# Order Statistics

## GATE 2021 (ST), Q.17 (STATISTICS SECTION)


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# Topics covered:

## Prerequisites:

- Definition of order statistics
- Range, median and an example
- Theorem 1 and its proof (CDF of $k^{th}$ order statistic)
- Theorem 2 and its proof (PDF of $k^{th}$ order statistic)
- Uniform order statistics
- Introducing Beta function
- Beta distribution and its features

## Gate Problem

- Problem
- Solution method 1
- Solution method 2

# Introduction to Order statistics

## Definition

- For a given statistical sample $\{X_1, X_2, \cdots X_n\}$, The order statistics is obtained by sorting the sample in ascending order
- The ordered sample values are denoted as $\{X_{(1)}, X_{(2)}, \cdots X_{(n)}\}$
- $X_{(1)}$ is the minimum and $X_{(n)}$ is the maximum of the given sample
- The $k^{th}$ smallest value $X_{(k)}$ is called $k^{th}$ order statistic

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)} \leq \cdots \leq X_{(n)} \tag{1}$$

- For a sample $\{X_1, X_2, \cdots X_n\}$ of size n:

$$X_{(k)} = \min\{\max\{X_j : j \in J\} \, : \, J \subset \{1, 2 \cdots n\} \, |J| = k\} \tag{2}$$

# Introduction to Order statistics

## Range and Median

- For a sample $\{X_1, X_2, \cdots X_n\}$, the range is the distance between the smallest and largest observations and is denoted by $R$

$$R = X_{(n)} - X_{(1)} \tag{3}$$

- Median is as the middle number of a sorted sample it is denoted by M it is defined using order statistics of a sample as

$$M = \begin{cases} X_{((n+1)/2)}, & \text{if } n \text{ is odd,} \\ \dfrac{X_{(n/2)} + X_{(n/2+1)}}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{4}$$

## Example Problem

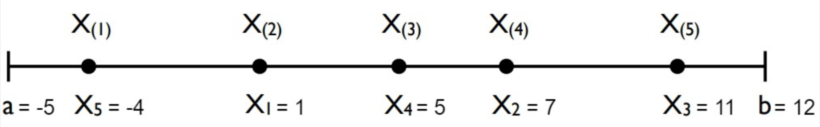
For the given data sample $\{1, 7, 11, 5, -4\}$ discuss the order statistics and find the value of $2^{nd}$ order statistic, range and median

$$X_{(1)} = -4, X_{(2)} = 1, X_{(3)} = 5, X_{(4)} = 7, X_{(5)} = 11 \tag{5}$$

$$2^{nd}\text{Order statistic} = 1 \tag{6}$$

$$\text{Range} = R = X_{(5)} - X_{(1)} = 15 \tag{7}$$

$$\text{Median} = M = X_{(3)} = 5 \tag{8}$$



# CDF of $k^{th}$ order statistic

## Theorem (1)

*Let $\{X_1, X_2, \cdots X_n\}$ be n i.i.d random variables with common CDF $= F(x)$ and common PDF $= f(x)$, then the marginal probability distribution of $k^{th}$ order statistic (CDF) is denoted by $F_{(k,n)}(x)$ and it is given by*

$$CDF = F_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \times (F(x))^j \times (1 - F(x))^{n-j} \tag{9}$$

## Extreme cases

Equation (9) gives $X_{(1)}$ and $X_{(n)}$ as:

$$\text{Distribution of minimum} = F_{(1,n)}(x) = 1 - (1 - F(x))^n \tag{10}$$

$$\text{Distribution of maximum} = F_{(n,n)}(x) = (F(x))^n \tag{11}$$

# Proof of theorem (1):

## Theorem (1) Proof.

Assume after order statistics, the sample becomes $\{X_{(1)}, X_{(2)} \cdots X_{(n)}\}$ we need to find $F_{(k,n)}(x) = \Pr\left(X_{(k)} \leq x\right)$

$$X_{(i)} \leq X_{(k)} \leq x \; \forall i \in \{1, \cdots k-1\} \tag{12}$$

$$X_{(i)} \leq x \text{ or } X_{(i)} > x \; \forall i \in \{k+1, \cdots n\} \tag{13}$$

