Discrete Structures Assignment 2

SURAJ-CS20BTECH11050

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Problem 1

problem 1(a)

Given, $f(n)\, a_n \;=\; g(n)\, a_{n-1} + h(n)$ for $n \geq 1$ and $a_0 \;=\; C$

Define, $Q(n) \;=\; \frac{(f(1)\, f(2) \ldots . f(n-1))}{(g(1)\, g(2) \ldots . g(n))} \;=\; \dfrac{\displaystyle\prod_{i=1}^{n-1} f(i)}{\displaystyle\prod_{i=1}^{n} g(i)}$

$$f(n) - g(n)a_{n-1} \;=\; h(n)$$

by Multiplying Q(n) on both sides of the above equation we get

$$(f(n)\, a_n - g(n)\, a_{n-1})\, Q(n) \;=\; h(n)\, Q(n)$$

$$f(n)\, a_n\, Q(n) - g(n)\, a_{n-1}\, Q(n) \;=\; h(n)\, Q(n)$$

$$\frac{a_n\, f(n) \displaystyle\prod_{i=1}^{n-1} f(i)}{\displaystyle\prod_{i=1}^{n} g(i)} - g(n)\, a_{n-1}\, Q(n) \;=\; h(n)\, Q(n)$$

by including $f(n)$ in numerator's product of above equation multiplying numerator and denominator with $g(n+1)$ we get

$$\frac{a_n\, g(n+1) \displaystyle\prod_{i=1}^{n} f(i)}{g(n+1)\, \displaystyle\prod_{i=1}^{n} g(i)} - g(n)\, a_{n-1}\, Q(n) \;=\; h(n)\, Q(n)$$

we know that $\dfrac{\prod_{i=1}^{n} f(i)}{g(n+1)\prod_{i=1}^{n} g(i)} = Q(n+1)$ by substituting $Q(n+1)$ in the above equation we get

$$Q(n+1)\, g(n+1)\, a_n - Q(n)\, g(n)\, a_{n-1} \;=\; h(n)\, Q(n)$$

Assume $Q(n+1)\, g(n+1)\, a_n \;=\; b_n$ hence the given recurrence relation is converted to Non-Homogeneous Recurrence Relation

$$b(n) - b(n-1) \;=\; h(n)\, Q(n)$$

$$b(n) \;=\; b(n-1) + h(n)\, Q(n)$$

problem1(b)

Given that $Q(1)\, g(1) \;=\; f(0) \;=\; 1$

Consider for some $i \in \mathbb{N}$ $b(i) - b(i-1) \;=\; h(i)\, Q(i)$ let us do the following summation to

$$\sum_{i=1}^{n}(b_i - b_{i-1}) \;=\; \sum_{i=1}^{n}(h(i)\, Q(i)$$

$$b_n - b_0 \;=\; \sum_{i=1}^{n}(h(i)\, Q(i)$$

$$Q(n+1)\, g(n+1)\, a_n - Q(1)\, g(1)\, a_0 \;=\; \sum_{i=1}^{n}(h(i)\, Q(i)$$

$$Q(n+1)\, g(n+1)\, a_n - C \;=\; \sum_{i=1}^{n}(h(i)\, Q(i)$$

$$a_n \;=\; \frac{C + \sum_{i=1}^{n}(h(i)\, Q(i)}{Q(n+1)\, g(n+1)}$$

Finally after solving the recurrence we get the value of a_n as

$$a_n \;=\; \frac{C + \sum_{i=1}^{n}(h(i)\, Q(i))}{Q(n+1)\, g(n+1)}$$

□

Problem 2

problem 2(a)

Given, $(n+1)\,a_n \;=\; (n+3)\,a_{n-1} + n$ for $n \geq 1$ and $a_0 \;=\; 1$

by previous exercise we know that

$$
\begin{aligned}
f(n) &= n+1 \\
g(n) &= n+3 \\
h(n) &= n \\
Q(n) &= \frac{1.2.3....n}{4.5....n+3} = \frac{6\,n!}{(n+3)!} \text{ (as per its definition)}
\end{aligned}
$$

Applying the a_n result from previous exercise we get

$$a_n = \frac{C + \sum_{i=1}^{n} (h(i)\,Q(i))}{Q(n+1)\,g(n+1)}$$

$$a_n = \frac{1 + \sum_{i=1}^{n} \left(i\,\frac{6i!}{(i+3)!} \right)}{\frac{6(n+1)!}{(n+4)!}\,(n+4)}$$

$$a_n = \frac{1 + \sum_{i=1}^{n} \left(\frac{6i}{(i+1)\,(i+2)\,(i+3)} \right)}{\frac{6}{(n+2)\,(n+3)}}$$

