Lecture 9 MA4020 (LINEAR ALGEBRA)

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Definition. The dimension of a finite-dimensional vector space V is the number of vectors in a basis.

Notation. $\dim V$ or $\dim_F V$

- dimension of V over F.

$\dim_F V < \infty$ means V is finite dimensional vector space.

Examples.

1. $V = M_n(\mathbb{R})$; $\dim_{\mathbb{R}} M_n(\mathbb{R}) = n^2$

$\mathcal{B} = \{ e_{ij} \mid 1 \leq i,j \leq n \}$

2. $V = \mathbb{R}^n$; $\dim_{\mathbb{R}} \mathbb{R}^n = n$

Basis $= \{ e_1, e_2, \dots, e_n \}$

3. $V = Sym_2(\mathbb{R}) = \{ A \ \ 2\times 2 \text{ matrix s.t. } A = A^t \}$

$$\dim_{\mathbb{R}} Sym_2(\mathbb{R}) =$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad (a_{12} = a_{21})$$

same

$$= a_{11} e_{11} + a_{12}(e_{12} + e_{21}) + a_{22} \cdot e_{22}$$

Basis $= \{ e_{11}, e_{22}, e_{12} + e_{21} \}$

$$\dim_{\mathbb{R}} Sym_2(\mathbb{R}) = 3$$

In general, $\dim_{\mathbb{R}} Sym_n(\mathbb{R}) = \dfrac{n(n+1)}{2}$

Basis for $Sym_n(\mathbb{R}) = \{ \underset{i=1,\dots,n}{e_{ii}}, \ \underset{i,j \text{ s.t } i \neq j}{e_{ij} + e_{ji}}, \}$

Proposition.

(a) If S spans V, then $|S| \geq \dim V$, and equality holds only if S is a basis.

(b) If $\mathcal{L}$ is linearly independent, then $|\mathcal{L}| \leq \dim V$, and equality holds if $\mathcal{L}$ is a basis.

Proof.

(a). S spans $V \Rightarrow S$ contains a basis

$$\Downarrow$$

$$|S| \geq \underset{\substack{\text{number of vectors in a} \\ \text{basis}}}{\underset{\shortparallel}{\dim V}}$$

If S itself is a basis

$$\Downarrow$$

$$|S| = \dim V.$$

(b) $\mathcal{L}$ is linearly independent, then

$\mathcal{L}$ can be extended by adding elements, to get a basis.

Thus $|\mathcal{L}| \leq \dim V$ and equality holds if $\mathcal{L}$ is a basis.

Proposition. If $W \subset V$ is a subspace of a finite-dimensional vector space, then W is finite-dimensional, and $\dim W \leqslant \dim V$.

Moreover, $\dim W = \dim V$ only if $W = V$.

Proof.

Claim. W is finite-dimensional vector subspace.

$$W \subset V$$

$W = \{0_V\}$ | $W \subsetneq V$ | $W = V$

$\operatorname{span}(\phi)$

$\Downarrow$ $\dim W = \dim V$ & W is finite-dimensional.

Non-trivial case. Let $W \subsetneq V$ be a proper subspace of V.

To show $\exists$ non-empty finite set $\mathcal{L}$ s.t.

$$\operatorname{Span} \mathcal{L} = W.$$

Start with $\mathcal{L} = \{ w \in W \mid w \neq 0_V \}$.

$\{w\}$ is linearly independent.

Step 1. $\operatorname{Span} \mathcal{L} = W$ — YES

NO $\rightarrow$ find $w \in W \setminus \operatorname{Span} \mathcal{L}$ s.t. $(\mathcal{L}, w)$ is L.I.

Step 2. $\operatorname{Span}(\mathcal{L}, w) = W$ — YES / NO.

$\Downarrow$ proceed as before stops in finite steps.

Discussion.

1.

$\mathbb{R}^2$

infinitely many basis → $\{e_1, e_2\}$

$\{e_1+e_2, e_1-e_2\}$

$\{2e_1+3e_2, 7e_1-5e_2\}$

and so on

V f.d. vector space over F

$\mathcal{B}_1 = \{u_1, \dots, u_m\}$ $\quad$ $\mathcal{B}_2 = \{v_1, \dots, v_m\}$ $\quad$ $\mathcal{B}_3 = \{w_1, \dots, w_m\}$

HOW TO RELATE THESE DIFFERENT BASES?

2. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V. Then for any vector $v \in V$,

$$v = x_1 v_1 + \cdots + x_n v_n \quad , \text{ where } x_i \in F, \ i = 1, \dots, n.$$

(in a unique way).

The scalars x_i are called the co-ordinates of v, and the column vector

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called the co-ordinate vector of v, with respect to $\mathcal{B}$.

HOW TO COMPUTE THE CO-ORDINATE VECTOR?

CASE STUDY. Assume that V is the space of column vectors F^n. $(V = F^n)$

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of F^n.

$$v_i = \begin{bmatrix} \vdots \end{bmatrix}_{n \times 1} \quad \text{column vector in } F^n$$

Thus $(v_1, \dots, v_n)$ forms an $n \times n$ matrix.

$$[\mathcal{B}] = \begin{array}{c} \begin{array}{ccc} v_1 & & v_n \end{array} \\ \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}_{n \times n} \end{array}$$

symbol for matrix of $\mathcal{B}$.

Example. if $\mathcal{B} = \{v_1, v_2\}$ is a basis, where

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \text{then}$$

$$[\mathcal{B}] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.$$

Recall e_i:

Column vector

$$\begin{array}{r} \\ i^{th}\text{ row} \\ \\ \end{array}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If $E = (e_1, \dots, e_n)$ is the standard basis, the matrix

E standard basis set

$$[E] = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \; (= I_n) \quad \text{identity matrix.}$$

Note that a linear combination

$$v = \text{"} x_1 v_1 + \cdots + x_n v_n \text{"}$$

can be written as the matrix product

$$[B] X = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v_1 x_1 + \cdots + v_n x_n$$

(block multiplication)

know, unknow, know

$$[B] X = Y$$

Call it $Y \rightarrow \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}_{n \times 1}$

If a vector $Y = (y_1, \ldots, y_n)^t$ is given, we can determine its coordinate vector with respect to the basis $\mathcal{B} = (v_1, \ldots, v_n)$ by solving the equation

$$[\mathcal{B}]X = Y \quad \text{or} \quad \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

invertible

$$X = [\mathcal{B}]^{-1} \cdot Y$$

for the unknown vector X.

Infact $X = [\mathcal{B}]^{-1} \cdot Y$

Example. $\mathcal{B} = (v_1, v_2)$, where $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$,

then $[\mathcal{B}] = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Suppose $Y = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, then

$$[\mathcal{B}]X = Y \quad \text{or} \quad X = [\mathcal{B}]^{-1} \cdot Y$$

$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Thus $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = Y$

$$\| $$

$$7v_1 - 2v_2 = Y$$

Proposition. Let A be an $n \times n$ matrix with entries in a field F. The columns of A form a basis of F^n if and only if A is invertible.

Proof. Write

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

For any column vector $X = (x_1, \cdots, x_n)^t$, the matrix product

$$A \cdot X = v_1 x_1 + \cdots + v_n x_n$$ is a linear combination of the set $(v_1, \cdots, v_n)$.

$(v_1, \cdots, v_n)$ is linearly independent in F^n

$\Updownarrow$

$AX = 0$ has the trivial solution, i.e. $X = 0$

$\Updownarrow$

A is invertible

Moreover $\dim F^n = n$

Thus $(v_1, \cdots, v_n)$ forms a basis of F^n.

Discussion. V vector space

Ordered set of vectors $(v_1, \ldots, v_m)$ in V.

$$(v_1, \ldots, v_m) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = v_1 x_1 + \cdots + v_m x_m$$

$$\mathcal{B} X = v_1 x_1 + \cdots + v_m x_m .$$

$$(v_1, \ldots, v_m) \cdot \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} = (w_1, \ldots, w_n).$$

Here $w_j = v_1 a_{1j} + \cdots + v_m a_{mj}$; $a_{ij} \in F$

Proposition. Let $S = (v_1, \ldots, v_m)$ and $U = (w_1, \ldots, w_n)$ be ordered elements of a vector space V. The elements of U are in the span of S if and only if there is an $m \times n$ scalar matrix A such that

$$(v_1, \ldots, v_m) \cdot A = (w_1, \ldots, w_n)$$

Definition. An isomorphism φ from a vector space V to a vector space V', both over the same field F, is a bijective map

$$\varphi : (V, +, \cdot) \longrightarrow (V', +', \cdot')$$

compatible with the addition and scaler multiplication map,

(a) $\varphi(v + v') = \varphi(v) +' \varphi(v')$ for all $v, v' \in V$

(b) $\varphi(c \cdot v) = c \cdot' \varphi(v)$ for all $v \in V$ and $c \in F$.