From the above equation (12), at least $k$ elements in $\{X_1, X_2 \cdots X_n\}$ should be $\leq x$ and remaining $(n-k)$ elements can be $\leq x$ or $> x$

$$F_{(k,n)}(x) = \Pr\left(\text{At least k elements have value} \leq x\right) \tag{14}$$

$$F_{(k,n)}(x) = \Pr\left(X_i \leq x \,:\, i \in I \,:\, I \subset \{1, \cdots n\} \, |I| \geq k\right) \tag{15}$$

# Proof of theorem (1):

## Theorem (1) Proof contd.

$$F_{(k,n)}(x) = {}^{n}\mathrm{C}_{k} \Pr\left(X \leq x\right)^{k} \Pr\left(X > x\right)^{n-k} + {}^{n}\mathrm{C}_{k+1} \Pr\left(X \leq x\right)^{k+1} \Pr\left(X > x\right)^{n-(k+1)} + \cdots\cdots {}^{n}\mathrm{C}_{n} \Pr\left(X \leq x\right)^{n} \Pr\left(X > x\right)^{0} \tag{16}$$

$$F_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_{j} \Pr\left(X \leq x\right)^{j} \Pr\left(X > x\right)^{(n-j)} \tag{17}$$

$$\therefore F_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_{j} \times \left(F(x)\right)^{j} \times \left(1 - F(x)\right)^{n-j} \tag{18}$$

$\square$

# PDF of $k^{th}$ order statistic

## Theorem (2)

*Let $\{X_1, X_2, \cdots X_n\}$ be n i.i.d random variables with common CDF $= F(x)$ and common PDF $= f(x)$, then the marginal probability density of $k^{th}$ order statistic (PDF) is denoted by $f_{(k,n)}(x)$ and it is given by*

$$PDF = f_{(k,n)}(x) = n \, {}^{n-1}\mathrm{C}_{k-1} \, f(x) \, (F(x))^{k-1} \, (1 - F(x))^{n-k} \tag{19}$$

## Extreme cases

Equation (19) gives $X_{(1)}$ and $X_{(n)}$ as:

$$\text{Density of minimum} = f_{(1,n)}(x) = n \, f(x) \, (1 - F(x))^{n-1} \tag{20}$$

$$\text{Density of maximum} = f_{(n,n)}(x) = n \, f(x) \, (F(x))^{n-1} \tag{21}$$

# Proof of theorem (2) in a simpler way:

## Theorem (2) Proof.

let us assume $X_{(k)} \in [x, x+dx]$ for $dx \to 0$. Solving using 4 sub -jobs

$$\text{Job 1} = \text{Choose } X_i = X_{(k)} \text{ and } \Pr\left(X_i \in [x, x+dx]\right) = {}^{n}\mathrm{C}_1 f(x) \tag{22}$$

$$\text{Job 2} = \text{Choose } k-1 \text{ elements from remaining} = {}^{n-1}\mathrm{C}_{k-1} \tag{23}$$

$$\text{Job 3} = \Pr\left(X \le x\right) \text{for chosen } (k-1) \text{ elements} = F(x)^{k-1} \tag{24}$$

$$\text{Job 4} = \Pr\left(X > x\right) \text{for } (n-k) \text{ elements} = (1 - F(x))^{n-k} \tag{25}$$

$$\Pr\left(X_{(k)} \in [x, x+dx]\right) = f_{(k,n)}(x) = \prod_{i=1}^{4} \text{Job i} \tag{26}$$

$$\therefore f_{(k,n)}(x) = n\, {}^{n-1}\mathrm{C}_{k-1}\, f(x) \left(F(x)\right)^{k-1} \left(1 - F(x)\right)^{n-k} \tag{27}$$

