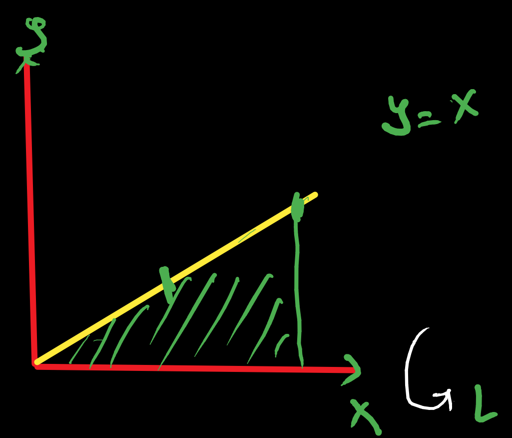
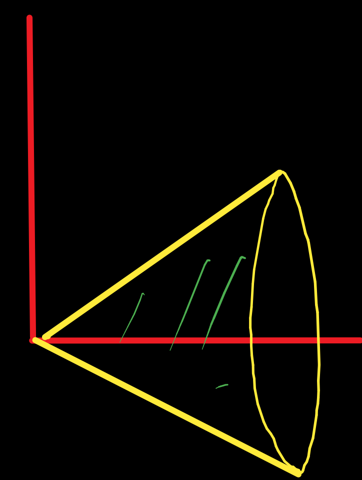
Surface of revolution

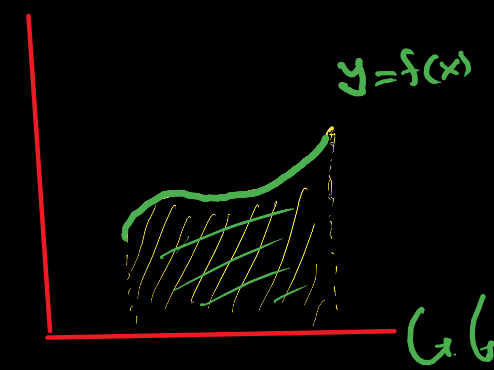
A surface of revolution is generated when a curve $C \subseteq \mathbb{R}^2$ is rotated around a line in $\mathbb{R}^2$.

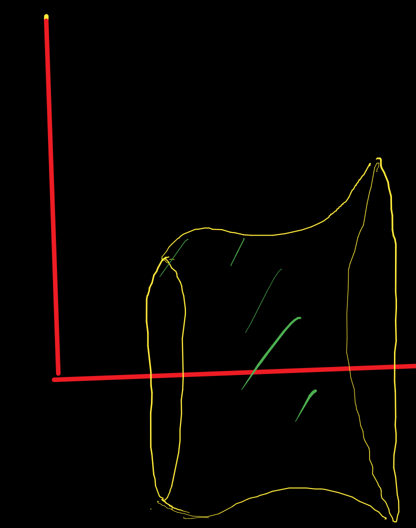
Examples:

(1)









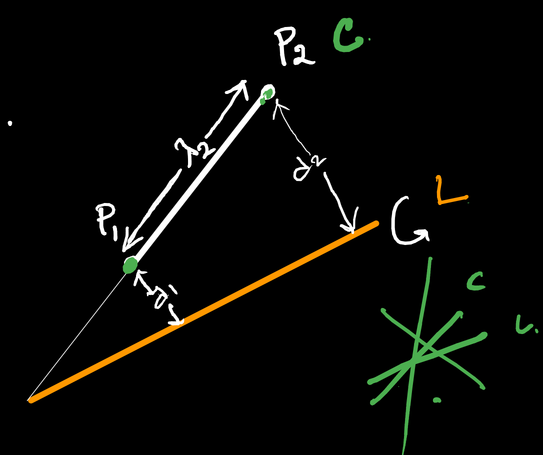
(2) Frustum of a cone:

Suppose that C is a slanted line segment P_1P_2 of length λ_2 and C does not cross L.

