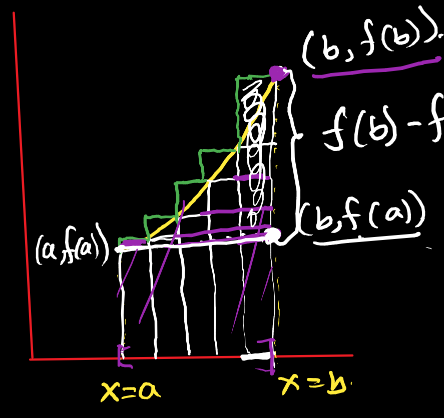
Generalizations:

$f:[0,1]\to\mathbb{R}$. $[a,b]$.

(1) $\boxed{f(x)=x^2}$ is an increasing function.

This idea applies straightaway to any monotonically increasing continuous functions $f:[a,b]\to\mathbb{R}$.



$[a,b]$

$$U_n - L_n = \left(f(b)-f(a)\right)\frac{(b-a)}{n} \xrightarrow{n\to\infty} 0$$

$$L_n \le A_R \le U_n \quad \forall n.$$

$\xrightarrow{n\to\infty} 0$

$$\boxed{A_R = \lim_{n\to\infty} U_n.}$$

$$f(a) \le f(x) \le f(b)$$

(2) This idea even generalize to the continuous functions $f:[a,b]\to\mathbb{R}$.

Divide the interval $[a,b]$ into n equal parts:

$$\underbrace{\left[a, a+\frac{b-a}{n}\right]}_{I_1},\ \underbrace{\left[a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}\right]}_{I_2},\ \dots,\ \underbrace{\left[a+\frac{(n-1)(b-a)}{n}, b\right]}_{I_n}$$

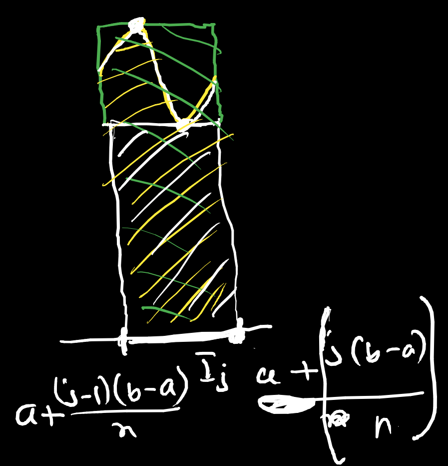
For the network of bigger rectangles:

on the base I_j construct rectangle with height $\max\{f(x) \mid x\in I_j\}$

$[a,b]$

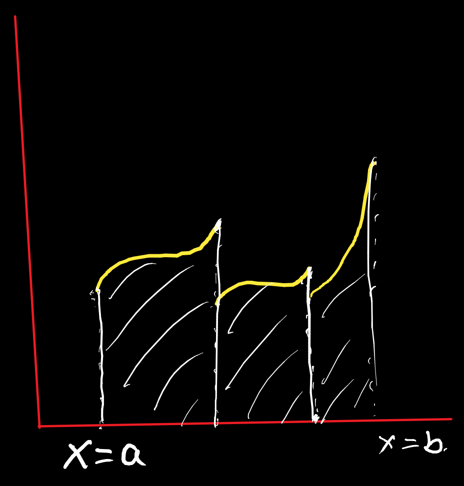
Smaller rectangles

on the base I_j draw a rectangle with height $\min\{f(x) \mid x\in I_j\}$



$$L_n \le A_R \le U_n$$

(3) Piecewise continuous bounded functions with finitely many points of discontinuity.



(4) We have seen that

$$A_R = \underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(M_j)}_{\lim_{n\to\infty} U_n} = \underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(m_j)}_{\lim_{n\to\infty} L_n}$$

$$f(M_j) = \max\{f(x) \mid x\in I_j\}$$
$$f(m_j) = \min\{f(x) \mid x\in I_j\}$$

What if we chose a random point $c_j \in I_j$ $\forall j=1,\dots,n.$ constructed the sum $\frac{1}{n}\sum_{j=1}^{n} f(c_j) \xrightarrow[n\to\infty]{?}$

for all j

$$\sum_{j=1}^{n}\frac{1}{n}f(m_j) \le \sum_{j=1}^{n}\frac{1}{n}f(c_j) \le \sum_{j=1}^{n}\frac{1}{n}f(M_j)$$

lim

A_R.

$y=f(x)$

(5) We would like to say that $\boxed{A_R = \int_a^b f(x)\,dx}$ "whenever it makes sense" !!.