Discrete Structures Assignment 2

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February 2021

Problem 1

problem 1(a)

Given, $f(n)\,a_n \,=\, g(n)\,a_{n-1} + h(n)$ for $n \geq 1$ and $a_0 \,=\, C$

Define, $Q(n) \,=\, \frac{(f(1)\,f(2)....f(n-1))}{(g(1)\,g(2)....g(n))} \,=\, \frac{\prod_{i=1}^{n-1} f(i)}{\prod_{i=1}^{n} g(i)}$

$$f(n) - g(n)a_{n-1} \,=\, h(n)$$

by Multiplying Q(n) on both sides of the above equation we get

$$(f(n)\,a_n - g(n)\,a_{n-1})\,Q(n) \,=\, h(n)\,Q(n)$$

$$f(n)\,a_n\,Q(n) - g(n)\,a_{n-1}\,Q(n) \,=\, h(n)\,Q(n)$$

$$\frac{a_n\,f(n)\prod_{i=1}^{n-1} f(i)}{\prod_{i=1}^{n} g(i)} - g(n)\,a_{n-1}\,Q(n) \,=\, h(n)\,Q(n)$$

by including $f(n)$ in numerator's product of above equation multiplying numerator and denominator with $g(n+1)$ we get

$$\frac{a_n\,g(n+1)\prod_{i=1}^{n} f(i)}{g(n+1)\,\prod_{i=1}^{n} g(i)} - g(n)\,a_{n-1}\,Q(n) \,=\, h(n)\,Q(n)$$

we know that $\frac{\prod_{i=1}^{n} f(i)}{g(n+1) \prod_{i=1}^{n} g(i)} = Q(n+1)$ by substituting $Q(n+1)$ in the above equation we get

$$Q(n+1)\, g(n+1)\, a_n - Q(n)\, g(n)\, a_{n-1} \,=\, h(n)\, Q(n)$$

Assume $Q(n+1)\, g(n+1)\, a_n \,=\, b_n$ hence the given recurrence relation is converted to Non-Homogeneous Recurrence Relation

$$b(n) - b(n-1) \,=\, h(n)\, Q(n)$$

$$b(n) \,=\, b(n-1) + h(n)\, Q(n)$$

problem1(b)

Given that $Q(1)\, g(1) \,=\, f(0) \,=\, 1$

Consider for some $i \in \mathbb{N}$ $b(i) - b(i-1) \,=\, h(i)\, Q(i)$ let us do the following summation to

$$\sum_{i=1}^{n} (b_i - b_{i-1}) \,=\, \sum_{i=1}^{n} (h(i)\, Q(i)$$

$$b_n - b_0 \,=\, \sum_{i=1}^{n} (h(i)\, Q(i)$$

$$Q(n+1)\, g(n+1)\, a_n - Q(1)\, g(1)\, a_0 \,=\, \sum_{i=1}^{n} (h(i)\, Q(i)$$

$$Q(n+1)\, g(n+1)\, a_n - C \,=\, \sum_{i=1}^{n} (h(i)\, Q(i)$$

$$a_n \,=\, \frac{C + \sum_{i=1}^{n} (h(i)\, Q(i)}{Q(n+1)\, g(n+1)}$$

Finally after solving the recurrence we get the value of a_n as

$$a_n \,=\, \frac{C + \sum_{i=1}^{n} (h(i)\, Q(i))}{Q(n+1)\, g(n+1)}$$

□

Problem 2

problem 2(a)

Given, $(n+1)\,a_n \;=\; (n+3)\,a_{n-1} + n$ for $n \geq 1$ and $a_0 \;=\; 1$

by previous exercise we know that

$$f(n) = n+1$$
$$g(n) = n+3$$
$$h(n) = n$$
$$Q(n) = \frac{1.2.3....n}{4.5....n+3} = \frac{6\,n!}{(n+3)!} \text{ (as per its definition)}$$

Applying the a_n result from previous exercise we get

$$a_n = \frac{C + \sum_{i=1}^{n}(h(i)\,Q(i))}{Q(n+1)\,g(n+1)}$$

$$a_n = \frac{1 + \sum_{i=1}^{n}\left(i\,\frac{6i!}{(i+3)!}\right)}{\frac{6(n+1)!}{(n+4)!}\,(n+4)}$$

$$a_n = \frac{1 + \sum_{i=1}^{n}\left(\frac{6i}{(i+1)\,(i+2)\,(i+3)}\right)}{\frac{6}{(n+2)\,(n+3)}}$$

