Exercise 5.4

(a) $\lim_{n\to\infty} \frac{1}{n^{17}} \sum_{i=1}^{n} i^{16}$

$S_n = \frac{1}{n^{17}} \sum_{i=1}^{n} i^{16}$

$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{16}$

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\}$.

Then P_n is a partition of $[0,1]$ with $\mu(P_n) = \frac{1}{n}$.

Now $\mu(P_n) \to 0$ as $n \to \infty$.

Clearly $S_n = S(P_n, f)$ where $f(x) = x^{16}$ for $x \in [0,1]$.

with the tag set $\{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$.

Since f is integrable, we see that $S_n \to \int_0^1 f(x)\,dx$ as $n \to \infty$.

Thus. $\lim_{n\to\infty} S_n = \int_0^1 x^{16}\,dx \underset{FTC(2)}{=} \left.\frac{x^{17}}{17}\right]_{x=0}^{x=1} = \frac{1}{17}$

Corollary 5.2.

Suppose f is integrable on $[a,b]$ and let (P_n) be a sequence of partitions of $[a,b]$ such that $\mu(P_n) \to 0$ as $n \to \infty$.

Then $U(P_n, f) - L(P_n, f) \to 0$ as $n \to \infty$

Furthermore, if $S(P_n, f)$ is a Riemann sum corresponding to P_n and f then $S(P_n, f) \to \int_a^b f(x)\,dx$. as $n \to \infty$

$F(x) = \frac{x^{17}}{17}$ is differentiable in $[0,1]$.

and $F'(x) = x^{16}$ which is integrable.

$[0,1]$ $\quad \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)$

$[a,b]$ $\quad P_n = \{a, a + \frac{b-a}{n}, \frac{2(b-a)}{n}, \cdots, b\}$

$a + \frac{b-a}{n}$

$\frac{b-a}{n}$ $\quad \frac{b-a}{n}$

$\frac{b-a}{n}$ $\left(f\left(a + \frac{b-a}{n}\right) + f\left(a + 2\frac{b-a}{n}\right) + \cdots + f(b)\right)$

$\int_0^{\frac{\sqrt{\pi}}{e}} \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^{\frac{\sqrt{\pi}}{e}}$

$\tan^{-1} \frac{\sqrt{\pi}}{e}$

$F(x) = \frac{x^{17}}{17}$ $\quad F'(x)$

$\pi = \frac{22}{7}$

3.14159

$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$

$S_n = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)$

$S_n \to \int_0^{\frac{\sqrt{\pi}}{e}} f$

$f(x) = \frac{1}{1+x^2}$

$a = 0$

$b = \frac{\sqrt{\pi}}{e}$

Exercise 6.3(a)

$\int_1^\infty e^{-x}\,dx$

Note that e^{-x} is continuous in $[1, a]$ for all $a > 1$.

Thus e^{-x} is integrable on $[1, a]$ for all $a > 1$

$[1, a]$

$\forall\, a \geq 1$

$f : [a, b] \to \mathbb{R}$

Thus $\int_1^\infty e^{-x}\,dx$ is convergent $\Longleftrightarrow \lim_{a\to\infty} \int_1^a e^{-x}\,dx$ exists.

Now, $\int_1^a e^{-x}\,dx \overset{FTC(2)}{=\!=} -e^{-x}\Big]_1^a = -e^{-a} - (-e^{-1}) = \frac{1}{e} - \frac{1}{e^a} \to 0$

$\lim_{a\to\infty} \int_1^a e^{-x}\,dx = \lim_{a\to\infty}\left(\frac{1}{e} - \frac{1}{e^a}\right) = \frac{1}{e}$. exists and hence the integral is convergent.

Exc 6.8 (a) $\int_1^\infty \frac{1}{1+x}\,dx$. (Note the change in lower limit).

We know that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and $\neq 0$.

Let $f(x) = \frac{1}{x}$

$g(x) = \frac{1}{x+1}$

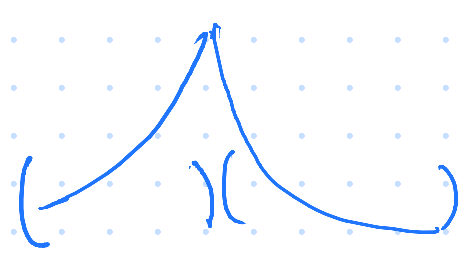
Then $\frac{f(x)}{g(x)} = \frac{x+1}{x} \to 1$ as $x \to \infty$. $\neq 0$

$(-3, 4)$

$(-3, -1)(-1, 4)$

By the limit comparison test, $\int_1^\infty \frac{1}{x+1}\,dx$ is divergent.

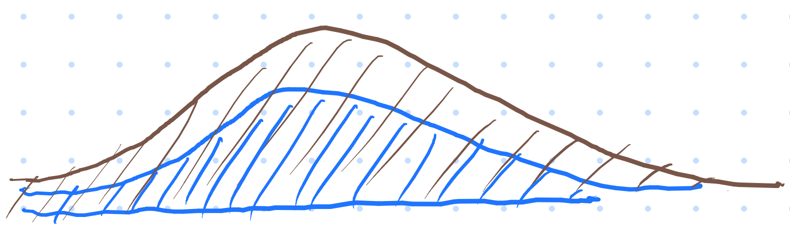
$[a, b]$



$\int_a^b f$ conv. iff $\int_a^c f$ & $\int_c^b f$

$(a, c)\,(c, b)$

$0 \leq f(x) \leq g(x)$



Exercise 7.3

Determine the area of the region bounded by

$y = -x^2 + 3x = f(x)$

$y = 2x^3 - x^2 - 5x = g(x)$.