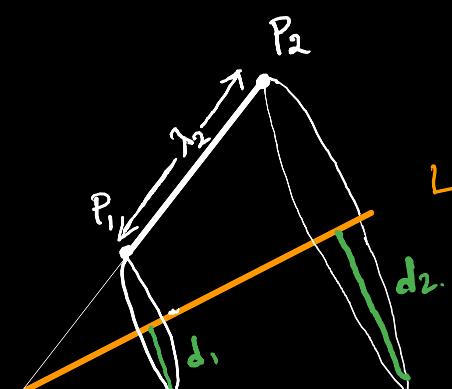
d_1 = distance of P_1 from L

d_2 = distance of P_2 from L.

$d_1 \leq d_2$



By rotating P_1P_2 around L we get a surface called the frustum F of a cone with base radii d_1 and d_2.



Suppose C is parametrized by $(x(t), y(t))$ with $t \in [\alpha, \beta]$.

If C is smooth (x, y are continuously differentiable)

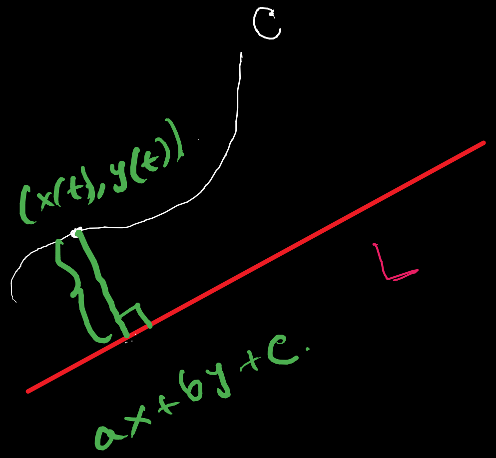
and C does not cross L (suppose $L: ax+by+c=0$),

then area of S, the surface of revolution obtained by rotating C around L, is given by:

$$\text{Area}(S) = 2\pi \int_{\alpha}^{\beta} \rho(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

$$\left[\begin{array}{l} \rho(t) = \text{the distance of } (x(t), y(t)) \\ \qquad\quad \text{from the line } L. \\ \qquad = \dfrac{|a\,x(t) + b\,y(t) + c|}{\sqrt{a^2+b^2}} \end{array}\right]$$

$$= 2\pi \int_{\alpha}^{\beta} \frac{|ax(t) + by(t) + c|}{\sqrt{a^2+b^2}} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$



Special cases:

Area of $S = 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2+y'(t)^2}\,dt$

$(x(t), y(t)) = (t, f(t))$

① $L = x$-axis.

$C: y = f(x)$. $x \in [a,b]$

f is continuously differentiable.

C does not cross L.!!

$\Rightarrow$ $f(x) \geq 0$ $\forall x \in [a,b]$

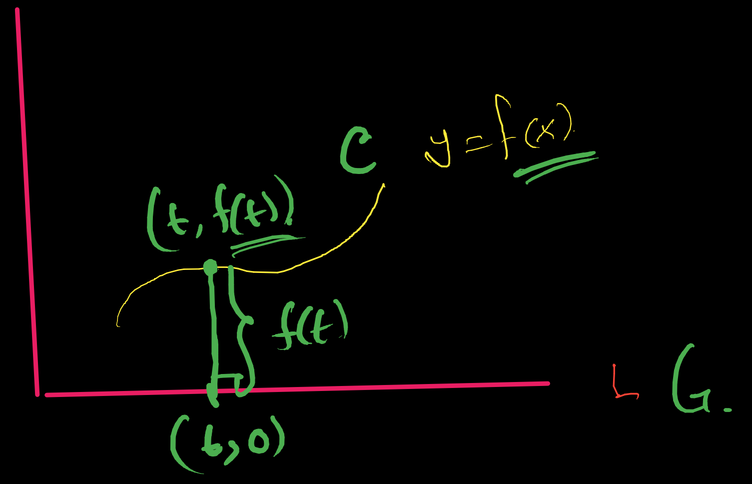
or $f(x) \leq 0$ $\forall x \in [a,b]$

Then.

$$\text{Area}(S) = 2\pi \int_a^b \rho(t)\sqrt{x'(t)^2+y'(t)^2}\,dt$$

$$= 2\pi \int_a^b |f(t)| \sqrt{1+(f'(t))^2}\,dt.$$

$$\text{Area}(S) = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\,dx.$$



② $L = y$-axis.

