Integral Calculus

Riemann Integration

Where we stopped !!

→ $f:[a,b]\to\mathbb{R}$ a bounded function.

→ Partition $P=\{x_0, x_1, \dots, x_n\}$ of $[a,b]$

→ Given a partition P → $m_i(f) = \inf\{f(x) \mid x\in[x_{i-1},x_i]\}$

→ $M_i(f) = \sup\{f(x) \mid x\in[x_{i-1},x_i]\}$

→ Lower sum: $L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$

→ Upper sum: $U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$.

→ $L(P,f) \le U(P,f)$.

Intuition: $L(P,f) \le \int_a^b f(x)\,dx \le U(P,f)$.

Next: Refine and look for better approximations!!

A lower bound for $L(P,f)$:

$$\boxed{m(f)\,(b-a) \leq L(P,f)}$$

Pf: For each $i=1,\dots,n$

$$m(f) \leq m_i(f).$$

$$\Rightarrow \quad m(f)(x_i - x_{i-1}) \leq m_i(f)\,(x_i - x_{i-1})$$

$$\Rightarrow \sum_{i=1}^{n} m(f)\,(x_i - x_{i-1}) \leq \sum_{i=1}^{n} m_i(f)\,(x_i - x_{i-1})$$

$$\Rightarrow m(f)\sum_{i=1}^{n}(x_i - x_{i-1}) \leq L(P,f).$$

$$\Rightarrow m(f)\,(b-a) \leq L(P,f) \qquad \square$$

An upper bound for $U(P,f)$.

Exercise!

$$\boxed{U(P,f) \leq M(f)\,(b-a)}$$

Pf: For each $i=1,\dots,n$

$$M_i(f) \leq M(f)$$

$$\Downarrow$$

$$M_i(f)(x_i - x_{i-1}) \leq \cdots$$

$$\Downarrow$$

Complete!!

$$\sum_{i=1}^{n}(x_i - x_{i-1}) = (b-a)$$

Refining a partition:

→ A partition P^* of $[a,b]$ is said to be a **refinement** of a partition P if $P \subseteq P^*$.

Claim ① Let $f:[a,b]\to\mathbb{R}$ be a bounded function. If P is a partition of $[a,b]$ and P^* is a refinement of P then

$$\boxed{L(P,f)\le L(P^*,f)} \quad \text{and} \quad \boxed{U(P^*,f)\le U(P,f).}$$

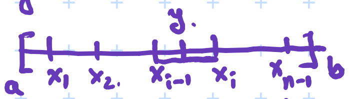
Exercise!!

Sketch of a proof: Suppose $P=\{x_0,\dots,x_n\}$ and $P^*=\{x_0,\dots,x_{i-1},y,x_i,\dots,x_n\}$

Define $m_i'(f)=\inf\{f(x)\mid x\in[x_{i-1},y]\}$

$m_i''(f)=\inf\{f(x)\mid x\in[y,x_i]\}$

Claim: It is enough to prove the assertion in the case when P^* contains exactly one more point than P. $P^*=\{y\}\cup P$



$L(P,f)$: $m_i(f)(x_i-x_{i-1})$

$L(P^*,f)$: $m_i'(f)(y-x_{i-1})+m_i''(f)(x_i-y)$

$[x_{i-1},y]$ $[y,x_i]$

$[x_{i-1},y]\subseteq[x_{i-1},x_i]$

$m_i(f)\le m_i'(f),\ m_i''(f)$

$$L(P,f)$$
$$= m_1(f)(x_1-x_0)+m_2(f)(x_2-x_1)+\cdots+m_i(f)(x_i-x_{i-1})+\cdots+m_n(f)(x_n-x_{n-1})$$
$$= m_1(f)(x_1-x_0)+\cdots+m_i(f)(x_i-y)+m_i(f)(y-x_{i-1})+\cdots+m_n(f)(x_n-x_{n-1})$$
$$\le m_1(f)(x_1-x_0)+\cdots+m_i'(f)(x_i-y)+m_i(f)(y-x_{i-1})+\cdots+m_n(f)(x_n-x_{n-1})$$
$$= L(P^*,f) \quad \square$$

Claim 2 Let P_1, P_2 be two partitions of $[a,b]$ and $f:[a,b]\to\mathbb{R}$ be a bounded function. Then

$$L(P_1,f) \leq U(P_2,f)$$

Pf: Let $P^* = P_1 \cup P_2$. Then P^* is a common refinement of P_1 and P_2. Then

$$\underbrace{L(P_1,f) \leq \boxed{L(P^*,f) \leq U(P^*,f)}}_{\text{claim 1}} \underbrace{\leq U(P_2,f)}_{\text{claim 1}}$$

Claim 1 $\boxed{L(P,f) \leq L(P^*,f).}$ $\boxed{U(P^*,f) \leq U(P,f).}$

Area of network smaller rectangles corresponding to P.