Discussion. $A \neq \phi$, $B \neq \phi$, any set A, B

Any map $\varphi : A \longrightarrow B$;

(when)

(i) φ is well-defined if

If $a = b$ in A then $\varphi(a) = \varphi(b)$

(ii) φ is one-one

(iii) φ is onto

$\varphi : \mathbb{Q} \longrightarrow \mathbb{Z}$

$p/q \longmapsto p+q$

$q \neq 0$

$\varphi(1/2) = 3$

$\varphi(2/4) = \varphi(1/2) = 6$

φ is one - one $\qquad \varphi : A \longrightarrow B$

If $\varphi(a) = \varphi(b) \Longleftrightarrow a = b$

$\varphi(a) \neq \varphi(b) \Longleftarrow a \neq b$

Onto map $\qquad f : A \longrightarrow B$

$$\begin{array}{c} \uparrow \\ b \end{array}$$

$\exists$ some $a \in A$

s.t. $f(a) = b$

Examples.

(a) The space F^n of n-dimensional row vector is isomorphic to the space of n-dimensional column vectors.

$$\varphi : (F^n, +, \cdot) \longrightarrow (F^n, +, \cdot)$$

$$X = [x_1, \dots, x_n] \longmapsto \varphi(X) = X^t$$

① φ is well-defined. ✓

② φ is 1-1 ✓

③ φ is onto. ✓

$$\varphi(X+Y) = (X+Y)^t = X^t + Y^t = \varphi(X) + \varphi(Y)$$

(b) $\mathbb{C}$ over $\mathbb{R}$ (as a vector space) and $\mathbb{R}^2$ over $\mathbb{R}$ (as a vector space) are isomorphic as vector space

$(\mathbb{C}, +, \cdot)$ $(\mathbb{R}^2, +, \cdot)$

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{C}$$

$$x = (a, b) \longmapsto a + bi$$

$$\varphi(x + y) = \_\_\_\_$$

Example. Let $S = (s_1, \dots, s_n)$ be a finite set whose elements are distinct. Define

$$V(S) = \{ a_1 s_1 + \dots + a_n s_n \mid a_i \in F \}.$$

$V(S)$, $+$:

Define

$$+ : V(S) \times V(S) \longrightarrow V(S)$$

$$\big((a_1 s_1 + \dots + a_n s_n), (b_1 s_1 + \dots + b_n s_n) \big) \longmapsto (a_1 + b_1) s_1 + \dots + (a_n + b_n) s_n$$

$$\cdot : F \times V(S) \longrightarrow V(S)$$

$$\big(c, (a_1 s_1 + \dots + a_n s_n) \big) \longmapsto (c a_1) s_1 + \dots + (c a_n) s_n$$

$(V(S), +, \cdot)$ is a vector space.

Claim. $\boxed{V(S) \cong F^n}$

$$\varphi : F^n \longrightarrow V(S)$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto a_1 s_1 + \dots + a_n s_n$$

Prove that φ is well-defined, one-one, onto &

(Exercise)
$$\begin{cases} \varphi(X+Y) = \varphi(X) + \varphi(Y), \text{ and} \\ \varphi(cX) = c \cdot \varphi(X) \end{cases}$$

V : vector space with $\mathcal{B} = (v_1, \cdots, v_n)$. $\dim V = n$

Define a map

$$\varphi : F^n \longrightarrow V$$

$$X \longmapsto (v_1, \cdots, v_n) \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \varphi(X) = \underbrace{\text{Basis } [\ \ ]}_{n \times n}$$

$$\underline{\underline{v_1 x_1 + \cdots + v_n x_n}} \in V$$

$$X \longmapsto \mathcal{B} X$$

(i) Is φ well-defined ?

