28/01/2021

# CS 1010 Discrete Structures Lecture 10: Counting Techniques Contd

Maria Francis

January 28, 2021

# Arrangements

- In how many ways can we select 3 students from a group of 5 students to stand in line for a picture?
  - Order in which students are selected matters.
  - Five ways to select the first student to stand at the start of the line, there are four ways to select the second student in the line, there are three ways to select the third student in the line, by the product rule $5 \cdot 4 \cdot 3 = 60$ ways to select 3 students from a group of 5.
- In how many ways can we arrange all five of these students in a line for a picture?
  - We select the first student in five ways,the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way.I.e. there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

# Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of $r$ elements of a set is called an $r$-permutation.
- The number of $r$-permutations of a set with $n$ elements is denoted by $P(n, r)$.

Theorem

*If $n$ is a positive integer $r$ is an integer with $1 \leq r \leq n$ then there are*

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1),$$

*$r$-permutations of a set with $n$ distinct elements.*

# $P(n, r)$

- Product rule is our tool.
- The first element of the permutation can be chosen in $n$ ways because there are $n$ elements in the set.
- There are $n-1$ ways to choose the second element because there are $n-1$ elements left in the set, similarly there are $n-2$ ways to choose the third element, and so on until there are exactly $n-(r-1)=n-r+1$ ways to choose the $r$th element.
- Therefore by product rule, there are $n(n-1)(n-2)\cdots(n-r+1)$ $r$-permutations of the set.
- $P(n,0)=1$ whenever $n$ is a nonnegative integer because there is exactly one way to order zero elements. I.e, there is exactly one list with no elements in it, namely the empty list.

# Formula for $P(n, r)$

- If $n$ and $r$ are integers with $0 \leq r \leq n$ then $P(n, r) = \frac{n!}{(n-r)!}$.
- $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$ which is equal to $n!/(n-r)!$.
- Because $n!/(n-0)! = n!/n! = 1$, the formula holds also when $r = 0$.
- Example: How many permutations of the letters ABCDEFGH contain the string ABC ?
  - Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of 6 objects, i.e. the block ABC and the individual letters D, E, F , G, and H .
  - Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.

# Combinations

- How many different committees of 3 students can be formed from a group of 4 students?
  - Here order doesn't matter, we need to find only the number of subsets with 3 elements from the set containing the 4 students.
  - There are four such subsets,one for each of the 4 students, because choosing three students is the same as choosing one of the four students to leave out of the group.
- An $r$-combination of elements of a set is an unordered selection of $r$ elements from the set.
- An $r$-combination is simply a subset of the set with $r$ elements.

# Combinations

- The number of $r$-combinations of a set with $n$ distinct elements is denoted by $C(n,r)$ - binomial coefficient, $\binom{n}{r}$.
- The number of $r$-combinations of a set with $n$ elements where $n$ is a non-negative integer and $r$ is an integer with $0 \leq r \leq n$ equals,

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

# Combinations

- The $P(n,r)$ $r$-permutations of the set can be obtained by forming the $C(n,r)$ $r$-combinations of the set and then ordering the elements in each $r$-combinations which can be done in $P(r,r)$ ways.
- Consequently by the product rule,

$$P(n,r) = C(n,r) \cdot P(r,r).$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}.$$

- We can also use division rule.

# Combinations

- Important Corollary : Let $n$ and $r$ be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.
- Proof: direct application of factorials property.
- How many bit strings of length n contain exactly $r$ 1s?
- The positions of $r$ 1s in a bit string of length $n$ form an $r$-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence there are $C(n, r)$ bit strings of length $n$ that contain exactly $r$ 1s.

# Combinatorial Proofs

- The previous corollary can also be proved using a combinatorial proof. A proof of an identity that uses counting arguments to prove that both sides of the identity count the same objects but in different ways
- Or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity.
- Also called double counting proofs and bijective proofs.
- T.S.T. $C(n, r) = C(n, n - r)$.
  - Suppose $S$ is a set with $n$ elements.
  - A function that maps a subset $A$ of $S$ to $\bar{A}$ is a bijection between subsets of $S$ with $r$ elements and $n - r$ elements. Verify!
  - Since its a bijection between two finite sets, the two sets should have the same number of elements and therefore we have the identity.

