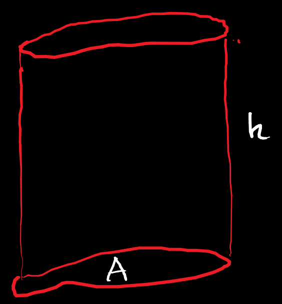
Volumes of Solids

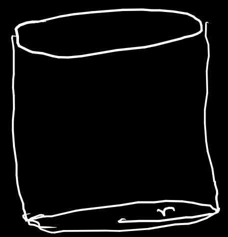
Axioms:

(1) $V_{cyl.} = Ah$ $\quad$ A = area of base.
h = height.

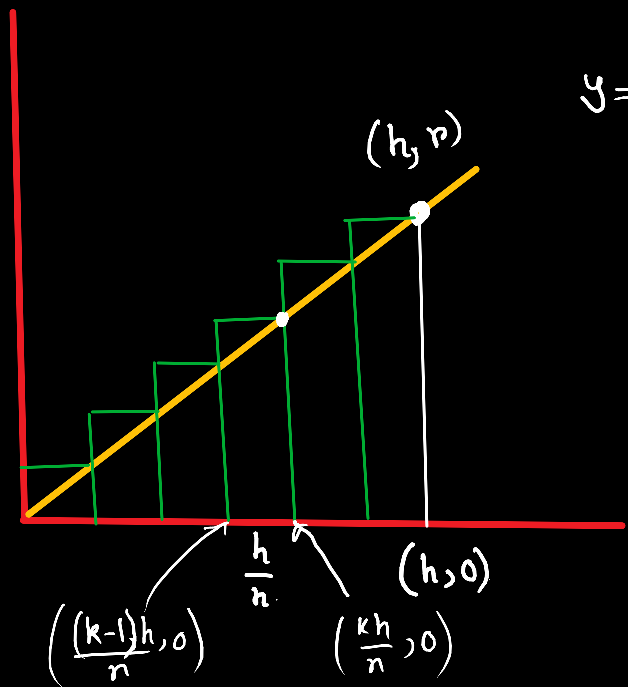


(2) $R \subseteq S \Rightarrow V_R \le V_S$.

(3) $R = \bigcup_{k=1}^{n} R_k \Rightarrow V_R = \sum_{k=1}^{n} V_{R_k}$.



Example:



$y = \frac{r}{h} x.$

Volume of cone $\le U_n$.

Volume of the k-th cyllinder V_k

height $= \frac{h}{n}$

radius of base $= \frac{rk}{n}$

Volume $= \pi \left(\frac{rk}{n}\right)^2 \frac{h}{n}$.

$$U_n = \sum_{k=1}^{n} \text{Volume}(V_k)$$

$$= \sum_{k=1}^{n} \frac{\pi r^2 h}{n^3} k^2.$$

$$= \pi r^2 h \frac{n(n+1)(2n+1)}{6n^3}$$

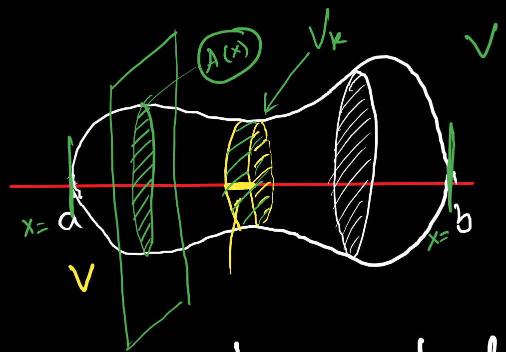
$$\to \frac{\pi r^2 h}{3}. \quad \text{as } n \to \infty$$

$$U_n - L_n = \pi r^2 \frac{h}{n} \to 0 \quad \text{as } n \to \infty$$

$$L_n \le \text{Volume of cone} \le U_n$$

Volume of the cone $= \lim_{n\to\infty} U_n = \frac{\pi r^2 h}{3}$

Generalization:



$V = V_1 \cup \cdots \cup V_k.$

$$\boxed{\text{Volume}(V) = \sum_{k=1}^{n} \text{Volume}(V_k)}$$

$A(m_k)\Delta x_k \leq \text{Vol}(V_k) \leq \boxed{A(M_k)\Delta x_k}$

inscribes V_k.

Volume of a r.c.c. that circumscribe V_k.

$\Delta x_k = x_k - x_{k-1}$

M_k m_k

$A(m_k)$ = Smallest cross-sectional area in $[x_{k-1}, x_k]$

$A(M_k)$ = Largest " " " $[x_{k-1}, x_k]$

$A(m_k)\Delta x_k \leq \text{Vol}(V_k) \leq A(M_k)\Delta x_k.$

$$\sum_{k=1}^{n} A(m_k)\Delta x_k \leq \text{Volume}(V) \leq \sum_{k=1}^{n} A(M_k)\Delta x_k$$