Claim 2

$L(P_1,f) \leq U(P_2,f)$

$\mathcal{L} = \{L(P,f) \mid P \text{ is a partition of } [a,b]\}$ ← bounded above

$L(P,f) \leq M(f)(b-a)$. $\forall P$.

$L(f) := \sup \{L(P,f) \mid P \text{ is a partition of } [a,b]\}$

$\leq$

$\mathcal{U} = \{U(P,f) \mid P \text{ is a partition of } [a,b]\}$

$U(P,f) \geq m(f)(b-a)$.

$U(f) := \inf \{U(P,f) \mid P \text{ is a partition of } [a,b]\}$. bounded below

Choose a partition Q of $[a,b]$. By Claim (2)

$$L(P,f) \leq U(Q,f) \quad \text{for each partition } P.$$

$$\Rightarrow \sup\{L(P,f) \mid P \text{ is a partition of } [a,b]\} \leq U(Q,f).$$

$$\Rightarrow \boxed{L(f) \leq U(Q,f).}$$

But this is true for any partition Q of $[a,b]$.

$$\Rightarrow L(f) \leq \inf\{U(Q,f) \mid Q \text{ is a partitions of } [a,b]\}$$

$$\Rightarrow \boxed{L(f) \leq U(f)}$$

$$L(f) \leq \int_a^b f(x)\,dx \leq U(f).$$

Definition (finally!!)

Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

Then f is said to be Riemann integrable if

$$\boxed{L(f) = \cancel{R}^{U}(f).}$$

$L(f) = U(f)$

In such a case, we write

$$\boxed{\int_a^b f(x)\,dx = L(f) = \cancel{R(f)}\ U(f).}$$

Example: (A trivial one!)

$c \in \mathbb{R}$.

$f : [a,b] \to \mathbb{R}$ $\quad f(x) = c$ for all $x \in [a,b]$.

$[x_{i-1}, x_i] \ni$ $f(x) = c \ \forall x$

let $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$.

$f(x) = c \quad \forall x \in [x_{i-1}, x_i]$

$$\Rightarrow \left\{ \begin{array}{l} m_i(f) = c \\ M_i(f) = c \end{array} \right\}$$

$$L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = c \sum_{i=1}^{n} (x_i - x_{i-1}) = c(b-a)$$

$$U(P,f) = \sum M_i(f)(x_i - x_{i-1}) = c \sum_{i=1}^{n} (x_i - x_{i-1}) = c(b-a).$$

True for all partitions P of $[a,b]$

f is integrable.

$$L(f) = c(b-a)$$

$$U(f) = c(b-a)$$

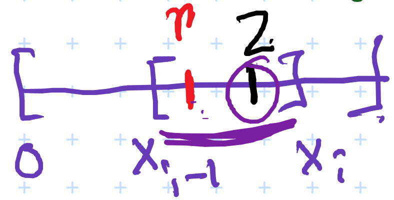
$$\Rightarrow \boxed{\int_a^b c\,dx = c(b-a)}$$

Question: Are all bounded functions integrable?

A non example: (Dirichlet function)

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \qquad f:[0,1] \to \mathbb{R}.$$

$\circledast$ between any two real numbers, there is a rational number and an irrational number.



Fix a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$.

By $\circledast$, there is a rational number in $[x_{i-1}, x_i]$

$\Rightarrow M_i(f) = 1. \ \forall i$

$M_i(f) = 1$

$m_i(f) = 0$

Similarly, $m_i(f) = 0 \ \forall i.$

$$U(P,f) = \sum_{i=1}^{n} \overset{1}{M_i(f)} (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$$

$$= x_n - x_0$$

$L(P,f) = 0$

$\Longrightarrow$ $U(f) = 1$, $L(f) = 0$

f is not integrable.