CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 13,14,15

Instructions

- Try these questions before class. Do not submit!

(1) Which of these collections of subsets are partitions of the set of integers?
  (a) the set of even integers and the set of odd integers
  (b) the set of positive integers and the set of negative integers
  (c) the set of integers not divisible by 3, the set of even integers and the set of integers that leave a remainder of 3 when divided by 6

  Answer:
  (a) the set of even integers and the set of odd integers:
  Yes since they are disjoint, nonempty subsets of $\mathbb{Z}$ whose union gives $\mathbb{Z}$. The corresponding equivalence relation is congruence modulo 2.
  (b) the set of positive integers and the set of negative integers
  No since 0 is not in any subset.
  (c) the set of integers not divisible by 3, the set of even integers and the set of integers that leave a remainder of 3 when divided by 6
  No since the set of integers not divisible by 3 and the set of even integers are not disjoint.

(2) For the given set and relations below, determine which define equivalence relations.
  (a) S is the set of all people in the world today, a is related to b if a and b have an ancestor in common.
  (b) S is the set of all people in the world today, a is related to b if a and b have the same father.
  (c) S is the set of real numbers a is related to b if $a = \pm b$.
  (d) S is the set of all straight lines in the plane, a is related to b if a is parallel to b.

  Answers:

  (a) S is the set of all people in the world today, a is related to b if a and b have an ancestor in common.
  No since it need not be transitive.
  (b) S is the set of all people in the world today, a is related to b if a and b have the same father.
  Yes
  (c) S is the set of real numbers a is related to b if $a = \pm b$.
  Yes

(d) S is the set of all straight lines in the plane, a is related to b if a is parallel to b.
Yes

(3) If G is a group of even order, prove that it has an element $a \neq e$, where e is the identity element satisfying $a^2 = e$, i.e. a is its own inverse.

We define a relation R on G by $g\ R\ g^{'}$ iff either $g = g^{'}$ or $g = g^{'^{-1}}$ for all $g, g' \in G$. This is an equivalence relation. Each equivalence class contains 2 elements $\{g, g^{-1}\}$, it contains less than 2 elements if $g = g^{-1}$.

Let $L_1, L_2, \ldots, L_k$ be the equivalence classes such that $G = L_1 \cup L_2 \cup \cdots \cap L_k$ and $\emptyset = L_1 \cap L_2 \cap \cdots \cap L_k$. Then $|L_1| + \cdots + |L_k| = |G|$, where $|G|$ is even and each $|L_i| \leq 2$. We have e is its own inverse and therefore the equivalence corresponding to e is of size 1. This implies there must be another equivalence class with exactly one element say $a \neq e$ since the total group size is even.

(4) Show that the complete graph of n vertices K_n has $n(n-1)/2$ edges.

A complete graph has an edge between any two vertices. You can get an edge by picking any two vertices. So it is $\binom{n}{2}$ edges, i.e. $n(n-1)/2$ edges.

(5) Show that the number of edges in $K_{m,n}$ is mn.

A complete bipartite graph with one set of vertices of size m and the other of size n implies there are $m \cdot n$ edges.

(6) Show that every regular bipartite graph has a perfect matching.

Let G be a regular bipartite graph with bipartition (A, B) and degree k. Let $X \subseteq A$ and let t be the number of edges with one end in X. Since every vertex in X has degree k, this means $k|X| = t$. Similarly, every vertex in $N(X)$ has degree k, so $t \leq k|N(X)|$, the neighbourhood of X. Thus $|X|$ is of at most the cardinality of $N(X)$. By Halls Theorem, this implies there is a complete matching from A to B. Analogously we can conclude that there is a complete matching from B to A. This implies there is a perfect matching from A to B.

(7) Every simple graph has a bipartite subgraph with at least $|E|/2$ edges.

Consider the graph G and two sets V_1 and V_2 where we will partition the vertices of G into V_1 and V_2 by looking at each vertex of G one by one. Use this criterion to make the choice: If the vertex has more edges going from V_1 to V_2 then assign it to V_2, otherwise assign it to V_1. If you assign a vertex v to V_i color each edge from v to V_i as red and every edge from v to V_{3-i} blue.

Then there are at least as many blue edges as there are red edges. When the process is finished, all edges will be colored, those within V_1 or V_2 will be red and those between V_1 and V_2 will be blue. 2-colorable implies bipartite.

Proof by induction:
Let $P(n)$ be that every graph on n vertices has a bipartite subgraph with at least $|E(G)/2|$ edges. We need to show that $P(n)$ implies $P(n+1)$. For a single vertex it is trivial. So we assume for $P(n)$ and consider a graph G with $n+1$ vertices.

Pick a vertex v of G and let H be the subgraph obtained from G by deleting v and all edges of G incident at v. H has fewer vertices than G and therefore by induction hypothesis H has a bipartite subgraph B with at least $|E(H)|/2$ edges. If $d = deg(v)$, $|E(H)| = |E(G)| - d$. Since B is a bipartite subgraph we can assume V_1 and V_2 as the bipartition of B. We can assume that B keeps all the vertices of H (We just have to remove the edges.) Now consider $v \in G$. Let $d_i, i = 1, 2$ be the number of edges between v and V_i in G. Choose $i \in \{1, 2\}$ so that $d_i \geq d/2$. Depending on the choice of i, add v to V_{3-i}. This also helps decide which of the d edges that are incident at v should be kept in order to extend B to another bipartitite subgraph of G with at least $|E(G)|/2$ edges.

(8) Prove that for a bipartite graph G on n vertices the number of edges in G is at most $\frac{n^2}{4}$.

In a bipartite graph the n vertices can be partitioned into two subsets of size i and $(n-i)$ $0 \leq i \leq n$ and the edges are from vertices of different subsets, so you have a maximum of $i(n-i)$ edges if every member of one subset is connected to every member of the other subset.

$f(i) = i(n-i)$, $0 \leq i \leq n$ is maximized by $i = n/2$ which leads to $n/2 \cdot n/2 = n^4/4$ - being the maximum number of edges.

(9) Show that for all graphs $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$.
We will first show $\kappa(G) \leq \min_{v \in V} \deg(v)$ and also analogously, $\lambda(G)$ is less than min degree.

When $G = (V, E)$ is a noncomplete connected graph with at least 3 vertices then vertex connectivity $\kappa(G) \leq min_{v \in V} deg(v)$ and edge connectivity $\lambda(G) \leq min_{v \in V} deg(v)$.
Let d be the minimum degree of a graph G. Then there is some vertex v with d neighbours. Provided that there are at least $d+2$ vertices in G, the removal of the d neighbours of v will disconnect v from the remainder of the graph. This will make G disconnected. Therefore there exists a vertex cut of size d , $\chi(G) \leq d$. If there are not at least $d+2$ vertices in G then there must be exactly $d+1$ vertices as otherwise the minimum degree of G cannot be d. Also we have $1 \leq \chi(G) \leq |G| - 2$ and $|G| = d+1$, $\chi(G) \leq d - 1 \lneq d$ so $\chi(G) \leq d$.

Edge connectivity : Consider a vertex v of min degree, and denote this degree as d. By removing the d edges that are adjacent to v, we disconnect the graph.

Now to show that $\kappa(G) \leq \lambda(G)$.

We use induction on $\lambda(G)$.

Basis step: If $\lambda = 0$, then we have a disconnected graph which implies κ is 0 too. If $\lambda = 1$ then removal of one edge disconnects the graph and it has end points a, b and this implies removal of one of these endpoints disconnects the graph and therefore $\kappa = 1$ too.

Induction Step:
Note that for $\lambda = n-1$, then $\kappa = n-1$ since the graph is the complete graph of n vertices. Therefore we need to show that the inequality is true for all $1 \lneq \lambda \lneq n-1$.

Let $\lambda = k$ such that $1 < k < n-1$ and we assume that the inequality is true for $k-1$.

Consider that the removal of $e_1, e_2, \ldots, e_k$ disconnects a graph G. Remove the edge e_k with endpoints a and b to form G_1 from G and now we a graph whose λ is $k-1$. By I.H. there are at most $k-1$ vertices $v_!, v_2, \ldots, v_j$ s.t. once we remove these vertices from G_1 we get a graph G_2 which is disconnected. Since $k < n-1$, we have that $k-1 \leq n-3$ and therefore G_2 has at least 3 vertices.

If both a, b are in G_2 and if adding e_k to G_2 gives us a connected graph G_3, then if we remove either a or b from G_3 we will disconnect it to get a new graph G_4. That is, removing at most k vertices disconnects G.

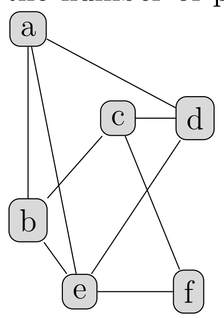
If a, b are vertices in G_2 s.t. adding e_k does not produce a connected graph then removing $v_1, v_2, \ldots, v_j$ disconnects G as well.

Finally if either a or b is not in G_2 then $G_2 = G \setminus \{v_1, v_2, \ldots, v_j\}$ and the connectivity of G is less than or equal to k. So for all cases we have $\kappa \leq k$. Hence we have shown the inductive step.

(10) Show that the existence of a simple circuit of a particular length is a graph invariant.

Suppose $G = \langle V_G, E_G \rangle$ and $H = \langle V_H, E_H \rangle$ are isomorphic graphs and suppose that G has a simple circuit of length m. Since G and H are isomorphic there is a bijection $h : V_G \to V_H$ s.t for each $u, v \in V_G$, $\{u, v\} \in E_G$ iff $\{h(u), h(v)\} \in E_H$. Let $\{\mathbf{v}_1, v_2, \ldots, v_m\}$ be the vertices of a simple circuit of size m in G s.t $\{v_k, v_{k+1}\} \in E_G$ for $k = 1, \ldots, m-1$ and $\{v_m, v_1\} \in E_G$. Then $\{h(v_k), h(v_{k+1})\} \in E_H$, for $k = 1, \ldots, m-1$, and $\{h(v_m), h(v_1)\} \in E_H$ and $h(v_1), \ldots, h(v_m)$ are distinct so they are the vertices of a simple m-cycle in H.

(11) Count the number of paths between c and d in the graph below of length 2 and 3:



Answer: We need to build the adjacency matrix for the graph w.r.t. to the vertices order (a, b, c, d, e, f): $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 0 & 3 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 2 & 0 & 2 \end{bmatrix}$ The third row and fourth column correspond to the number of paths from c and d of length 2 and that is 0.

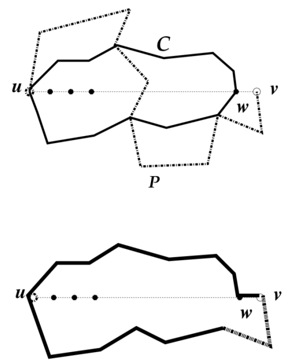
$A^3 = \begin{bmatrix} 4 & 7 & 3 & 7 & 6 & 4 \\ 7 & 2 & 8 & 2 & 9 & 1 \\ 3 & 8 & 0 & 8 & 2 & 6 \\ 7 & 2 & 8 & 2 & 9 & 1 \\ 6 & 9 & 2 & 9 & 4 & 7 \\ 4 & 1 & 6 & 1 & 7 & 0 \end{bmatrix}$ The third row and fourth column correspond to the number of paths from c and d of length 3 and that is 8.

(12) Show that K_n has a Hamilton circuit whenever $n \geq 3$.

Follows from Dirac's theorem that says if G is a simple graph with n vertices $n \geq 3$ s.t. the degree of every vertex is at least $n/2$ then G has a Hamilton circuit.

(13) If G is a connected planar simple graph then G has a vertex of degree not exceeding 5.
If G has one or two vertices the result is true. If G has at least three vertices then $e \leq 3v - 6$, so $2e \leq 6v - 12$ (Result stated in class). If the degree of every vertex were at least 6 then by handshaking theorem, $2e = \sum_{v \in V} deg(v)$,

FIGURE 0.1. 2-connectivity implies cycle



that is $2e \geq 6v$. But this contradicts the inequality $2e \leq 6v - 12$. It follows that there must be a vertex with degree no greater than 5.

(14) A graph with at least 3 vertices is 2-connected iff every pair of vertices lie in a cycle.
A connected graph is called 2-connected if for every vertex $x \in V(G)$, $G - x$ is connected.

Sufficient condition: If every two vertices belong to a cycle, no removal of one vertex can disconnect the graph.

Necessary condition that needs to proved : If G is 2-connected every two vertices belong to a cycle.

We will prove it by induction on the distance $dist(u, v)$ between two vertices in the graph.

Basis case: Since the vertices are distinct, the smallest distance is 1. This means u and v are adjacent. Let z be any vertex in G other than u and v. Because of the removal of u (or v) does not disconnect G. There is a path P_1 (or P_2) that connects u (or v) with z and that does not contain v (or u).

The cycle containing u and v consists of the edge (u, v) and a path from u to v obtained from the walk from v to z using P_2 and the reverse of P_1 from z to u.

Inductive step: Let the proposition be true for all pairs of vertices on the distance $\leq k$ and let $dist(u, v) = k + 1$. Consider the shortest path from u to v and let w be the vertex on the path which is adjacent to v. Since $dist(u, w) = k$ there is a cycle C containing u and w. Since the removal of w does not disconnect u from v there is a path P that connects u and v that does not contain w. A cycle containing u and v can be constructed from C and P and edge between w and v. Look at Figure 0.1 for details.

(15) If G_1 and G_2 are two connected subgraphs of G having at least one vertex in common then $G_1 \cup G_2$ is connected.

Proof: Let $v \in V(G_1) \cap V(G_2)$. Let $a \in V(G_1)$ and $b \in V(G_2)$ but $a, b \notin V(G_1) \cap V(G_2)$ Then there is a path a to v P_1 in G_1. Let $P_1 : a =$

$x_0, x_1, \cdots, x_k = v$. Let i be the smallest such that $x_i \in G_2$. $i \geq 1$. Let Q be the path from x_i to b in G_2. Then $x_0, x_1, x_{i-1}Q$ is a path from a to b in $G_1 \cup G_2$ as no x_j can occur in Q for $j < i$.

(16) The complementary graph $\hat{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\hat{G}$ if and only if they are not adjacent in G. If a graph G is not connected, prove that its complement graph is connected.

Let $G_1, \cdots, G_k$ be the connected components of G. Let $\hat{G}$ be the complement graph of G. As there is no edge in G between a vertex in G_i and a vertex in G_j, there is an edge between any vertexin G_i and any vertex in G_j.

Lets consider an edge in G, such as the edge $\{v, w\}$. They are in the same component of G. Since G is disconnected, we can find a vertex u in a different component such that neither uw nor uv are edges of G. Then vuw is a path from v to w in $\hat{G}$. Thus, $\hat{G}$ is connected.

(17) Show that the property that a graph is bipartite is an isomorphic invariant. If G and H are isomorphic and G is a bipartite graph, we show that H is also a bipartite graph.

Since G is bipartite graph, there is a bipartition (V_1, V_2). Let f be the isomorphism between G and H. Then let $W_1 = f(V_1)$ and $W_2 = f(V_2)$. As f is a bijective function, W_1 and W_2 are disjoint since V_1 and V_2 are. Also the union of W_1 and W_2 gives the vertex set of H.

We only need to verify that every edge in H has an endpoint in W_1 and the other one in W_2. As G and H are isomorphic then for every distinct vertices a and b in G, they are adjacent iff $f(a)$ and $f(b)$ are adjacent. Therefore, for any edge $e = \{a, b\}$ in G we can find a corresponding one $e^{'} = \{f(a), f(b)\}$ in H. As G is bipartite one of the vertices is in V_1 and the other one is in V_2 meaning one of $f(a)$ or $f(b)$ is in W_1 and the other is in W_2. Therefore, H is bipartite.

(18) How many distinct Hamiltonian cycles are there in a complete graph $K_n, n \geq 3$?

$\frac{(n-1)!}{2}$. Since it is the same as number of cyclic permutations where clockwise and anti-clockwise arrangements are considered the same.

(19) What is the height of a full and balanced 7-ary tree with 340 leaves?

Answer: The height of a full and balanced m-ary tree is $\lceil log_m l \rceil$ where l is the number of leaves. Here we have $h = \lceil log_7 340 \rceil = 3$ since $7^3 = 343$.

Actually any of the following answers you could have written: (i) $\lceil log_7 340 \rceil$ (ii) 3 (iii) There cannot be a full 7-ary tree with 340 leaves since that would mean one of the nodes at a level just above leaf level will have > 1 child but < 7 children.

(20) Consider a simple graph G.

(a) If G has k connected components and each of these components have $n_1, n_2, \ldots, n_k$ vertices respectively, then the number of edges of G does not exceed $\sum_{i=1}^{k} C(n_i, 2)$. Prove. *(2 marks)*
Proof: Each connected component with n_i vertices can have at most $C(n_i, 2)$ edges – the case when there is an edge between every distinct vertices and there can only be one edge between any distinct vertices since it is a simple graph.

(b) Use the previous result to show that a simple graph with n vertices and k connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges. *(4 marks)*
Proof: We have $\sum_{i=1}^{k}(n_i - 1) = n - k$. Squaring on both sides we get,

$$\sum_{i=1}^{k}(n_i - 1)^2 + A = n^2 - 2nk + k^2,$$

where A represents the remaining sum of terms which is always a non-negative sum since $(n_i - 1) \geq 0$, for all i.
Consider $\sum_{i=1}^{k}(n_i - 1)^2$. It is equal to,

$$\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} 2n_i + k = \sum_{i=1}^{k} n_i^2 - 2n + k.$$

This implies,

$$\sum_{i=1}^{k}(n_i)^2 \leq n^2 - 2nk + k^2 + 2n - k = n^2 - (k-1)(2n-k).$$

Note that we removed A since it is a positive sum.
From above result we have the number of edges is at most,

$$\begin{aligned}
\sum_{i=1}^{k} C(n_i, 2) = \sum_{i=1}^{k}(n_i - 1)n_i/2 &= \frac{1}{2}\sum_{i=1}^{k}(n_i)^2 - \frac{n}{2} \\
&\leq \frac{n^2 - (k-1)(2n-k) - n}{2} \\
&= \frac{n^2 - 2nk + k^2 + n - k}{2} \\
&= \frac{(n-k)(n-k+1)}{2}
\end{aligned}$$

(c) Use previous result to show that a simple graph with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges. *(2 marks)*
The value of $\frac{(n-k)(n-k+1)}{2}$ decreases as k increases. If a simple graph with n vertices is not connected it will have at least 2 connected components.

There, $k \geq 2$. Then there are at most $(n-2)(n-1)/2$ edges in the graph. But here it is said the graph has more than $(n-1)(n-2)/2$ edges and therefore the graph is connected.

(21) Ore's theorem : If G is a simple graph with n vertices $n \geq 3$, s.t. $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G then G has a Hamilton circuit.

Answer: Assume for a contradiction that G has no Hamiltonian circuit.

(a) Pick any two vertices of G which does not have an edge between them and add a new edge between them. Keep doing this until we get a graph G_m which has a Hamiltonian circuit. The process is assured to stop since we will reach a complete graph on n vertices which has a Hamiltonian circuit.

(b) Let G_{m-1} be the graph obtained just before adding edge $\{x, y\}$ to gt G_m. Let $(z_1, \ldots, z_n, z_1)$ be the Hamiltonian circuit in G_m. It will have $\{x, y\}$ at some point of time else G_{m-1} would have been the graph we considered. $\{x, y\}$ could be $\{z_n, z_1\}$ in which case $(z_1, \ldots, z_n)$ is a Hamiltonian path in G_{m-1}. Otherwise, there is some r, s.t. $1 \leq r < n$ and $z_r = x, z_{r+1} = y$ such that $(z_{r+1}, \ldots, z_n, z_1, \ldots, z_r)$ is a Hamiltonian path in G_{m-1}.
In both cases, all the edges used in this path appear in G_{m-1} and only $\{x, y\}$ appear in G_n and not in G_{m-1}. Let us relabel the vertices so that this path is $(x_1, \ldots, x_n)$.

(c) Suppose we find a vertex x_i s.t. x is adjacent to x_i and y is adjacent to x_{i-1} then,

$$(x, x_i, \ldots, x_n, y, x_{i-1}, \ldots, x)$$

is a Hamiltonian circuit in G_{m-1} a contradiction.
Note that here we need $n \geq 3$ since if $n = 2$ then the first step is (x, y) and the second s (y, x) which means we have used an edge twice, not possible in paths where edges are not repeated.

(d) Does there exists such a i? We have not used the hypothesis on degrees yet. Since G_{m-1} is obtained from G by adding edges it still satisfies the following hypothesis on G:

$$A = \{i : 2 \leq i \leq n \text{ and } x_i \text{ is adjacent to } x\}$$
$$B = \{i : 2 \leq i \leq n \text{ and } a_{i-1} \text{ is adjacent to } y\}.$$

$|A| = deg(x)$ and $|B| = deg(y)$. As x and y are not adjacent to each other in G_{m-1} we have that $deg(x) + deg(y) \geq n$. So we have A, B subsets of $\{2, \ldots, n\}$ containing at least n elements between them. Therefore they intersect non-trivially and x_i, $i \in A \cap B$ is the desired vertex.

(22) Design recursive algorithms for preorder, inorder, postorder traversals for binary trees.

Algorithm for Preorder:
Preorder(tree)
(a) Visit the root.
(b) Preorder(left-subtree)
(c) Preorder(right-subtree)
Algorithm for Inorder:
Inorder(tree)
(a) Inorder(left-subtree)
(b) Visit the root.
(c) Inorder(right-subtree)
Algorithm for Postorder:
Postorder(tree)
(a) Postorder(left-subtree)
(b) Postorder(right-subtree)
(c) Visit the root.

(23) What does the inorder traversal of a BST give rise to? Ascending order of nodes/sorted list of elements.

(24) Let x and y be two nodes of a binary tree B. Prove that x is an ancestor of y iff x stands before y in the pre-order traversal of B and x stands after y in the post-order traversal of B. Proof: If x is an ancestor of y then in a preorder traversal of B since x will be visited before y since every node is visited before its children/descendants are visited. x will stand after y in the postorder traversal since the node is visited only after the all the descendants/given by its subtrees are visited.

Now to show converse: That is x stands before y in the preorder traversal of B and x stands after y in the postorder traversal of B. Let us assume that x is not an ancestor of y. There can be two cases here:

(a) y is an ancestor of x: but if it was then that would mean in preorder traversal y would have been visited before x, contradiction
(b) y is not an ancestor of x. Since either of them is not ancestor of the other, this implies neither x nor y is the root. So there is atleast a common ancestor. Let the lowest common ancestor (lca) be a - that is the ancestor you encounter on the paths from x to root and y to root.
    (i) Both x and y are in the same subtree of T of a - since a is the lca either x or y is the root r of a subtree- or else that root would have been the lca. But that means either x is ancestor of y or vice-versa.
    (ii) x is in the left subtree of a and y is in the right subtree of a. But that means in post-order traversal x will appear before y since the left subtrees are exhausted before right subtree.
    (iii) x is in the right subtree of a and y is in left subtree of a then in that case y would be visited before x in preorder traversal and that is a contradiction.

Therefore x is not an ancestor of y leads to a contradiction when we assume that x stands before y in the preorder traversal of B and x stands after y in the postorder traversal of B. So the negation shoudl be true - x is an ancestor of y.

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