Lecture 11 – MA4020 (LINEAR ALGEBRA) Sep 28, 2021.

Discussion.

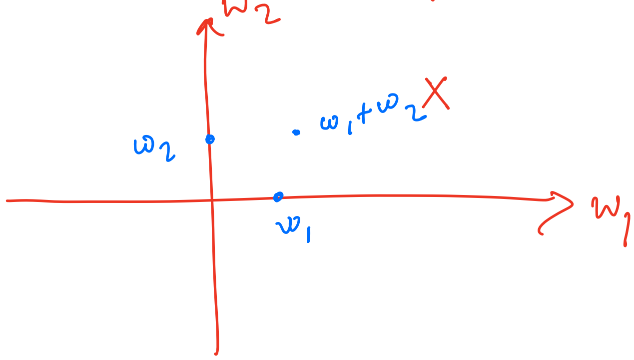
Let V be a finite dimensional vector space over $\mathbb{R}$.

Assume that W_1 and W_2 are subspaces of V

(i) $W_1 \cup W_2$ is a subspace of V ?

NO

$V = \mathbb{R}^2$



Define

$W_1 + W_2$

$$= \{ w_1 + w_2 \mid w_1 \in W_1, \; w_2 \in W_2 \}$$

(ii) $W_1 + W_2$ is a subspace of V ?

YES

v_1 v_2

$$v_1 = w_1 + w_2 \quad ; \quad w_1, w_1' \in W_1$$

$$v_2 = w_1' + w_2' \qquad w_2, w_2' \in W_2$$

$$v_1 + v_2 = (w_1 + w_1') + (w_2 + w_2') \in W_1 + W_2$$

$$V$$

$$W_1 \quad W_2 \quad W_3 \quad - - \quad W_\ell \quad \text{subspaces of } V$$

Define $W_1 + W_2 + \cdots + W_\ell$

$$= \{ w_1 + \cdots + w_\ell \mid w_i \in W_i \} \text{ subspace of } V$$

Example. $V = \mathbb{R}^3$

$$W_1 = \{ (x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R} \}$$

$$W_2 = \{ (0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R} \}$$

$$W_3 = \{ (0, y, y) \in \mathbb{R}^3) \; y \in \mathbb{R} \}$$

Is it true that $\mathbb{R}^3 = W_1 + W_2 + W_3$?

✓ YES / NO

Is it true that $\mathbb{R}^3 = W_1 + W_2$?

YES.

$$W_1 + W_2 = \{ (x,y,z) \in \mathbb{R}^3 \text{ s.t. } x,y,z \in \mathbb{R} \}$$

$$= \mathbb{R}^3$$

Question. Can we describe every vector in $\mathbb{R}^3$ in a unique way as a sum

$v = w_1 + w_2 + w_3$; where $w_1 \in W_1$, $w_2 \in W_2$ and $w_3 \in W_3$

Clearly $(0,0,0) = (0,0,0) + (0,0,0) + (0,0,0)$

$\parallel$

$(0,1,0) + (0,0,1) + (0,-1,-1)$

Answer: NO.

$\mathbb{R}^3 \neq W_1 \oplus W_2 \oplus W_3$

Let us discuss class of vector spaces and subspaces, where a vector can be written as a sum of elements of subspaces in a unique way.

Section 6. DIRECT SUMS

Let V be a vector space, and let $W_1, \dots, W_n$ be subspaces of V.

Consider vectors $v \in V$ which can be written as a sum

$$v = w_1 + \dots + w_n \quad ; \quad \text{where } w_i \in W_i .$$

The set of all such vectors is called the sum of the subspaces, and is denoted by

$$W_1 + \dots + W_n = \{ v \in V \text{ s.t. } v = w_1 + \dots + w_n , \text{ with } w_i \in W_i \}.$$

Observation. (i) $W_1 + \dots + W_n$ is a subspace of V

(Easy).

Definition. The subspaces $W_1, \dots, W_n$ are called independent if no sum $w_1 + \dots + w_n$ with $w_i \in W_i$ is zero, except for the trivial sum, i.e.

$$w_1 + \dots + w_n = \bar{0} \text{ and } w_i \in W_i \Rightarrow w_i = 0 \ \forall i.$$

Definition. If subspaces $W_1, \dots, W_n$ are independent and their span is the whole space V, then we say that V is the *direct sum* of $W_1, \dots, W_n$.

direct sum

$$V = W_1 \oplus \dots \oplus W_n$$

if $V = W_1 + \dots + W_n$ and if $W_1, \dots, W_n$ are independent.

