Lecture 08, LINEAR ALGEBRA (MA 4020)

Sep 14, 2021

[We started with problems from Artin : See recording of lecture]

V : vector space over F (or over $\mathbb{R}$)

$S = (v_1, \cdots, v_n)$ ordered set of vectors of V

$$\mathcal{L}(S) = \left\{ \sum_{i=1}^{n} c_i v_i \;\middle|\; v_i \in V,\ c_i \in F,\ n \in \mathbb{N} \right\}$$

linear

finite linear combination of elements of S.

Linear span of S, and is denoted by $\underline{\underline{\text{Span}(S)}}$.

$(v_1, \cdots, v_n)$ is L.I, then if (L.I)

$$c_1 v_1 + \cdots + c_n v_n = 0$$

$$\Rightarrow \quad c_i = 0 \quad \text{for all } i = 1, \cdots, n.$$

A set which is not linearly independent is called linearly dependent (L.D)

Definition. A set of vectors $v_1, \dots, v_n$ which is linearly independent and which also spans V is called a **basis**.

Discussion.

① $\mathbb{R}^2$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbb{R}^2 \ni v = \begin{bmatrix} a \\ b \end{bmatrix}$

$\mathrm{Span}(e_1, e_2) = \mathbb{R}^2$

$\begin{bmatrix} a \\ b \end{bmatrix} = \underset{a}{c_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underset{b}{c_2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

② $\mathbb{R}^n$

$e_1, \dots, e_n$ s

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$e_i = \begin{array}{r} \\ \\ \\ i^{th} \text{ row} \\ \\ \\ \end{array}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$(e_1, \dots, e_n)$ is L.I.

$\mathrm{Span}(e_1, \dots, e_n) = \mathbb{R}^n$

③ $V = (M_n(\mathbb{R}), +, \cdot)$

Linearly independent set $= \{ -, -, \dots \}$

$e_{ij} = \begin{array}{c} j^{th} \\ \\ \end{array}$ i^{th} $\begin{bmatrix} 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Basis $= \{ e_{ij} \mid 1 \leq i, j \leq n \}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= a_{11} e_{11} + a_{12} e_{12} + a_{21} e_{21} + a_{22} e_{22}$$

$\{ e_{11}, e_{12}, e_{21}, e_{22} \}$ Linearly independent.

$$a_{11} e_{11} + a_{12} e_{12} + a_{21} e_{21} + a_{22} e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\Rightarrow a_{ij} = 0$

$\Updownarrow$

$a_{ij} = 0$

(vector space)

4. $\mathcal{P}_n = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R} \}$

polynomials of degree $\leq n$.

Basis $= \{ \quad \}$?

$= \{ 1, x, x^2, \ldots, x^n \}$ $\underline{\underline{L \cdot I}}$

$c_0 1 + c_1 \cdot x + c_2 x^2 + \cdots + c_n x^n = 0$

$\forall x \in \mathbb{R}$.

$\Longleftrightarrow c_i = 0$.

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of a vector space V.

$$\operatorname{Span} \mathcal{B} = V$$

$$\uparrow$$

$$w$$

$$\exists \underbrace{c_1 v_1 + \dots + c_n v_n}_{\in \operatorname{Span}\mathcal{B}} \quad \text{s.t.} \quad \boxed{w = c_1 v_1 + \dots + c_n v_n}$$

$\uparrow$ $c_i \in F$ for some $c_1, c_2, \dots, c_n \in F$.

Is this unique? (YES)

$\exists\ c_1' v_1 + \dots + c_n' v_n$ s.t. $w = c_1' v_1 + \dots + c_n' v_n$

Proposition. The set $\mathcal{B} = (v_1, \dots, v_n)$ is a basis if and only if every vector $w \in V$ can be written in a unique way in the form: $w = c_1 v_1 + \dots + c_n v_n$, $c_i \in F$.

Proof. Suppose $\mathcal{B}$ is a basis and w can be written as a linear combination in two ways

$$w = c_1 v_1 + \dots + c_n v_n$$
$$\parallel \qquad\qquad \parallel$$
$$w = c_1' v_1 + \dots + c_n' v_n.$$

Then

$$\underline{\underline{0}} = w - w = \underbrace{(c_1 - c_1')}_{\overset{\|}{0}} v_1 + \cdots + \underbrace{(c_n - c_n')}_{\overset{\|}{0}} v_n .$$

Since $\mathcal{B}$ is a basis, we get $\boxed{c_i = c_i'}$ for all $i = 1, \ldots, n$.