# Binomial Coefficients

- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.
- A binomial expression is simply the sum of two terms $x + y$. The terms can be products of constants and variables but there is a sum that separates them.
- Expand $(x + y)^3$ using combinatorial method.
  - Terms of the form $x^3$, $x^2y$, $xy^2$ and $y^3$ arise.
  - To get $x^3$, an $x$ must be chosen from each of the sums $(x + y)(x + y)(x + y)$, this can be done only in one way. Thus the coefficient of $x^3$ (and also $y^3$) is 1.
  - To obtain a term $x^2y$ an $x$ must be chosen in two of the three sums and therefore $\binom{3}{2}$ and the $y$ comes from the other sum.
  - As for $xy^2$ - it can be considered as the number of ways to pick one $x$ from three sums or a $y$ from two of the three sums $C(3, 1) = C(3, 2)$.
  - Therefore $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

# Binomial Theorem

- Let $x$ and $y$ be variables and let $n$ be a nonnegative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j$$
$$= C(n,0)x^n + C(n,1)x^{n-1}y +$$
$$\cdots + C(n,n-1)xy^{n-1} + C(n,n)y^n.$$

- The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$.
- Note that to obtain such a term, $x^{n-j}y^j$, it is necessary to choose $n-j$ $x$s from the $n$ sums so that the other $j$ terms in the product are $y$s.
- The coefficient of $x^{n-j}y^j$ is $C(n, n-j)$ which is equal to $C(n,j)$.

# Examples

- Coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- From the binomial theorem we have this coefficient is $C(25,13) = \frac{25!}{13!12!}$.
- Coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?
- This is the same as $(2x+(-3y))^{25}$.
- From the binomial theorem we have this coefficient is $C(25,13)2^{12}(-3)^{13} = -\frac{25!}{13!12!} \cdot 2^{12} \cdot 3^{13}$.

# Identities

- Let $n \in \mathbb{N}$.

$$\sum_{k=0}^{n} C(n,k) = 2^n.$$

  - Using binomial theorem, with $x = 1$ and $y = 1$ we see that $2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k) 1^k 1^{n-k} = \sum_{k=0}^{n} C(n,k)$.
- There is also a combinatorial proof using number of subsets.

# Identities

- Let $n \in \mathbb{N}$.

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0.$$

- Consider $x = -1, y = 1$, we see that $0 = 0^n = ((-1)+1)^n$ $= \sum_{k=0}^{n} C(n,k)(-1)^k 1^{n-k} = \sum_{k=0}^{n} C(n,k)(-1)^k$.
- This also implies that $C(n,0) + C(n,2) + C(n,4) + \cdots = C(n,1) + C(n,3) + C(n,5) + \cdots$.

# Identities

- Let $n \in \mathbb{N}$.

$$\sum_{k=0}^{n} 2^k C(n,k) = 3^n.$$

- It is the expansion of $(1+2)^n$.

# Pascal's Identity

- The binomial coefficients satisfy many different identities.
- Let $n$ and $k$ be positive integers with $n \geq k$. Then,