This is equivalent to saying that

every vector $v \in V$ can be written as $v = w_1 + \dots + w_n$ in exactly one way.

Discussion.

$W_1 + \cdots + W_n$ is a subspace of V.

If $W_1, \ldots, W_n$ are subspaces (independent) and

$\boxed{W_1 + \cdots + W_n \neq V,}$ then let

Let $U = W_1 + \cdots + W_n$, subspace of V.

Here, U is the direct sum of $W_1, \ldots, W_n$,

$$U = W_1 \oplus \cdots \oplus W_n.$$

Proposition.

(a). A single subspace W_1 is independent.

(b). Two subspaces W_1, W_2 are independent if and only if $W_1 \cap W_2 = (0)$.

Proof. (a) "Easy".

(b) W_1 and W_2 are independent $\iff$ $W_1 \cap W_2 = (0)$.

[(A) $\iff$ (B).]

Proof.

$(\Rightarrow)$ We will prove this by $\neg(B) \Rightarrow \neg(A)$.

$W_1 \cap W_2 \neq (0)$

Take $\underline{\underline{v \overset{\neq 0}{\in} W_1 \cap W_2}}$. $\Rightarrow$ $v \in W_1$ and $v \in W_2$

$-v \in W_2$

Note that we can always write

$$\bar{0} = \overset{\in W_1}{v} + \overset{\in W_2}{(-v)}$$

$$\parallel$$

$$\overset{\in W_1}{(-v)} + \underset{\in W_2}{v}$$

Thus 0 vector is written in two different ways,

$\Rightarrow$ W_1 and W_2 are not independent.

$(\Leftarrow)$ $(A) \Leftarrow (B)$ $W_1 \cap W_2 = (0)$

$\parallel$

W_1 & W_2 are independent

Let $w_1 + w_2 = 0$; $w_1 \in W_1$ and $w_2 \in W_2$.

This implies $\left\{ \begin{array}{l} w_1 = -w_2 \\ \text{and} \quad w_2 = -w_1 \end{array} \right\} \Rightarrow$ $w_1 \in W_1 \cap W_2$ and $w_2 \in W_1 \cap W_2$

But $W_1 \cap W_2 = (0)$ $\Rightarrow$ $\underline{\underline{w_1 = 0 \text{ and } w_2 = 0}}$.

Hence W_1 and W_2 are independent.

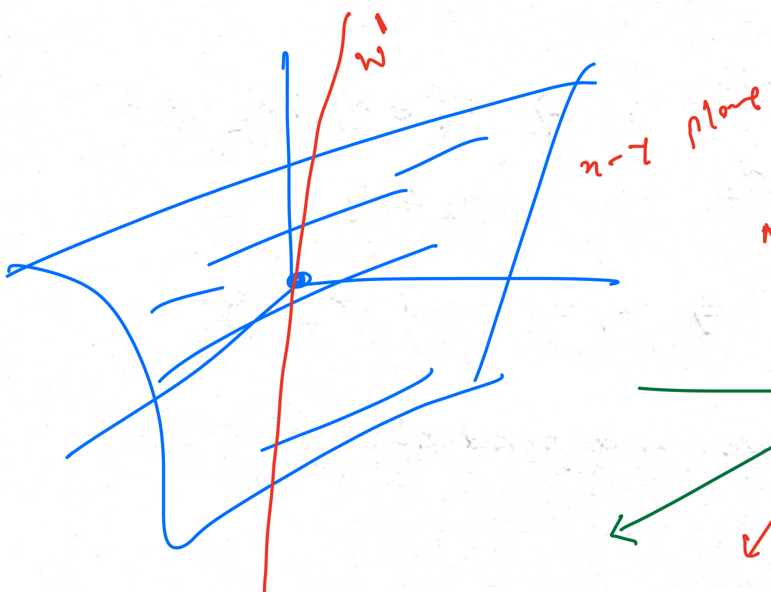
Proposition. Let $W_1, \ldots, W_n$ be subspaces of a finite-dimensional vector space V, and let $\mathcal{B}_i$ be a basis for W_i.

