RUSSELL'S ANTINOMY

A CHALLENGE TO FREGE'S SET THEORY AND CONTEMPORARY SET THEORIES

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1 Introduction:

Russell's Antinomy or Russell's Paradox is a prominent set theoretical paradox which is also known as Russell-Zermelo Paradox.It was discovered by Bertrand Arthur William Russell,British Logician,Philosopher and Mathematician in 1901 it was first observed by Ernst Zermelo in 1899 but was not thought much important.While working on Principles of Mathematics Russell encountered the paradox studying cantor power class theorem: Carnality of set strictly less than its power set. Gottlob Frege whose lifelong project was to demonstrate that Foundation of Mathematics was Logic made him research many years and published "Grundgesetze der Arithmetik(The Basic laws of Arithmetic)" where he proposed a set theory which was thought to be ideal and perfect but Russell wrote his problem in terms of both logic and set theory to Frege which lead his theory and his idea of logic to be inconclusive. Let us study the paradox,its consequences and how we have avoided this problem in detail.

2 Paradox statement in different forms:

2.1 Barber Paradox

Lets think about the paradox in elementary English form The barber Paradox which was derived from Russell's Paradox **"Barber in the village shaves the beard of every person in the village who do not shave his beard himself now, the question is <u>should the barber shave himself or not?</u>"** Clearly we cannot answer the question if we think he shaves himself the according to the rule he should not if he does not shave himself then he should we consider any of the two possibility hence its a paradox.

2.2 Russell's Paradox in Set Theory

Naive idea of set theoretical concept of paradox is if we take a set of all people in the village who does not shave themselves then by the above paradoxical idea should we include barber in the set or not. More generally, lets consider the following set in set builder form and define the paradox symbolically:

$$Let\, S = \{x \,|\, x \notin x\}\, then\, S \in S \iff S \notin S$$

"Let S be Set of all sets which do not contain themselves should S contain the Set S in it i.e should it contain itself or not?"
If we assume $S \notin S$ then the set S should be added in it as per its definition then it contradicts its own definition that sets which do not contain themselves so we conclude that we are now again in the similar situation as of Barber Paradox and cannot decide S should be added in S or not in either way it leads to a contradiction.

2.3 Russell's Paradox in Logical view

From the theory of predicate logic consider the following binary predicate where $\varphi(x)$ is substituted with $x \notin x$

$$\exists y \forall x(x \in y \iff \varphi(x))$$

After substituting $\varphi(x)$ with $x \notin x$ followed by by existential instantiation of variable y with some constant set Y and universal instantiation of x with Y we get

$$Y \in Y \iff Y \notin Y$$

Hence a contradiction and again we observe similar situation as observed above.

3 Consequences and effects on Mathematics:

Frege's analogies were proved to be incorrect but that's not the end no other mathematical analogy or theory provided an explanation to the paradox.Many mathematicians at that time really did not care about Russell's paradox but to form any new theory in set theory and logic the paradox was a great hurdle, they soon realised no mathematical proof was trustworthy and only by eliminating paradox mathematics can prove its consistency.

Any Axiomatic foundation in Mathematics has to be precise that there should be 0 probability of proving the axiom wrong, but Frege's axioms were proved to be wrong by Russell through his paradox and Frege was not able provide any justification to overcome it so he mentioned Russell Paradox in his book's appendix his work on finding Foundation of Mathematics was not complete Though we still use Frege's Axioms.There is no theory that provides an answer to paradox besides the paradox is being avoided to prove the consistency of mathematics.

4 Reason for Frege's Failure:

Although Frege was extremely careful designing his theory he simply gave permission to form any list of distinct elements as set such paradox is basically possible due to this.From principle of Pseudo-Scotus if such paradoxes or contractions are appeared in a theory any statement can be proven to be true and thus destroys the significance of entire theory erasing the difference between truth and false thus Frege's theory was unsuccessful because his theory made Russell's Paradox possible but it did not provide any solution or avoided the Russell Paradox.

5 Possible Solutions to the Paradox:

There is no rigorous solution to the paradox until now.There are mainly two counter measures made to avoid Paradox.They are Russell's Type Theory, Zermelo's Axioms. Zermelo's Axioms were widely accepted which further evolved into Zermelo-Frankel Set Theory which uses Axiom of Choices famously known as ZFC's accepted by all mathematicians and considered as Foundation of on going Mathematics. Russell has changed the idea of Logic and tried to avoid Paradox though Types and Set Theory based on Type Theory.

5.1 How ZFC'S Successfully Avoided Russell Paradox

Zermelo's Theory was an updated version of Frege's set theory and his axioms.He brilliantly avoided the paradox without actually changing the fundamental idea of Logic.The change of following Axiom has made solution possible.

