Assignment 4

Suraj - CS20BTECH11050

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1 GATE 2021 (ST), Q.17 (Statistics Section)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f_{(k,8)}(x) = \begin{cases} \frac{7}{32} x^6 (2-x), & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases} \tag{1.0.1}$$

then k equals ______________

2 Solution

Definition 2.1. Order Statistics
For given statistical sample $\{X_1, X_2, \cdots X_n\}$, the order statistics is obtained by sorting the sample in ascending order. It denoted as $\{X_{(1)}, X_{(2)}, \cdots X_{(n)}\}$. The k^{th} smallest value $X_{(k)}$ is called k^{th} order statistic

Theorem 2.1. Let $\{X_1, X_2, \cdots X_n\}$ be n i.i.d random variables with common CDF = $F_X(x)$ and common PDF = $f_X(x)$, then the marginal probability distribution of k^{th} order statistic (CDF) is denoted by $F_{X_{(k)}}(x)$ and it is given by

$$F_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \times (F_X(x))^j \times (1 - F_X(x))^{n-j} \tag{2.0.1}$$

Proof.

$$F_{(k,n)}(x) = \Pr\left(X_{(k)} \le x\right) \tag{2.0.2}$$

$$F_{(k,n)}(x) = \Pr\left(\text{At least k elements have value} \le x\right) \tag{2.0.3}$$

Since $\Pr(X \le x) = F_X(x)$, Let $Q \sim Bern(F_X(x))$

$$\Pr(Q = 1) = F_X(x) \tag{2.0.4}$$

$$\Pr(Q = 0) = 1 - F_X(x) \tag{2.0.5}$$

Let $P \sim B(n, F(x))$ taking n trails from $Bern(F(x))$

$$\Pr(P = i) = {}^{n}\mathrm{C}_i \Pr(Q = 1)^i \Pr(Q = 0)^{n-i} \tag{2.0.6}$$

$$\Pr(P = i) = {}^{n}\mathrm{C}_i F_X(x)^i (1 - F_X(x))^{n-i} \tag{2.0.7}$$

Equation (2.0.7) is probability of exactly i R.V of given sample have values $\le x$

$$F_{(k,n)}(x) = \Pr(P \ge k) = \sum_{j=k}^{n} \Pr(P = j) \tag{2.0.8}$$

$$\therefore F_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (F_X(x))^j \, (1 - F_X(x))^{n-j} \tag{2.0.9}$$

□

Theorem 2.2. Let $\{X_1, X_2, \cdots X_n\}$ be n i.i.d random variables with common CDF = $F_X(x)$ and common PDF = $f_X(x)$, then the marginal probability density of k^{th} order statistic (PDF) is denoted by $f_{(k,n)}(x)$ and it is given by

$$f_{(k,n)}(x) = n \, {}^{n-1}\mathrm{C}_{k-1} \, f_X(x) \, (F_X(x))^{k-1} \, (1 - F_X(x))^{n-k} \tag{2.0.10}$$

Proof.

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx}\left(\sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (1 - F_X(x))^{n-j} \, F_X(x)^j\right) \tag{2.0.11}$$

$$f_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (j) \, (1 - F_X(x))^{n-j} \, F_X(x)^{j-1} \, f(x) - \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (n-j) \, (1 - F_X(x))^{n-j-1} \, F_X(x)^j \, f_X(x) \tag{2.0.12}$$

$$S_1 = \sum_{j=k}^{n} \frac{n!\,(1 - F_X(x))^{n-j}\, F_X(x)^{j-1} f_X(x)}{(n-j)!\,(j-1)!} \tag{2.0.13}$$

$$S_2 = \sum_{j=k}^{n} \frac{n!\,(1 - F_X(x))^{n-j-1}\, F_X(x)^{j} f_X(x)}{(n-j-1)!\,j!} \tag{2.0.14}$$

let $i = j + 1$ in equation (2.0.14) (changing limits)

$$S_2 = \sum_{i=k+1}^{n} \frac{n!\,(1 - F_X(x))^{n-i}\, F_X(x)^{i-1} f_X(x)}{(n-i)!\,(i-1)!} \tag{2.0.15}$$

$$f_{(k,n)}(x) = S_1 - S_2 \tag{2.0.16}$$

$$f_{(k,n)}(x) = \frac{n!\, f_X(x)\,(1 - F_X(x))^{n-k}\, F_X(x)^{k-1}}{(n-k)!\,(k-1)!} \tag{2.0.17}$$

$$\therefore f_{(k,n)}(x) = n\, {}^{n-1}\mathrm{C}_{k-1}\ (1 - F_X(x))^{n-k}\, F_X(x)^{k-1} f_X(x) \tag{2.0.18}$$