$$\int_a^b A(x)\,dx$$

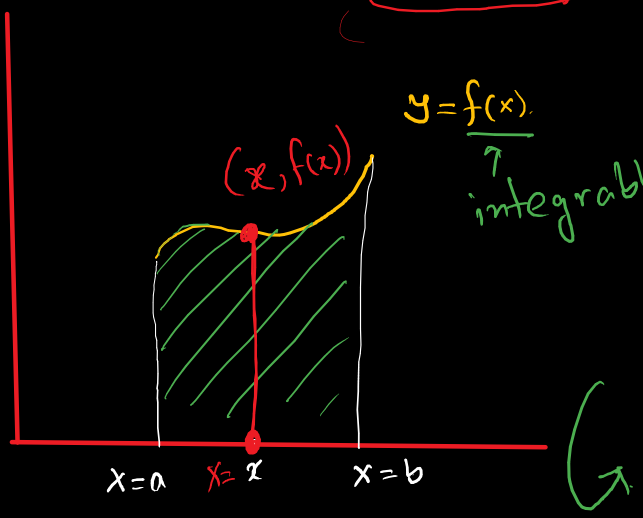
$$\lim_{n\to\infty} \sum_{k=1}^{n} A(M_k)\Delta x_k = \int_a^b A(x)\,dx.$$

If the "area" function is ~~continuous.~~ integrable

$$\boxed{\text{Volume}(V) = \int_a^b A(x)\,dx}$$

Solids of revolutions:

$f(x)$



Rotate this region about x axis

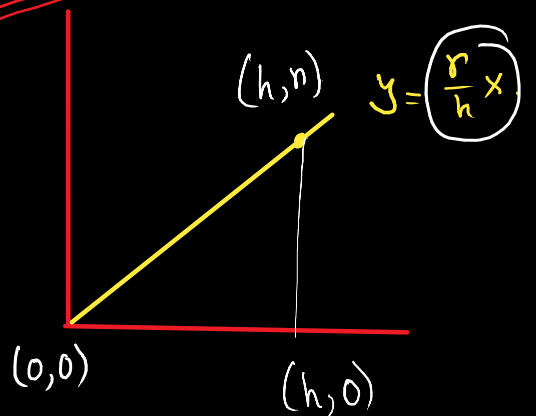
$A(x) = \pi \cdot f(x)^2$ Disk method

$$\text{Volume} = \int_a^b \pi f(x)^2 \, dx$$

$$= G(b) - G(a) \quad (\text{FTC})$$

where $G' = \pi f(x)^2$.

Cone



$\frac{x^3}{3}$

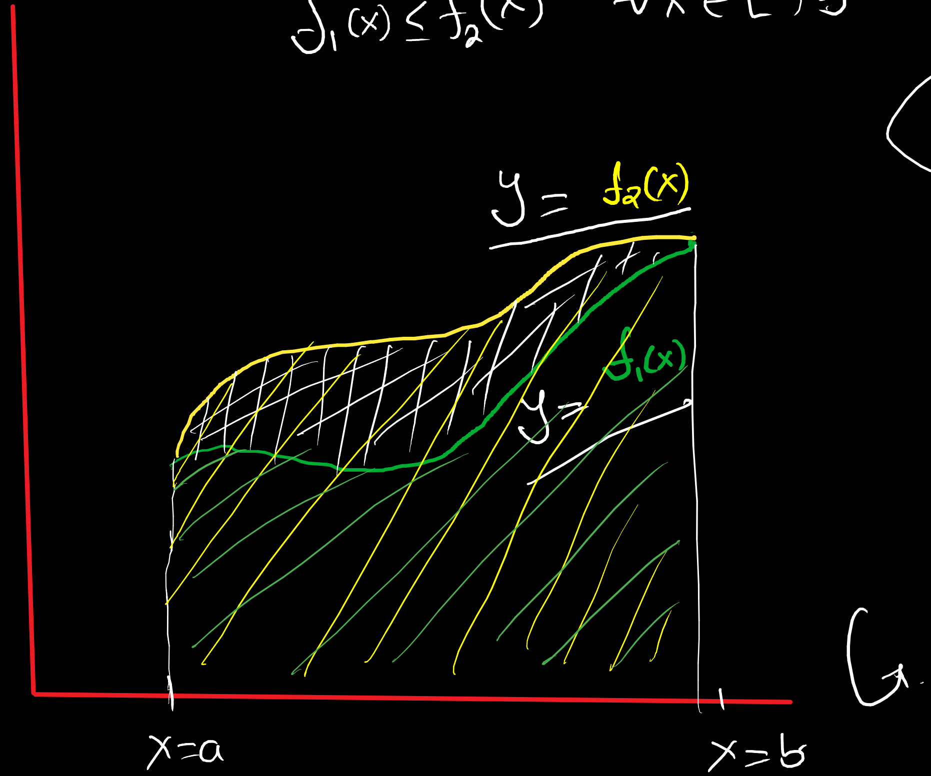
Volume of the cone

$$= \int_0^h \pi \left(\left(\frac{r}{h}\right)x\right)^2 dx.$$

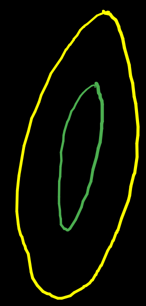
$$= \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx = \frac{\pi r^2}{h^2} \left. \frac{x^3}{3} \right]_{x=0}^{x=h}$$

$$= \frac{\pi r^2 h}{3}.$$

$$f_1(x) \le f_2(x) \quad \forall x \in [a,b]$$



Washer method:



$$V_2 = \pi \int_a^b f_2(x)^2 \, dx.$$

$$V_1 = \pi \int_a^b (f_1(x))^2 \, dx$$

$$\boxed{V = \pi \int_a^b \left(f_2(x)^2 - f_1(x)^2\right) dx.}$$