Lecture 16 MA4020 (LINEAR ALGEBRA)

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Assume that X, Y, Z, and W are vector spaces over the same field F. Let α, β, γ are linear transformations.

$$\begin{tikzcd}
X \arrow[r, "\alpha", "A"'] \arrow[rrr, bend left, no head, "\gamma\circ(\beta\circ\alpha)"] \arrow[rr, bend right, no head, "\beta\circ\alpha"'] & Y \arrow[r, "\beta", "B"'] & Z \arrow[r, "\gamma", "C"'] & W
\end{tikzcd}$$

$C(BA)$

$\gamma\circ(\beta\circ\alpha) : X \longrightarrow W$

$\|$ some linear transformation

$(\gamma\circ\beta)\circ\alpha : X \longrightarrow W$

$(CB)A$

$$\begin{tikzcd}
X \arrow[r, "\alpha", "A"'] & Y \arrow[r, "\beta", "B"'] \arrow[rr, bend right, no head, "\gamma\circ\beta"'] & Z \arrow[r, "\gamma", "C"'] & W
\end{tikzcd}$$

By associativity; we have

$$(\gamma\circ\beta)\circ\alpha = \gamma\circ(\beta\circ\alpha).$$

Hence, $(CB)A = C(BA)$.

This proves $(M_n(\mathbb{R}), +, \cdot)$ having associativity law.

$$\begin{tikzcd}
 & v & A & T(v) & \\
\mathcal{B}_V \arrow[d, squiggly] & V \arrow[rr, "T"'] \arrow[d, "\varphi", "P"'] & & W \arrow[d, "\psi"', "Q"] & \mathcal{B}_W \arrow[d, squiggly] \\
\mathcal{B}'_V & V \arrow[rr, "T"'] \arrow[rr, bend left=60] & & W & \mathcal{B}'_W \\
 & & QAP^{-1} & &
\end{tikzcd}$$

$$\psi \circ T = T' \circ \varphi \qquad\qquad Q, P \in GL_*(F)$$

$$\underset{Q}{\underset{\uparrow}{\psi}} \circ \underset{A}{\underset{\uparrow}{T}} \circ \underset{P^{-1}}{\varphi^{-1}} = T'$$

Special Case.

$$\begin{tikzcd}
 & & A & & \\
\mathcal{B}_V & V \arrow[rr, "T"'] \arrow[d, "\varphi", "P"'] & & V \arrow[d, "\psi"', "P"] & \mathcal{B}_V \\
\mathcal{B}'_V & V \arrow[rr, "T"'] & & V & \mathcal{B}'_V \\
 & & PAP^{-1} & &
\end{tikzcd}$$

Proposition.

(a) Vector Space form: Let $T : V \longrightarrow W$ be a linear transformation. Bases $\mathcal{B}, \mathcal{C}$ can be chosen so that the matrix of T takes the form

$$A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \cdots \circledast$$

where I_r is the $r \times r$ identity matrix, and $r =$ rank T.

(b) Matrix form: Given any $m \times n$ matrix A, there are matrices $Q \in GL_m(F)$ and $P \in GL_n(F)$ so that QAP^{-1} has the form $\circledast$.

Proof

(a) $\implies$ (b) "Easy"

Let us prove (a).

Proof of (a).

$T : V \longrightarrow W$ linear transformation.

Let $(u_1, \ldots, u_k)$ be a basis for $\ker T$.

Extend to a basis $\mathcal{B}$ for V : $(v_1, \ldots, v_r ; u_1, \ldots, u_k)$

where $r + k = n = \dim_F V$.

Let $w_i = T(v_i)$, $i = 1, \ldots, r$.

Then $(w_1, \ldots, w_r)$ is a basis for $\operatorname{im} T \subseteq W$.

