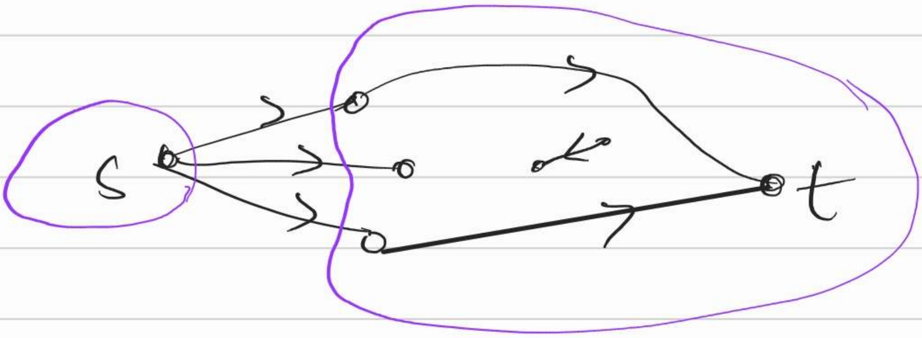
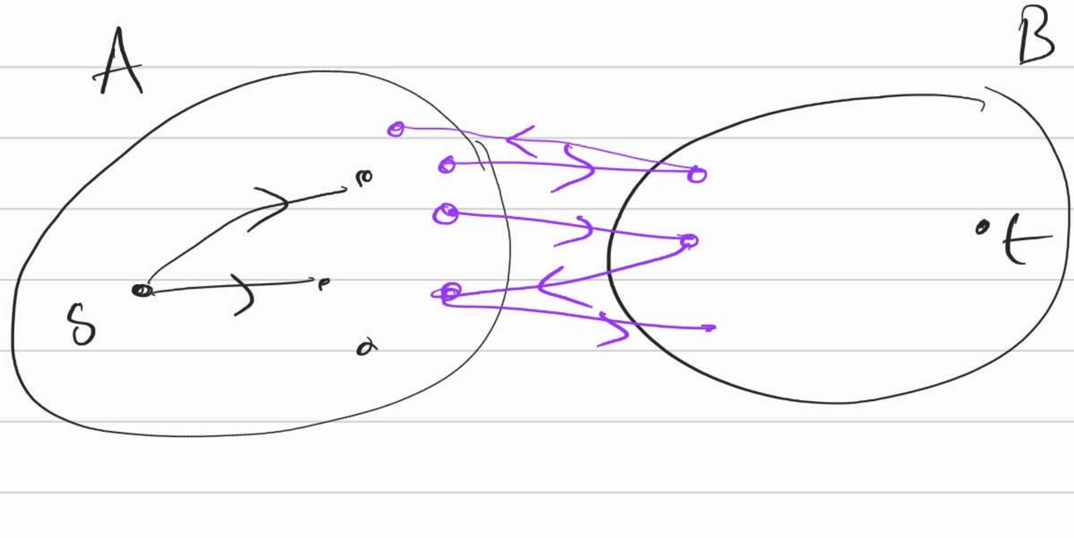
07/04/2022

**Obs 1:** $$v(f) \le \sum_{e \text{ out of } s} c(e)$$




**Defn:-** A s–t cut is a partition of vertices into two sets A, B s.t. $s \in A$ and $t \in B$.




Any flow has to cross the cut (A, B).

$$v(f) \le \sum_{e \text{ going out of } A} c(e)$$

Defn:- For any s-t cut $(A, B)$ the capacity of the cut, denoted $c(A, B)$, is equal to $\sum_{e \text{ going from } A \text{ to } B} c(e)$

Min-cut problem: Find a s-t cut of minimum capacity.

Defn: Let $A \subseteq V$. Then for any flow f define $f^{out}(A) = \sum_{e \text{ going out of } A} f(e)$

and $f^{in}(A) = \sum_{e \text{ coming into } A} f(e)$.

For example. $f^{out}(s) = \nu(f)$

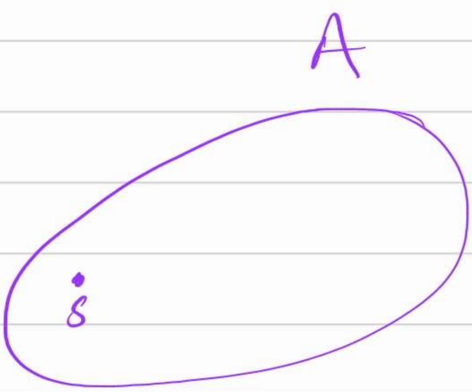
$f^{in}(s) = 0.$

$f^{out}(t) = 0 \qquad f^{in}(t) = v(f)$

<u>Lemma:</u> Let $f$ be a flow and $(A, B)$ be any s-t cut.