$C: x = g(y)$ $y \in [a,b]$.

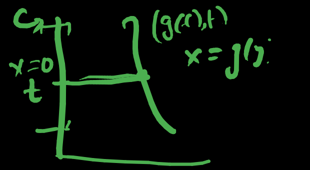
g continuously differentiable.

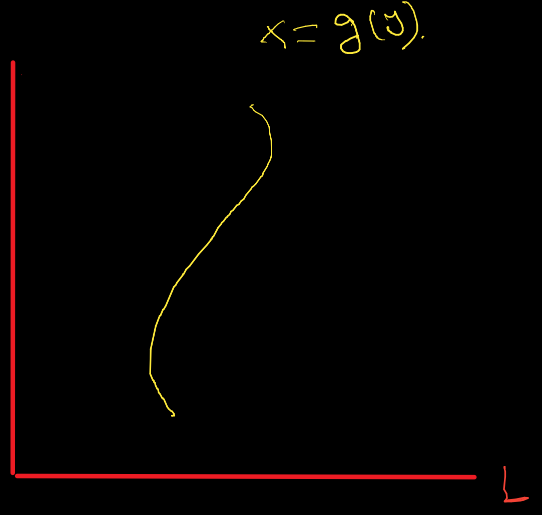
C does not cross L;

$\Rightarrow$ Either $g \geq 0$ or $g \leq 0$

$$\text{Area}(S) = 2\pi \int_a^b |g(y)| \sqrt{1+g'(y)^2}\,dy.$$

$(g(t), t)$





① **Rotating a curve about x axis:**

$$2\pi\int_a^b |f(x)|\sqrt{1+f'(x)^2}\,dx$$

Example:

Find the area of the surface generated by revolving the curve

$C:\ y=x^3 \quad 0\le x\le \frac{1}{2}$

about x axis.

$f(x)=x^3\ge 0$ in $[0,\frac{1}{2}]$

$[-\frac{1}{2},0]$

$x\in[-\frac{1}{2},\frac{1}{2}]$.

The surface area is given by

$$2\pi\int_0^{1/2} x^3\sqrt{1+9x^4}\,dx$$

$$=\frac{\pi}{27}\left(\left(1+\frac{9}{16}\right)^{3/2}-1\right).$$

Proposition:

Let $f:[a,b]\to\mathbb{R}$ be a continuous function.
Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be a continuously differentiable function (ϕ' is continuous).
Let $\phi([\alpha,\beta])=[a,b]$.
Then $(f\circ\phi)\cdot\phi':[\alpha,\beta]\to\mathbb{R}$ is integrable. Also

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx=\int_\alpha^\beta (f\circ\phi)(t)\cdot\phi'(t)\,dt$$

$[\alpha,\beta]\xrightarrow{\phi}[a,b]\xrightarrow{f}\mathbb{R}$, $f\circ\phi$

$\phi':[\alpha,\beta]\to\mathbb{R}$.

$$2\pi\int_0^{1/2} x^3\sqrt{1+9x^4}\,dx$$

Let $f:[0,\frac{1}{2}]\to\mathbb{R}$. $\quad f(x)=\sqrt{1+x}$.

$\phi:[0,\frac{1}{2}]\to\mathbb{R}$. $\quad \phi(x)=9x^4$. $\quad \phi(0)=0$, $\phi(\frac{1}{2})=\frac{9}{16}$