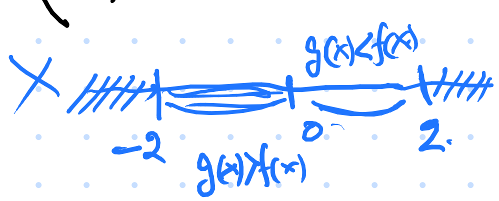
$$g(x) - f(x) = (2x^3 - x^2 - 5x) - (-x^2 + 3x)$$
$$= 2x^3 - 8x$$
$$= 2x(x^2 - 4)$$
$$= 2x(x+2)(x-2)$$

The curves $y = f(x)$, $y = g(x)$ intersect at the points

$(0, 0)$ $x=0$

$(2, 2)$ $x=2$

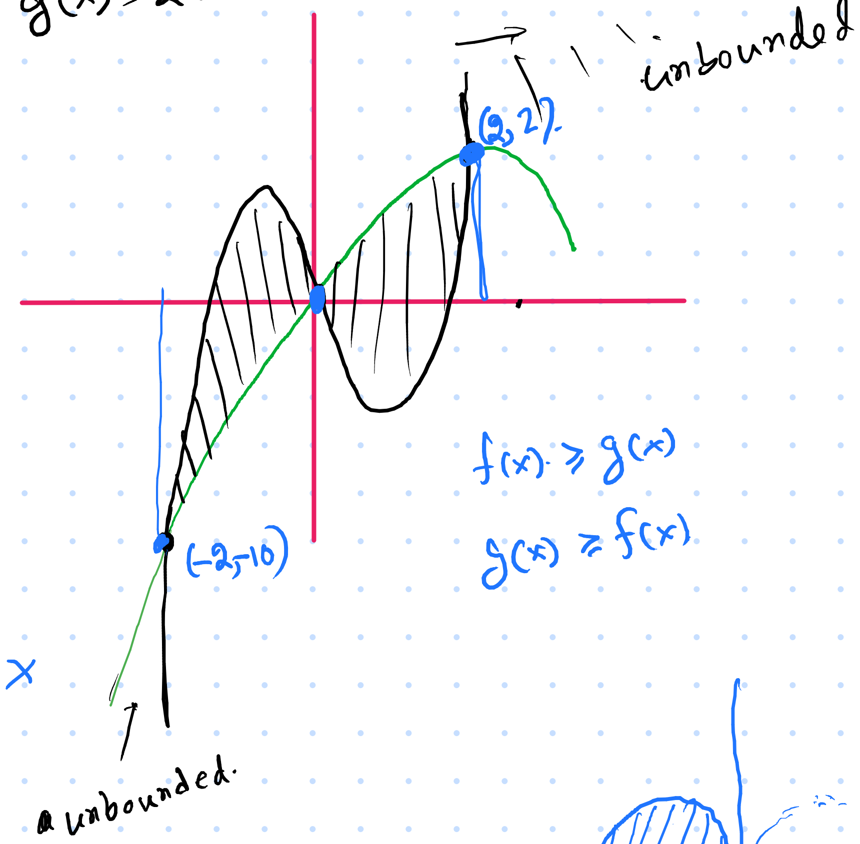
$(-2, -10)$ $x=-2$



x	<-2	$x>2$	$0<x<2$	$-2<x<0$
$g(x) > f(x)$		✓		✓
$g(x) < f(x)$	✓		✓	

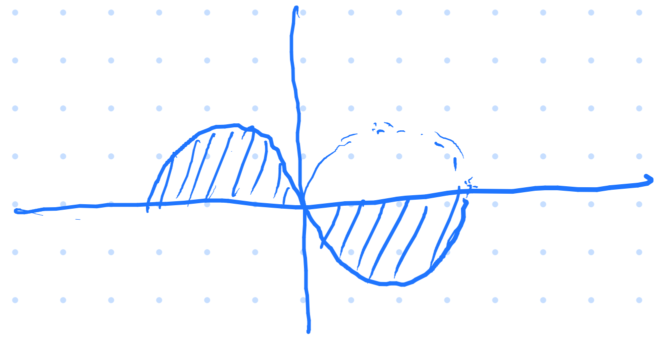
$f(x) = -x^2 + 3x$

$g(x) = 2x^3 - x^2 - 5x$



$f(x) \geq g(x)$

$g(x) \geq f(x)$



Area of the required region

$$= \int_{-2}^{0} (g(x) - f(x))\,dx + \int_{0}^{2} (f(x) - g(x))\,dx.$$

$$= \int_{-2}^{0} (2x^3 - 8x)\,dx + \int_{0}^{2} (8x - 2x^3)$$

$$\overset{FTC(2)}{=} 2 \cdot \frac{x^4}{4}\Big]_{-2}^{0} - 8 \cdot \frac{x^2}{2}\Big]_{-2}^{0} + 8\,\frac{x^2}{2}\Big]_{0}^{2} - 2 \cdot \frac{x^4}{4}\Big]_{0}^{2}$$

$$= 16$$