Error Probability for Multilevel Digital Systems in Presence of Intersymbol Interference and Additive Noise

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***Abstract*—Generalization of a recently published technique for the evaluation of error probability in fiber-optic communication systems is described. The crux of the method is a minimax approximation of the cumulative distribution function of the additive noise. The additive noise is not constrained to be Gaussian. Examples and comparisons with previously published techniques are presented.**

I. Introduction

The PRIMARY objective in digital communication system design is to maximize repeater spacing while maintaining a specified error performance. The most meaningful and appropriate criterion in evaluating system performance is the average error probability (bit error rate). The use of computer-aided design (CAD) tools to estimate the error probability as function of the specific characteristics of system components has become increasingly important. These computer-aids can also be utilized as efficient tools for margin allocation, for sensitivity analysis to identify components which affect the system performance most and for cost-performance tradeoffs. Obviously, the success of any CAD tool depends to a large extent on the efficiency and accuracy of the technique it uses to compute the bit-error rate. Consequently, the evaluation of such measure has received considerable attention [1]–[13].

The exhaustive technique is based on evaluating the conditional error probability for each of the possible sequences of data and computing the average over all possible sequences. The computational cost of this technique is usually expensive and this limits the number of interfering samples which can be taken into account. Another approach is to bound the error probability [12], [13].

Two techniques which have been seriously considered in the literature are the series expansion method [5]–[7] and the use of Gauss quadrature rules [10], [11]. The first method is a simple one. However, its convergence is slow and it provides oscillating results when the channel distortion, signal to noise ratio, or the number of levels increases. The second method is more accurate than the series expansion. However, the numerical procedure becomes increasingly ill-conditioned as the number of moments of the random variable representing the intersymbol interference is increased.

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Recently, a new method has been published for the evaluation of error probability in fiber-optic communication systems [3]. In this paper, the technique described in [3] is generalized for multilevel digital systems. The proposed computational algorithm is based on deriving a best approximation (in a minimax sense) for the cumulative distribution function of the additive noise. This guarantees that for a given number of moments the error in evaluating the error probability is minimum.

In Section II, a mathematical formulation of the problem is given. The proposed minimax approach is introduced in Section III. Examples and comparison with published techniques are presented in Section IV. An upper bound for the truncation error in the case of Gaussian additive noise is derived in the Appendix.

II. Problem Statement

The system to be considered is shown in Fig. 1. The input to the decision circuit at time t is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k h(t - kT) + n(t) \tag{1}$$

where $n(t)$ represents the equivalent additive noise at the receiver output, $h(t)$ is the impulse response of the overall linear time-invariant system, $1/T$ is the bit rate, $\{a_k\}$ represents a sequence of random symbols. Each a_k assumes one of two values d_1 and $d_2 > d_1$ with probabilities p_1 and p_2, respectively.

The received signal at the decision time t_0 is given by

$$y_0 = a_0 h_0 + n_0 + x \tag{2}$$

where

$$y_0 = y(t_0),\ n_0 = n(t_0),\ h_0 = h(t_0),\quad \text{and } x = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k h(t_0 - kT).$$

The error probability is given by

$$P_e = p_1 P[y_0 > S \mid a_0 = d_1] + p_2 P[y_0 < S \mid a_0 = d_2] \tag{3}$$

where S is the threshold level and $P[|.]$ is the conditional probability with respect to the source symbols.

Equation (3) can be rewritten as

$$P_e = p_1 \int_R g(x)[1 - D(S - x - d_1 h_0)]\, dx + p_2 \int_R g(x) D(S - x - d_2 h_0)\, dx \tag{4}$$

