

Assignment 4

Suraj - CS20BTECH11050

Download all python codes from

<https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment%204/Python%20codes>

Download Latex-tikz codes from

<https://github.com/Suraj11050/Assignments-AI1103/blob/main/Assignment%204/Assignment%204.tex>

1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f_{(k,8)}(x) = \begin{cases} \frac{7}{32} x^6 (2-x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (1.0.1)$$

then k equals _____

2 SOLUTION

Definition 2.1. For given statistical sample $\{X_1, X_2, \dots, X_n\}$, the order statistics is obtained by sorting the sample in ascending order. It denoted as $\{X_{(1)}, X_{(2)}, \dots, X_{(n)}\}$. The k^{th} smallest value $X_{(k)}$ is called k^{th} order statistic

Theorem 2.1. Let $\{X_1, X_2, \dots, X_n\}$ be n i.i.d random variables with common CDF $= F_X(x)$ and common PDF $= f_X(x)$, then the marginal probability distribution of k^{th} order statistic (CDF) is denoted by $F_{(k,n)}(x)$ and it is given by

$$F_{(k,n)}(x) = \sum_{j=k}^n {}^nC_j \times (F_X(x))^j \times (1 - F_X(x))^{n-j} \quad (2.0.1)$$

Proof.

$$F_{(k,n)}(x) = \Pr(X_{(k)} \leq x) \quad (2.0.2)$$

$$F_{(k,n)}(x) = \Pr(\text{At least } k \text{ elements have value } \leq x) \quad (2.0.3)$$

Since $\Pr(X \leq x) = F_X(x)$, Let $Q \sim \text{Bern}(F_X(x))$

$$\Pr(Q = 1) = F_X(x) \quad (2.0.4)$$

$$\Pr(Q = 0) = 1 - F_X(x) \quad (2.0.5)$$

Let $P \sim B(n, F_X(x))$ taking n trails from $\text{Bern}(F_X(x))$

$$\Pr(P = i) = {}^nC_i \Pr(Q = 1)^i \Pr(Q = 0)^{n-i} \quad (2.0.6)$$

$$\Pr(P = i) = {}^nC_i F_X(x)^i (1 - F_X(x))^{n-i} \quad (2.0.7)$$

Equation (2.0.7) is probability of exactly i R.V of given sample have values $\leq x$

$$F_{(k,n)}(x) = \Pr(P \geq k) = \sum_{j=k}^n \Pr(P = j) \quad (2.0.8)$$

$$\therefore F_{(k,n)}(x) = \sum_{j=k}^n {}^nC_j (F_X(x))^j (1 - F_X(x))^{n-j} \quad (2.0.9)$$

□

Theorem 2.2. Let $\{X_1, X_2, \dots, X_n\}$ be n i.i.d random variables with common CDF $= F_X(x)$ and common PDF $= f_X(x)$, then the marginal probability density of k^{th} order statistic (PDF) is denoted by $f_{(k,n)}(x)$ and it is given by

$$f_{(k,n)}(x) = n {}^{n-1}C_{k-1} f_X(x) (F_X(x))^{k-1} (1 - F_X(x))^{n-k} \quad (2.0.10)$$

Proof.

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx} \left(\sum_{j=k}^n {}^nC_j (1 - F_X(x))^{n-j} F_X(x)^j \right) \quad (2.0.11)$$

$$\begin{aligned} f_{(k,n)}(x) &= \sum_{j=k}^n {}^nC_j (j) (1 - F_X(x))^{n-j} F_X(x)^{j-1} f_X(x) \\ &\quad - \sum_{j=k}^n {}^nC_j (n-j) (1 - F_X(x))^{n-j-1} F_X(x)^j f_X(x) \end{aligned} \quad (2.0.12)$$

$$S_1 = \sum_{j=k}^n \frac{n!}{(n-j)!(j-1)!} (1 - F_X(x))^{n-j} F_X(x)^{j-1} f_X(x) \quad (2.0.13)$$

$$f_{(k,8)}(x) = \frac{8}{2^{(1+(k-1)+(8-k))}} \times {}^7C_{k-1} x^{k-1} (2-x)^{8-k} \quad (2.0.24)$$

$$S_2 = \sum_{j=k}^n \frac{n!}{(n-j-1)!j!} (1 - F_X(x))^{n-j-1} F_X(x)^j f_X(x) \quad (2.0.14)$$

$$f_{(k,8)}(x) = {}^7C_{k-1} \frac{1}{32} x^{k-1} (2-x)^{8-k} \quad (2.0.25)$$

let $i = j + 1$ in equation (2.0.14) (changing limits)

