Assignment 4

Suraj - CS20BTECH11050

Download all python codes from

https://github.com/Suraj11050/Assignments–AI1103/tree/main/Assignment%204/Python%20codes

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1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f(x) = \begin{cases} \frac{7}{32} x^6 \, (2 - x), & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

then k equals _______________

2 SOLUTION

Let $X \in [0, 2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x) \, dx = 1 \tag{2.0.1}$$

$$\Pr(x) = \frac{1}{2} \quad (\because \text{Uniform order}) \tag{2.0.2}$$

The PDF for X is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.0.3}$$

The CDF for X is

$$F(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.0.4}$$

Let $X_{(k)}$ be vector of order statistic of $(X_1, X_2 \ldots, X_n)$

Lemma 2.1. Marginal probability density (PDF) for a k_{th} order statistic of a random sample of size n given CDF= $F(x)$ and PDF= $f(x)$ is given by

$$f_{(k,n)}(x) = n \, {}^{n-1}\mathrm{C}_{k-1} \, (1 - F(x))^{n-k} F(x)^{k-1} f(x)$$

Proof. The CDF of the k^{th} order statistic from a sample of size n is:

$$F_{(k,n)}(x) = \Pr\left(X_{(k)} \le x\right)$$
$$= \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (1 - F(x))^{n-j} F(x)^j \tag{2.0.5}$$

Deriving PDF of k^{th} order statistic from a sample of size n:

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx} \left(\sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (1 - F(x))^{n-j} F(x)^j \right)$$

$$f_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (j) \, (1 - F(x))^{n-j} F(x)^{j-1} f(x) - \sum_{j=k}^{n} {}^{n}\mathrm{C}_j \, (n - j) \, (1 - F(x))^{n-j-1} F(x)^j f(x)$$

$$S_1 = \sum_{j=k}^{n} \frac{n!}{(n - j)! \, (j - 1)!} (1 - F(x))^{n-j} F(x)^{j-1} f(x) \tag{2.0.6}$$

$$S_2 = \sum_{j=k}^{n} \frac{n!}{(n - j - 1)! \, j!} (1 - F(x))^{n-j-1} F(x)^j f(x) \tag{2.0.7}$$

let $i = j + 1$ change the limits for the summation in equation (2.0.7)

$$S_2 = \sum_{i=k+1}^{n} \frac{n!}{(n - i)! \, (i - 1)!} (1 - F(x))^{n-i} F(x)^{i-1} f(x)$$

$$S_1 - S_2 = \frac{n!}{(n-k)!\,(k-1)!}\,(1-F(x))^{n-k}\,F(x)^{k-1} f(x) \quad (2.0.8)$$

$$f_{(k,n)}(x) = n\,{}^{n-1}\mathrm{C}_{k-1}\,(1-F(x))^{n-k}\,F(x)^{k-1} f(x) \quad (2.0.9)$$

□

Using Lemma (2.1) given $n = 8$ substituting n, equation (2.0.3) and equation (2.0.4) in above equation (2.0.9) we get:

$$f_{(k,n)}(x) = 8\,{}^{7}\mathrm{C}_{k-1}\left(1-\frac{x}{2}\right)^{8-k}\left(\frac{x}{2}\right)^{k-1}\frac{1}{2}$$

$$f_{(k,n)}(x) = \frac{1}{32}\,{}^{7}\mathrm{C}_{k-1}\,(2-x)^{8-k}\,x^{k-1} \quad (2.0.10)$$

Comparing the marginal probability density function obtained in equation (2.0.10) with the equation given in question

$$\frac{1}{32}\,{}^{7}\mathrm{C}_{k-1}\,(2-x)^{8-k}\,x^{k-1} = \frac{7}{32}\,(2-x)\,x^{6}$$

$$\therefore k = 7 \quad (2.0.11)$$

Hence the marginal probability density in the given problem is 7^{th} order uniform statistic and **the value of k is 7**