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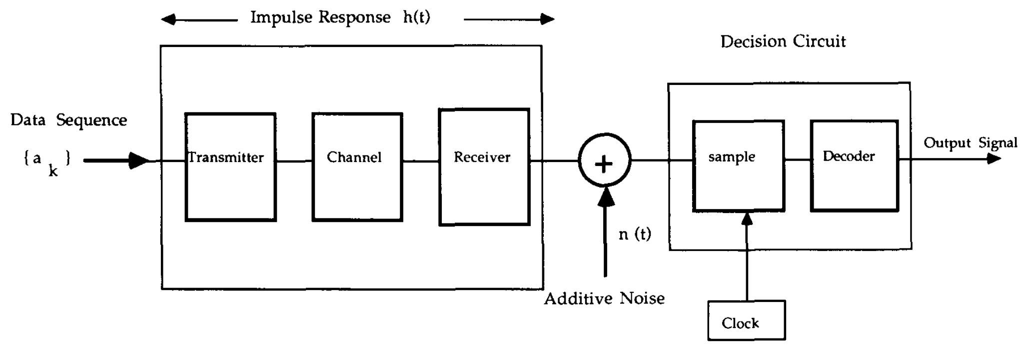

Fig. 1. Model of a data transmission system.

TABLE I
ERROR PROBABILITY FOR A BINARY DIGITAL SYSTEM (EXAMPLE 1)

Exhaustive Method [10] $P(e) = 2.7614 \times 10^{-3}$			
Order of highest moment used	Quadrature Rule [10] $P(e)$	Series Expansion [6], [10] $P(e)$	Minimax Approximation $P(e)$
4	3.77×10^{-5}	4.2×10^{-6}	2.9758×10^{-3}
6	8.86×10^{-4}	5.98×10^{-5}	2.7650×10^{-3}
8	2.4×10^{-3}	4.71×10^{-4}	2.7573×10^{-3}
10	2.9×10^{-3}	2.14×10^{-3}	2.7610×10^{-3}
12	2.8×10^{-3}	5.46×10^{-3}	2.7610×10^{-3}
14	2.74×10^{-3}	6.56×10^{-3}	2.761446×10^{-3}
16	2.75×10^{-3}	5.06×10^{-4}	2.761442×10^{-3}
18	2.766×10^{-3}	—	2.761425×10^{-3}
20	2.7617×10^{-3}	—	2.761425×10^{-3}
22	2.7615×10^{-3}	—	2.761425×10^{-3}
24	2.76164×10^{-3}	—	2.761425×10^{-3}

where $g(.)$ is the probability density functions of n_0, $D(.)$ is the cumulative distribution function of the additive noise and R is the range of definition of x. For mutilevel signals where a_k can take one of the values $\pm 1, \pm 3, \ldots, \pm(2L-1)$ with equal probabilities, the error probability is given by [10]

$$P_e = K(L)\int_R g(x)[1 - D(h_0 - x)]\,dx \tag{5}$$

where $K(L) = 2(1 - 1/2L)$ for pulse-amplitude modulation systems. For partial response coded (PRC) signaling schemes, values of $K(L)$ for all classes of PRC can be found in [10, Table I]. Equation 5 assumes slicing levels $0, \pm 2h_0, \cdots, \pm(2L-2)h_0$ and even probability density function for the additive noise.

In the special case that the additive noise is Gaussian, (5) can be reduced to

$$P_e = K(L)\int_R g(x)\,\mathrm{erfc}\left(\frac{H_0 + x}{\sqrt{2}\mu}\right)dx \tag{6}$$

where μ is the standard deviation of the additive noise and erfc(.) is the complementary error function

$$\mathrm{erfc}\,(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-w^2}\,dw.$$

From (4)–(6), the average error probability can be expressed in terms of the integral

$$I = \int_R f(x)g(x)\,dx \tag{7}$$

where the probability density function $g(x)$ of the intersymbol interference (ISI) is not known explicitly unless a direct enumeration of all possible sequences is performed, which requires large amount of computational CPU time.

