11/04/2022

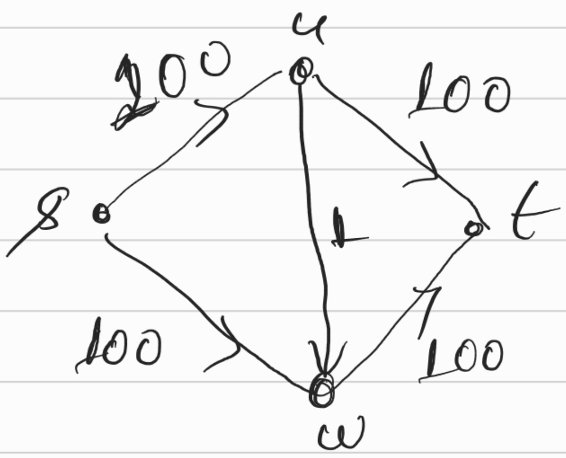
Assumption : Capacities are integers.

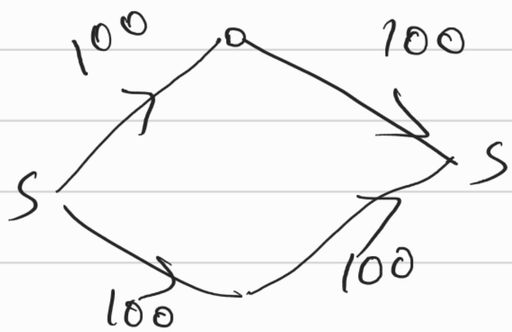
Can we do Rational numbers?
Yes by reducing to integer case.

Can we do Real numbers?

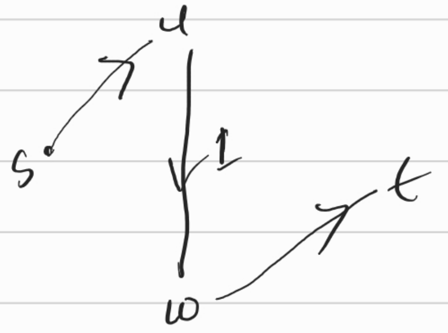
How to choose augmenting paths?

$$v(f') \geq v(f) + \text{bottleneck}(P, f)$$

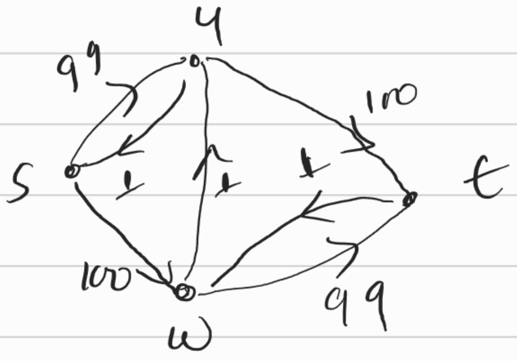




residual graph



→ increment by 1



→ $s \to w \to u \to t$

→ again increment by 1.

Runtime of the Ford-Fulkerson Algo: $O(C \cdot m)$ where

$$C = \sum_{e \text{ going out of } s} c(e)$$

To represent C how many bits do you need?

$(\log C)$

\# edges $= m$ each capacity requires $(\log C)$-bits.

poly time run-time $\equiv$ poly$(m, \log C)$

$\not\equiv$ poly(m, C)

Choosing Augmenting Paths.

natural choice :- augment with a s-t path that has the largest bottle-neck among all s-t paths

Let $\Delta \geq 1$ be a parameter.

Define residual graph wrt Δ, $G_f(\Delta)$, is obtained by keeping those edges in G_f that has residual capacity $\geq \Delta$.

Scaling Max-flow

Initialize with flow equals 0.

Set Δ = largest power of 2 less than equal to $C := \max_{e \text{ going out of } s} c(e)$

$(\Delta \leq C < 2\Delta)$

While $\Delta \geq 1$

Δ-Scaling phase

While there is a s-t path P in $G_f(\Delta)$

$f' = \text{Augment}(P, f)$

Update f to f' and $G_f(\Delta)$ to $G_{f'}(\Delta)$.