Hence, expression for w is unique.

Note. $\mathcal{B} = \{v_1, \ldots, v_n\}$ basis of V, then

$$\text{Span } \mathcal{B} = V$$

0 vector

unique representation for 0 vector,

Converse Part. If every vector $w \in V$ can be written in a unique way, then claim $\underset{\mathcal{B}}{(v_1, \ldots, v_n)}$ is a basis.
$(c_1 v_1 + \cdots + c_n v_n)$

$$0 = c_1 v_1 + \cdots + c_n v_n$$

$\Rightarrow$ $c_i = 0$ for all $i = 1, \ldots, n$

$\Rightarrow$ $(v_1, \ldots, v_n)$ are linearly independent

$\underset{c_1 v_1 + \cdots + c_n v_n}{\overset{}{w}} \in V$, then $w \in \text{Span}(\mathcal{B}) \Rightarrow V \subseteq \text{Span}(\mathcal{B})$

But $\boxed{\text{Span}(\mathcal{B}) \subseteq V \text{ (always)}}$

Thus $V = \text{Span}(\mathcal{B})$.

(or $\mathbb{R}^n$)

Example. Let $V = F^n$ the space of column vectors.

e_i : column vector with 1 in the i^{th} position and zero elsewhere. Then n vectors

$e_1, e_2, \ldots, e_n$ form a basis for F^n.

[often called standard basis]

$E = (e_1, e_2, \ldots, e_n)$
(Notation)

$V = F^n$

Note that every vector $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ can be expressed

uniquely in the form

$X = x_1 e_1 + \cdots + x_n e_n$ as a linear combination of E.

Proposition. Let $L \overset{= (v_1, \ldots, v_r)}{}$ be a linearly independent ordered set in V, and $\underline{\underline{v \in V}}$ be any vector. Then the ordered set $L' = (L, v)$ obtained by adding v to L is linearly independent if and only if v is not in the subspace spanned by L.

$$\left[\begin{array}{c} A \iff B \\ \neg A \iff \neg B \end{array} \right]$$

Proof. Assume that $L = (v_1, \cdots, v_r)$ for some r.

If $v \in \operatorname{Span} L$, then

$v = c_1 v_1 + \cdots + c_r v_r$ for some $c_i \in F$.

Hence $c_1 v_1 + \cdots + c_r v_r + (-1) v = 0$ is a linear relation among the vectors of L', and the co-efficient -1 is not zero.

Thus L' is linearly dependent.

Conversely, suppose that L' is linearly dependent.

Then there is some linear relation

$$\left(c_1 v_1 + \cdots + c_r v_r\right) + b\, v = 0, \qquad \text{— } (*)$$

in which not all co-efficients are zero.

$\boxed{\text{At least } b \neq 0.}$

If b were zero, the expression would

reduce to $c_1 v_1 + \cdots + c_r v_r = 0$

$$\Downarrow$$

$$\Rightarrow c_1 = \cdots = c_r = 0 \qquad \left\{ \because \mathcal{L} \text{ is linearly independent} \right.$$

Since $b \neq 0$, we can re-write $(*)$ as

$$v = \left(\frac{-c_1}{b}\right) v_1 + \cdots + \left(-\frac{c_r}{b}\right) v_n$$

$v \in$

$\Rightarrow v \in \underline{\operatorname{Span} \mathcal{L}}$.

$\operatorname{Span}(\mathcal{L}') = V$

if $\operatorname{Span}(\mathcal{L}') \neq V$

$\subsetneq$ (if)

spanned by finitely many elements

$\mathcal{L}' = (\mathcal{L}, v)$

$(\mathcal{L}', w)$

$(\mathcal{L}, v_1, v_2, \ldots, v_k)$ L.I.