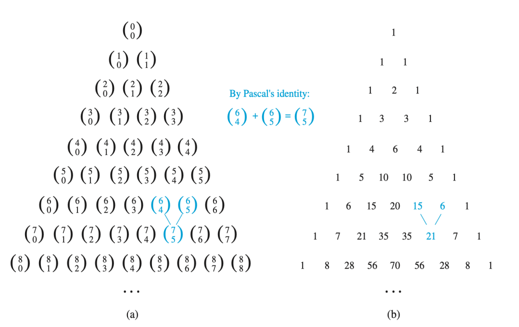
$$C(n+1, k) = C(n, k-1) + C(n, k).$$

- Let $T$ be a set with $n+1$ elements.
- Let $a \in T$ and let $S = T \setminus \{a\}$.
- There are $C(n+1, k)$ subsets of $T$ containing $k$ elements.
- A subset of $T$ with $k$ elements either contains $a$ together with $k-1$ elements of $S$ or contains $k$ elements of $S$ and does not contain $a$.
- There are $C(n, k-1)$ subsets of $S$ with $k-1$ elements, $\rightarrow$ there are $C(n, k-1)$ subsets of $k$ elements of $T$ that contain $a$.
- There are $C(n, k)$ subsets of $k$ elements of $T$ with no $a$, because there are $C(n, k)$ subsets of $k$ elements of $S$.

# Pascal's Identity

- Pascal's identity + $C(n, 0) = C(n, n) = 1$ defines binomial coefficients recursively.
- Useful in the computation of binomial coefficients because we only need addition, and not multiplication, of integers.

# Pascal's Triangle




The $n$th row: $C(n, k), k = 0, 1, 2, \ldots, n$. When two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

# Vandermonde's Identity

Let $m, n, r \in \mathbb{N}$ with $r \leq m, n$.
$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$

- Suppose that there are $m$ items in one set and $n$ items in a second set.
- The total number of ways to pick $r$ elements from the union of these sets is $C(m+n, r)$.

# Vandermonde's Identity

- Another way to pick $r$ elements from the union is to pick $k$ elements from the second set and then $r - k$ elements from the first set.
- $C(n, k)$ ways to choose $k$ elements from the second set and $C(m, r - k)$ ways to choose $r - k$ elements from the first set,
- The product rule tells us this can be done in $C(m, r - k)C(n, k)$ ways.
- Two expressions for the number of ways to pick $r$ elements from the union of a set with $m$ items and a set with $n$ items.

# Other Identities

If $n \in \mathbb{N}$, $C(2n, n) = \sum_{k=0}^{n} C(n, k)^2$.

- $m = r = n$, we get
  $C(2n, n) = \sum_{k=0}^{n} C(n, n-k) C(n, k) = \sum_{k=0}^{n} C(n, k)^2$.

# Other Identities

Let $n, r \in \mathbb{N}$ with $r \leq n$.
$C(n+1, r+1) = \sum_{j=r}^{n} C(j, r)$.

- LHS counts the bit strings of length $n+1$ containing $r+1$ ones.
- RHS counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.
- This final one must occur at position $r+1, r+2, \ldots, n+1$.

# Other Identities

Let $n, r \in \mathbb{N}$ with $r \leq n$.
$C(n+1, r+1) = \sum_{j=r}^{n} C(j, r)$.

- If the last one is the $k$th bit there must be $r$ ones among the first $k-1$ positions, there are $C(k-1, r)$ such bit strings.
- Last bit $k$ can be chosen from any value between $r+1$ and $n+1$, so we have

$$\sum_{k=r+1}^{n+1} C(k-1, r) = \sum_{j=r}^{n} C(j, r)$$

  bit strings of length $n$ containing exactly $r+1$ ones.
- The last summation you get from setting $j = k - 1$.

# Generalized Permutations and Combinations

- In many counting problems, elements may be used repeatedly. For eg: a letter or digit may be used more than once on a license plate.
- Also, some counting problems involve indistinguishable elements. For eg, to count the number of ways the letters of the word SUCCESS can be rearranged, the placement of identical letters must be considered.

# Permutations with Repetition

- How many strings of length $r$ can be formed from the uppercase letters of the English alphabet?
  - By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are $26^r$ strings of uppercase English letters of length $r$.
- The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$.
  - There are $n$ ways to select an element of the set for each of the $r$ positions in the $r$-permutation when repetition is allowed, because for each choice all $n$ objects are available.
  - By the product rule there are $n^r$ $r$-permutations when repetition is allowed.