Comparing the PDF obtained in equation (2.0.25) with the equation (1.0.1)

$$\frac{{}^7C_{k-1}}{32} (2-x)^{8-k} x^{k-1} = \frac{7}{32} (2-x) x^6 \quad (2.0.26)$$

$$\therefore k = 7 \quad (2.0.27)$$

$$S_2 = \sum_{i=k+1}^n \frac{n!}{(n-i)!(i-1)!} (1 - F_X(x))^{n-i} F_X(x)^{i-1} f_X(x) \quad (2.0.15)$$

Hence the marginal probability density given is 7^{th} order statistic and **the value of k is 7**

$$f_{(k,n)}(x) = S_1 - S_2 \quad (2.0.16)$$

$$f_{(k,n)}(x) = \frac{n! f_X(x) (1 - F_X(x))^{n-k} F_X(x)^{k-1}}{(n-k)!(k-1)!} \quad (2.0.17)$$

$$\therefore f_{(k,n)}(x) = n {}^{n-1}C_{k-1} (1 - F_X(x))^{n-k} F_X(x)^{k-1} f_X(x) \quad (2.0.18)$$

□

Method 1:

Let $X \in [0, 2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x) dx = 1 \quad (2.0.19)$$

$$\Pr(x) = \frac{1}{2} \quad (\because \text{Uniform order}) \quad (2.0.20)$$

The PDF for X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.21)$$

The CDF for X is

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{2}, & 0 < x < 2, \\ 1, & x \geq 2 \end{cases} \quad (2.0.22)$$

Using theorem (2.2) PDF of k^{th} order statistic of given sample from equation (2.0.10)

$$f_{(k,n)}(x) = n {}^{n-1}C_{k-1} \frac{1}{2} \left(\frac{x}{2}\right)^{k-1} \left(1 - \frac{x}{2}\right)^{n-k} \quad (2.0.23)$$

Definition 2.2. Beta function

The **Beta function** is defined for $r, s \in \mathbb{R}^+$ by

$$B(r, s) = \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} \quad (2.0.28)$$

Definition 2.3. Uniform order statistics

Let $\{X_1, \dots, X_n\}$ be i.i.d form a uniform distribution on $[0, 1]$ such that $f_X(x) = 1$ and $F_X(x) = x$, from theorem (2.2), equation (2.0.10)

$$f_{(k,n)}(x) = n {}^{n-1}C_{k-1} x^{k-1} (1-x)^{n-k} \quad (2.0.29)$$

Lemma 2.3. Since equation (2.0.29) is PDF its definite integral can be given by Beta function as

$$\int_0^1 n {}^{n-1}C_{k-1} x^{k-1} (1-x)^{n-k} dx = 1 \quad (2.0.30)$$

$$\int_0^1 x^{k-1} (1-x)^{n-k} dx = \frac{(k-1)!(n-k)!}{n!} \quad (2.0.31)$$

$$\int_0^1 x^{k-1} (1-x)^{n-k} dx = \frac{\Gamma(k)\Gamma(n-k+1)}{\Gamma(k+(n-k+1))} \quad (2.0.32)$$

$$\int_0^1 x^{k-1} (1-x)^{n-k} dx = B(k, n-k+1) \quad (2.0.33)$$

Definition 2.4. Beta Distribution

The Beta distribution is a continuous distribution defined on the range (0, 1) whose PDF given by

$$f(x) = \frac{1}{B(r, s)} x^{r-1} (1-x)^{s-1} \quad (2.0.34)$$

where $\int_0^1 f(x) = 1$ as per definition (2.2)