Proposition. Let S be an ordered set of vectors, let $v \in V$ be any vector, and let $S' = (S, v)$. Then

$$\mathrm{Span}(S) = \mathrm{Span}(S') \iff v \in \mathrm{Span}(S).$$

$(A \iff B)$

$(\neg A \iff \neg B)$

Proof. Since $S' = (S, v)$

$\Rightarrow \boxed{v \in \mathrm{Span}\, S'.}$ ✓

Claim. $v \in \mathrm{Span}(S)$.

If $v \notin \mathrm{Span}(S)$

$\Rightarrow \mathrm{Span}(S) \neq \mathrm{Span}(S')$

Conversely, if $v \in \mathrm{Span}(S)$,

then $\underline{\underline{S'}} \subset \mathrm{Span}(S)$

$\Rightarrow \mathrm{Span}(S') \subset \mathrm{Span}(S)$

But $\mathrm{Span}(S) \supset \mathrm{Span}(S')$

$\Rightarrow \boxed{\mathrm{Span}(S') = \mathrm{Span}(S)}.$

Definition. A vector space V is called finite-dimensional if there is some finite set S which spans V.

① $(\mathbb{R}^n, +, \cdot)$ $S = (e_1, \ldots, e_n)$

② $M_n(\mathbb{R})$

$$\begin{cases} S = \{v_1, \ldots, v_n\} \\ \text{Span}(S) = V \end{cases}$$

Proposition. Any finite set S which spans V $\overset{\{0\}}{\underset{\|}{}}$ contains a basis. In particular, any finite-dimensional vector space has a basis.

Proof. Suppose $S = (v_1, \ldots, v_n)$ and that S is not linearly independent. Then

$\text{Span}(S) = V$.

$\{v_1, \ldots, v_n\}$ L.D

$$c_1 v_1 + \cdots + c_n v_n = 0, \quad \text{some } c_i \text{ is non-zero}$$

$\Downarrow$

$\{v_1, \ldots, v_{n-1}\}$ L.D.

Assume that $c_n \neq 0$. Then

$\Downarrow$

$$v_n = \frac{-c_1}{c_n} v_1 + \cdots + \frac{-c_{n-1}}{c_n} \cdot v_{n-1}$$

$c_n \neq 0$

This implies that

$$v_n \in \mathrm{Span}(v_1, \cdots, v_{n-1})$$

~~we~~ set $v = v_n$ and $S = (v_1, \cdots, v_{n-1})$ before,

we get $\mathrm{Span}(v_1, \cdots, v_{n-1}) = \mathrm{Span}(v_1, \cdots, v_n) = V.$

$\mathrm{Span}(v_1)$

* $\mathrm{Span}(0)$

0_v is this L.I. ?

NO

$\mathrm{Span}(v_1, \cdots, v_k)$

If $v_1, \cdots, v_{n-1}$ are still linearly dependent,

$\mathrm{Span}(v_1, \cdots, v_{n-2})$,

continue the same process of elimination,

$= V$

until ~~we~~ we reach a stage

$v_1, \cdots, v_n$

$$\mathrm{Span}(v_1, \cdots, v_\ell)$$

s.t. $v_1, \cdots, v_\ell$ is linearly independent

Since $\mathrm{Span}(v_1, \cdots, v_\ell) = V$, thus $v_1, \cdots, v_\ell$ is a basis.

Convention.

(a) The empty set is linearly independent.

ϕ

(b) $\mathrm{Span}(\phi) = \{0\}$

zero subspace.

With this convention, we complete the proof.

Proposition. Let V be a finite-dimensional vector space.

Any linearly independent set $\mathcal{L}$ can be extended by adding elements, to get a basis.

$\mathcal{L}, v$ $\quad$ $\text{Span}(\mathcal{L}, v) = V$

$(\mathcal{L}, v, w)$ $\quad$ $\text{Span}(\mathcal{L}, v, w) = V$

Proof.

Given $\quad V$: f.d. vector space

By def^n [$\exists$ a finite set S which spans V].

By Proposition [Any finite set S which spans V contains a basis]

Let S be a finite set which spans V.

If all elements of S are in $\text{Span}\,\mathcal{L}$, then $\mathcal{L}$ spans V, and so it contains a basis.

If not, choose $v \in S$, which is not in $\text{Span}\,\mathcal{L}$.

then $(\mathcal{L}, v)$ is linearly independent.

