Integral Calculus

Riemann Integration

Riemann Condition:

(Running assumption: $f : [a,b] \to \mathbb{R}$ bounded funct.)

f is integrable on $[a,b]$

$\iff$ For every $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ such that

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$$

Pf:

$(\Rightarrow)$ Suppose that f is integrable. Let $\varepsilon > 0$ be given.

There is a partition P_1 such that

$U(P_1, f) < U(f) + \frac{\varepsilon}{2}$ ✓

[$U(f) = \inf\{U(P,f) \mid P \text{ is a partition of } [a,b]\}$

$\Rightarrow U(f) + \frac{\varepsilon}{2}$ is not a lower bound for $\{U(P,f) \mid P \text{ is a partition of } [a,b]\}$]

Similarly, there exists a partition P_2 s.t.

$L(P_2, f) > L(f) - \frac{\varepsilon}{2}$ ✓

$L(f)$

Take $P_\varepsilon := P_1 \cup P_2$. Now P_ε is a refinement of P_1 and P_2

$\Rightarrow L(P_\varepsilon, f) \geq L(P_2, f)$

$U(P_\varepsilon, f) \leq U(P_1, f)$

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) \leq U(P_1, f) - L(P_2, f)$$
$$< U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2}$$
$$= \underbrace{U(f) - L(f)}_{0} + \varepsilon$$
$$= \varepsilon.$$

Since f is integrable
$L(f) = U(f)$

$(\Leftarrow)$ Given any $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ such that

$0 \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$

Now, $U(f) \leq U(P_\varepsilon, f)$ and $L(f) \geq L(P_\varepsilon, f)$.

$\Rightarrow U(f) - L(f) \leq U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$

$\rightsquigarrow$ For every $\varepsilon > 0$, $0 \leq U(f) - L(f) < \varepsilon.$

$\Rightarrow U(f) - L(f) = 0$

$\Rightarrow U(f) = L(f).$ $\square$

$0 \leq a < \varepsilon$
$\forall \varepsilon > 0$

$\Rightarrow a = 0$

f is integrable.

(Running assumption: $f : [a,b] \to \mathbb{R}$ bounded funct.)

Corollary:

f is integrable on $[a,b]$ $\Leftrightarrow$ there is a sequence (P_n) of partitions of $[a,b]$ such that $U(P_n, f) - L(P_n, f) \to 0$ as $n \to \infty$. In such a case $L(P_n, f) \to \int_a^b f$ and $U(P_n, f) \to \int_a^b f$.

Proof:

($\Rightarrow$) Suppose f is integrable. By Riemann's Condition, given $\varepsilon > 0$, there exists a partition P_ε such that $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon = \frac{1}{n}$

take $\varepsilon = \frac{1}{n}$.

For $n = 1, 2, \ldots$

For each n, there exists a partition P_n s.t. $U(P_n, f) - L(P_n, f) < \frac{1}{n} \to 0$ as $n \to \infty$.

$\forall n$.

($\Leftarrow$) There is a sequence (P_n) of partitions of $[a,b]$ such that $U(P_n, f) - L(P_n, f) \to 0$ as $n \to \infty$

$\Rightarrow$ Given $\varepsilon > 0$, there exists an $n_0 \geq 0$ s.t. $U(P_n, f) - L(P_n, f) < \varepsilon$ whenever $n \geq n_0$

$a_n \to 0$, $n \to \infty$

Put $P_\varepsilon = P_{n_0}$ to get the Riemann condition.

$\Rightarrow f$ is integrable on $[a,b]$

To show that $L(P_n, f) \to \int_a^b f$.

Let (P_n) be a sequence of partitions such that $U(P_n, f) - L(P_n, f) \to 0$.

Now

$$0 \leq L(f) - L(P_n, f) \leq U(f) - L(P_n, f) \leq U(P_n, f) - L(P_n, f)$$

$$U(P_n, f) - L(P_n, f) \to 0.$$

$\Rightarrow L(P_n, f) \to L(f) = \int_a^b f$.

Exercise: $\boxed{U(P_n, f) \to \int_a^b f.}$

Exercise!!

Example: $f(x) = x$ for $x \in [a,b]$.

Take a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$. $[x_{i-1}, x_i]$ $i=1,\dots,n$.