□

Method 1:
Let $X \in [0, 2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x)\ dx = 1 \tag{2.0.19}$$

$$\Pr(x) = \frac{1}{2}\ (\because \text{Uniform order}) \tag{2.0.20}$$

The PDF for X is

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.0.21}$$

The CDF for X is

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \frac{x}{2}, & 0 < x < 2, \\ 1, & x \ge 2 \end{cases} \tag{2.0.22}$$

Using theorem (2.2) PDF of k^{th} order statistic of given sample from equation (2.0.10)

$$f_{(k,n)}(x) = n\, {}^{n-1}\mathrm{C}_{k-1} \frac{1}{2} \left(\frac{x}{2}\right)^{k-1} \left(1 - \frac{x}{2}\right)^{n-k} \tag{2.0.23}$$

$$f_{(k,8)}(x) = \frac{8}{2^{(1+(k-1)+(8-k))}} \times {}^{7}\mathrm{C}_{k-1}\, x^{k-1}\ (2 - x)^{8-k} \tag{2.0.24}$$

$$f_{(k,8)}(x) = {}^{7}\mathrm{C}_{k-1} \frac{1}{32}\, x^{k-1}\ (2 - x)^{8-k} \tag{2.0.25}$$

Comparing the PDF obtained in equation (2.0.25) with the equation (1.0.1)

$$\frac{{}^{7}\mathrm{C}_{k-1}}{32} (2 - x)^{8-k}\ x^{k-1} = \frac{7}{32}\ (2 - x)\ x^6 \tag{2.0.26}$$

$$\therefore k = 7 \tag{2.0.27}$$

Hence the marginal probability density given is 7^{th} order statistic and **the value of k is 7**

Definition 2.2. Beta function
The **Beta function** is defined for $r, s \in \mathbb{R}^+$ by

$$B(r, s) = \int_0^1 x^{r-1}\ (1 - x)^{s-1}\ dx = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r + s)} \tag{2.0.28}$$

Definition 2.3. Beta Distribution
The Beta distribution is a continuous distribution defined on the range $(0, 1)$ whose PDF given by

$$f(x) = \frac{1}{B(r, s)}\, x^{r-1}\ (1 - x)^{s-1} \tag{2.0.29}$$

where $\int_0^1 f(x) = 1$ as per definition (2.2)

CDF, Mean value and Variance of Beta distribution

$$F(x) = \frac{\int_0^x x^{r-1}\ (1 - x)^{s-1}}{B(r, s)} = \frac{B_x(r, s)}{B(r, s)} \tag{2.0.30}$$

$$E(x) = \frac{r}{r + s} \tag{2.0.31}$$

$$Var(x) = \frac{rs}{(r + s)^2\,(r + s + 1)} \tag{2.0.32}$$

Definition 2.4. Uniform Order Statistics
Let $\{X_1, \cdots X_n\}$ be i.i.d form a uniform distribution on $[0, 1]$ such that $f_X(x) = 1$ and $F_X(x) = x$, from theorem (2.2), equation (2.0.10)

$$f_{(k,n)}(x) = n\, {}^{n-1}\mathrm{C}_{k-1}\, x^{k-1}\ (1 - x)^{n-k} \tag{2.0.33}$$

Lemma 2.3. Uniform order statistics on [0,1] the PDF of k^{th} order statistic follows Beta distribution with $r = k,\ s = n - k + 1$ and PDF is given by

$$f_{(k,n)}(x) = \frac{1}{B(k, n - k + 1)}\, x^{k-1}\ (1 - x)^{(n-k+1)-1} \tag{2.0.34}$$