$\square$

# Uniform order statistics

## Introduction

Let $\{X_1, \cdots X_n\}$ be i.i.d form a uniform distribution on $[0, 1]$ such that $f(x) = 1$ and $F(x) = x$, from theorem (2), equation (19)

$$f_{(k,n)}(x) = n \, {}^{n-1}\mathrm{C}_{k-1} \, f(x) \, (F(x))^{k-1} \, (1 - F(x))^{n-k} \tag{28}$$

$$f_{(k,n)}(x) = n \, {}^{n-1}\mathrm{C}_{k-1} \, x^{k-1} \, (1 - x)^{n-k} \tag{29}$$

Since equation (29) is marginal probability density (PDF)

$$\int_0^1 n \, {}^{n-1}\mathrm{C}_{k-1} \, x^{k-1} \, (1 - x)^{n-k} \, dx = 1 \tag{30}$$

$$\int_0^1 x^{k-1} \, (1 - x)^{n-k} \, dx = \frac{1}{n \, {}^{n-1}\mathrm{C}_{k-1}} \tag{31}$$

# Uniform order statistics

## Introduction contd

$$\int_0^1 x^{k-1} (1-x)^{n-k} \, dx = \frac{(k-1)! \, (n-k)!}{n!} \tag{32}$$

$$\int_0^1 x^{k-1} (1-x)^{n-k} \, dx = \frac{\Gamma(k) \, \Gamma(n-k+1)}{\Gamma\left((n-k+1)+k\right)} \tag{33}$$

## Definition

let $r = k$ and $s = n - k + 1$ The **Beta function** is defined for r, s $> 0$

$$B(r,s) = \int_0^1 x^{r-1} (1-x)^{s-1} \, dx = \frac{\Gamma(r) \, \Gamma(s)}{\Gamma(r+s)} \tag{34}$$

# Beta function

## Substituting in equation (28) and (29)

By using the above definition and equation

$$f_{(k,n)}(x) = \frac{1}{B(k, n-k+1)} f(x) \left(F(x)\right)^{k-1} \left(1 - F(x)\right)^{n-k} \tag{35}$$

$$f_{(k,n)}(x) = \frac{x^{k-1} \left(1-x\right)^{n-k}}{B(k, n-k+1)} \text{ (Uniform order statistics)} \tag{36}$$

## Beta distribution

The Beta distribution is a continuous distribution defined on the range $(0, 1)$ whose PDF given by

$$f(x) = \frac{1}{B(r, s)} x^{r-1} \left(1-x\right)^{s-1} \tag{37}$$

# Beta Distribution

## Beta distribution contd

Let $B_x(r,s) = \int\limits_0^x x^{r-1}\,(1-x)^{s-1}\,dx$, CDF of Beta distribution:

$$F(x) = \int\limits_0^x f(x)\,dx = \int\limits_0^x \frac{1}{B(r,s)}\,x^{r-1}\,(1-x)^{s-1}\,dx \tag{38}$$

$$F(x) = \frac{\int\limits_0^x x^{r-1}\,(1-x)^{s-1}\,dx}{B(r,s)} = \frac{B_x(r,s)}{B(r,s)} \tag{39}$$

$$\tag{40}$$

$$\therefore \text{CDF} = F(x) = \frac{B_x(r,s)}{B(r,s)} \tag{41}$$

## Mean value of Beta distribution

$$E(x) = \int_0^1 x \times f(x)\,dx = \int_0^1 \frac{x}{B(r,s)}\,x^{r-1}\,(1-x)^{s-1}\,dx \tag{42}$$

$$= \frac{\int_0^1 x^{(r+1)-1}\,(1-x)^{s-1}\,dx}{B(r,s)} = \frac{B(r+1,s)}{B(r,s)} \tag{43}$$

$$= \frac{\Gamma(r+1)\,\Gamma(s)}{\Gamma(r+s+1)} \times \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} \tag{44}$$

