E1 - Linear Algebra (MA 4020)

Deadline: Sunday, September 12, 2021; 10:00 PM — Maximum Marks 12

Instructions. (Answer any four questions)

1. Write your name and roll number on the answered pages/papers.
2. Scan the document in the pdf file format.
3. Rename the scanned document with your name-E1.
4. Upload the **pdf file** on the Google classroom. (No .jpeg or .jpg file please)

Note. Write answers carefully. Anyone found copying, even for a single question, will be awarded zero marks. Late submission by default will get zero marks. Upload your answers in time, with as many questions as you have done.

1. Find all solutions of the system of equations $AX = B$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & -2 \end{bmatrix}$$

and B has the following value:

$$(a) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (b) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

2. Answer the following:

(i) Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, \ldots, E_k$ such that $E_k \ldots E_1 A$ either is the identity or has its bottom row zero.

(ii) Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

3. Compute the determinant of the following $n \times n$ matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots n \\ 2 & 2 & 3 & \cdots n \\ 3 & 3 & 3 & \cdots n \\ \vdots & \vdots & \vdots & \vdots \\ n & n & n & \cdots n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix},$$

4. Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

5. Consider a system of n linear equations in n unknowns: $AX = B$, where A and B have integer entries. Prove or disprove the following:

(a) The system has a rational solution if $\det A \neq 0$.

(b) If the system has a rational solution, then it also has an integer solution.

6. Answer the following:

(i) Let A be a real 2×2 matrix, and let A_1, A_2 be the rows of A. Let P be the parallelogram whose vertices are $O, A_1, A_2, A_1 + A_2$. Prove that the area of P is the absolute value of the determinant $\det A$ by comparing the effect of an elementary row operation on the area and on $\det A$.

(ii) Prove an analogous result for $n \times n$ matrices..