Calculus - Assignment 1 - Sequence

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1. Using $(\epsilon - N)$ definition, prove the following.

(i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

(ii) $\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$

(iii) $\lim_{n\to\infty} \frac{n^{3/4}\sin(n!)}{n+1} = 0.$

(iv) $\lim_{n\to\infty} r^n = 0$, where $r \in \mathbb{R}$ and $|r| < 1$.

2. Show the existence of the following limits, and also find the limits.

(i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right).$

(ii) $\lim_{n\to\infty} \left(\frac{n^2+2n+3}{n^3+5n^2+1} \right).$

(iii) $\lim_{n\to\infty} n^{1/n}.$

(iv) $\lim_{n\to\infty} \frac{\cos(\pi\sqrt{n})}{\sqrt{n}}.$

3. Show that the following sequences are divergent (i.e., not convergent).

(i) $\left\{ \frac{n^2}{n+1} \right\}.$

(ii) $\left\{ (-1)^n \left(\frac{1}{2} + \frac{1}{n} \right) \right\}.$

4. Determine whether the following sequences $\{x_n\}$ are monotone (i.e., monotone increasing or decreasing). Find $\sup\{x_n\}$ and $\inf\{x_n\}$.

(i) $\left\{ \frac{n}{n^2+1} \right\}.$

(ii) $\left\{ (-1)^n \left(1 + \frac{1}{n} \right) \right\}.$

(iii) $\left\{ \sin\left(\frac{-1}{n} \right) \right\}.$

If any of the above is monotone and bounded, then conclude that the sequence is convergent, also find the limit of that sequence. Justify your answer.

5. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $x_n = y_n$ for all $n \geqslant n_0$ for some $n_0 \in \mathbb{N}$. Prove that $\{x_n\}$ converges (to a limit l) if and only if $\{y_n\}$ is so.

6. Let $\lim_{n\to\infty} x_n = l$. So that if $l \neq 0$, then there exists $N \in \mathbb{N}$ such that

$$|x_n| > \tfrac{|l|}{2} \quad \text{for all } n > N.$$

7. For a sequence $\{x_n\}$ of real numbers, if $\{|x_n|\}$ converges to 0, then $\{x_n\}$ also converges to 0.

Deduce that $\left\{\frac{(-1)^n}{n}\right\}$ converges to 0.

8. Find all the subsequential limits of $\left\{(-1)^n\left(1+\dfrac{1}{n}\right)\right\}$, and then lim sup and lim inf of the sequence. Conclude whether the sequence is convergent.

9. Prove that a sequence $\{x_n\}$ converges (to a limit l) if and only if both the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the same limit (that is l).

Deduce that $\{(-1)^n\}$ is divergent.

10. Prove that $\left\{\dfrac{n}{n+1}\right\}$ is a Cauchy sequence. Deduce that it is convergent.

Hints

1. For 1iv, consider two cases: $r = 0$ and $r \neq 0$. In the 2nd case, since $\frac{1}{|r|} > 1$, let $\frac{1}{|r|} = 1 + a$ for some $a > 0$. Show that $|r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$. Next verify $\epsilon - N$ condition.

2. Do not forget 'Sandwich Theorem (Lecture 6)", and 'the relation between limits and algebraic operations on sequences (Lecture 5)'.

For 2iii, note that $n^{1/n} > 1$ for all $n \in \mathbb{N}$. Set $n^{1/n} = 1 + x_n$. Then taking power to n, show that $|x_n| < \frac{\sqrt{2}}{\sqrt{n-1}}$. Verifying $\epsilon - N$ condition, show that $\lim_{n\to\infty} x_n = 0$. Complete the solution.

5. **Remark**. If one changes finitely many terms of a sequence, it does not affect the convergence and boundedness properties of the sequence.

6. Note that $||a| - |b|| \leqslant |a - b|$ for any real numbers a and b. Moreover, the limit l satisfies the $\epsilon - N$ condition. Consider $\epsilon = \frac{|l|}{2}$.