Research Paper Presentation

Error Probability for Multilevel Systems in Presence of Intersymbol Interference and Additive Noise

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Abstract

- In **"Performance evaluation of optical fiber transmission systems, by M Nakhla"** a technique for evaluation of error probability in fiber-optic communication was proposed
- This technique is generalised for Multilevel Digital Systems
- First, We will discuss the prerequisites and previous methods in evaluating error probability
- The crux of this method is a **Minimax Approximation** of cumulative distribution function of additive noise will be explained
- Finally, examples and comparisons with previously published techniques are presented along with simulations

Introduction

Digital Communication System

- **Primary Objectives** :
  1. Maximize repeater spacing
  2. Maintaining a specified error performance
- Evaluating system performance - Average error probability (BER)
- Bit error rate(BER) estimated by Computer-aided design(CAD) tools

Role of CAD Tools

- Estimates BER as a function of specific characteristics of system
- Margin allocation, Sensitivity analysis
- Identification of specified components
- Cost performance tradeoff

Successful CAD : Efficient and accurate to compute error probability

Introduction

Techniques used to estimate Error Probability

Exhaustive Technique :

- Evaluating the conditional error probability for each of the possible sequences of data and computing their average
- Average Error Probability = Mean(Conditional Error Probabilities)
- Computational cost is highly expensive
- Limits the number of interfering samples

Bounding Error Probability :

- Series Expansion Method
- Gauss Quadrature
- New Method : **Minimax approach**

Introduction

Series Expansion Method

- Simple method with less computational cost
- Slow Convergence
- Provides oscillating results during channel distortion, increase in Signal to Noise Ratio(SNR) or increase in the number of levels

Use of Gaussian Quadrature Rules

- More accurate than series expansion
- Numerical procedure becomes increasingly ill-conditioned as no of moments of R.V representing intersymbol interference is increased

ILL-Conditioned Problem : A problem where, for a small change in the independent variables there is a large change in the dependent variable. This means that correct solution to the equation becomes hard to solve.

Introduction

Minimax approach

- The proposed Computational algorithm is based on deriving a best approximation(in minmax sense) for CDF of the additive noise
- This method guarantees that for a given number of moments error in evaluating error probability is minimum

Problem Statement

Digital system to be considered as shown in Figure (1). Input to the decision circuit at time t is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k\, h(t - kT) + n(t) \tag{1}$$

Problem Statement

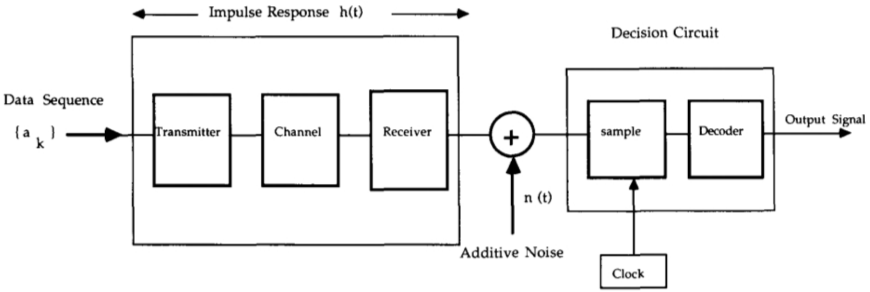

Figure: Model of a data transmission system

Problem Statement

NOTATIONS

SYMBOL	DENOTES
$y(t)$	Input at Decision Circuit
a_k	Sequence of Random Variables
$h(t)$	Impulse Response
$n(t)$	Equivalent Additive Noise
$1/T$	Bit Rate

Calculating Error Probability

Each $a_k \in \{d_1, d_2\} : d_2 > d_1$ with probabilities p_1, p_2 respectively,
Received Signal at decision time t_0 is given by

$$y_0 = a_0 h_0 + n_0 + x \tag{2}$$

$$y_0 = y(t_0),\ h_0 = h(t_0),\ n_0 = n(t_0) \tag{3}$$

Problem Statement

Calculating Error Probability

$$x = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k h(t_0 - kT) \;\; \text{(In equation(2))} \tag{4}$$

