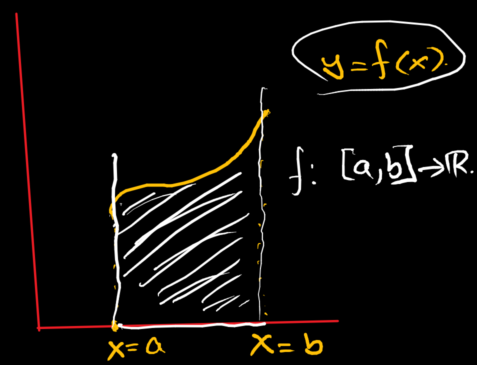
Integral calculus

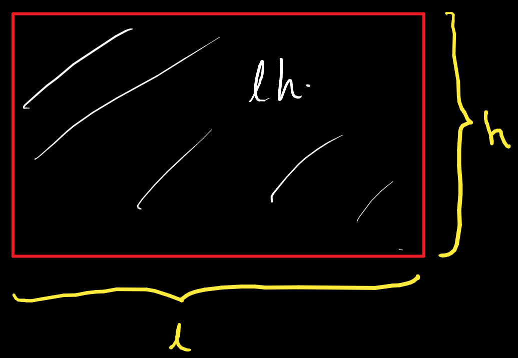
600 - 300 BC (Ancient Greece)

Area of non rectilinear regions of the form

$f : [a,b] \to \mathbb{R}$

"nice"





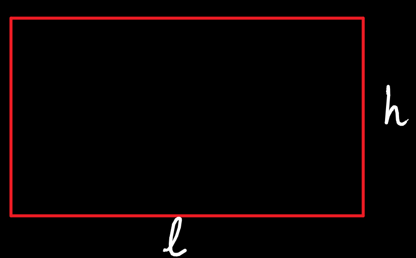
Differential calculus

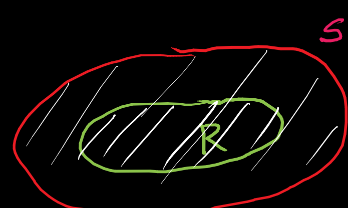
1680 AD.

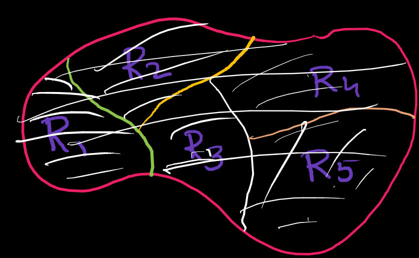
Three axioms:

- Area of a rectangle $l \times h$
- $R \subseteq S \Rightarrow A_R \leq A_S$
- $R = \underline{R_1 \cup \cdots \cup R_n} \Rightarrow A_R = A_{R_1} + A_{R_2} + \cdots + A_{R_n}$

  provided R_i, R_j possibly intersect at the boundary points

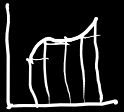


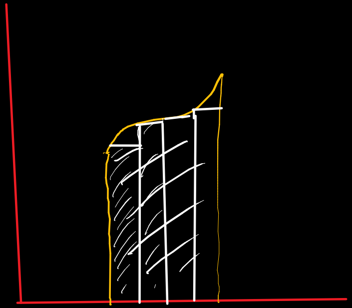


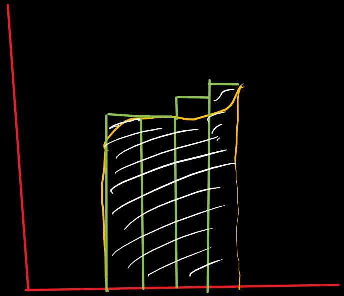


Method of exhaustion:

Goal: "Squeeze" the region between two networks of rectangles:



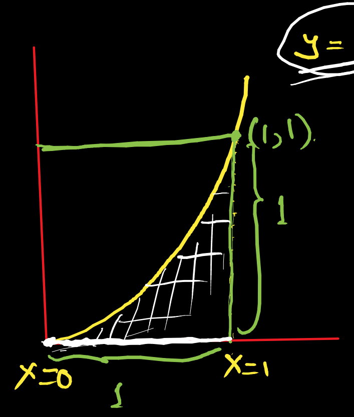




~~**Exhaustion**~~

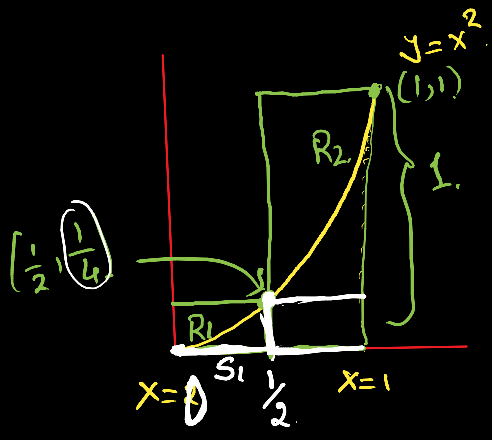
~~A state of extreme physical or mental tiredness !!~~

Example: $y=f(x)=x^2 \quad x\in[0,1]$.