$$= \frac{r!}{(r+s)!} \times \frac{(r+s-1)!}{(r-1)!} = \frac{r}{r+s} \tag{45}$$

$$\therefore \text{Mean value of X } (E(x)) = \frac{r}{r+s} \tag{46}$$

# Beta distribution

## Variance of Beta distribution

$$
\begin{aligned}
Var(x) &= E(x^2) - (E(x))^2 && (47)\\
&= \frac{\int_0^1 x^{(r+2)-1} \ (1-x)^{s-1} \ dx}{B(r,s)} - \left(\frac{r}{r+s}\right)^2 && (48)\\
&= \frac{B(r+2,s)}{B(r,s)} - \frac{r^2}{(r+s)^2} && (49)\\
&= \frac{\Gamma(r+2)\,\Gamma(s)}{\Gamma(r+s+2)} \times \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} - \frac{r^2}{(r+s)^2} && (50)\\
&= \frac{(r+1)!}{(r+s+1)!} \times \frac{(r+s-1)!}{(r-1)!} - \frac{r^2}{(r+s)^2} && (51)
\end{aligned}
$$

## Variance of Beta distribution contd

$$= \frac{r(r+1)}{(r+s)(r+s+1)} - \frac{r^2}{(r+s)^2} \tag{52}$$

$$= \frac{r(r+1)(r+s) - r^2(r+s+1)}{(r+s)^2(r+s+1)} \tag{53}$$

$$= \frac{(r^3 + r^2 s + r^2 + rs) - (r^3 + r^2 s + r^2))}{(r+s)^2(r+s+1)} \tag{54}$$

$$Var(x) = \frac{rs}{(r+s)^2(r+s+1)} \tag{55}$$

$$\therefore \text{Variance of X } (Var(x)) = \frac{rs}{(r+s)^2(r+s+1)} \tag{56}$$

# Beta Distribution

## Beta distribution summary

PDF, CDF, Expectation value and Variance of Beta distribution

$$f(x) = \frac{1}{B(r,s)} x^{r-1} (1-x)^{s-1} \tag{57}$$

$$F(x) = \frac{B_x(r,s)}{B(r,s)} \tag{58}$$

$$E(x) = \frac{r}{r+s} \tag{59}$$

$$Var(x) = \frac{rs}{(r+s)^2 (r+s+1)} \tag{60}$$

In Uniform order statistics on [0,1] the PDF of $k^{th}$ order statistic follows Beta distribution with $r = k$, $s = n - k + 1$ and PDF $f_{(k,n)}(x)$ from the above equation (36)

# Gate problem

**GATE 2021 (ST), Q.17 (STATISTICS SECTION)**

If the marginal probability density function of the $k^{th}$ order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f(x) = \begin{cases} \dfrac{7}{32} x^6 \, (2 - x)\,, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

then $k$ equals ________________

# Solution (Using Theorem(2))

**Method 1:**

The problem involves the uniform order statistics on $[0, 2]$, sample size 8, we begin the solution by finding the PDF of the given sample:

$$\int_0^2 f(x)\,dx = 1 \tag{61}$$

$$f(x) = \frac{1}{2} \tag{62}$$

CDF of the given sample:

$$F(x) = \int_0^x f(x)\,dx = \frac{x}{2} \tag{63}$$

# Solution (Using Theorem(2))

Using Theorem (2) PDF of $k^{th}$ order statistic is given by

$$f_{(k,n)}(x) = n\,{}^{n-1}\mathrm{C}_{k-1}\,f(x)\,(F(x))^{k-1}\,(1-F(x))^{n-k} \tag{64}$$

$$f_{(k,n)}(x) = n\,{}^{n-1}\mathrm{C}_{k-1}\,\frac{1}{2}\,\left(\frac{x}{2}\right)^{k-1}\,\left(1-\frac{x}{2}\right)^{n-k} \tag{65}$$

$$f_{(k,8)}(x) = \frac{8}{2^{(1+(k-1)+(8-k))}} \times {}^{7}\mathrm{C}_{k-1}\,x^{k-1}\,(2-x)^{8-k} \tag{66}$$

$$f_{(k,8)}(x) = {}^{7}\mathrm{C}_{k-1}\,\frac{1}{32}\,x^{k-1}\,(2-x)^{8-k} \tag{67}$$