Solving Summation using telescopic addition

$$\sum_{i=1}^{n} \left(\frac{6i}{(i+1)\,(i+2)\,(i+3)} \right) = \sum_{i=1}^{n} \frac{3}{(i+2)} \left(\frac{3}{(i+3)} - \frac{1}{(i+1)} \right)$$

$$= \sum_{i=1}^{n} 9 \left(\frac{1}{(i+2)} - \frac{1}{(i+3)} \right) - \sum_{i=1}^{n} 3 \left(\frac{1}{(i+1)} - \frac{1}{(i+2)} \right)$$

$$= 9\left(\frac{1}{3} - \frac{1}{(n+3)}\right) - 3\left(\frac{1}{2} - \frac{1}{(n+2)}\right)$$

$$= \frac{3}{2} - \frac{9}{(n+3)} + \frac{3}{(n+2)}$$

$$\textbf{Summation} = \frac{3}{2} - \frac{(6n+9)}{(n+2)(n+3)}$$

By substituting value of summation in a_n we get

$$a_n = \frac{1 + \frac{3}{2} - \frac{(6n+9)}{(n+2)(n+3)}}{\frac{6}{(n+2)(n+3)}}$$

$$a_n = \frac{\frac{5(n^2+5n+6) - 2(6n+9)}{2\,(n+2)(n+3)}}{\frac{6}{(n+2)(n+3)}}$$

$$a_n = \frac{5n^2 + 13n + 12}{12}$$

$$\therefore\ a_n = \frac{n\,(5n+13)}{12} + 1$$

problem 2(b)

Problem 3

Theorem: *Let c_1 , $c_2 \in \mathbb{R}$ with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2....$, where α_1 and α_2 are constants*

Proof: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a recurrence relation whose character equation is $r^2 - c_1 r - c_2 = 0$ has one root r_0

We have $\Delta = c_1^2 + 4 c_2 = 0$, $r_0 = \dfrac{c_1}{2}$ and $r_0^2 = c_1 r_0 + c_2$ from quadratic equation

Let solution of above recurrence be of form $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ where α_1 and α_2 are constants

$$
\begin{aligned}
a_n &= c_1 a_{n-1} + c_2 a_{n-2} \\
&= c_1 (\alpha_1 r_0^{n-1} + \alpha_2 (n-1) r_0^{n-1}) + c_2 (\alpha_1 r_0^{n-2} + \alpha_2 (n-2) r_0^{n-2}) \\
&= (c_1 r_0 + c_2) \alpha_1 r_0^{n-2} + (c_1 r_0 + c_2) n r_0^{n-2} \alpha_2 - (c_1 r_0 + 2c_2) r_0^{n-2} \alpha_2 \\
&= (c_1 r_0 + c_2) \alpha_1 r_0^{n-2} + (c_1 r_0 + c_2) n r_0^{n-2} \alpha_2 - \left(\frac{c_0^2 + 4 c_2}{2} \right) r_0^{n-2} \alpha_2 \\
&= (c_1 r_0 + c_2) \alpha_1 r_0^{n-2} + (c_1 r_0 + c_2) n r_0^{n-2} \alpha_2 \\
&= \alpha_1 r_0^n + \alpha_2 r_0^n \\
&= a_n
\end{aligned}
$$

$\because$ $2 r_0 = c_1$ and $\Delta = c_1^2 + 4 c_2 = 0$ and $r_0^2 = c_1 r_0 + c_2$

To show every solution of the Recurrence has the same form as above consider the following statements

$$a_0 = c_0 = \alpha_1$$

$$a_1 = c_1 = (\alpha_1 + \alpha_2) r_0$$

$$\alpha_1 = c_0$$

$$\alpha_2 = \frac{c_1 - r_0\, c_0}{r_0}$$

With the values of α_1 and α_2 we obtain a sequence $\{a_n\}$ such that

$$a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$$

satisfy same initial conditions as the given recurrence relation

We know that for a linear homogeneous recurrence of degree 2 Unique solution is obtained for two given initial conditions

$\therefore\ a_n = \alpha_1\, r_0^n + \alpha_2\, n\, r_0^n$ is the only possible solution for the given recurrence relation $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$

$\square$

Problem 4