Proof. Since equation (2.0.33) is PDF its definite integral can be given by Beta function as

$$\int_0^1 n\, {}^{n-1}C_{k-1}\, x^{k-1}\,(1-x)^{n-k}\,dx = 1 \tag{2.0.35}$$

$$\int_0^1 x^{k-1}\,(1-x)^{n-k}\,dx = \frac{(k-1)!\,(n-k)!}{n!} \tag{2.0.36}$$

$$\int_0^1 x^{k-1}\,(1-x)^{n-k}\,dx = \frac{\Gamma(k)\,\Gamma(n-k+1)}{\Gamma\,(k+(n-k+1)))} \tag{2.0.37}$$

$$\int_0^1 x^{k-1}\,(1-x)^{n-k}\,dx = B(k, n-k+1) \tag{2.0.38}$$

$$\int_0^1 \frac{x^{k-1}\,(1-x)^{(n-k+1)-1}}{B(k, n-k+1)}\,dx = 1 \tag{2.0.39}$$

from definition (2.3) with $r = k$ and $s = n-k+1$ equation (2.0.33) follows beta distribution □

Method 2:
From lemma (2.3), PDF of k^{th} order statistic of a uniform distribution on $[0, 1]$ follows beta distribution

$$\int_0^2 f_{(k,8)}(x)\,dx = \int_0^2 \frac{7}{32}\,x^6\,(2-x)\,dx \tag{2.0.40}$$

$$\int_0^2 f_{(k,8)}(x)\,dx = \int_0^2 56\left(\frac{x}{2}\right)^6\left(1-\frac{x}{2}\right)d\left(\frac{x}{2}\right) \tag{2.0.41}$$

Let new random variable be t such that $t = x/2$, New sample be $\{T_1, \cdots T_8\}$ such that $T_i = X_i/2$.

$$f_{(k,8)}(t) = 56\,t^6\,(1-t) \tag{2.0.42}$$

$$\int_0^2 f_{(k,8)}(x)\,dx = \int_0^1 f_{(k,8)}(t)\,dt = 1 \tag{2.0.43}$$

The Uniform distribution of new random sample is on $[0, 1]$ such that PDF $= 1$ and CDF $= t$

$f(k, 8)(x)$ in equation (1.0.1) (after conversion)

$$f_{(k,8)}(t) = \begin{cases} 56\,t^6\,(1-t)\,, & 0 < t < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.0.44}$$

Since equation (2.0.44) is a Beta distribution with $r = k,\ s = n-k+1$

$$r - 1 = k - 1 = 6 \tag{2.0.45}$$

$$\therefore k = 7 \tag{2.0.46}$$

Hence the **value of k is 7**

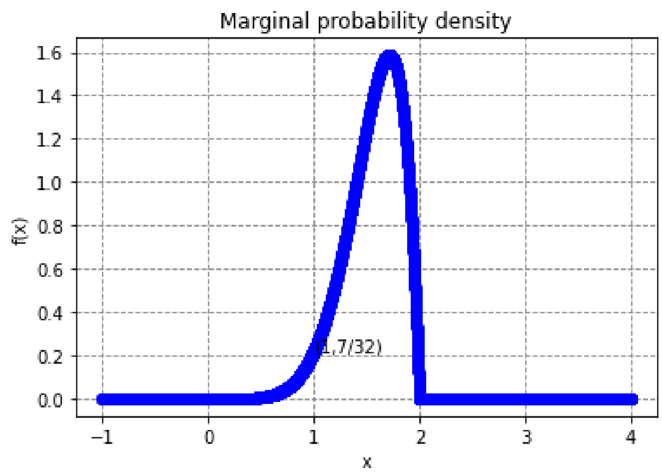

Fig. 1: PDF of $f_{(7,8)}(x)$

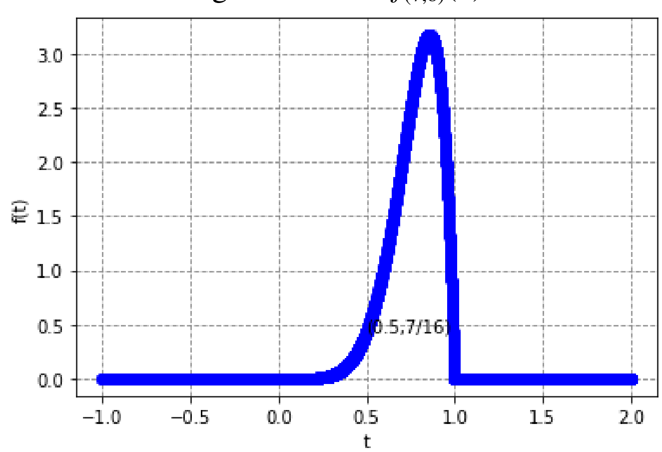

Fig. 2: PDF of $f_{(7,8)}(t)$

Presentation link:
https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment4presentation