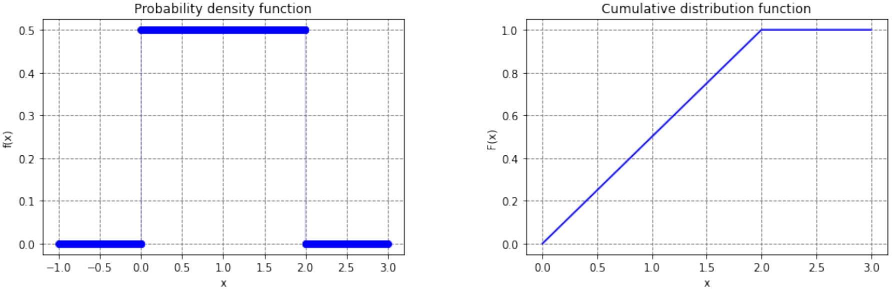
Comparing equation (67) with the given PDF of $k^{th}$ order statistic

$$\frac{{}^{7}\mathrm{C}_{k-1}}{32}\,x^{k-1}\,(2-x)^{n-k} = \frac{7}{32}\,x^{6}\,(2-x) \tag{68}$$

$$\therefore k = 7 \tag{69}$$

# PDF and CDF of Uniform Distribution

The plots of PDF, CDF of uniform statistics in equations (62) and (63):




(a) PDF $f(x)$ (b) CDF $F(x)$

Figure: PDF and CDF of given uniform statistics

# Solution(Using Beta distribution)

**Method 2:**
Since the $k^{th}$ order statistic of a uniform distribution on $[0,1]$ follows Beta distribution, convert the random variables, given PDF in $[0,1]$ range

$$\int_0^2 f(x)\,dx = \int_0^2 \frac{7}{32} x^6 \,(2-x)\,dx \tag{70}$$

$$\int_0^2 f(x)\,dx = \int_0^2 56\,\left(\frac{x}{2}\right)^6 \left(1-\frac{x}{2}\right)\,d\left(\frac{x}{2}\right) \tag{71}$$

$$\text{Let, } f_{(k,8)}(t) = 56\,t^6\,(1-t) \tag{72}$$

Let new random variable be $t$ such that $t = x/2$, New sample be $\{T_1, \cdots T_8\}$ such that $T_i = X_i/2$.

# Solution(Using Beta distribution)

The Uniform distribution of new random sample is on $[0, 1]$ such that $f(t) = 1$ and $F(t) = t$

$$\int_0^2 f(x)\,dx = \int_0^1 56\,t^6\,(1-t)\,dt = \int_0^1 f(t)\,dt \tag{73}$$

Given $k^{th}$ order statistic (after conversion)

$$f_{(k,8)}(t) = \begin{cases} 56\,t^6\,\,(1-t)\,, & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{74}$$

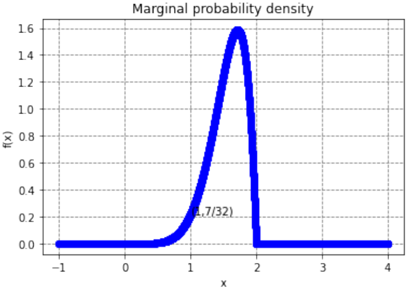
Since equation (74) is a Beta distribution with $r = k$, $s = n - k + 1$

$$k - 1 = 6 \tag{75}$$

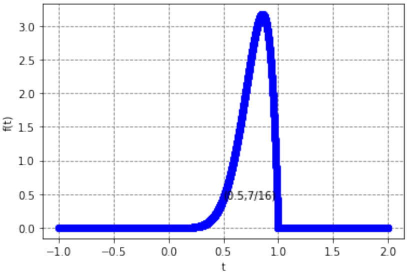
$$\therefore k = 7 \tag{76}$$

# Plots of PDF of $k^{th}$ order statistic

The plots for given PDF and PDF in equation (74) are shown below:




(a) PDF of $f_{(7,8)}(x)$




(b) PDF of $f_{(7,8)}(t)$

Figure: Plots of $k^{th}$ order statistic

# THANK YOU

**Assignment link for reference:**

https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment4