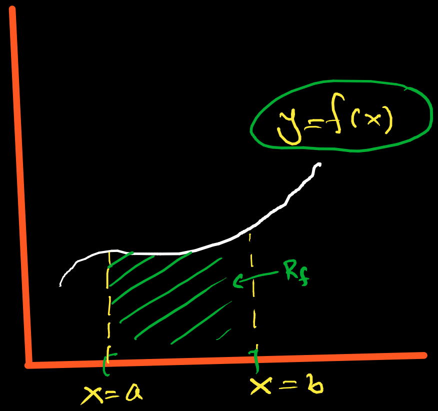
Application of Integration

Area "under" a curve:

Recall: Let $f:[a,b]\to\mathbb{R}$ be a bounded function. If $f\geq 0$, then we say that the region

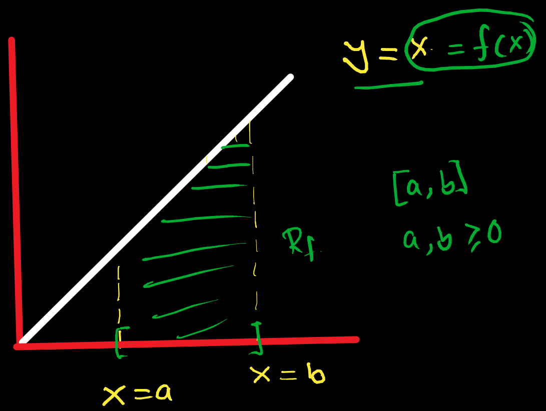
$$R_f := \{(x,y) \mid a\leq x\leq b,\ 0\leq y\leq f(x)\}$$

has an area if f is integrable on $[a,b]$.



We "define"

$$\text{Area}(R_f) := \int_a^b f(x)\,dx.$$

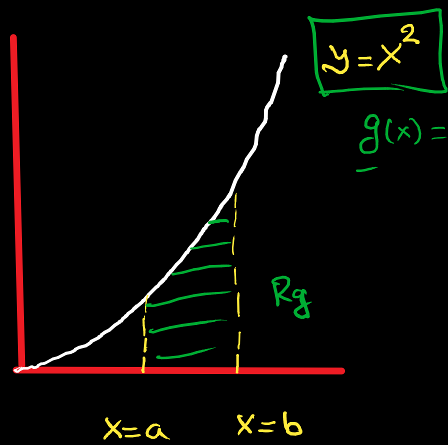


$f(x)\geq 0$

$[a,b]$

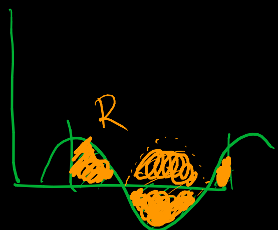
$a,b\geq 0$

$$\int_a^b x\,dx = \frac{b^2-a^2}{2}$$



$g(x)=x^2\geq 0$

$$\text{Area}(R_g) = \int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$



Furthermore, if $f\ngeq 0$, then we divide the region into two parts, namely, we consider the nonnegative functions:

$$f^+ := \max\{f,0\} \quad\text{and}\quad f^- := -\min\{f,0\}$$

(we proved that f^+, f^- are integrable)

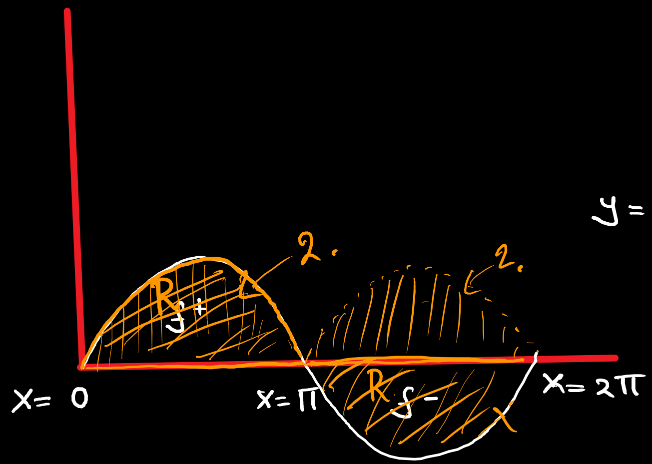
and define,

$$\text{Signed area} = \boxed{\int_a^b f(x)\,dx} = \text{Area}(R_{f^+}) - \text{Area}(R_{f^-})$$

$$= \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx.$$

$$\text{Total area}\ (R) = \text{Area}(R_{f^+}) + \text{Area}(R_{f^-})$$

Example:



$y = f(x) = \sin x$

$[0, 2\pi]$

$f^+(x) = \begin{cases} \sin x & \text{when } x \in [0,\pi] \\ 0 & \text{when } x \in [\pi, 2\pi] \end{cases}$

Area(R_{f^+})

$$= \int_0^{2\pi} f^+(x)\,dx = \int_0^{\pi} f^+(x)\,dx + \int_{\pi}^{2\pi} \underbrace{f^+(x)}_{0}\,dx = 0$$

$f^+ \geq 0$

$$= \int_0^{\pi} \sin x\,dx$$

$$\text{FTC (II)} = -\cos x\Big]_0^{\pi} = -(\cos\pi - \cos 0)$$

$$= -(-1-1) = 2.$$

Signed area:

$$\int_0^{2\pi} \sin x\,dx. = \int_0^{2\pi} f^+(x)\,dx - \int_0^{2\pi} f^-(x)\,dx$$

$$= 2 - 2 = 0$$

$$= -\cos(x)\Big]_0^{2\pi} \quad \text{FTC (II)}$$

$$= -(\cos 2\pi - \cos 0) = 0$$

$f^- = -\min\{f, 0\} = \begin{cases} 0 & \text{if } x \in [0,\pi] \\ -\sin x & \text{if } x \in [\pi, 2\pi] \end{cases}$

Area(R_{f^-})

$$= \int_0^{2\pi} f^-(x)\,dx$$

$$= \int_0^{\pi} f^-(x)\,dx + \int_{\pi}^{2\pi} f^-(x)\,dx$$

$$= \int_{\pi}^{2\pi} (-\sin x)\,dx = \cos x\Big]_{\pi}^{2\pi}$$

$$= 1 - (-1) = 2.$$

Signed area $\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} f^+(x)\,dx - \int_0^{2\pi} f^-(x)\,dx. = 0$

Total area $\int_0^{2\pi} f^+(x)\,dx + \int_0^{2\pi} f^-(x)\,dx = 2+2 = 4.$

Area between curves

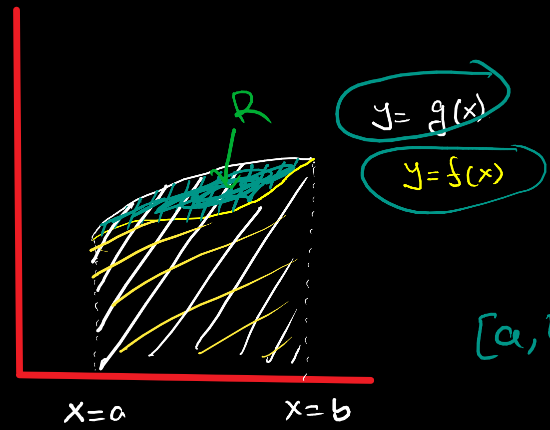
Suppose $f, g : [a,b] \to \mathbb{R}$ be integrable functions.

How do we compute the area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ when $a \le x \le b$.

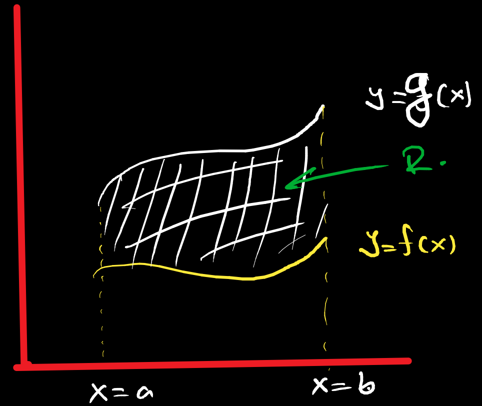
Area (R)

$= \text{Area}(R_g)$

$- \text{Area}(R_f)$



[a, b]



$\forall x \in [a,b]$

$$\boxed{g(x) \ge f(x)}$$

$R = \{(x,y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}$

$$\boxed{\text{Area}(R) := \text{Area}(R_{g-f})}$$

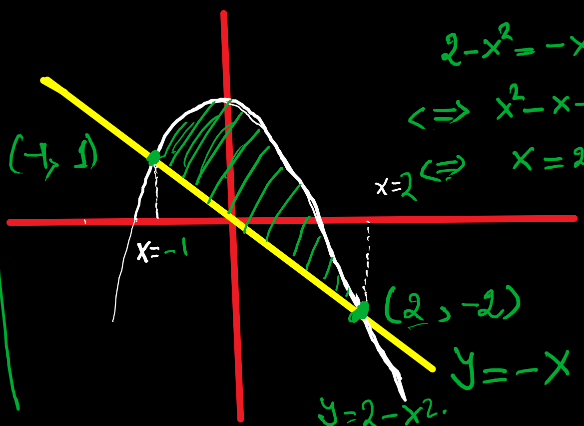
Example: Find the area of the region enclosed by $\underset{g(x)}{y = 2 - x^2}$ and $\underset{f(x)}{y = -x}$.

$2 - x^2 \ge -x$

$\Leftrightarrow x^2 - x - 2 \le 0$

$\Leftrightarrow (x-2)(x+1) \le 0$

$\Leftrightarrow -1 \le x \le 2$



$2 - x^2 = -x$

$\Leftrightarrow x^2 - x - 2 = 0$

$\Leftrightarrow x = 2$ or $x = -1$

$$\text{Area}(R) = \text{Area}(R_{g-f})$$

$$= \int_{-1}^{2} (g(x) - f(x))\,dx$$

$$= \int_{-1}^{2} ((2 - x^2) - (-x))\,dx$$

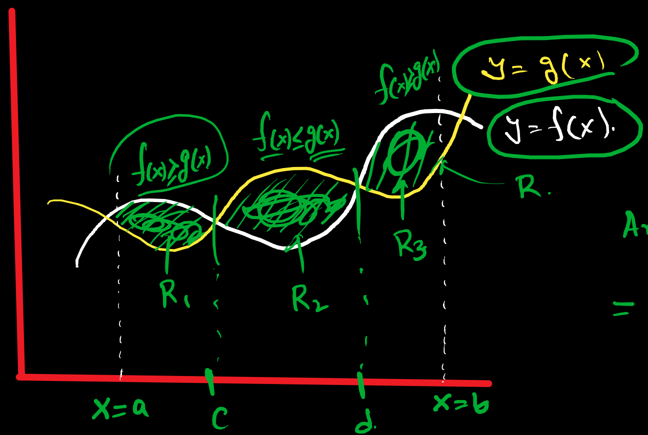
$$= \frac{9}{2} \text{ sq. units.}$$

What if we are not in such a nice situation?

If we can divide a planar region bounded by two curves $y=f(x)$ and $y=g(x)$ into finitely many subregions of the above type, then we simply add up the areas of these subregions.

For example, if $f,g:[a,b]\to\mathbb{R}$ are continuous functions such that the curves $y=f(x)$ and $y=g(x)$ intersect each other at finitely many points, then the area of the region bounded by $y=f(x)$, $y=g(x)$, $x=a$, $x=b$

is given by $$\int_a^b |f(x)-g(x)|\,dx.$$



$$\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2) + \text{Area}(R_3)$$

$$= \int_a^c (f-g)(x)\,dx + \int_c^d (g-f)(x)\,dx + \int_d^b (f-g)(x)\,dx.$$

Compute the area of the region bounded by $y = x^2 - 4$

$y = -x^2 - 2x$

$x = -3$

$x = 1$