Extend to a basis $\mathcal{C}$ of W

$(w_1, \ldots, w_r ; \underline{x_1, \ldots, x_s})$, where $r + s = m = \dim W$.

$$A = \begin{pmatrix} 1 & | & & | & | & & | \\ 0 & & & & & & \\ 0 & T(v_1) & \cdots & T(v_r) & T(u_1) & \cdots & T(u_k) \\ \vdots & | & & | & | & & | \\ 0 & | & & | & | & & | \end{pmatrix}$$

$u_1, \ldots u_k \in \ker(T)$

$T(v_1) = w_1$

$T(u_1) = 0 = 0 \cdot w_1 + \cdots + 0 \cdot w_r$

$\qquad + 0 \cdot x_1 + \cdots + 0 \cdot x_s$

Observe that $= 1 \cdot w_1 + 0 \cdot (/ + 0 \cdot (/ \cdots$

$$A = \left[\begin{array}{c|c} \begin{matrix} 1 & \\ 0 & \\ 0 & I_r \\ \vdots & \\ 0 & \\ 0 & \\ \vdots & \\ 0 & \end{matrix} & 0 \end{array}\right]$$

Linear Transformation $T : V \longrightarrow V$

[linear operator]

Recall:

Proposition. Let A be the matrix of a linear operator T w.r.t. a basis $\mathcal{B}$. The matrices A' which represent T for different bases are those of the form

$$A' = PAP^{-1}, \quad \text{for } P \in GL_n(F) \text{ arbitrary.}$$

A square matrix A is similar to A' if $A' = PAP^{-1}$ for some $P \in GL_n(F)$.

Notation. The word conjugate is also used for the similar matrices.

Invariant subspace.

Let $T : V \longrightarrow V$ be a linear operator on a vector space. A subspace W of V is called an invariant subspace or a T- invariant subspace if it is carried to itself by the operator

$$T(W) \subset W.$$

$$T(w) \in \underset{\substack{\uparrow \\ \text{Subspace}}}{W} \quad \forall w \in W$$

In other words, W is T-invariant if $T(w) \in W$ for all $w \in W$.

Then we may define linear operator on W

$$\underset{\substack{\uparrow \\ (\text{restriction of } T \text{ to } W)}}{T|_W} : W \longrightarrow W$$

$$\text{Matrix of } T = \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$\mathcal{B} = (\underbrace{w_1, \ldots, w_k}_{\text{Basis of } W}, v_1, \ldots, v_{n-k})$$ basis for V.

$$T : V \longrightarrow V$$

$$\left[\begin{array}{c|c|c|c|c|c} | & & | & | & & | \\ T(w_1) & \cdots & T(w_k) & T(v_1) & \cdots & T(v_{n-k}) \\ | & & | & | & & | \end{array}\right]$$

Since ~~Ts~~ W is T-invariant subspace, the matrix of linear transformation w.r.t. $\mathcal{B}$ has the form

$$\left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right]$$

where A is the matrix of the restriction

$$T|_W : W \longrightarrow W.$$

Case study.

Assume that $V = W_1 \oplus W_2$ is the direct sum of two T-invariant subspaces, and let

$\mathcal{B}_1$: basis of W_1

$\mathcal{B}_2$: basis of W_2.

Then $\mathcal{B} = (\mathcal{B}_1 ; \mathcal{B}_2)$ is a basis of V

$$T : V \longrightarrow V$$

$$\left[\begin{array}{cc} T(\mathcal{B}_1) & T(\mathcal{B}_2) \end{array} \right]$$