CDF, Mean value and Variance of Beta distribution

$$F(x) = \frac{\int_0^x x^{r-1} (1-x)^{s-1}}{B(r, s)} = \frac{B_x(r, s)}{B(r, s)} \quad (2.0.35)$$

$$E(x) = \frac{r}{r+s} \quad (2.0.36)$$

$$Var(x) = \frac{rs}{(r+s)^2 (r+s+1)} \quad (2.0.37)$$

Lemma 2.4. Uniform order statistics on [0,1] the PDF of k^{th} order statistic follows Beta distribution with $r = k$, $s = n - k + 1$ and PDF is given by

$$f_X(x) = \frac{1}{B(k, n-k+1)} x^{k-1} (1-x)^{(n-k+1)-1} \quad (2.0.38)$$

Method 2:

we know that, PDF of k^{th} order statistic of a uniform distribution on [0, 1] follows beta distribution

$$\int_0^2 f_X(x) dx = \int_0^2 \frac{7}{32} x^6 (2-x) dx \quad (2.0.39)$$

$$\int_0^2 f_X(x) dx = \int_0^2 56 \left(\frac{x}{2}\right)^6 \left(1 - \frac{x}{2}\right) d\left(\frac{x}{2}\right) \quad (2.0.40)$$

Let new random variable be t such that $t = x/2$,
New sample be $\{T_1, \dots, T_8\}$ such that $T_i = X_i/2$.

$$f_T(t) = 56 t^6 (1-t) \quad (2.0.41)$$

$$\int_0^2 f_X(x) dx = \int_0^1 f_T(t) dt = 1 \quad (2.0.42)$$

The Uniform distribution of new random sample is on [0, 1] such that PDF = 1 and CDF = t

$f(k, 8)(x)$ in equation (1.0.1) (after conversion)

$$f_{(k,8)}(t) = \begin{cases} 56 t^6 (1-t), & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.43)$$

Since equation (2.0.43) is a Beta distribution with $r = k$, $s = n - k + 1$

$$r - 1 = k - 1 = 6 \quad (2.0.44)$$

$$\therefore k = 7 \quad (2.0.45)$$

Hence the value of k is 7

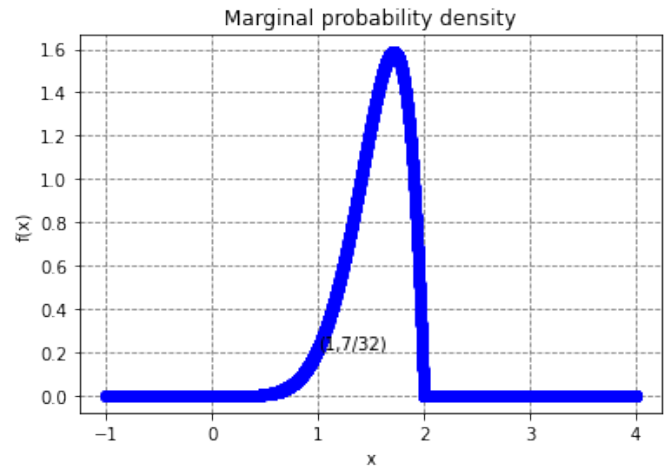


Fig. 1: PDF of $f_{(7,8)}(x)$

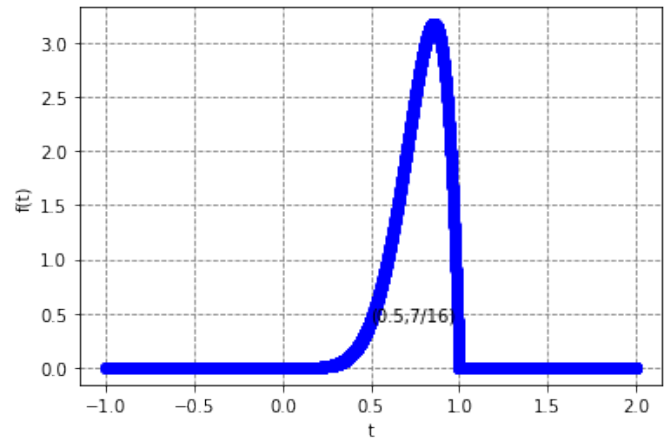


Fig. 2: PDF of $f_{(7,8)}(t)$

Presentation link:

https:

[//github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment4presentation](https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment4presentation)