Then $v(f) = f^{out}(A) - f^{in}(A)$

<u>Proof:</u>




$f^{out}(s) = v(f)$ and $f^{in}(s) = 0$

$v(f) = f^{out}(s) - f^{in}(s)$

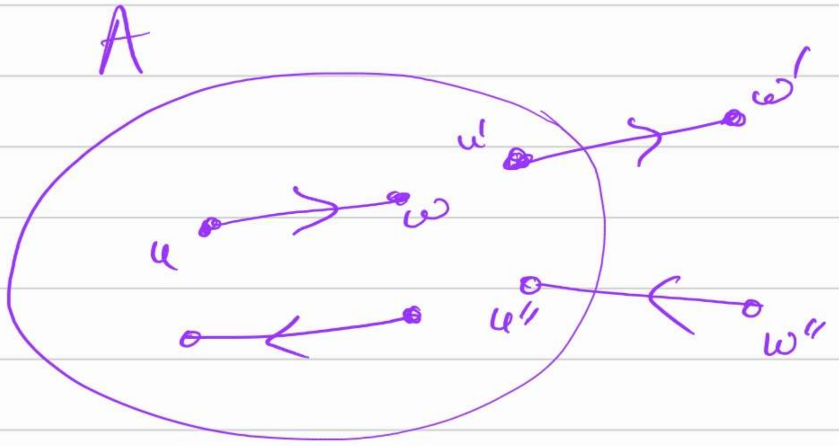
Consider the following sum.

$$\sum_{u \in A} \left(f^{out}(u) - f^{in}(u)\right)$$

Since $s \in A$, $f^{out}(s) - f^{in}(s) = v(f)$

for $u \in A$, $u \neq s$, $f^{out}(u) - f^{in}(u) = 0$

$$v(f) = \sum_{u \in A} \left(f^{out}(u) - f^{in}(u)\right)$$




if both the endpoint of edge are inside A

$$+f(e) - f(e) = 0$$

if only the tail of the edge lies inside A

$+f(e)$

if only the head of the edge lies inside A

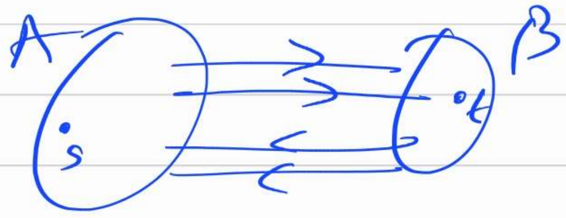
$-f(e)$

$$\sum_{u \in A} \left(f^{out}(A) - f^{in}(A)\right) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= f^{out}(A) - f^{in}(A)$$

Lemma :- $v(f) = f^{in}(B) - f^{out}(B)$

Proof :- $f^{out}(A) = f^{in}(B)$




$$f^{in}(A) = f^{out}(B)$$

$$v(f) = f^{out}(A) - f^{in}(A) = f^{in}(B) - f^{out}(B)$$

<u>Lemma</u>: Let $f$ be any flow and $(A, B)$ be any s-t cut.

Then $v(f) \leq c(A, B)$.

<u>Proof</u>:- $v(f) = f^{out}(A) - f^{in}(A)$

$$\leq f^{out}(A)$$

$$= \sum_{e \text{ out of } A} f(e)$$

$$\leq \sum_{e \text{ out of } A} c(e) = c(A, B)$$

Suppose there is a cut of capacity $K$.

$\Rightarrow$ maximum-flow $\leq K$.

Suppose $\exists$ a flow of value $v$.

then can the capacity of a cut $< v$?

NO!

Now Suppose $f$ is a flow s.t.

$$v(f) = c(A^*, B^*)$$

where $(A^*, B^*)$ is an $st$-cut.

Thm: If $f$ is a $s$-$t$ flow such that there is no $s$-$t$-path in the residual graph $G_f$.

Then $\exists$ a $s$-$t$-cut $(A^*, B^*)$

s.t. $v(f) = c(A^*, B^*)$.

Proof:- Consider $G_f$

$\nexists$ no $s \rightsquigarrow t$ path in $G_f$.

Define $A^*$ to be the set of vertices which are reachable from $s$ in $G_f$.

Clearly $s \in A^*$. Does $t \in A^*$?

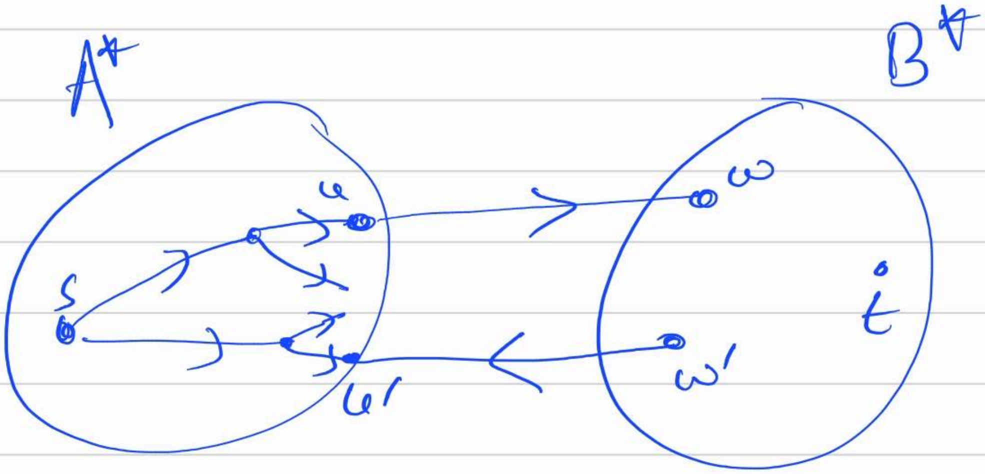
No.