$f(x)$ is an increasing function.

$$\Rightarrow \begin{rcases} m_i(f) = x_{i-1} \\ M_i(f) = x_i \end{rcases} \forall i \Rightarrow \begin{cases} L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \\ \leq \\ U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}). \end{cases}$$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (x_i - x_{i-1})^2.$$

$$U(P,f) + L(P,f) = \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) = x_n^2 - x_0^2 = b^2 - a^2.$$

$$\Rightarrow \quad U(P,f) = \frac{(b^2-a^2)}{2} + \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})^2$$

$$L(P,f) = \frac{(b^2-a^2)}{2} - \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})^2$$

For $n \in \mathbb{N}$, let P_n denote the partition of $[a,b]$ into n equal parts. $x_i - x_{i-1} = \frac{b-a}{n}$.

$$\Rightarrow \quad U(P_n,f) = \frac{b^2-a^2}{2} + \frac{1}{2}\sum_{i=1}^{n}\frac{(b-a)^2}{n^2} \to \frac{b^2-a^2}{2}$$

$$L(P_n,f) = \frac{b^2-a^2}{2} - \frac{1}{2}\sum_{i=1}^{n}\frac{(b-a)^2}{n^2} \to \frac{b^2-a^2}{2}$$

as $n \to \infty$

$\to 0$

$$U(P_n,f) - L(P_n,f) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \begin{cases} f \text{ is integrable} \\ \int_a^b f = \lim_{n\to\infty} L(P_n,f) = \frac{b^2-a^2}{2}. \ \square \end{cases}$$

Results on integrable functions (proofs omitted).

If $f : [a,b] \to \mathbb{R}$ is monotone, then f is integrable.

$f(x) = x$
$f(x) = x^2$.

Examples:

(a) $f(x) = [x]$. ↑ for $x \in [a,b]$.

(b) $g(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0 & \text{if } x = 0 \end{cases}$ ↓

$g : [0,1] \to \mathbb{R}$.

For $x \in [0,1]$

Exercise!! there exists $n \in \mathbb{N}$.

$\frac{1}{n+1} < x \leq \frac{1}{n}$.

Produce 10 more examples!!

If $f : [a,b] \to \mathbb{R}$ (bounded) has at most a finite number of discontinuities, then f is integrable.

Examples:

(a) Any continuous function is integrable.

(b) A polynomial is integrable.

(c) ....

(d) ....

10 more examples!!

Algebraic Properties

Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions, $c \in \mathbb{R}$ a constant.

ⓐ $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

ⓑ cf is integrable and $\int_a^b cf = c\int_a^b f$.

ⓒ $f \cdot g$ is integrable. $\left(\int_a^b fg \neq \int_a^b f \cdot \int_a^b g.\right)$

ⓓ If there exists $\delta > 0$ s.t. $|f(x)| \geq \delta$ $\forall x \in [a,b]$, then $\frac{1}{f}$ is integrable.

$\Rightarrow \frac{1}{f}$ is bounded.

Order properties

ⓐ If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

ⓑ $|f|$ is integrable and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Area of planar regions

$f : [a,b] \to \mathbb{R}$ a bounded function. $\quad f(x) \geq 0 \quad \forall x \in [a,b]$

If $f \geq 0$, then we say that the region R_f given by

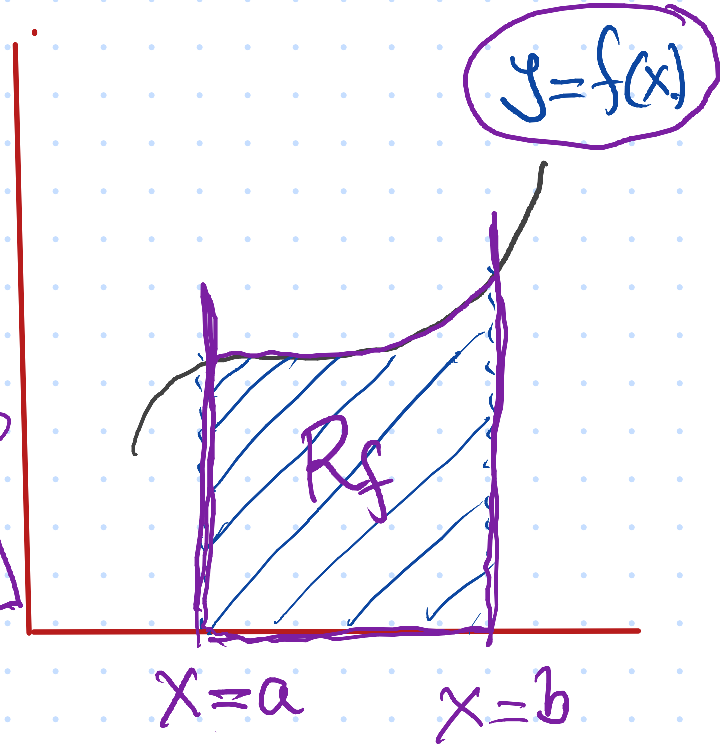
$$R_f = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

has an **area** if f is integrable on $[a,b]$ and in such a case we define

$$\text{Area}(R_f) = \int_a^b f(x)\,dx.$$

$$\int_a^b f \geq \boxed{\int_a^b 0 = 0}$$

$f(x) = c = 0$
$\forall x \in [a,b]$



Note:

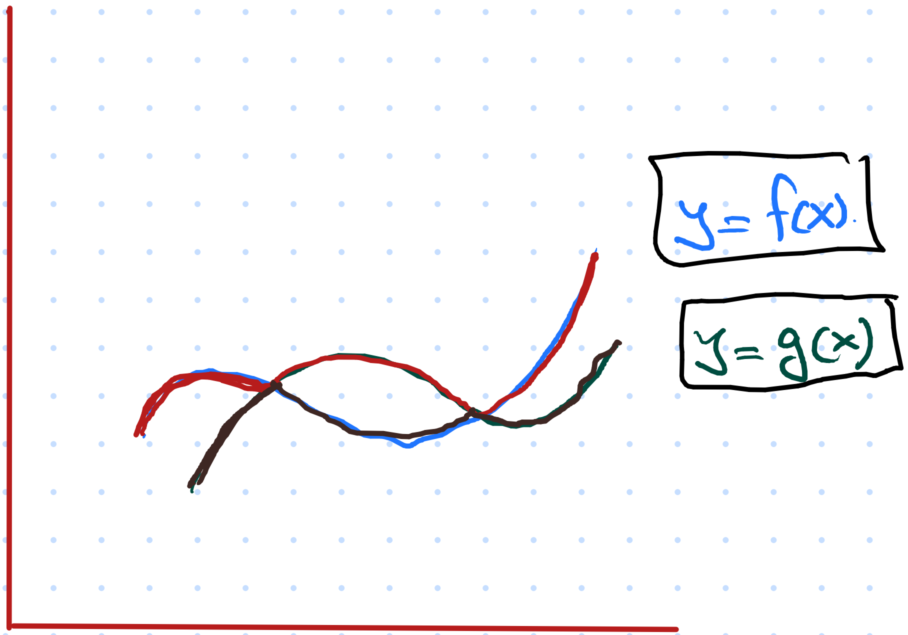
$\text{Area}(R_f) \geq 0.$

What happens if $f \not\geq 0$?

Note: If f, g are integrable, then so are $\max(f,g)$ and $\min(f,g)$.

$$\max(f,g) = \frac{f+g+|f-g|}{2}.$$

$$\min(f,g) = \frac{f+g-|f-g|}{2}$$



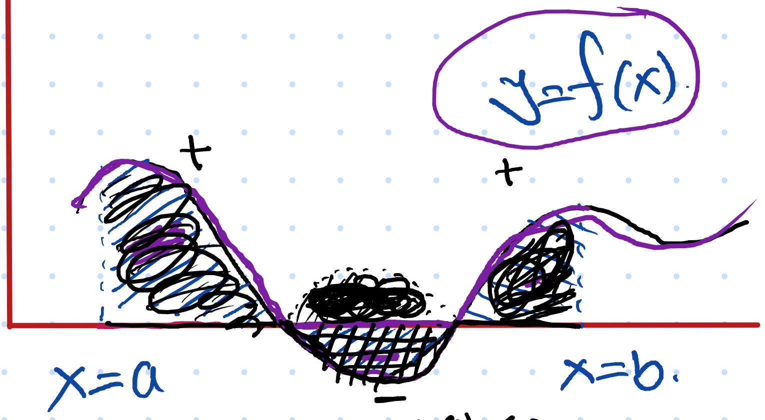
f,g are integrable

$\Rightarrow f+g,\ f-g,\ |f-g|$ "

$\Rightarrow \boxed{f+g+|f-g|}$ integrable.

$\Rightarrow \dfrac{f+g+|f-g|}{2}$ "

$f \not\geq 0$



$M_i(f) \leq 0$

$m_i(f) \leq 0$

Define $f^+ = \max(f,0)$

$f^- = -\min(f,0)$

We have

$$\int_a^b f(x)\,dx = \text{Area}(R_{f^+}) - \text{Area}(R_{f^-})$$

What follows:

Fundamental Theorem of Calculus

Differential calculus

$\updownarrow$

Integration