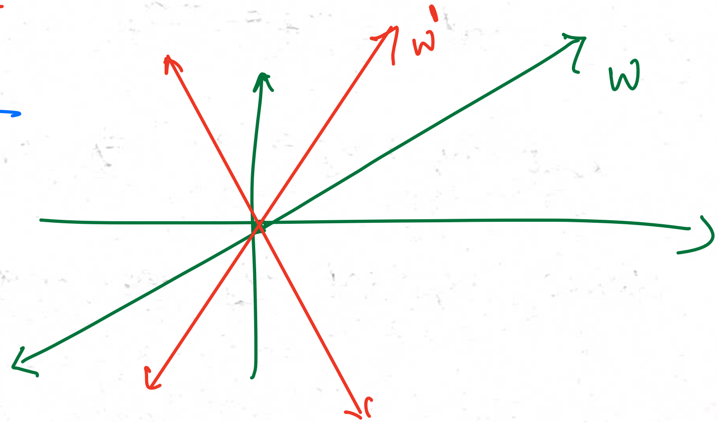
(a) The ordered set $\mathcal{B}$ obtained by listing the bases $\mathcal{B}_1, \ldots, \mathcal{B}_n$ in order is a basis of V if and only if $V = W_1 \oplus \cdots \oplus W_n$.

(b) $\dim(W_1 + \cdots + W_n) \leqslant (\dim W_1) + \cdots + (\dim W_n)$,

Proof.

with equality if and only if the spaces are independent.





Corollary. Let W be a subspace of a finite-dimensional vector space V. There is another subspace W' such that $V = W \oplus W'$.

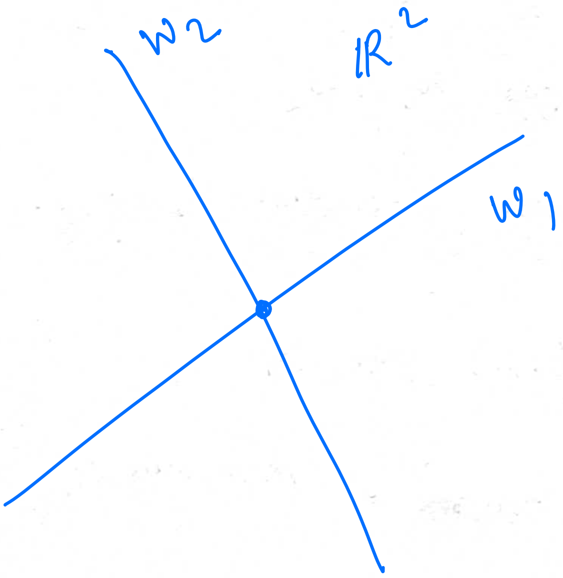
Proof.

Let $(w_1, \dots, w_d)$ be a basis for W.

We extend to a basis $(w_1, \dots, w_d, \underbrace{v_1, \dots, v_{n-d}})$ for V.

$\mathrm{Span}(v_1, \dots, v_{n-d})$ is the required subspace W'.

$$V = W + \overset{\parallel}{W'}$$



$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$\underset{\underset{2}{\parallel}}{\dim(\mathbb{R}^2)} = 1 + 1 - 0$$

$$= 2$$

Discussion. $\mathbb{R}^3$

subspace of V

Basis

$W_1 = x\text{-}y$ plane

$W_2 = x\text{-}z$ plane

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

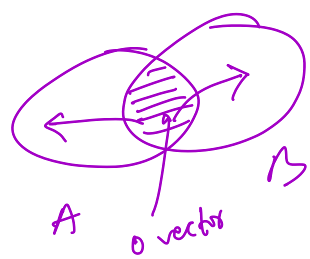
$$= 2 + 2 - 1 = 3$$

$W_1 \cap W_2 \subseteq W_1$

Basis $\rightarrow$ {extend to W_1}

$\cap\, W_2$

{extend basis to W_2}



$$\dim W_1 + \dim W_2 = (\underset{\geq 0}{\ }) + \dim(W_1+W_2)$$

$$\boxed{\dim(W_1+W_2) \leq \dim W_1 + \dim W_2}$$

Proposition. Let W_1, W_2 be subspaces of a finite-dimensional vector space V. Then

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Proof. Re-write above relation as

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \underbrace{\dim(W_1 \cap W_2)}_{\geq 0}. \quad (\leq \dim W_1 + \dim W_2$$

Assume that $\boxed{\dim W_1 = m}$ and $\boxed{\dim W_2 = n}$, for some $m, n \in \mathbb{N}$.

