g(t) = 1+t^2 \neq 0$ are continuous and $g(t_0) \neq 0$ $\forall t_0 \in \mathbb{R}$

H $[0, 2]$ $x^2 \cdot 4$

Thus $h(t) = \frac{\sin \pi (t)}{1+t^2}$ is continuous and hence integrable on $[0, a]$ $\forall a > 0$.

$\int_0^{x^2} \frac{f(t)}{g(t)}\, dt$.

$x \in [0,1]$ $x^2 \in [0,1]$

Choose $a \geq 0$ $x^2 \in [0, a]$ $[a, b]$

By Fundamental theorem of Calculus (part 1), using the continuity of h at $x \in [0, a]$, we see that H is differentiable and

$$H'(x) = h(x) \text{ for all } x \in [0, a]. \longrightarrow (1)$$

Let $\phi : [0, \sqrt{a}] \to [0, a]$ be given by

$$\phi(x) = x^2.$$

Note that ϕ is differentiable on $[0, \sqrt{a}] \longrightarrow (2)$

By chain rule, $H \circ \phi$ is differentiable at x for all $x \in [0, \sqrt{a}]$

and $(H \circ \phi)'(x) = H'(\phi(x)) \cdot \phi'(x)$.

Thus
$$\left(\int_0^{x^2} \frac{\sin \pi t}{1+t^2}\, dt \right)' = \left(H(x^2) \right)' = (H \circ \phi)'(x)$$
$$= H'(\phi(x)) \cdot \phi'(x)$$
$$= h(x^2) \cdot \phi'(x)$$
$$= 2x \frac{\sin(\pi x^2)}{1+x^4}$$

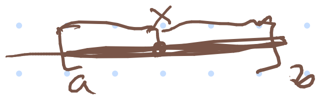
$[a, b]$ $\int_{\mathbb{R}}$

Exercise 4.5

Let $f:[a,b]\to\mathbb{R}$ be an integrable function. Define $G:[a,b]\to\mathbb{R}$ by

$$G(x):=\int_x^b f(t)\,dt.$$

Then G is continuous on $[a,b]$. If f is continuous at $c\in[a,b]$, then G is differentiable at c with $G'(c)=-f(c)$.



Ans. Let $F(x):=\int_a^x f(t)\,dt.$

By domain additivity, $\boxed{\int_a^b f(t)\,dt} = \underbrace{\int_a^x f(t)\,dt}_{F(x)} + \int_x^b f(t)\,dt.$ c

$$\Rightarrow \quad G(x) = \int_a^b f(t)\,dt - F(x)$$

Since, f is intergable, by FTC ①, F is continuous, Since $\int_a^b f(t)\,dt$ is a constant, we see that $G(x)$ is continuous.

If f is continuous at c, then by FTC (1), F is differentiable at c and $F'(c)=f(c)$. Again, since $\int_a^b f(t)\,dt$ is constant. G is differentiable at c and.

$$G'(c) = -F'(c) = -f(c).$$

$$0 = \int_x^b f + \int_b^x f$$ X

$$F(x) = \int_a^x f(t)\,dt.$$