CS 1010 DISCRETE STRUCTURES

QUIZ 4
LECTURE 9-10

(1) How many ways are there for a horse race with four horses to finish if ties are possible? [Note: Any number of the four horses may tie.]
 (a) 24
 (b) 75
 (c) 8
 (d) 45
Answer: b, 75.
Solution: The final race results:
No ties: $1, 2, 3, 4 = 4! = 24$ ways
$1, 2, 3, 3 - 4!/2! = 12$ ways
$1, 2, 2, 3 - 4!/2! = 12$ ways
$1, 2, 2, 2 - 4!/3! = 4$ ways
$1, 1, 2, 3 - 12$ ways
$1, 1, 2, 2 - 4!/2!2! = 6$ ways
$1, 1, 1, 2 - 4$ ways
$1, 1, 1, 1 - 1$ way
Adding them you get 75.

(2) Wilson's theorem states that for $n \in \mathbb{N}$ and $n > 1$, n is prime if and only if

$$(n-1)! \equiv -1 \bmod n.$$

Can this result be used in primality testing algorithms?
 (a) The theorem does not help us check if n is prime or not, it only tells us one of the properties that a prime number satisfies.
 (b) The result is a characterization of a prime number and computing $(n-1)!$ mod n is just successive multiplications and therefore the theorem can be used in primality tests.
 (c) The result is theoretically a primality test but from a computational point of view it is not efficient since computing $(n-1)!$ takes too many steps.
 (d) The result is true for n's that are not primes, i.e. there exists a composite n such that the product of positive integers less than n is -1 mod n, and the test using this result may report a composite number as prime and therefore should not be used.
Ans: c

(3) Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. *(3 marks)*

Answer -: We count in two ways the number of ways to choose a committee with a leader from a set of n people. On one hand we can choose the leader first in any of the n ways. We can then choose the rest of the committee, which can be any subset of the remaining $n-1$ people; this can be done in 2^{n-1} ways since there are 2^{n-1} subsets, thus we can choose a committee with a leader in $n2^{n-1}$ ways. Therefore the right hand side of the proposed identity counts the number of ways the committee can be formed.
On the other hand we can organize our count by the size of the committee. Let k be the number of people who will serve on the committee. The number of ways to select a committee with k people is clearly $\binom{n}{k}$, and once we have chosen the committee, there are clearly k ways in which to choose its leader. By the sum rule the left-hand side of the proposed identity therefore also counts the number of such committees. Since both L.H.S and R.H.S count the same quantity, they must be equal.

(4) The number of diagonals which can be drawn by joining the vertices of a heptagon (7-sided polygon) is
(a) 21
(b) 14
(c) 12
(d) 10
Ans -: 14

(5) A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?
(a) 56
(b) 49
(c) 120
(d) 792
Ans-: 56

(6) A bag contains 24 balls such that 9 balls are red, 7 are white and 8 are blue. What is the minimum number of balls that must be picked up from the bag blindfolded (without replacing any of it) to be assured of picking at least one ball of each colour?
(a) 18
(b) 10
(c) 9
(d) 3
Answer a

(7) How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to guarantee that at least one pair of these numbers add up to 9?

(a) 4
(b) 5
(c) 6
(d) 2
Answer b

(8) When four identical coins are tossed simultaneously, in _____ number of the outcomes at most two of the coins will turn up as heads.
(a) 11
(b) 10
(c) 6
(d) 14
Answer: a)11