Let threshold value of y_0 be S, error probability in the form of conditional probability with respect to source symbols given as

$$P_e = p_1 \Pr\left(y_0 > S | a_0 = d_1\right) + p_2 \Pr\left(y_0 < S | a_0 = d_2\right) \tag{5}$$

$$P_e = p_1 \int_R g(x)\left(1 - D(S - x - d_1 h_0)\right) + p_2 \int_R g(x) D(S - x - d_2 h_0) \tag{6}$$

Where $g(x)$,$D(x)$ is PDF and CDF of additive noise in equation(6). R is range of definition of x

Problem Statement

Calculating Error Probability of Multilevel Signals

For Multilevel Signals a_k can take values from $\{\pm 1, \pm 3, \cdots \pm (2L-1)\}$ with equal probabilities error probability given by

$$P_e = K(L) \int_R g(x) \left(1 - D(h_0 - x)\right) dx \tag{7}$$

Where $K(L) = 2(1 - 1/2L)$ for pulse amplitude modulation(PAM) system
Equation(7) assumes slicing levels $0, \pm 2h_0, \cdots \pm (2L-2)h_0$ and even PDF for additive noise

Level Slicing

An enhancement technique where the Digital Numbers (DN) distributed along the x-axis of an image histogram is divided into a series of analyst-specified intervals of "slices".

Problem Statement

Special Case for Additive noise Guassian

$$P_e = K(L) \int_R g(x)\, Q\left(\frac{h_0 + x}{\mu}\right) dx \tag{8}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-w^2}{2}}\, dw \tag{9}$$

μ standard deviation of additive noise and $Q(x)$ is Gaussian Q function

Problem Statement

General Average Error Probability

From Equation(6)-(8) Average error probability is given by

$$P_e = \int_R f(x)g(x)\,dx \tag{10}$$

Where PDF $g(x)$ of the intersymbol interference(ISI) is not known unless a direct enumeration of all possible sequences is performed,which requires a large amount of computational CPU time

Intersymbol Interference(ISI)

This is a form of distortion of a signal, in which one or more symbols interfere with subsequent signals, causing more noise or delivering a poor output and thus makes communication less reliable

Evaluation of Error Probability (Minimax approach)

If $f(x)$ is continuous in the interval $[-a, a]$ then

$$\int_{-a}^{+a} f(x)g(x)\,dx \cong \boldsymbol{F^T B^{(m)}}\ \boldsymbol{AM^{(m)}} \tag{11}$$

$\boldsymbol{F}$ is n-dimensional vector which has scaled derivatives of $f(x)$ at $x = 0$

$$\boldsymbol{F^T} = \left[f(0), \frac{f^{(1)}(0)a}{1!}, \cdots \frac{f^{(n-1)}(0)a^{n-1}}{(n-1)!} \right] \tag{12}$$

$\boldsymbol{M^{(m)}}$, m-dimensional vector $(m \leq n)$ which has scaled derivatives of $g(x)$

$$\mathrm{M^{(m)}} = \left[M_0, \frac{M_1}{a}, \cdots \frac{M_{m-1}}{a^{n-1}} \right]^T \tag{13}$$

$$\text{Where,}\quad M_k = \int_{-a}^{a} x^k g(x)\,dx \tag{14}$$

Evaluation of Error Probability (Minimax approach)

$\boldsymbol{A} = \{a_{i,j}\}$ and $\boldsymbol{B^{(m)}} = \{b_{i,j}^{(m)}\}$ are $(m \times m)$ and $(n \times m)$ **constant** matrices (independent of f,g) which are recursively generated

For a given number of moments and $n >> 1$:
The equation(11) is the minimax approximation of (10) and best approximation in Chebyshev sense,of $f(x)$ in the interval $[-a, a]$

Using minimax approach on equation(6)

$$P_e \cong \left[p_1 F_1^T + p_1 F_1^T\right] B^{(m)}\ AM^{(m)} \tag{15}$$

$\boldsymbol{F}_1, \boldsymbol{F}_2$ contain the scaled derivatives of $(1 - D(s - x - d_1 h_0))$ and $D(s - x - d_2 h_0)$ respectively, $\boldsymbol{M^m}$ contains the scaled moments of the ISI

Evaluation of error probability (Minimax approach)

Using minimax approach on equation(7)