Exhaustion!!

$$\underset{\substack{\parallel\\0}}{L_1} \le A_R \le \underset{\substack{\parallel\\1}}{U_1}$$



$$\mathrm{Area}(S_1\cup S_2)\le A_R \le \mathrm{Area}(R_1\cup R_2)$$

$$=\mathrm{Area}(S_1)+\mathrm{Area}(S_2) \qquad = \mathrm{Area}(R_1)+\mathrm{Area}(R_2)$$

$$=0+\frac{1}{2}\times\frac{1}{4} \qquad = \frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times 1$$

$$=\frac{1}{8} \qquad = \frac{1}{8}+\frac{1}{2}=\frac{5}{8}$$

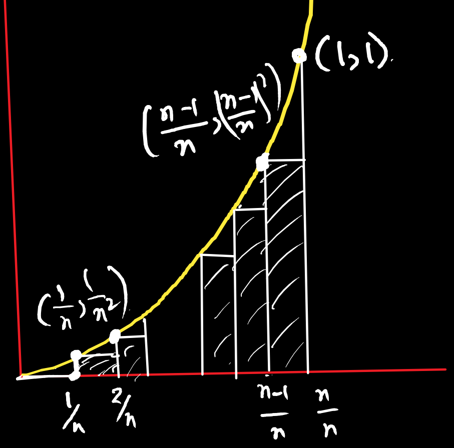
$$\boxed{\frac{1}{8}\le A_R\le\frac{5}{8}}$$

$$\boxed{0\le A_R\le 1}$$

$\boxed{n=1000}$

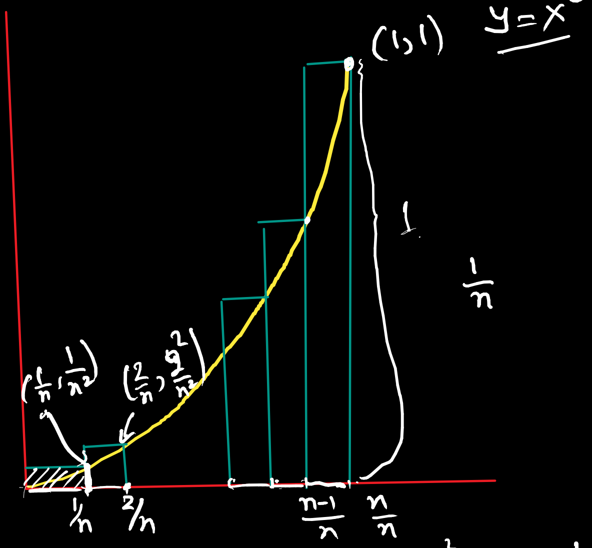
$$0.3328335 \le A_R \le 0.3338335$$

$A_R \approx 0.33$ (two decimal places)



$$L_n\le A_R\le U_n$$

$$\forall n=1,2,\dots$$

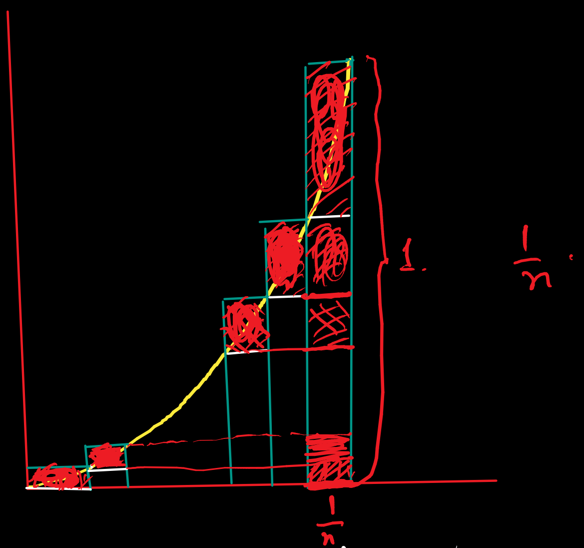


$$L_n=\frac{1}{n}\times 0+\frac{1}{n}\times\frac{1}{n^2}+\frac{1}{n}\times\frac{2^2}{n^2}+\dots+\frac{1}{n}\times\frac{(n-1)^2}{n^2}$$

$$=\frac{1}{n^3}\left(1^2+2^2+\dots+\frac{(n-1)^2}{n^2}\right)$$

$$U_n=\frac{1}{n}\times\frac{1}{n^2}+\frac{1}{n}\times\frac{2^2}{n^2}+\dots+\frac{1}{n}\times\frac{n^2}{n^2}$$

$$=\frac{1}{n^3}\left(1^2+2^2+\dots+n^2\right)$$



$$U_n-L_n=\frac{1}{n^3}\left(\cancel{1^2}+\cancel{2^2}+\cancel{\dots}+n^2\right)-\frac{1}{n^3}\left(\cancel{1^2}+\cancel{2^2}+\dots+\cancel{(n-1)^2}\right)=\frac{1}{n}$$

$$\lim_{n\to\infty}(U_n-L_n)=\lim_{n\to\infty}\frac{1}{n}=0$$

$$\boxed{A_R=\frac{1}{3}}$$

$$L_n\le A_R\le U_n \quad \forall n\ge 1.$$

$$\boxed{\lim_{n\to\infty}U_n=\lim_{n\to\infty}L_n}$$

$\lim_{n\to\infty}$ $\lim_{n\to\infty}$

$$\lim_{n\to\infty}L_n\le A_R\le\lim_{n\to\infty}U_n$$

$$A_R=\lim_{n\to\infty}L_n=\lim_{n\to\infty}U_n$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\left(1^2+2^2+\dots+n^2\right)$$

$$1^2+2^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Exercise.

$f(x)=x^3$.

$$=\lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{1}{6}\lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)=\frac{1}{6}\lim_{n\to\infty}\left(\underset{\substack{\downarrow\\1}}{1+\frac{1}{n}}\right)\left(\underset{\substack{\downarrow\\2}}{2+\frac{1}{n}}\right)=\frac{1}{3}$$