Define $B^* = V \setminus A^*$

Clearly $t \in B^*$.

<u>Claim</u>: $v(f) = c(A^*, B^*)$

Consider $G_f$




$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

Consider an edge $e = (u, w) \in G$
where $u \in A^*$ and $w \in B^*$

What is the value of $f(e)$?

$$0 \leq f(e) \leq c(e)$$

Can $f(e) < c(e)$ ?

Suppose it is the case that $f(e) < c(e)$.

then consider the residual graph $G_f$

$$u \in A^* \xrightarrow{c(e) - f(e) > 0} w \in B^*$$

then the edge from $u$ to $w$ will be present in $G_f$ with residual capacity $c(e) - f(e)$.

But this is a contradiction to the fact that $w \in B^*$.

$$s \rightsquigarrow u \to w$$

This implies that $f(e) = c(e)$.

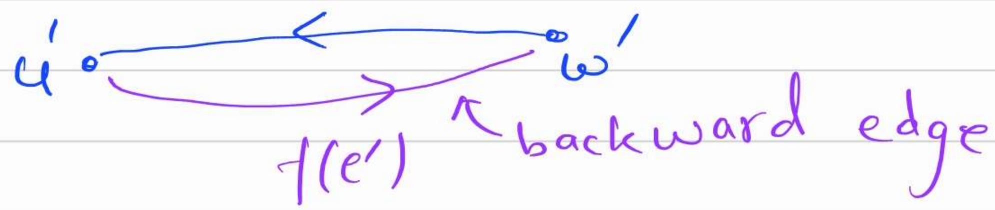
for an edge $e$ going from $A^*$ to $B^*$.

$e' = (w', u')$


what is the value of $f(e')$ ?

Can $f(e') > 0$ ?

Suppose it does, then consider the residual graph $G_f$.




$\Rightarrow$ $w'$ is reachable from $s$

$\Rightarrow$ which is a contradiction to the fact that $w' \in B^*$,

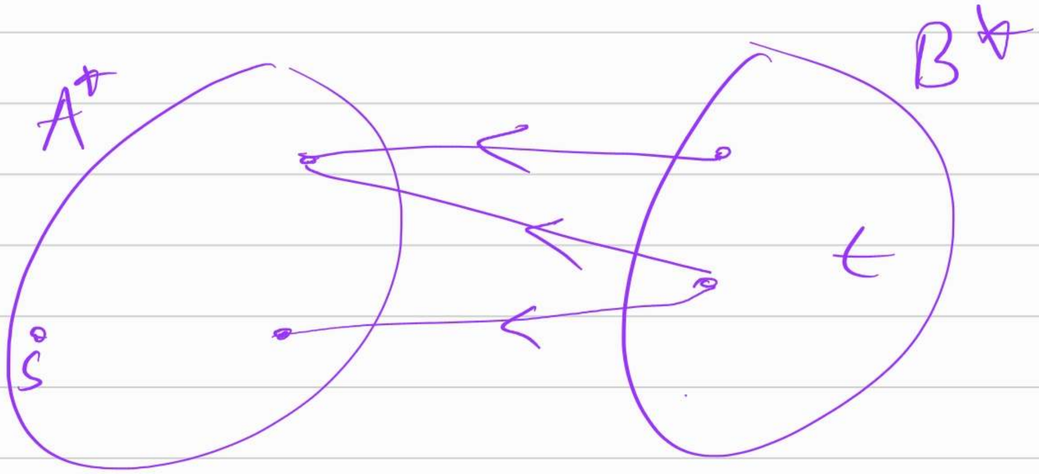
$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

$$= \sum_{e \text{ going out of } A^*} f(e) - \sum_{e \text{ coming into } A^*} f(e)$$

$$= \sum_{e \text{ going out of } A^*} c(e) - 0$$

$$= c(A^*, B^*).$$

residual graph




**Max-flow- Min- Cut thm**

In any flow network, the maximum flow is equal the minimum capacity of a s-t- cut.

Q: Given a maximum flow $f$,

Can you find a minimum cut $(A^*, B^*)$?

Algo: Construct $G_f$.

Set $A^*$ = the vertices reachable from $s$ in $G_f$.

$B^* = V \setminus A^*$.

You can implement it in $O(m+n)$