Observe that $\begin{cases} W_1 \cap W_2 \subseteq W_1, \text{ and} \\ W_1 \cap W_2 \subseteq W_2. \end{cases}$

Also, $W_1 \cap W_2$ is a subspace of V, hence finite-dimensional.

Choose $\mathcal{B}_1 = (u_1, .., u_r)$, basis for $W_1 \cap W_2$, $r = \dim(W_1 \cap W_2)$.

Extend $\mathcal{B}_1$ to get a basis for W_1:

$$\mathcal{B}_1' = (u_1, \dots, u_r; x_1, \dots x_{m-r}), \quad m = \dim W_1$$

Similarly, extend $\mathcal{B}_1$ to get a basis for W_2

$$\mathcal{B}_1'' = (u_1, \dots, u_r; y_1, \dots, y_{n-r}), \quad n = \dim W_2$$

To prove the proposition, it is enough to show

that $\mathcal{B} = (u_1, \ldots, u_r ; x_1, \ldots, x_{m-r} ; y_1, \ldots, y_{n-r})$ is a basis for $W_1 + W_2$.

We need to show

(i) $\mathcal{B}$ is linearly independent;

(ii) $Span(\mathcal{B}) = W_1 + W_2$.

Proof of (i)

Suppose $\mathcal{B}$ is linearly dependent, then

$$\underbrace{a_1 u_1 + \cdots + a_r u_r}_{u} + \underbrace{b_1 x_1 + \cdots + b_{m-r} x_{m-r}}_{x} + \underbrace{c_1 y_1 + \cdots + c_{n-r} y_{n-r}}_{y} = 0 \tag{A}$$

where some scalars are non-zero.

In short,

$$u + x + y = 0.$$

$$\Rightarrow y = \underline{\underline{-u - x}} \in W_1.$$

Also, $y \in W_2 \Rightarrow y \in W_1 \cap W_2$

Then y is a linear combination of $(u_1, \ldots, u_r)$

$y = d_1 u_1 + \cdots + d_r u_r$ for some d_i ; $i = 1, \ldots, r$

$$y - (d_1 u_1 + \cdots + d_r u_r) = 0$$

or, $c_1 y_1 + \cdots + c_{n-r} y_{n-r} + (-d_1) u_1 + (-d_2) u_2 + \cdots + (-d_r) u_r = 0$

Recall $(y_1, \ldots, y_{n-r}; u_1, \ldots, u_r)$ is a basis for W_2

$\Rightarrow$ $y = 0$

Thus our original relation reduces to

$$u + x = 0.$$

Again, since $(u_1, \ldots, u_r; x_1, \ldots, x_{m-r})$ is a basis for W_1

$\Rightarrow$ all scalars are zero

$\Rightarrow$ $u = 0$ and $x = 0$

Thus whole relation in equ. (A) was trivial,

and hence $\mathcal{B}$ is a basis.

Proof of (ii).

For any vector v in $W_1 + W_2$ is of the

form : $v = w_1 + w_2$, $w_1 \in W_1$
$w_2 \in W_2$.

$$w_1 = a_1 u_1 + \cdots + a_r u_r + b_1 x_1 + \cdots + b_{m-r} x_{m-r},$$

$$w_2 = a_1' u_1 + \cdots + a_r' u_r + c_1 y_1 + \cdots + c_{n-r} y_{n-r}$$

Then

$$\begin{aligned} w_1 + w_2 = (a_1 + a_1') u_1 + \cdots &+ (a_r + a_r') u_r \\ &+ b_1 x_1 + \cdots + b_{m-r} x_{m-r} \\ &+ c_1 y_1 + \cdots + c_{n-r} y_{n-r}. \end{aligned}$$

Thus any $v \in W_1 + W_2$ is a linear combination of $\mathcal{B}$.

Discussion on problems from the Artin book, and S. Axler book,

please see recording of lecture !!

- We discussed many problems today.

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CHAPTER END