$T(W_1) \subset W_1$

$T(W_2) \subset W_2$

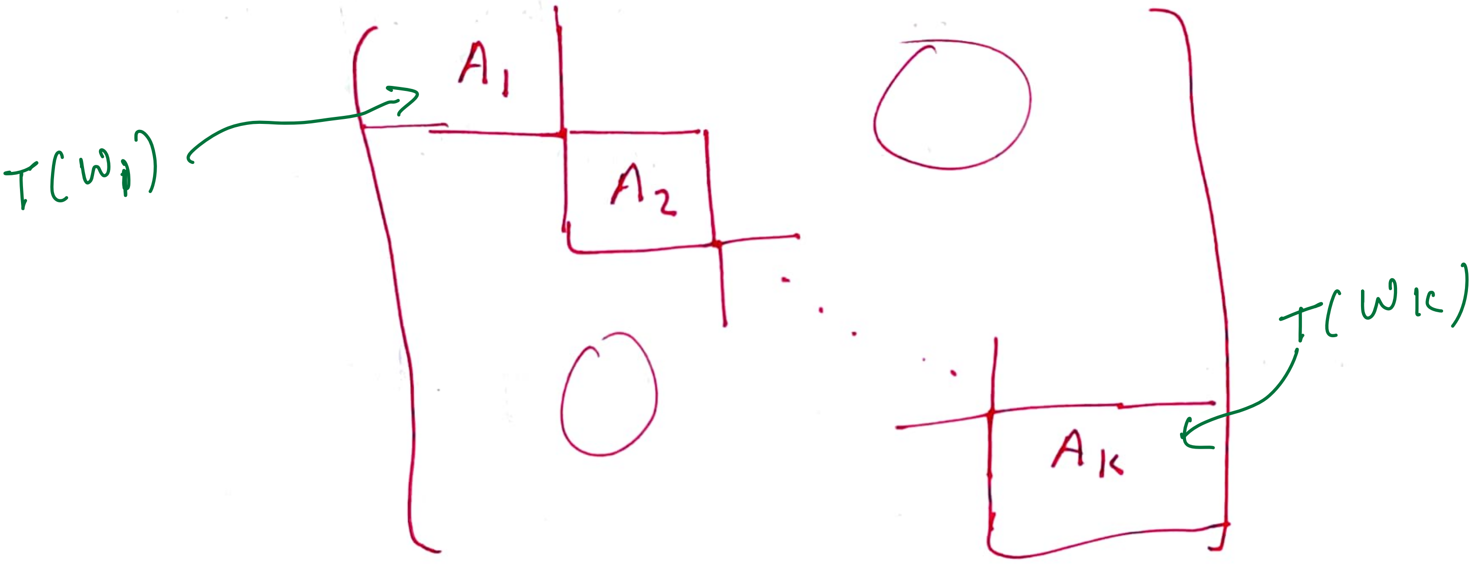
$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right]$$

Matrix of T is block diagonal matrix, where A_i is the matrix of T restricted to W_i.

In general, say $V = \underset{\substack{\uparrow \\ \mathcal{B}_1}}{W_1} \oplus \cdots \oplus \underset{\substack{\uparrow \\ \mathcal{B}_k}}{W_k}$

$\{ \mathcal{B}_1, \ldots \mathcal{B}_k \}$ basis of V.

Then matrix of linear operator $T : V \to V$ has the form



How to find an eigenvector?

How many eigen vector T can have?

$$\begin{aligned} Id : V &\longrightarrow V \\ v &\longmapsto v \end{aligned}$$

$Id(v) = 1 \cdot v$ true for all $v \in V$

$T : V \longrightarrow V$

want $v \in V$ s.t. $T(v) = \lambda v$

Eigen-vector. $T: V \longrightarrow V$

An eigenvector v for a linear operator T is a nonzero vector such that $T(v) = \lambda v$ for some scalar $\lambda \in F$.

The scalar λ is called the eigen value associated to the eigenvector v.

Convention. Eigenvalue of a linear operator T, we mean a scalar $\lambda \in F$ which is the eigenvalue associated to some eigenvector.

Notation. Sometimes eigenvectors and eigenvalues are called characteristic vectors and characteristic values.

Discussion. Let v be an eigenvector for a linear operator T. The subspace W spanned by v is T- invariant.

A ← Graph Theory

finite simple graph G

Adjacency matrix ← ↑ (connected)

"linear algebraic graph Theory"

$$T \underset{\text{correspondence}}{\overset{1-1}{\frown}} A$$

Definition/Notation.

By an eigenvector for an $n \times n$ matrix A, we mean a vector which is an eigenvector for ~~left~~ left multiplication by A, a non-zero column vector X such that

$$A X = \underset{\substack{\uparrow \\ \text{Eigen value}}}{\lambda} X \quad \text{for some } \lambda \in F$$

Suppose that A is the matrix of T w.r.t. a basis $\mathcal{B}$, and let X denote the co-ordinate vector of a vector $v \in V$. Then $T(v)$ has co-ordinate AX.

Hence,

X is an eigenvector for A $\Leftrightarrow$ v is an eigenvector for T.

Moreover, T and A have the same eigen values.

Conversely, if the subspace W ~~is~~ spanned by v is invariant, then v is an eigenvector.

Thus an eigenvector can be described as a basis of a <u>one-dimensional T-invariant subspace.</u>

If v is an eigenvector, and if we extend it to a basis $(v = v_1, \dots, v_n)$ of V, then the matrix of T will have the block form

$T(v) = cv = c \cdot 1 + 0 \cdot (\ ) + \dots + 0 \cdot (\ )$

$$\begin{bmatrix} c & B \\ 0 & D \end{bmatrix} \qquad \left[\begin{array}{c|ccc} c & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right]$$

$T(v) = cv$

Question. What can we say about eigenvalues of similar matrices?

