# Calculus - Assignment 2 - Series

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Before trying to solve the assignments, you should see all the video lectures that I have uploaded in YouTube (particularly the lectures on series of real numbers):
Click here to get the YouTube Link for the playlist.
You may use the theorems (without giving the proofs) from the video lectures, but if I ask you to do in a particular method, then you should do in that way to understand that particular theory better.

**1.** Verify whether the series $\sum_{n=1}^{\infty} x_n$, where $x_n = \frac{n^2}{n^2+1}$, satisfies the necessary condition (in terms of the limit of $x_n$) for convergence. Deduce that the series is divergent.

**2.** Using Cauchy's criterion for a series to be convergent, prove that

$$\text{the series} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{is convergent.}$$

**3.** A series $\sum_{n=1}^{\infty} x_n$ is called **conditionally convergent** if $\sum_{n=1}^{\infty} x_n$ is convergent, but it is not absolutely convergent, i.e., $\sum_{n=1}^{\infty} |x_n|$ is not convergent. Give examples of two conditionally convergent series, and justify why these are conditionally convergent series.

**4.** Determine whether the following series are convergent or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3/4}\sin(n!)}{n^2+1}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

First you should try on your own. If you need, then see the next page for some hints.

## Hints

**1.** Can you verify the necessary condition for a series to be convergent done in Lecture 12?

**2.** First see Lecture 12. Observe that Cauchy's criterion for a series to be convergent can be stated as follows: A series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if for any $\epsilon > 0$, there exists a natural number $N$ (depending on $\epsilon$) such that

$$|x_{n+1} + x_{n+2} + \cdots x_{n+p}| < \epsilon \text{ for all } n > N \text{ and for all } p \in \mathbb{N}.$$

For Question 2, you just have to verify the above $\epsilon - N$ condition. Do it as follows:

(a) Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$. Then

$$\begin{aligned}
|s_{n+p} - s_n| &= |x_{n+1} + x_{n+2} + \cdots x_{n+p}| \\
&= \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \frac{1}{n+5} - \cdots + (-1)^{p-1} \frac{1}{n+p} \right| \\
&= \left| \frac{1}{n+1} - \left( \frac{1}{n+2} - \frac{1}{n+3} \right) - \left( \frac{1}{n+4} - \frac{1}{n+5} \right) - \cdots \right| \\
&< \frac{1}{n+1}
\end{aligned}$$

(b) Next prove that for every $\epsilon > 0$, there exists $N$ satisfying the desired condition.

**4.** You may use various tests for convergence and divergence of a series discussed in the video lectures. For some series, you may use the fact about $p$-series:

For a real number $p$, the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

(4i) Can you use the bound $\sin(x) \leqslant 1$ for all $x$?

(4iii) Note that $\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) = \dfrac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$.