For 2L-level system, the error probablity is evaluated by

$$P_e \cong K(L)F^T B^{(m)} AM^{(m)} \tag{16}$$

Where $\boldsymbol{F}$ contains scaled derivatives of $[1 - D(h_0 - x)]$

Minmax approach for Gaussian Additive noise

In equation(15), F contains scaled derivatives given by

$$\boldsymbol{F} = \left\{ \frac{f^{(i)}(0)a^i}{i!} \right\};\ i \in \{1, 2, \cdots, n\} \tag{17}$$

$$f^{(i)}(0) = \sqrt{\frac{2}{\pi}} \frac{(-1)^i}{\mu} e^{-\left[\frac{h_0}{\sqrt{2}\mu}\right]^2} H_{i-1}\left(\frac{h_0}{\mu}\right) \tag{18}$$

Examples and Comparisons

Hermite Polynomial

$H_k(x)$ is a Hermite polynomial of degree k which can be generated using the recursive relation

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x) \tag{19}$$

$$\text{With } H_0(x) = 1\ ,\ H_1(x) = x \tag{20}$$

Example 1

Considers the binary PAM transmission, the received pulse is assumed to have the form below, For a truncated 11-pulse train approximation

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{21}$$

$$\text{Sampling time}(t_0) = 0.2T \ \ \text{Signal to Noise Ratio(SNR)} = 16\ dB$$

Examples and Comparisons

Example 1 simulation data

Error Probability for a Binary Digital System ($P(e)$)			
Exhaustive method $P(e) = 2.7614 \times 10^{-3}$			
Order of highest moment used	Quadrature Rule	Series Expansion	Minmax Approximation
4	3.77×10^{-5}	4.2×10^{-6}	2.9758×10^{-3}
6	8.86×10^{-4}	5.98×10^{-5}	2.7650×10^{-3}
8	2.4×10^{-3}	4.71×10^{-4}	2.7573×10^{-3}
10	2.9×10^{-3}	2.14×10^{-3}	2.7610×10^{-3}
12	2.8×10^{-3}	5.46×10^{-3}	2.7610×10^{-3}
14	2.74×10^{-3}	6.56×10^{-3}	2.761446×10^{-3}
16	2.75×10^{-3}	5.06×10^{-4}	2.761442×10^{-3}
18	2.766×10^{-3}	-	2.761425×10^{-3}
20	2.7617×10^{-3}	-	2.761425×10^{-3}
22	2.7615×10^{-3}	-	2.761425×10^{-3}
24	2.76164×10^{-3}	-	2.761425×10^{-3}

Table: Example 1

Simulation Graph - Example 1

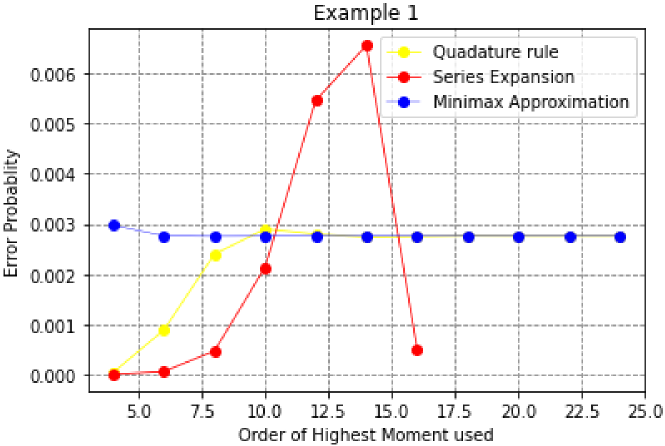

Figure: Error Probability for a Binary Digital System

Examples and Comparisons

Conclusions from Table(1)

- Series expansion has low accuracy, provides oscillating results and for moments>16 approximation gives negative value for error probability
- Gaussian Quadrature has low accuracy for moments<6 but for higher moments it gives accurate results but with high computational cost
- Minimax approximation provides with precise and accurate results with low computational cost

Example 2

Consider a four-level PAM signal. The received pulse is given below, for a truncated 5-pulse train

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad t_0 = 0.05T \quad \text{SNR} = 24\ dB \tag{22}$$

