Endsem Exam (Online Mode) - Linear Algebra (MA 4020)

Date: November 30, 2021 — Maximum Marks 20

Time: 50 minutes, 04 : 15 PM - 5 : 05 PM — Extra uploading time 8 minutes

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Instructions

1. There are **two sections, A, and B**. Depending upon your roll number ending with even or odd integer, answer the respective section. All questions are compulsory.
2. In **both sections, A, and B, the Q.2 have negative marking**.
3. Write your **name and roll number on each answered pages**.
4. Scan the document in the pdf file format.
5. Upload the **pdf file** on the Google classroom (No .jpeg or .jpg file please.).

SECTION A (Roll numbers ending with 1,3,5,7,9)

Q.1 Is $v = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ an eigenvector of $A = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$? If yes, find an eigenvalue of v.

(2 Marks)

Q.2 Let $A \in M_3(\mathbb{R})$ which is **not** a diagonal matrix. Pick out the cases when A is diagonalizable (or not diagonalizable) over $\mathbb{R}$:
(No justification required. For each correct answer $+1$, and wrong answer -1)

(a) When $A^2 = A$;
(b) When $(A - 3I)^2 = 0$,
(c) When $A^2 + I = 0$.

(3 Marks)

Q.3 Let A be a 3×3 upper triangular matrix whose diagonal entries are $2, 3$ and 4. Express A^{-1} as a polynomial of A and I. (3 Marks)

Q.4 Determine the eigenvalues and eigenvectors of (4 Marks)

$$A = \begin{pmatrix} 7 & 1 \\ 5 & 11 \end{pmatrix} \in M_2(\mathbb{R}).$$

Q.5 Suppose V is a real inner-product space and $(v_1, \ldots, v_m)$ is a linearly independent list of vectors in V. Write orthonormal list of vectors $(w_1, \ldots, w_m)$ in V using Gram-Schmidt procedure. (No proof required till this stage).

Prove that there exist exactly 2^m orthonormal lists $(w_1, \ldots, w_m)$ of vectors in V such that

$$\operatorname{Span}(v_1, \ldots, v_j) = \operatorname{Span}(w_1, \ldots, w_j)$$

for all $j \in \{1, 2, \ldots, m\}$. (3 Marks)

Q.6 Let V be a finite dimensional vector space. Let $T \in \mathcal{L}(V, V)$ be a linear transformation.

(a) Prove that T is invertible if and only if the constant term of the minimal polynomial of T is non-zero.

(b) Assume that T is invertible. Show that λ is an eigenvalue of T if and only if $\lambda \neq 0$ and λ^{-1} is an eigenvalue of T^{-1}. (5 Marks)

SECTION B

Q.1 Is $w = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ an eigenvector of $A = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$? If yes, find an eigenvalue of w.

(2 Marks)

Q.2 Let $A = \begin{pmatrix} 21 & 2021 \\ 2021 & 1 \end{pmatrix}$ Which of the following statements are true/false? (No justification required. For each correct answer $+1$, and wrong answer -1)

(i) The matrix A is diagonalizable over $\mathbb{R}$.

(ii) There exists a basis of $\mathbb{R}^2$ consisting of eigenvectors $\{v_1, v_2\}$ of the matrix A such that $v_1^T v_2 = 0$.

(iii) There exists an invertible 2×2 matrix B such that $B^3 = A$. (3 Marks)

Q.3 Let A be a 2021×2021 matrix whose characteristic polynomial is given by

$$(t-2)^{n_1}(t-3)^{n_2}(t-4)^{n_3}.$$

such that $n_1 + n_2 + n_3 = 2021$ and each $n_i > 1$. If A is diagonalizable, find A^{-1} as a polynomial of A and I. (3 Marks)

Q.4 Determine the eigenvalues and eigenvectors of (4 Marks)

$$A = \begin{pmatrix} 10 & 1 \\ 4 & 13 \end{pmatrix} \in M_2(\mathbb{R}).$$

Q.5 Suppose V is a real inner-product space and $(v_1, \ldots, v_m)$ is a linearly independent list of vectors in V. Write orthonormal list of vectors $(w_1, \ldots, w_m)$ in V using Gram-Schmidt procedure. (No proof required till this stage).

Prove that there exist exactly 2^m orthonormal lists $(w_1, \ldots, w_m)$ of vectors in V such that

$$\mathrm{Span}(v_1, \ldots, v_j) = \mathrm{Span}(w_1, \ldots, w_j)$$

for all $j \in \{1, 2, \ldots, m\}$. (3 Marks)

Q.6 Let V be a finite dimensional vector space. Let $T \in \mathcal{L}(V, V)$ be a linear transformation.

(a) Prove that T is invertible if and only if the constant term of the minimal polynomial of T is non-zero.

(b) Assume that T is invertible. Show that λ is an eigenvalue of T if and only if $\lambda \neq 0$ and λ^{-1} is an eigenvalue of T^{-1}.

(5 Marks)