# Combinations with Repetition

- Let $S = \{A, B, C\}$, there are three ways to choose two distinct elements of $S$ where order does not matter - $\{A, B\}, \{A, C\}, \{B, C\}$.
- But what if we are not required to choose distinct elements of $S$ but can choose same element repeatedly.
- The resulting sets are called the *r*-combinations *with repetition* of the set $S$. They are $\{A, B\}, \{A, C\}, \{B, C\}, \{A, A\}, \{B, B\}, \{C, C\}$.
- They are actually multisets/bags.

# Combinations with Repetition

- The number of $r$-combinations with repetition of an $n$-element set is $C(n+r-1, r)$.
- The proof of this theorem uses an important trick called stars and bars.
- Let $S$ be a set with $n$ elements that are ordered in some way.
- We will establish a bijection between $r$-combinations with repetition of the set $S$ and string of stars and bars.

# Combinations with Repetition

- Let $R$ be a particular $r$-combination with repetition of $S$.
- Write down $n-1$ bars. These $n-1$ bars divide the line into $n$ regions.
- Put one star in the $i$th region for each time that the $i$th element of $S$ appears in $R$.
- We now have a mapping from a $r$-combination with repetition to a string with $r$ stars and $n-1$ bars.
    - $S = \{A, B, C, D, E\}$ with elements ordered alphabetically.
    - Let $R$ be the 7-combination with repetition $\{A, B, B, B, D, E, E\}$.
    - The correspondence to $R$ is as follows: $*|***||*|**$.
    - The two bars with no stars between indicate that element $C$ never appears in $R$.

# Combinations with Repetition

- The mapping is a bijection because it has an inverse,i.e. given any stars and bars string we can construct the corresponding $r$-combination with repetition.
- The number of stars in the first region determines the number of times that the first element of $S$ appears in the $r$-combination,the stars in the second region determine the number of times that the second element appears, etc.
- I.e the number of $r$-combinations with repetition of an $n$-element set is equal to the number of strings containing $n-1$ bars and $r$ stars.
- The number of such strings is equal to the number of ways to choose $r$ distinct positions for the stars in a string of $n+r-1$ possible positions, i.e. number of ordinary $r$-combinations of a set with $n+r-1$ elements which is $C(n+r-1, r)$.

# Examples

Baskin-Robbins is an ice cream store that has 31 different flavors. How many different triple-scoop ice cream cones are possible at Baskin-Robbins? Two ice cream cones are considered the same if one can be obtained from the other by reordering the scoops. Of course, we are permitted to have two or even three scoops of the same flavor.

- We are looking for precisely the number of 3-combinations with repetition of the set of 31 flavors.
- $n = 31$ and $r = 3$. This gives: $C(31 + 3 - 1, 3) = C(33, 3)$.

# Examples

How many solutions does the equation $x_1 + x_2 + x_3 = 11$, have where $x_1, x_2$ and $x_3$ are nonnegative integers?

- Note that a solution corresponds to a way of selecting 11 items from a set with three elements so that $x_1$ items of type one, $x_2$ items of type two and $x_3$ items of type three are chosen.
- The number of solutions is equal to the number of 11-combinations with repetition allowed from a set with 3 elements.
- Therefore we have $C(3 + 11 - 1, 11) = C(13, 11)$ solutions.

# Permutations with Indistinguishable Objects

- The number of different permutations of $n$ objects, where there are $n_1$ indistinguishable objects of type 1, $n_2$ indistinguishable objects of type 2, ..., and $n_k$ indistinguishable objects of type $k$, is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- The $n_1$ objects can be placed among the $n$ positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions free.
- Then the objects of type two can be placed in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions free.
- Continue placing the objects until at the last stage $n_k$ objects of type $k$ can be placed in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.

# Permutations with Indistinguishable Objects

- By product rule,

$$
\begin{aligned}
&C(n, n_1)C(n-n_1, n_2)\cdots C(n-n_1-\cdots-n_{k-1}, n_k)\\
&= \frac{n!}{n_1!(n-n_1)!}\,\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{n-n_1-\cdots-n_{k-1}!}{n_k!0!}\\
&= \frac{n!}{n_1!n_2!\cdots n_k!}.
\end{aligned}
$$