Examples and Comparisons

Example 2 simulation data

Error Probability for a Four level System ($P(e)$)	
Exhaustive method $P(e) = 5.2 \times 10^{-7}$	
Order of highest moment used	Minimax Approximation
8	1.4×10^{-7}
10	2.7×10^{-7}
12	4.0×10^{-7}
14	5.0×10^{-7}
16	5.27×10^{-7}
18	5.26×10^{-7}
20	5.21×10^{-7}
22	5.199×10^{-7}
24	5.204×10^{-7}
26	5.205×10^{-7}
28	5.204×10^{-7}
30	5.204×10^{-7}

Table: Example 2

Simulation Graph - Example 2

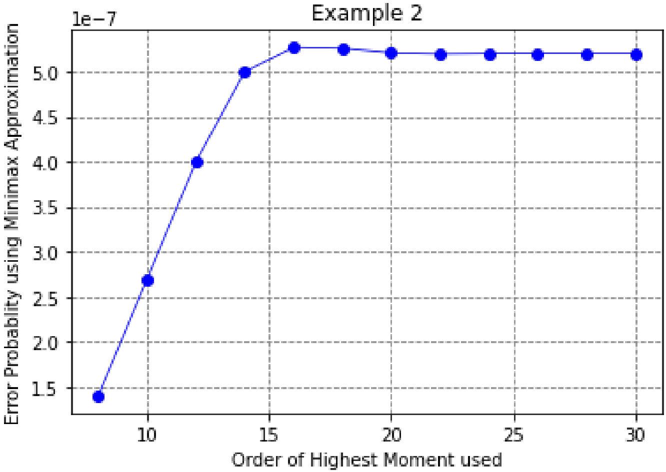

Figure: Error Probability for a Four Level System

Example 3

This example considers a binary system with non-Gaussian additive noise. The received pulse is given by

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad t_0 = 0.2T \quad \text{SNR} = 24\ dB \tag{23}$$

Additive noise is assumed to be Cauchy distributed with PDF

$$\phi(x) = \frac{10^{-2}}{x^2 + 10^{-4}} \tag{24}$$

For a truncated 11-pulse train, Table 3 shows the minimax approach

Examples and Comparisons

Example 3 simulation data

Error Probability for a Binary System ($P(e)$) With Non-Gaussian Additive Noise	
Order of highest moment used	Minimax Approximation
2	3.19×10^{-3}
4	4.17×10^{-3}
6	4.115×10^{-3}
8	4.119×10^{-3}
10	4.118×10^{-3}
12	4.119×10^{-3}
14	4.119×10^{-3}
16	4.119×10^{-3}

Table: Example 3

Simulation Graph - Example 3

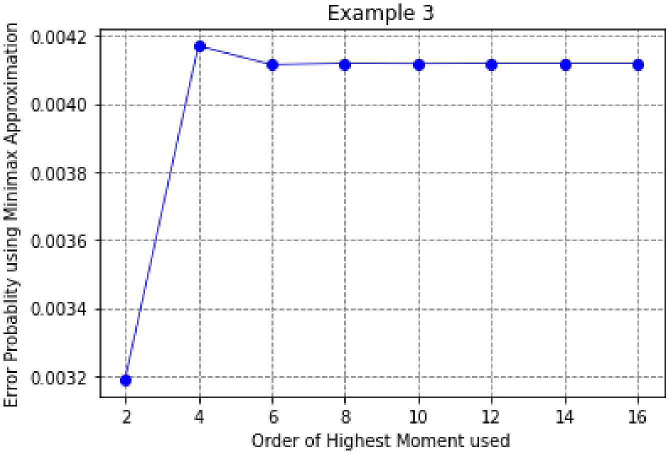

Figure: Error Probability for a Binary System with Non-Gaussian Additive Noise

Conclusion

Computational Cost

- CPU time required to obtain the results above examples using minimax approximation was under 0.4 s on IBM3081KX

Conclusions from Results

- Series expansion works for small range of moments, though Gaussian Quadrature seems accurate its computational cost is high and numerical procedure is ill conditioned
- Minmax approach provides accurate and precise results for a wide number of moments with low computational cost
- This method guarantees that for a number of moments the error in evaluating error probability is minimum
- The additive noise is not constrained to be Gaussian function

THANK YOU

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Indian Institute of Technology Hyderabad