If $X = Y$, then $\varphi(X) = \mathcal{B} X$
$\qquad \qquad \qquad \qquad \| \quad$ (since $X = Y$)
$\qquad \qquad \qquad \qquad \mathcal{B} Y$
$\qquad \qquad \qquad \qquad \|$
$\qquad \qquad \qquad \qquad \varphi(Y)$

(ii) Is φ one-one ?

Suppose $X \neq Y$ in F^n, we want to show $\varphi(X) \neq \varphi(Y)$.

for any $X, Y \in F^n$

If $\varphi(X) = \varphi(Y)$, then

$$\begin{matrix} \varphi(X) & = & \varphi(Y) \\ \| & & \| \\ \mathcal{B} X & & \mathcal{B} Y \end{matrix}$$

$$v_1 x_1 + \cdots + v_n x_n = v_1 y_1 + \cdots + v_n y_n$$

$$\Rightarrow \quad v_1 \underbrace{(x_1 - y_1)}_{=0} + \cdots + v_n \underbrace{(x_n - y_n)}_{=0} = 0$$

Since $(v_1, \ldots, v_n)$ are L.I, $\boxed{x_i = y_i \; ; \; i = 1, \ldots, n}$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

$$X = Y$$

(iii) Is φ onto?

$$\varphi : F^n \longrightarrow V \ni w$$
$$X \longmapsto \underset{v_1 x_1 + \cdots + v_n x_n}{\overset{\shortparallel}{B \cdot X}}$$

If for every $w \in V$, there is one $X \in F^n$ s.t. $\varphi(X) = w$, then φ is onto map.

Let $w = v_1 \lambda_1 + \cdots + v_n \lambda_n$; $\lambda_i \in F$

Choose $X = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$, then $\varphi(X) = w$.

Thus, $\varphi : F^n \longrightarrow V$ is bijective.

$$\varphi : F^n \longrightarrow V$$

(a) $\varphi(X+Y) = \mathcal{B}\cdot(X+Y)$

$$= \mathcal{B}\cdot X + \mathcal{B}Y$$

$$= \varphi(X) + \varphi(Y) \qquad \forall\ X, Y \in F^n$$

(b) $\varphi(cX) = \mathcal{B}(cX)$

$$= c\mathcal{B}X$$

$$= c\,\varphi(X) \qquad \forall\ X \in F^n \text{ and } c \in F.$$

Hence φ is an isomorphism of vector spaces F^n and V.

$(\mathbb{R}^n)$

Discussion

$$\begin{aligned} \varphi : F^n &\longrightarrow V \\ X &\longmapsto \mathcal{B}\cdot X \end{aligned}$$ is an isomorphism

w

$\varphi^{-1}(w) \in F^n$

$\parallel$

$$X = \begin{bmatrix} \vdots \\ \\ \end{bmatrix}.$$

Corollary. Every vector space V of dimension n is isomorphic to the space F^n of column vectors.

Remark. V is finite dimensional vector space over F, then $V \cong F^m$ for some integer $m \geq 0$.

Discussion.

Assume that

$$V \overset{\text{isomorphic}}{\sim} F^n$$

$\mathcal{B}_V = (v_1, \dots, v_n)$ old basis

$$\mathcal{B}_V \longleftrightarrow \mathcal{B}'_V$$

$\mathcal{B}'_V = (v'_1, \dots, v'_n)$ new basis

Question. How are the two bases related?

Question. How are co-ordinate vectors related?

(w.r.t. $\mathcal{B}_V$
w.r.t. $\mathcal{B}'_V$)

Suppose V is finite dimensional vector space over F

i.e. $V \cong F^n$ for some fixed n.

$\mathcal{B}_V = (v_1, \cdots, v_n)$

Bases of V

$\mathcal{B}'_V = (v'_1, \cdots, v'_n)$

$\varphi : F^n \longrightarrow V$, then

$$(v'_1, \cdots, v'_n) \cdot [\, P \,] = (v_1, \cdots, v_n)$$

$$\mathcal{B}'_V \cdot P = \mathcal{B}_V$$

Proposition. Let $S = (v_1, \cdots, v_n)$ and $U = (w_1, \cdots, w_n)$ be ordered set of vectors in V. The elements of U are in the span of S if and only if there is an $m \times n$ matrix A s.t.

$$(v_1, \cdots, v_n) \cdot A = (w_1, \cdots, w_n)$$

The matrix P is called the matrix of change of basis.

Note. P is invertible.