III. EVALUATION OF THE ERROR PROBABILITY

If $f(x)$ has n continuous derivatives in the interval $[-a,\ a]$ then

$$\int_{-a}^{+a} f(x)g(x)\,dx \cong \boldsymbol{F}^\dagger \boldsymbol{B}^{(m)}\boldsymbol{A}\boldsymbol{M}^{(m)} \tag{8}$$

where † denotes transpose, $\boldsymbol{F}$ is n-dimensional vector $(m \le n)$ which contains the scaled derivatives of $f(x)$ at $x = 0$,

$$\boldsymbol{F}^\dagger = \left[f(0), \frac{f^{(1)}(0)a}{1!}, \cdots, \frac{f^{(n-1)}(0)a^{n-1}}{(n-1)!}\right] \tag{8a}$$

$\boldsymbol{M}^{(m)}$ is m-dimensional vector $(m \le n)$ which contains the scaled derivatives of $g(x)$,

$$\boldsymbol{M}^{(m)} = \left[M_0, \frac{M_1}{a}, \cdots, \frac{M_{m-1}}{a^{m-1}}\right]^\dagger \tag{8b}$$

where

$$M_k = \int_{-a}^{a} x^k g(x)\,dx$$

$\boldsymbol{A} = \{a_{i,j}\}$ and $\boldsymbol{B}^{(m)} = \{b_{i,j}^{(m)}\}$ are $m \times m$ and $n \times m$ **constant** matrices (independent of f or g) which are recursively generated.

For a given number of moments and for $n \gg 1$, the integral approximation in (8) corresponds to the best approximation, in a Chebyshev sense, of $f(x)$ in the interval $[-a,\ a]$. Proof of (8) is given in [3]. An upper bound for the truncation error is derived in the Appendix.

Using (8), the integrals in (4) can be approximated as follows:

$$P_e \cong [p_1\boldsymbol{F}_1^\dagger + p_2\boldsymbol{F}_2^\dagger]\boldsymbol{B}^{(m)}\boldsymbol{A}\boldsymbol{M}^{(m)} \tag{9}$$

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where $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ contain the scaled derivatives of $[1-D(S-x-d_1h_0)]$ and $D(S-x-d_2h_0)$; respectively. $\boldsymbol{M}^{(m)}$ contains the scaled moments of the ISI.

For $2L$-level system, the error probability is evaluated using (5) and (8):

$$P_e \cong K(L)\boldsymbol{F}^\dagger \boldsymbol{B}^{(m)}\boldsymbol{A}\boldsymbol{M}^{(m)} \tag{10}$$

where $\boldsymbol{F}$ contains the scaled derivatives of $[1-D(h_0-x)]$.

In the special case of Gaussian additive noise, the scaled derivatives are given by

$$\boldsymbol{F} \equiv \left\{\frac{f^{(i)}(0)}{i!}a^i\right\}; \quad i \in \{1, 2, \ldots, n\}$$

$$f^{(i)}(0) = \sqrt{\frac{2}{\pi}}\frac{(-1)^i}{\mu^i}e^{-\left[\frac{h_0}{\sqrt{2}\mu}\right]^2}H_{i-1}\left(\frac{h_0}{\mu}\right). \tag{11}$$

$H_k(x)$ is a Hermite polynomial of degree k which can be generated using the recursive relation

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x) \quad \text{with } H_0(x) = 1 \quad \text{and } H_1(x) = x.$$

IV. Examples and Comparisons

Example 1: This example considers the case of binary PAM transmission where the received pulse is assumed to have the form

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

The sampling time is $0.2T$ and the signal-to-noise ratio is taken to be 16 dB. For a truncated 11 pulse train approximation, Table I shows the results obtained using the minimax approximation as compared to those reported in [10] using the exhaustive technique, the series expansion method and the Gauss quadrature rule. The series expansion method ends with the 16 moments because the succeeding approximation gives a negative value for the error probability.

Example 2: In this example, a four-level PAM signal is considered. The received pulse is given by

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

For a truncated five-pulse train, a signal-to-noise ratio of 24 dB, and a sampling time $= 0.05T$, Table II shows the results obtained using the minimax approach as compared to those reported in [6] using the exhaustive technique.

Example 3: This example considers a binary system with *non-Gaussian* additive noise. The received pulse is given by

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}; \quad t_0 = 0.2T.$$

The additive noise is assumed to be Cauchy distributed with density function

$$\phi(x) = \frac{10^{-2}}{(x^2+10^{-4})}.$$

For a truncated 11-pulse train, Table III shows the convergence of the minimax approach as a function of the number of moments. It should be mentioned that CPU time required to obtain the results of any of the above examples using the minimax approximation was under 0.4 s on IBM3081KX.

