Integral Calculus

Riemann Integration

→ $a, b \in \mathbb{R}$ with $a<b$.

→ a bounded real-valued function

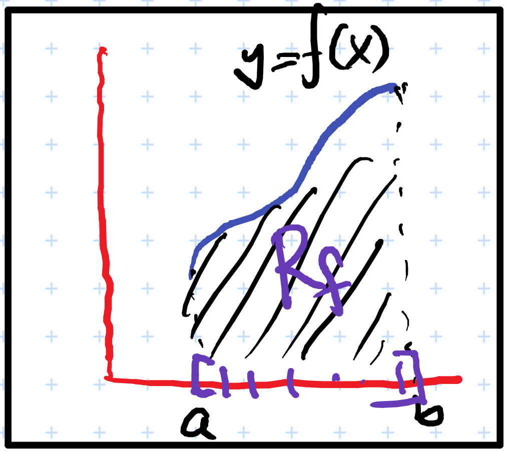
$$f : [a,b] \to \mathbb{R}.$$

Goal: To define the definite integral

$$\int_a^b f(x)\,dx$$

in a way such that if f is non-negative and continuous, then

$$\boxed{\int_a^b f(x)\,dx = \text{Area}(R_f)}$$



$$R_f := \{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$$

Partitions:

A partition of an interval $[a,b]$ is a finite ordered set $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$..

A partition $\{x_0, x_1, \ldots, x_n\}$ divides the interval $[a,b]$ into n parts:

$$[\underset{x_0}{\overset{\shortparallel}{a}}, x_1], \ [x_1, x_2], \ \ldots, \ [x_{n-1}, \underset{x_n}{\overset{\shortparallel}{b}}]$$

$[x_{i-1}, x_i] \leftarrow i^{th}$ subinterval

Example: Interval $[a,b]$

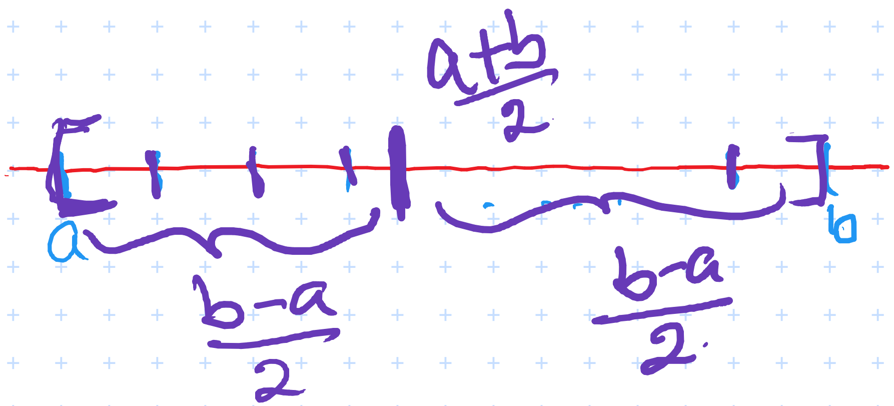
Partition $P_n := \left\{a, a+\frac{(b-a)}{n}, a+\frac{2(b-a)}{n}, \dots, a+\frac{(n-1)(b-a)}{n}, b\right\}$

P_n divides $[a,b]$ into n subintervals

$n=1$ $\quad P_1 = \{a, b\} \leftarrow$ trivial partition

$P_2 = \left\{a, \frac{a+b}{2}, b\right\}$

We have used this partition during the previous lecture.



$x_i - x_{i-1} = \frac{b-a}{n}$

$\forall i$

Moving forward: definitions:

Ingredients:

- $f:[a,b]\to\mathbb{R}$ bounded
- Partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$.

Definitions:

independent of partitions.