Zermelo method of avoiding the Paradox is replacing the following axiom
$\forall P(x) \, \exists \, a \, set \, y = \{x \mid P(x)\}$ from Frege's set of Axioms with the following Axiom

$$(\forall P(x) \wedge \, \forall set \, b) \, \exists sety = \{x \, |x \in \, set \, b \, \wedge P(x)\}$$

The difference between above two axioms is that in Frege's Axiom there was no proper domain defined for variable for x which unknowingly freed x and makes such paradoxes possible it gives liberty to the set definer and one can define any type of set from a predicate and a free variable thus Russell Paradox is defined in such Axiomatic mathematics and Maths becomes Inconsistent
While the second Axiom strictly says x has to be itself contained in a set to be used to define a set restricts the usage of variable x in whatever way we can cleverly avoid the paradox in both set theory and logical view. A set of x can only be used for a predicate not any x makes the $R = \{x \mid x \notin x\}$ not a set but a class.As it is not a set concept of Russell paradox does not show up. Naive set theory and Frege's set theory failed in avoiding Russell's Paradox where ZFC's have efficiently avoided Russell Paradox instead of solving it in order to maintain Consistency of Mathematics.

5.2 Solution by Russell Using Theory of Types

Russell used Theory of Types which uses the concept of Existential hierarchy. There were 3 levels of distinction which are Objects, Predicates, Predicates of Predicates.Russell's idea was to define sets that are not member of themselves and individual sets are on different level i.e they are of different type and thus avoiding the paradox in logical view. Russell further developed the concept to sets by considering Classes and individuals so on and and defining an Axiom that classes cannot be members of themselves and so as individuals so on.Russell simply avoided the paradox by saying set R is class and the other sets are like individuals.

$$5^{th} Axiom \, \varepsilon P = \varepsilon Q \equiv \forall x \, (P(x) \equiv Q(x))$$

This axiom is used to escape from Russell's Paradox in gives a relation between same type entities

$$R(x) \iff \exists S\, [x = \varepsilon S \wedge \neg S(x)]$$

Russell's Paradox representation in type theory

$$R(\varepsilon R) \equiv \neg R(\varepsilon R)$$

Thus Russell's Paradox both Sets has been successfully shown as different types and circumvent Russell Paradox.Though This theory used by advanced mathematics to prove some results it is portrayed more like definition based and does not involve logic hence it was not used by logicians and was less famous among Mathematicians.

5.3 Some more Solutions:

Even W.V Quine who coined the term Russell Antinomy to the Russell Paradox has tried to solve the paradox in his way and suggested a solution(method to escape) which also involved Type Theory.He introduced a new concept Stratification which assigns natural number values to individuals of a class and does not allow new member to join easily.thus blocking Russell's Paradox.

Another possible counter measure to the Paradox given by Aussonderung who avoids Russell's paradox by another axiom similar to Zermelo's Axiom.x is a subset of P that satisfies condition C if and only if x satisfies condition C.

6 Further Developments and Applications:

Many new Paradoxes have been developed by simple modifications of Russell Paradox i.e "The ______ er of all ______ that do not contain themselves" Grelling–Nelson paradox, Richard's paradox are developed from Russell's Paradox which fills the blank with Describe and Denote respectively. Russell's paradox and some other paradoxes have challenged mathematicians to provide new theories and led to the birth of Type Theory and ZFC'S. Still active research is going on to provide more rigorous solution to the Paradox. There are not much applications to the paradox but it is definitely avoided in any newly formed theory otherwise entire theory would be invalid and marked to be inconsistent.

7 Conclusion to Russell's Antinomy:

I worked on this report to provide some basic idea about Russell Paradox, its significance and some solutions provided to avoid it through this report. I slighted went through the history and provided English analogical definition to Russell's Paradox and also thoroughly went through its solutions,Finally gone through further developments in the paradox. This Report gives an idea how mathematicians who try to build new theories worked to avoid Russell Paradox. Though Russell Paradox has not helped to solve any new problems avoiding it in our theory mainly avoids all other issues and provides an ideal Axiomatic Theory which helps to solve all the problems.

With greater Axioms and complexity of theory greater Paradoxes or Antinomies may also be possible a theory should be strict enough to avoid such Paradoxes and thus providing an ideal Theory.Formation of any new Axiomatic theory one should consider the possibility of Russell Paradox and try avoiding it.The Paradox has devastated Frege's work and also proved Naive set theory inconsistent.Avoiding such paradoxes helps us to solve further results instead of trying to solve it. Hence both Zermelo and Russell have tried to avoid the Paradox instead of solving it.Therefore we can conclude that Paradoxes in Mathematics can be dangerous but avoiding them would provide us compatible results.

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Thank you for giving me an opportunity to explore new things

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