Convergence Tests

$$\int_1^\infty \frac{\sin x}{x^2}\,dx$$

$$\lim_{x\to\infty} \int_1^x \frac{\sin t}{t^2}\,dt.$$

Direct Comparison test:

Assume that the integral $\int_a^b f(x)\,dx$ exists for all $b \geq a$ and $0 \leq f(x) \leq g(x)$ for all $x \in [a,\infty)$. Then

$\int_a^\infty f(t)\,dt$

$\int_a^\infty g(x)\,dx$ converges $\Longrightarrow$ $\int_a^\infty f(x)\,dx$ converges.

$\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow$ $\int_a^\infty g(x)\,dx$ diverges.

Examples:

① $\int_1^\infty \frac{\sin x}{x^2}\,dx$ convergent by comparison test.

$\Rightarrow \frac{\sin x}{x^2} \leq \frac{1}{x^2}$. $\int_1^\infty \frac{1}{x^2}\,dx$ Converges

② $\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\,dx$ divergent.

$x^2 - 0.1 \leq x^2$.

$\Rightarrow \sqrt{x^2-0.1} \leq \sqrt{x^2}$

$\Rightarrow \frac{1}{x} = \frac{1}{\sqrt{x^2}} \leq \frac{1}{\sqrt{x^2-0.1}}$

$\int_1^\infty \frac{1}{x}\,dx$ diverges

Limit comparison test:

Assume that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist $\forall\, b \geq a$, where $f(x) \geq 0$ and $g(x) > 0$ $\forall\, x \geq a$.

If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c$ where $c \neq 0$. $c \in \mathbb{R}$.

then either both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge.

or both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ diverge.

Examples

① $\int_1^\infty \frac{1}{1+x^2}\,dx$

$f(t) = \frac{1}{1+t^2}$

$g(t) = \frac{1}{t^2}$.

$\int_1^\infty \frac{1}{t^2}\,dt$ convergent

$t \geq 1$
$\Downarrow$
$f(t) \geq 0$
$g(t) > 0$

$$\frac{f(t)}{g(t)} = \frac{\frac{1}{1+t^2}}{\frac{1}{t^2}} = \frac{t^2}{1+t^2}.$$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t^2}{1+t^2}$$

$$= \lim_{t\to\infty} \frac{1}{1+\frac{1}{t^2}} = 1. \neq 0$$

② $\int_1^\infty \frac{3}{e^x+5}\,dx$. convergent!!.

$f(t) = \frac{3}{e^t+5}$ $\quad g(t) = \frac{1}{e^t}$.

$\underline{\underline{\text{Exc}}}$ $\int_1^\infty \frac{1}{e^t}\,dt$ is convergence.

$$\frac{f(t)}{g(t)} = \frac{\frac{3}{e^t+5}}{\frac{1}{e^t}} = \frac{3e^t}{e^t+5}.$$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{3e^t}{e^t+5} = \lim_{t\to\infty} \frac{3}{1+\frac{5}{e^t}} = 3 \neq 0$$