Examples: Riemann sums

Key idea: Let $f:[a,b]\to\mathbb{R}$ be an integrable function and (P_n) be a sequence of partitions of $[a,b]$ such that $\mu(P_n)\to 0$ as $n\to\infty$. Then $U(P_n,f)-L(P_n,f)\to 0$ as $n\to\infty$.

and $S(P_n,f)\to\int_a^b f$ as $n\to\infty$

① $\int_a^b f(x)\,dx$ known. $a_n := S(P_n,f)$ where $\mu(P_n)\to 0$ as $n\to\infty$ then we can compute $\lim_{n\to\infty} a_n$.

② $\int_a^b f(x)\,dx$ is NOT known, then by defining the sequence $a_n := S(P_n,f)$ $\mu(P_n)\to 0$ as $n\to\infty$ we can approximate $\int_a^b f(x)\,dx$.

Examples: ① $a_n = \sum_{i=1}^n \frac{i^p}{n^{p+1}}$ where p non-negative rational number

$$= \frac{1}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^p$$
$$= \sum_{i=1}^n \left(\frac{i}{n}\right)^p\left(\frac{i}{n}-\frac{i-1}{n}\right)$$
$$= \sum_{i=1}^n f(t_i)\left(\frac{i}{n}-\frac{i-1}{n}\right)$$
$$= S(P_n,f)$$
$$\to \int_0^1 x^p\,dx.$$

$P_n=\{0,\frac{1}{n},\dots,1\}$

$[0,\frac{1}{n}]\ [\frac{1}{n},\frac{2}{n}],\dots,[\frac{n-1}{n},\frac{n}{n}]$

$f(x)=x^p$ on $[0,1]$.

$\boxed{F(x)=\frac{x^{p+1}}{p+1}}$

$\mu(P_n)=\frac{1}{n}\to 0$ as $n\to\infty$

$$FTC(2) = F(1)-F(0)=\frac{1}{p+1}.$$

② $a_n = \sum_{i=1}^n \frac{1}{\sqrt{n^2+in}}$ $\qquad b_n=\sum_{i=1}^n \frac{1}{\sqrt{n^2+(i-1)n}}$

$$= \frac{1}{n}\sum_{i=1}^n \frac{1}{\sqrt{1+\frac{i}{n}}}$$
$$= \sum_{i=1}^n \frac{1}{\sqrt{1+(\frac{i}{n})}}\left(\frac{i}{n}-\frac{i-1}{n}\right)$$
$$= \sum_{i=1}^n f(t_i)\left(\frac{i}{n}-\frac{i-1}{n}\right).$$
$$\to \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$
$$= F(1)-F(0).$$
$$= 2\sqrt{2}-2.$$

$P_n=\{0,\frac{1}{n},\dots,\frac{n}{n}\}$

$[0,\frac{1}{n}],[\frac{1}{n},\frac{2}{n}],\dots,[\frac{n-1}{n},\frac{n}{n}]$

$\frac{1}{n}$

$\mu(P_n)=\frac{1}{n}\to 0$ as $n\to\infty$

$f(x)=\frac{1}{\sqrt{1+x}}$ is integrable

$t_i=\frac{i}{n}$ as $i=1,\dots,n$

f also has an antiderivative $F(x)=2\sqrt{1+x}$

③ $\int_0^1 \frac{dx}{1+x^2}$ $\qquad f(x)=\frac{1}{1+x^2}.$

$P_n=\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n}{n}\}$

$[0,\frac{1}{n}],\dots,[\frac{n-1}{n},\frac{n}{n}]$

$\mu(P_n)=\frac{1}{n}\to 0$ as $n\to\infty$.

$T=\{t_1,\dots,t_n\}$

$t_i=\frac{i-1}{n}$ for $i=1,\dots,n$.

$$a_n := S(P_n,f)_T$$
$$= \sum_{i=1}^n f(t_i)\left(\frac{i}{n}-\frac{i-1}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^n \frac{1}{1+(\frac{i-1}{n})^2}$$
$$= \frac{1}{n}\sum_{i=1}^n \frac{n^2}{n^2+(i-1)^2} \to \int_0^1 \frac{1}{1+x^2}\,dx \text{ as } n\to\infty$$

$a_n \quad n\gg 0$

$S=\{s_1,\dots,s_n\}$

$s_i=\frac{i}{n}$ for $i=1,\dots,n$.

$$b_n = S(P_n,f)_S$$
$$= \sum_{i=1}^n f(s_i)\left(\frac{i}{n}-\frac{i-1}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^n \frac{1}{1+(\frac{i}{n})^2}$$
$$= \sum_{i=1}^n \frac{n}{n^2+i^2} \xrightarrow[n\to\infty]{} \int_0^1 \frac{1}{1+x^2}\,dx$$