TABLE II
ERROR PROBABILITY FOR A FOUR LEVEL SYSTEM (EXAMPLE 2)

Exhaustive Method [6] $P(e) = 5.2 \times 10^{-7}$

Order of Highest Moment Used	Minimax Approximation
8	1.4×10^{-7}
10	2.7×10^{-7}
12	4.0×10^{-7}
14	5.0×10^{-7}
16	5.27×10^{-7}
18	5.26×10^{-7}
20	5.21×10^{-7}
22	5.199×10^{-7}
24	5.204×10^{-7}
26	5.205×10^{-7}
28	5.204×10^{-7}
30	5.204×10^{-7}

TABLE III
ERROR PROBABILITY FOR A BINARY SYSTEM WITH NON-GAUSSIAN ADDITIVE NOISE (EXAMPLE 3)

Order of Highest Moment Used	Error Probability
2	3.19×10^{-3}
4	4.17×10^{-3}
6	4.115×10^{-3}
8	4.119×10^{-3}
10	4.118×10^{-3}
12	4.119×10^{-3}
14	4.119×10^{-3}
16	4.119×10^{-3}

V. Appendix
An Upper Bound for the Truncation Error

Equation (7) can be written as

$$I = \int_{-a}^{a}\boldsymbol{F}^\dagger[\boldsymbol{B}^{(m)}|\boldsymbol{B}^{(n-m)}]\begin{bmatrix}\boldsymbol{T}^{(m)}\\ \boldsymbol{T}^{(n-m)}\end{bmatrix}g(x)\,dx + R_n \tag{A-1}$$

where

$$R_n = \frac{f^{(n)}(\eta)}{n!}M_n; \quad -a \le \eta \le a$$

$$M_n = \int_{-a}^{a}g(x)x^n\,dx.$$

From (8) and (A-1); the truncation error is given by

$$\varepsilon = \boldsymbol{F}^\dagger\boldsymbol{B}^{(n-m)}\int_{-a}^{a}\boldsymbol{T}^{(n-m)}g(x)\,dx + R_n. \tag{A-2}$$

Since,

$$T_k\left(\frac{x}{a}\right) \le 1; \quad -a \le x \le a$$

hence,

$$|\varepsilon| \le |\boldsymbol{F}^\dagger\boldsymbol{S}| + |R_n| \tag{A-3}$$

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where $S = \{S_i\}$ where $S_i = 0;\ i \in \{1, 2, \ldots, m\}$ and $S_i = \sum_{k=m+1}^{i} b_{i,k};\ i \in \{m+1, m+2, \ldots, n\}$.

From (A-2) and (A-3)

$$|\varepsilon| \leq \sum_{i=m}^{n-1} \frac{a^i |f^{(i)}(0)}{i!} S_{i+1} + |R_n|. \tag{A-4}$$

In the special case of Gaussian additive noise

$$|f^{(n)}(\eta)| = \sqrt{\frac{2}{\pi}} \frac{1}{\mu^n} e^{-\left[\frac{h_0+\eta}{\sqrt{2}\mu}\right]^2} H_{n-1}\left(\frac{h_0+\eta}{\mu}\right). \tag{A-5}$$

An upper limit for the Hermite polynomial is given by [14, p. 787]

$$\left| H_{n-1}\left(\frac{h_0+\eta}{\mu}\right)\right| < B\, e^{\left[\frac{h_0+\eta}{\sqrt{2}\mu}\right]^2} \sqrt{(n!)} 2^{\frac{n}{2}} \tag{A-6}$$

where $B \approx 1.086435$.

Using (A-4)–(A-6)

$$|R_n| \leq \sqrt{\frac{2}{\pi(n!)}} \frac{B}{\mu^n} 2^{\frac{n}{2}} M_n \tag{A-7}$$

and

$$|\varepsilon| \leq \sqrt{\frac{2}{\pi}} B\, e^{-\left[\frac{h_0}{2\mu}\right]^2} \sum_{i=m}^{n-1} \frac{1}{\sqrt{(i!)}} \left(\frac{a}{\mu}\right)^i S_{i+1} + |R_n|. \tag{A-8}$$

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