Solving Summation using telescopic addition

$$\sum_{i=1}^{n}\left(\frac{6i}{(i+1)\,(i+2)\,(i+3)}\right) = \sum_{i=1}^{n}\frac{3}{(i+2)}\left(\frac{3}{(i+3)} - \frac{1}{(i+1)}\right)$$

$$= \sum_{i=1}^{n} 9\left(\frac{1}{(i+2)} - \frac{1}{(i+3)}\right) - \sum_{i=1}^{n} 3\left(\frac{1}{(i+1)} - \frac{1}{(i+2)}\right)$$

$$= 9\left(\frac{1}{3} - \frac{1}{(n+3)}\right) - 3\left(\frac{1}{2} - \frac{1}{(n+2)}\right)$$

$$= \frac{3}{2} - \frac{9}{(n+3)} + \frac{3}{(n+2)}$$

$$\textbf{Summation} = \frac{3}{2} - \frac{(6n+9)}{(n+2)(n+3)}$$

By substituting value of summation in a_n we get

$$a_n = \frac{1 + \frac{3}{2} - \frac{(6n+9)}{(n+2)(n+3)}}{\frac{6}{(n+2)(n+3)}}$$

$$a_n = \frac{\frac{5(n^2+5n+6) - 2(6n+9)}{2\,(n+2)(n+3)}}{\frac{6}{(n+2)(n+3)}}$$

$$a_n = \frac{5n^2+13n+12}{12}$$

$$\therefore\ a_n = \frac{n\,(5n+13)}{12} + 1$$

problem 2(b)

Execution code:

```
from sympy import Function, rsolve
from sympy.abc import n
g = Function('g')
Func = g(n-1) - (n+1) * g(n)
print 'Solving the Recurrence', Func
soln = rsolve(Func, g(n), g(0): 1)
print soln
```

Output:

```
>>> from sympy import Function, rsolve
... from sympy.abc import n
... g = Function('g')
... Func = g(n − 1) − (n + 1) ∗ g(n)
... print 'Solving the Recurrence', Func
... soln = rsolve(Func, g(n), g(0) : 1)
... print soln
Solving the Recurrence −(n + 1) ∗ g(n) + g(n − 1)
```

$$\frac{1}{\Gamma(n+2)}$$

$$\text{Which is further equals to } \left(\frac{1}{(n+1)!}\right)$$

Hence the problem solved using **SymPy**

□

Problem 3

Theorem: *Let* $c_1\,, c_2 \in \mathbb{R}$ *with* $c_2 \neq 0$. *Suppose that* $r^2 - c_1 r - c_2 = 0$ *has only one root* r_0. *A sequence* $\{a_n\}$ *is a solution of the recurrence relation* $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ *iff* $a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$ *for* $n = 0, 1, 2....$, *where* α_1 *and* α_2 *are constants*

Proof: Let $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$ be a recurrence relation whose character equation is $r^2 - c_1 r - c_2 = 0$ has one root r_0

We have $\Delta = c_1^2 + 4\, c_2 = 0$, $r_0 = \dfrac{c_1}{2}$ and $r_0^2 = c_1\, r_0 + c_2$ from quadratic equation

Let solution of above recurrence be of form $a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$ where α_1 and α_2 are constants

$$
\begin{aligned}
a_n &= c_1\, a_{n-1} + c_2\, a_{n-2} \\
&= c_1\,(\alpha_1\, r_0^{n-1} + \alpha_2\,(n-1)\, r_0^{n-1}) + c_2\,(\alpha_1\, r_0^{n-2} + \alpha_2\,(n-2)\, r_0^{n-2}) \\
&= (c_1\, r_0 + c_2)\,\alpha_1\, r_0^{n-2} + (c_1\, r_0 + c_2) n\, r_0^{n-2}\,\alpha_2 - (c_1\, r_0 + 2c_2) r_0^{n-2}\,\alpha_2 \\
&= (c_1\, r_0 + c_2)\,\alpha_1\, r_0^{n-2} + (c_1\, r_0 + c_2) n\, r_0^{n-2}\,\alpha_2 - \left(\frac{c_0^2 + 4\, c_2}{2}\right) r_0^{n-2}\,\alpha_2 \\
&= (c_1\, r_0 + c_2)\,\alpha_1\, r_0^{n-2} + (c_1\, r_0 + c_2) n\, r_0^{n-2}\,\alpha_2 \\
&= \alpha_1\, r_0^n \,+\, \alpha_2\, r_0^n \\
&= a_n
\end{aligned}
$$

$\because\ 2\, r_0 = c_1$ and $\Delta = c_1^2 + 4\, c_2 = 0$ and $r_0^2 = c_1\, r_0 + c_2$

To show every solution of the Recurrence has the same form as above consider the following statements

$$a_0 = c_0 = \alpha_1$$

$$a_1 = c_1 = (\alpha_1 + \alpha_2)\, r_0$$

$$\alpha_1 = c_0$$

$$\alpha_2 = \frac{c_1 - r_0\, c_0}{r_0}$$

With the values of α_1 and α_2 we obtain a sequence $\{a_n\}$ such that

$$a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$$

satisfy same initial conditions as the given recurrence relation

We know that for a linear homogeneous recurrence of degree 2 Unique solution is obtained for two given initial conditions

