Calculus - Assignment 3 - Limit, Continuity and Differentiability

Last modified: December 14, 2020

Before trying to solve the assignments, you should see all the video lectures that I have uploaded in YouTube (particularly the lectures on Limit, Continuity and Differentiability):
Click here to get the YouTube Link for the playlist.
You may use the theorems (without giving the proofs) from the video lectures, but if I ask you to do in a particular method, then you should do in that way to understand that particular theory better.

1. Let $a, b, c \in \mathbb{R}$ with $a < c < b$. Let $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements. In case of disproving a statement, show that in general it is false, by providing particular example.

(i) $\lim_{x\to c}(fg)(x) = 0$.

(ii) $\lim_{x\to c}(fg)(x) = 0$, provided the function $g : (a, b) \to \mathbb{R}$ is bounded, i.e., there exists $M \in \mathbb{R}_{>0}$ such that $|g(x)| \leqslant M$ for all $x \in (a, b)$.

(iii) $\lim_{x\to c}(fg)(x) = 0$, if $\lim_{x\to c} g(x)$ exists.

2. Prove the following:

(i) $\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$ does not exist.

(ii) $\lim_{x\to 0}\left(x \sin\frac{1}{x}\right) = 0$.

(iii) $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

3. Prove the following statements.

(i) The function $f(x) = \sin(x)$ is continuous at every point $c \in \mathbb{R}$.

(ii) From 2iii and 3i, deduce that the function

$$g(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

from $\mathbb{R}$ to itself is continuous.

4. Let $f(x) = x^2\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$, but f' is not continuous on $\mathbb{R}$.

5. Compute $\frac{dy}{dx}$, given $y = f\left(\frac{2x-1}{x+1}\right)$ and $f'(x) = \sin(x^2)$.

6. Show that the cubic $x^3 - 6x + 3$ has all roots real.

7. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. Suppose that $f(x)$ has three distinct real roots. Prove the following statement.

(i) $p < 0$.

(ii) The function f attains maximum/minimum at $x = \pm\sqrt{(-p)/3}$.

(iii) The maximum/minimum values are of opposite signs.

(iv) $4p^3 + 27q^2 < 0$.

8. If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} = 0$, where $c_0, c_1, \ldots, c_n$ are real numbers, show that the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ has at least one real solution between 0 and 1.

9. Use the Mean Value Theorem to prove that $|\sin(a) - \sin(b)| \leqslant |a - b|$ for all $a, b \in \mathbb{R}$.

First you should try on your own. If you need, then see the next page for some hints.

Hints

1. For (i), find counterexamples. The statement (ii) can be proved using $\epsilon - \delta$ definition, while (iii) is proved in class (Lecture 20).

2. **2i.** Use the definition of limit in terms of sequences (The sequential criterion).

2ii Sandwich Theorem!

2iii. If $0 < x < \frac{\pi}{2}$, then $\sin(x) < x < \tan(x)$, hence $1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$. On the other hand, if $-\frac{\pi}{2} < x < 0$, then $\tan(x) < x < \sin(x)$, hence $1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$ as $\sin(x)$ is negative in this case. Thus if $x \in N'(0, \frac{\pi}{2})$ (deleted neighborhood of 0), then $1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$. Hence use Sandwich Theorem. You may assume the result that $\lim_{x \to 0} \cos(x) = \cos(0) = 1$ (which can be proved similarly as the proof of (3i) below).

3i For every $c \in \mathbb{R}$, prove that $\lim_{x \to c} \sin(x) = \sin(c)$. You may use the $\epsilon - \delta$ definition. Note that

$$\begin{aligned} |\sin(x) - \sin(c)| &= 2\left|\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right| \\ &\leqslant 2\left|\sin\left(\frac{x-c}{2}\right)\right| \quad [\text{Since } |\cos(y)| \leqslant 1] \\ &\leqslant |x-c| \quad [\text{Since } |\sin(y)| \leqslant |y|]. \end{aligned}$$

4.

5. Use the chain rule of differentiation.

6.

7.

8. Consider the suitable function $f : [0,1] \to \mathbb{R}$, and apply Rolle's Theorem.