Continue until you get a basis (This process stops in finite steps)

$(\mathcal{L}, v_1, w_2)$

$\vdots$

$(\mathcal{L}, v_1, v_2, \ldots, v_n)$

$\square$

Proposition. Let S, L be finite subsets of V. Assume that S spans V and that L is linearly independent. Then S contains at least as many elements as L does.

$$\underbrace{\operatorname{Span}((1,1),(1,0),(2,1))}_{S} = \mathbb{R}^2$$

$$L = \{(2,0),(0,1)\}$$

Discussion.

$$\underset{\text{finite set}}{S} \subseteq V \supseteq \underset{\text{finite set}}{L}$$

$\operatorname{Span}(S) = V$ $\qquad$ L is linearly independent

Claim: $\# S \geq \# L$ $\quad (|S| \geq |L|)$

$S = (v_1, \ldots, v_m)$ $\qquad\qquad$ $L = (w_1, \ldots, w_n)$

$\operatorname{Span} S = V$ $\qquad$ Claim: $m \geq n$.

Since $w_j \in V$ $(= \operatorname{Span} S)$, we can write w_j as a linear combination of S,

$$w_j = a_{1j} v_1 + \cdots + a_{mj} v_m$$

$$= \sum_{i=1}^{m} a_{ij} v_i$$

Let $u = c_1 w_1 + \cdots + c_n w_n$

$$= c_1 \left(\sum_{i=1}^{m} a_{i1} v_i \right) + \cdots + c_n \cdot \left(\sum_{i=1}^{m} a_{in} v_i \right)$$

$$= \sum_{i=1}^{m} c_1 a_{i1} v_i + \cdots + \sum_{i=1}^{m} c_n a_{in} v_i$$

$$= \sum_{i,j} (c_j a_{ij}) v_i$$

$$\underset{\substack{\parallel \\ 0}}{u} = (\text{—}) v_1 + \cdots + (\text{—}) v_m$$

The co-efficient of $v_i = \sum_j a_{ij} c_j$.

If the co-efficient of v_i is zero for every i, then $u = 0$

To find a linear relation among the vectors of $\mathcal{L}$, it suffices to solve the system

$\sum_j a_{ij} x_j = 0$ of m equations in n unknowns.

If $m < n$, then system has non-trivial solution, then $\mathcal{L}$ is Linearly dependent. Hence $m \geq n$.

$$V$$
$$\mathcal{B}_1 \quad \mathcal{B}_2 \quad \mathcal{B}_3 \quad \cdots \quad \mathcal{B}_m$$

Proposition. Two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of the vector space V have the same number of elements.

Proof. Set $\mathcal{B}_1 = S$ and $\mathcal{B}_2 = L$ in previous proposition to get

$$|\mathcal{B}_1| \geq |\mathcal{B}_2| .$$

By symmetry,

$$|\mathcal{B}_2| \geq |\mathcal{B}_1|$$

Hence, $|\mathcal{B}_1| = |\mathcal{B}_2|$.

Definition. The dimension of a finite-dimensional vector space V is the number of vectors in a basis.

Notation. $\dim V$ or $\dim_F V$ dimension of a vector space over a field F.

" $\boxed{\dim_F V < \infty}$ means V is f.d. vector space over F "

$V = M_n(\mathbb{R})$

$\dim_{\mathbb{R}} V = n^2$

$\mathcal{B} = \{ e_{ij} \mid 1 \le i, j \le n \}$

$V = \mathbb{R}^n$

$\dim_{\mathbb{R}} V = n$ ✓

$V = Sym_n(\mathbb{R})$

$\dim_{\mathbb{R}} V = \frac{n(n+1)}{2}$

Describe Basis $= \{ \quad , \ldots \}$

$Sym_2(\mathbb{R})$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

$\dim_{\mathbb{R}} Sym_2(\mathbb{R})$

$= 3$

Basis $= \{ e_{11}, \; e_{22}, \; e_{12}+e_{21} \}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = a_{11}\, \underline{\underline{e_{11}}} + a_{12}\, (\underline{\underline{e_{12} + e_{21}}}) + a_{22} \cdot \underline{\underline{e_{22}}}$$

$\mathbb{R}^3 \qquad \{0\}, \qquad \mathbb{R}^3$