$f\circ\phi(x)=f(9x^4)=\sqrt{1+9x^4}$

$\phi:[0,\frac{1}{2}]\to[0,\frac{9}{16}]$ $\quad \phi'(x)=36\cdot x^3$

$$=\frac{2\pi}{36}\int_0^{1/2} x^3\sqrt{1+9x^4}\,dx$$

$$=\frac{\pi}{18}\int_0^{9/16}\sqrt{1+t}\,dt$$

$$=\frac{\pi}{18}\cdot\frac{2}{3}\left.(1+t)^{3/2}\right]_0^{9/16}$$

$$=\frac{\pi}{27}\left(\left(1+\frac{9}{16}\right)^{3/2}-1\right)$$ //

② **Rotating a curve about y axis:**

$$2\pi\int_a^b |g(y)|\sqrt{1+g'(y)^2}\,dy.$$

$x=2\sqrt{4-y} \qquad 0\le y\le\frac{15}{4}$

$$\text{Surface area}=2\pi\int_0^{15/4}2\sqrt{4-y}\sqrt{1+\frac{1}{4-y}}\,dy$$

$$g'(y)=-2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{4-y}}$$

$$=2\pi\int_0^{15/4}2\cancel{\sqrt{4-y}}\cdot\frac{1}{\cancel{\sqrt{4-y}}}\sqrt{5-y}\,dy.$$

$$=4\pi\int_0^{15/4}\sqrt{5-y}\,dy.$$

$$=4\pi\cdot\frac{2}{3}\left.\left(-(5-y)^{3/2}\right)\right]_0^{15/4}$$

$$=\frac{8\pi}{3}\left[-\left(5-\frac{15}{4}\right)^{3/2}+5^{3/2}\right]$$

$$=\frac{8\pi}{3}\cdot 5^{3/2}\cdot\frac{7}{8}=\boxed{\frac{35\pi\sqrt{5}}{3}}$$

③ **Surface area of a sphere.** radius r.

$x^2+y^2+z^2=r^2$.

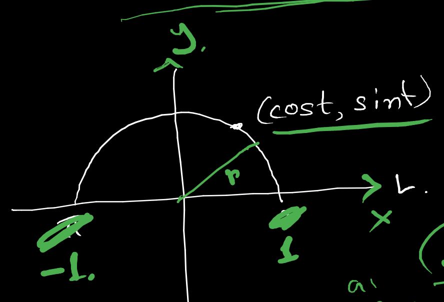
$x=r\cos t \qquad y=r\sin t$

$0\le t\le\pi$

$x=x(t)$

$y=y(t)$

$$\int_a^b \frac{|ax(t)+by(t)+c|}{\sqrt{a^2+b^2}}\sqrt{x'(t)^2+y'(t)^2}\,dt.$$



xy.

$\frac{1.y(t)}{\sqrt{1^2}}=y(t)$.

L

$y=0$

$0\cdot x+1\cdot y+c=0$

$$2\pi\int_0^\pi\frac{r\sin t}{\sqrt{1}}\sqrt{r^2\sin^2 t+r^2\cos^2 t}\,dt.$$

$x'(t)^2+y'(t)^2$.

$$=2\pi r^2\cdot\int_0^\pi\sin t$$

$$=2\pi r^2\left.(-\cos t)\right|_0^\pi$$

$$=2\pi r^2\left((-\cos\pi)-(-\cos 0)\right).$$

$$=2\pi r^2(1-\cos\pi)=4\pi r^2.$$

Aliter:

$y=f(x)=\sqrt{r^2-x^2}$.

$0\le x\le r$.

$$y=2\pi\int_0^r\sqrt{r^2-x^2}\cdot\sqrt{1+\left(\frac{1}{2}(r^2-x^2)^{-1/2}(-2x)\right)^2}\,dx$$

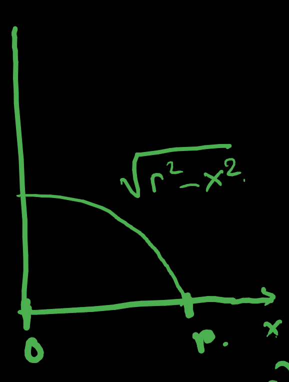
$$=2\pi\int_0^r\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$

$$=2\pi\int_0^r\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$=2\pi\int_0^r r\,dx$$

$$=2\pi r\cdot\left.x\right]_0^r$$

$$=2\pi r^2.$$



$2\pi r^2 + 2\pi r^2$

$=4\pi r^2$.