$\therefore\ a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$ is the only possible solution for the given recurrence relation $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$

$\square$

Problem 4

Given f is an increasing function and a Recurrence relation

$$f(n) = a\,f\left(\frac{n}{b}\right) + c\,n^d$$

where $a \geq 1$, $b > 1$ and $c\,, d\ \in \mathbb{R}$

(a)

$$(a = b^d) \wedge (n = b^k) \implies f(n) = f(1)\,n^d + c\,n^d\,\log_b n$$

Proof: Let $a = b^d$ and $n = b^k$ we know that,

$$f(n) = f(1)\,a^k + \sum_{i=0}^{k-1} a^i\,g\left(\frac{n}{b^i}\right)$$

$$f(n) = f(1)\,(b^d)^k + \sum_{i=0}^{k-1} (b^d)^i\,\left(c\left(\frac{n}{b^i}\right)^d\right)$$

$$f(n) = f(1)\,(b^k)^d + \sum_{i=0}^{k-1} (b^{di})\,\left(\frac{c\,n^d}{b^{di}}\right)$$

$$f(n) = f(1)\,n^d + \sum_{i=0}^{k-1}\,c\,n^d\ \ (\because\ n = b^k)$$

$$f(n) = f(1)\,n^d + k\,c\,n^d\ \ (\because\ c, n \text{ are independent of } i)$$

$$f(n) = f(1)\,n^d + c\,n^d\,\log_b n\ \ (\because\ k = \log_b n)$$

$\therefore$ The solution of given recurrence is $f(n) = f(1)\,n^d + c\,n^d\,\log_b n$

$\square$

(b)

$(a \neq b^d) \wedge (n = b^k) \implies f(n) = c_1\, n^d + c_2\, n^{\log_b a}$ where $c_1 = \dfrac{b^d\, c}{b^d - a}$ and $c_2 = f(1) + \dfrac{b^d\, c}{a - b^d}$

proof: Let $a \neq b^d$ and $n = b^k$ we know that,

$$f(n) = f(1)\, a^k + \sum_{i=0}^{k-1} a^i\, g\left(\frac{n}{b^i}\right)$$

$$f(n) = f(1)\, a^{\log_b n} + \sum_{i=0}^{k-1} a^i\, c \left(\frac{n}{b^i}\right)^d$$

$$f(n) = f(1)\, n^{\log_b a} + \sum_{i=0}^{k-1} c\, n^d \left(\frac{a}{b^d}\right)^i$$

$$f(n) = f(1)\, n^{\log_b a} + c\, n^d \sum_{i=0}^{k-1} \left(\frac{a}{b^d}\right)^i$$

$$f(n) = f(1)\, n^{\log_b a} + c\, n^d\, \frac{\left(\frac{a}{b^d}\right)^k - 1}{\left(\frac{a}{b^d}\right) - 1}$$

$$f(n) = f(1)\, n^{\log_b a} + c\, b^d\, n^d\, \frac{\frac{a^{log_b n}}{(b^{log_b n})^d} - 1}{a - b^d}$$

$$f(n) = f(1)\, n^{\log_b a} + c\, b^d\, n^d\, \frac{\frac{n^{log_b a} - n^d}{n^d}}{a - b^d}$$

$$f(n) = f(1)\, n^{\log_b a} + \frac{b^d\, c\, n^{log_b n}}{a - b^d} - n^d\, \frac{b^d\, c}{a - b^d}$$

$$f(n) = \left(\frac{b^d\, c}{b^d - a} \right) n^d + \left(f(1) + \frac{b^d\, c}{a - b^d} \right) n^{log_b a}$$

Let $c_1 = \dfrac{b^d\, c}{b^d - a}$ and $c_2 = \left(f(1) + \dfrac{b^d\, c}{a - b^d} \right)$ we finally get the solution for given recurrence such that $a \neq b^d$ as

$$f(n) = c_1\, n^d + c_2\, n^{log_b a}$$

$\because$ We used logarithmic identities $b^{log_b n} = n$ and $a^{log_b n} = n^{log_b a}$ we achieved the solution of the given recurrence as

$$f(n) = \begin{cases} f(n) = f(1)\, n^d + c\, n^d\, \log_b n & \text{if } a = b^d \\ f(n) = \left(\dfrac{b^d\, c}{b^d - a} \right) n^d + \left(f(1) + \dfrac{b^d\, c}{a - b^d} \right) n^{log_b a} & \text{if } a \neq b^d \end{cases}$$

$\square$