(Same)

Corollary. Similar matrices have the same eigenvalues.

Proof. Note that similar matrices represent the same linear transformation, and hence eigenvalues are same.

Remark. The basis vector v_j is an eigenvector of T with eigenvalue λ,

$\iff$ the j^{th} column of A has the form λe_j

$A = [a_{ij}]$

$T(v_j) = v_1 a_{1j} + v_2 a_{2j} + \cdots + v_n a_{nj}$

$$A \begin{array}{c} j^{th}\text{ column} \\ \left[\begin{array}{ccc} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ & a_{nj} & \end{array}\right] \end{array}$$

If

$T(v_j) = \lambda v_j$

$\Rightarrow a_{jj} = \lambda$ and

$a_{ij} = 0$ if $i \neq j$

Corollary. $T: V \longrightarrow V$ linear operator

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ matrix of T

Then

$T(v_j) = \lambda \cdot v_j$

A is a diagonal matrix

$\iff$ every basis vector v_j is an eigenvector.

(Proof is straightforward)

$$\left(\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_k \end{bmatrix} \cong \underline{\underline{P \cdot A \cdot P^{-1}}} \right.$$

Corollary. The matrix A of a linear transformation is similar to a diagonal matrix

$\iff$ there is a basis $\mathcal{B}' = (v_1', \ldots, v_n')$ of V made up of eigenvectors.

(Proof is easy).

THE CHARACTERISTIC POLYNOMIAL.

A non-zero vector v is an eigenvector for linear operator $(T : V \to V)$ if

$$Tv = \lambda v \quad \text{for some } \lambda \in F.$$

Question. How to find such v ?

Note that it is not clear how to pick v such that $T(v) = \lambda v$.

We may try equivalent form:

v is an eigenvector of T $\Longleftrightarrow$ X is an eigenvector for A.

↑

If A is complecated, then finding X is difficult.

Suppose we know λ for some eigenvector v (assuming that exists.)

then we may solve a linear equation $Tv = \lambda v$ to find v !

Assume that eigenvalue λ is determined, then

write $\quad T v = \lambda v \quad$ as

$$(T - \lambda I)(v) = 0$$

$\uparrow$

Identity operator

$$\left(\begin{array}{l} I : V \longrightarrow V \\ \quad v \longmapsto v \end{array} \right)$$

Then $\quad T - \lambda I : V \longrightarrow V$

$\qquad\qquad\qquad\quad v \longmapsto$

$T - \lambda I$ is also a linear operator

defined by $\quad (T - \lambda I)(v) = T(v) - \lambda v$

If A is the matrix of T w.r.t. some basis,

then the matrix ~~To~~ of linear

operator $T - \lambda I$ is $A - \lambda I$.

$$T - \lambda I : V \longrightarrow V$$

$v \in \ker(T - \lambda I)$.

Lemma. The following conditions on a linear operator $T : V \longrightarrow V$ on a finite-dimensional vector space are equivalent:

$\{0\}$ $\neq$ $\qquad$ $\{0\} \underset{\neq}{\overset{\text{subspace}}{\subset}} \ker T$ proper

(a) $\ker T > 0$

proper subspace

(b) $\underset{\neq V}{\operatorname{im} T} < V$

(c) If A is the matrix of the operator w.r.t. an arbitrary basis, then $\det A = 0$.

(d) 0 is an eigenvalue of T.

Proof.

$$\boxed{\dim V = \dim(\ker T) + \dim(\operatorname{im} T)}$$

(Dimension formula)

(a) $\Longleftrightarrow$ (b) $\qquad$ $\operatorname{im}(T) < \neq V$

$\Updownarrow$

T is not an isomorphism

i.e. A is not an invertible matrix

$\Updownarrow$

(c) $\det A = 0$

Part (d) $\quad$ Let $\underset{\neq 0}{v} \in \ker T$, v is eigenvector of T

$$\underset{\overset{\|}{0}}{T(v)} = \lambda \cdot v$$

$$0 = \lambda \cdot v \implies \lambda = 0$$