$\rightarrow m(f)=\inf\{f(x)\mid x\in[a,b]\}$

$\rightarrow M(f)=\sup\{f(x)\mid x\in[a,b]\}$

$\rightarrow m_i(f)=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}$

$A=[x_{i-1},x_i]\subseteq[a,b]=B.$

$\forall i=1,\dots,n$

$\rightarrow M_i(f)=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$

A,B satisfying $A\subseteq B.$

If $f:B\to\mathbb{R}$ bounded

$m_B=\inf\{f(x)\mid x\in B\}$

$m_A=\inf\{f(x)\mid x\in A\}$ $\Longrightarrow m_B\le m_A.$

$\forall x\in B$

$m_B\le f(x)$

$\Rightarrow$ ~~for~~ $m_B\le f(x)$ $\forall x\in A$

$\Rightarrow m_B\le m_A.$

$\Longrightarrow \boxed{m(f)\le m_i(f)}$

$\Longrightarrow \boxed{M_i(f)\le M(f).}$

the area of a network of rectangles ($\leftrightarrow P$) contained inside R_f.

$$\boxed{m(f) \le m_i(f) \le M_i(f) \le M(f) \quad \forall i=1,\dots,n.}$$

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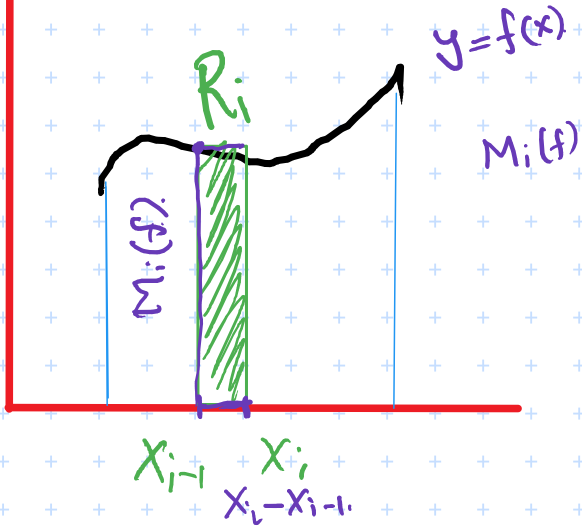
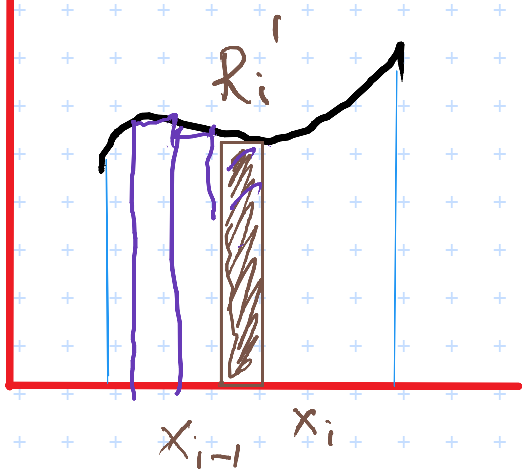
$$L(P,f) = \sum_{i=1}^{n} m_i(f)\,(x_i - x_{i-1}) \longleftarrow$$ Lower sum of f w.r.t. P

$$\le$$

$$U(P,f) = \sum_{i=1}^{n} M_i(f)\,(x_i - x_{i-1}) \longleftarrow$$ upper sum of f w.r.t. P.

$\longleftrightarrow$ the area of a network ($\leftrightarrow P$) of rectangles containing R_f.





$$L(P,f) = \sum_{i=1}^{n} m_i(f)\,(x_i - x_{i-1})$$

$$U(P,f) = \sum_{i=1}^{n} M_i(f)\,(x_i - x_{i-1})$$

$$\boxed{L(P,f) \leq U(P,f)}$$

Proof:

$$m_i(f) \leq M_i(f) \quad \forall i$$

$$\Downarrow$$

$$(x_i - x_{i-1})\,m_i(f) \leq (x_i - x_{i-1})\,M_i(f).$$

$$\Downarrow$$

$$\sum_{i=1}^{n} (x_i - x_{i-1})\,m_i(f) \leq \sum_{i=1}^{n} (x_i - x_{i-1})\,M_i(f) \quad \square$$

Summary

→ $f:[a,b]\to\mathbb{R}$ a bounded function.

→ Partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$

→ Given a partition P
- → $m_i(f)=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}$
- → $M_i(f)=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$

→ Lower sum: $L(P,f)=\sum_{i=1}^{n} m_i(f)(x_i-x_{i-1})$

→ Upper sum: $U(P,f)=\sum_{i=1}^{n} M_i(f)(x_i-x_{i-1})$.

→ $L(P,f)\leq U(P,f)$. holds for every partition of $[a,b]$

Intuition: $L(P,f)\leq \int_a^b f(x)\,dx \leq U(P,f)$.

$\boxed{\text{Next: Refine and look for better approximations!!}}$