CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 7

Instructions

- Try these questions before class. Do not submit!

(1) Show that the proof we saw for the powerset of a set is strictly larger than the set is actually a diagonal argument.

(2) S.T. principle of mathematical induction, strong induction and well-ordering principle are equivalent.

(3) P.T. every amount of postage of 12 cents or more can be formed using 4-cent and 5-cent stamps. (Worked out example in Rosen textbook.The idea is to see how both strong induction and weak (its not really weak or incomplete) induction can be useful proof techniques for the same problem.)

(4) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The arithmetic mean of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and geometric mean is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

(5) Suppose that we want to prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n}}$$

for all positive integers n.

(a) S.T. if we try to prove this inequality using mathematical induction the basis step works but the inductive step fails.

(b) S.T. mathematical induction can be used to prove the stronger inequality,

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

for all integers greater than 1 which together with a verification for the case where $n = 1$ establishes the weaker inequality we originally tried to prove using mathematical induction.

(6) Use strong induction to prove that $\sqrt{2}$ is irrational. Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.

(7) Picks theorem says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P, respectively. Use strong induction on the number of vertices of P to prove Picks theorem.

[Hint: For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three tri- angles subtracted. For the inductive step, take advantage of Lemma 1 in the textbook : "*Every simple polygon with at least four sides has an interior diagonal.*" This is a continuation of the topic "Using Strong Induction in Computational Geometry" in the textbook. Please read up the definitions of polygon, convex polygon, diagonal, interior diagonal, triangulation from the textbook. Also read how the textbook proves the theorem " A simple polygon with n sides where n is an integer with $n \geq 3$ can be triangulated into $n - 2$ triangles. "]

(8) Use structural induction to show that $n(T) \geq 2h(T) + 1$ where T is a full binary tree, $n(T)$ equals the number of vertices of T and $h(T)$ is the height of T.

To prove results about full binary trees using structural induction we need to complete the following basis step and recursive step.
Basis Step:S.T. the result is true for the tree consisting of a single vertex.
Recursive Step:S.T. if the result is true for the trees T_1 and T_2 then it is true for the tree $T_1 \cdot T_2$ consisting of a root r, which has T_1 as its left subtree and T_2 as its right subtree.

The set of leaves and the set of internal vertices of a full binary tree can be defined recursively.
Basis Step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.
Recursive Step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and of T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

For definitions and results on $h(T)$ and $n(T)$ refer to the Rosen textbook.