End while

$\Delta = \frac{\Delta}{2}$

End while

Return (f).

Runtime analysis :

① The no. of iterations of Outer While loop is $1 + \lceil \log C \rceil$

$< 1 \leftarrow C$

Lemma: Let f be a flow at the end of Δ-scaling phase.

Then $\exists$ an st-cut (A, B) s.t.

$$c(A, B) \le v(f) + m\Delta.$$

edges.

Proof:-

$\exists$ no s-t path in $G_f(\Delta)$.

Consider (A, B) where

$A = \{ u \mid u \text{ reachable from } s \text{ in } G_f(\Delta) \}$

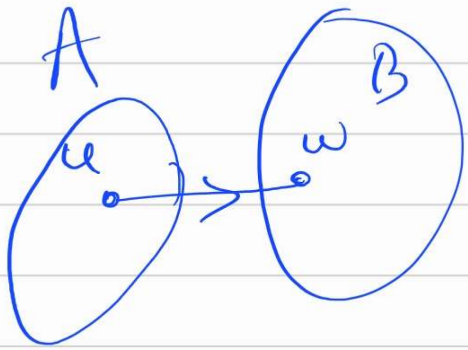
$B = V \setminus A$

$s \in A$ and $t \in B$?

Since there is no s-t path in $G_f(\Delta)$.

$v(f) = f^{out}(A) - f^{in}(B)$ — 1

$e = (u, w)$



What is residual capacity of e in $G_f(\Delta)$

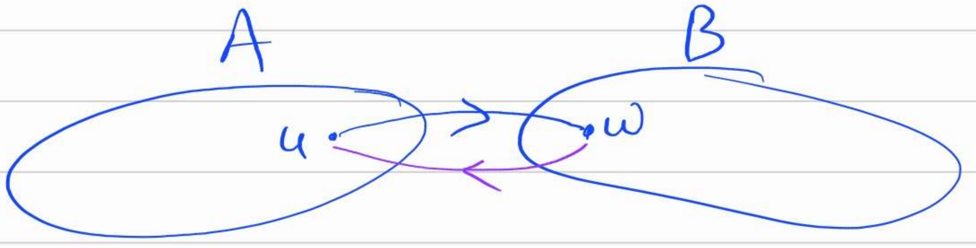
if (u, w) is a forward edge.
residual capacity $= c(e) - f(e)$

if (u,w) exists then $w \in A$.

So we cannot have this forward edge in $G_f(\Delta)$.

$$\Rightarrow \quad c(e) - f(e) < \Delta$$

$$\Rightarrow \quad c(e) < f(e) + \Delta.$$



Can (u,w) exist as a backward edge in $G_f(\Delta)$?

No.

What is the residual capacity of this edge? $f(e) < \Delta$

$$v(f) = f^{out}(A) - f^{in}(A)$$

$$= \sum_{e \text{ going out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= \sum_{e \text{ forward edges.}} f(e) - \sum_{\substack{e \text{ s.t. } e \text{ gives} \\ \text{a backward edge}}} f(e)$$

$$\geq \sum \left(c(e) - \Delta\right) - \sum \Delta$$

$$\geq \underbrace{\sum c(e)}_{} - \sum \Delta - \sum \Delta$$

$$\geq c(A,B) - m \cdot \Delta$$

$\blacksquare$

Lemma: The no. of augmentations in a fixed Δ-scaling phase is $\leq 2 \cdot m$.

Runtime : $O(\log C \cdot 2m \cdot m)$

$= O(\log C \cdot m^2)$

Proof : In the beginning

iteration $\leq 2m$

$\frac{C}{2} \leq \Delta \leq C$

$c(e) - \Delta < \Delta$

Consider: at the beginning of a Δ-scaling phase

Let f^* be the optimum - flow.

Let $\Delta' = 2\Delta$ be the previous scaling phase.

Let f' be the flow at end of Δ'-scaling phase.

$v(f^*) \leq v(f') + m \cdot \Delta' \leq v(f') + 2m\Delta$

$\Rightarrow$ # augmentations in Δ-scaling phase $\leq 2m$.