CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 12

Instructions

- Try these questions before class. Do not submit!

(1) Suppose that the function f satisfies the recurrence relation

$$f(n) = 2f(\sqrt{n}) + log\ n,$$

whenever n is a perfect square greater than 1 and $f(2) = 1$.
(a) Find $f(16)$.
(b) Find a big-$\mathcal{O}$ estimate for $f(n)$. [Hint: Make the substitution $m = log\ n$.]
(a) Find $f(16)$.

$$\begin{aligned} f(16) &= 2f(4) + log\ 16 \\ &= 2(2f(2) + log\ 4) + log\ 16 \\ &= 2(2+2) + 4 = 12 \end{aligned}$$

(b) Find a big-$\mathcal{O}$ estimate for $f(n)$. [Hint: Make the substitution $m = log\ n$.]
Let $m = log\ n$. Therefore $2^m = n$.

$$\begin{aligned} f(n) = f(2^m) &= 2f(\sqrt{2^m}) + log\ 2^m \\ &= 2f(2^{m/2}) + m. \end{aligned}$$

Thus we have $f(2^m) = 2f(2^{m/2}) + m$. Replacing $f(2^m)$ with $h(m)$ we get,

$$h(m) = 2h(m/2) + m,$$

a divide and conquer recurrence with $g(m) = m$. This means we have to consider the general case, $f(n) = af(n/b) + g(n)(= cn^d)$ and the Master theorem gives us an estimate of this case.
We have, $a = 2, b = 2, c = 1, d = 1$. Here, $a = 2 = b^d = 2^1$ so we consider the second case of Master Theorem. $h(m)$ is $\mathcal{O}(mlog\ m)$. Therefore $f(2^m) = h(m) = \mathcal{O}(mlog\ m)$. Replacing $m = log\ n$ we get $f(n) = \mathcal{O}(log\ nlog\ log\ n)$.

(2) Find a closed form for the generating function for each of these sequences. A closed form means an algebraic expression not involving a summation over a range of values or the use of ellipses. For each sequence, use the most obvious choice of a sequence that follows the pattern of the initial terms listed.
(a) $0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, \ldots$
(b) $0, 0, 0, 1, 1, 1, 1, 1, \ldots$

(c) $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$
(d) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0, 0, 0, 0, 0, \ldots$
(e) $2, -2, 2, -2, 2, -2, 2, -2, \ldots$
(f) $1, 1, 0, 1, 1, 1, 1, 1, 1, \ldots$
(g) $0, 0, 0, 1, 2, 3, 4, \ldots$
(a) $0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, \ldots$
It is a polynomial not a series. $f(x) = 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6$.
(b) $0, 0, 0, 1, 1, 1, 1, 1, \ldots$
After the first three terms everything else is 1s. We know if all are ones then, $1/(1-x) = 1+x+x^2+\cdots$. So we just subtract from this $1+x+x^2$ or we can write it as product of x^3 and $1/(1-x)$ which is the same as starting the sum after the second power, $x^3/(1-x)$.
(c) $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$
GF is $x + x^4 + x^7 + x^{10} + \cdots$ which is $x(1 + x^3 + (x^3)^2 + \cdots$ which is $x/(1-x^3)$.
(d) $2, 4, 8, 16, 32, 64, 128, \ldots$
Take 2 out of GF to get $2(1 + (2x) + (2x)^2 + \cdots)$ which we know from the table is $2(1/(1-2x))$.
(e) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0, 0, 0, 0, 0, \ldots$
$(1+x)^7$, again a polynomial.
(f) $2, -2, 2, -2, 2, -2, 2, -2, \ldots$
Take 2 out and then we get the expansion for $1/(1+x)$.
(g) $1, 1, 0, 1, 1, 1, 1, 1, 1, \ldots$
It is $1/(1-x)$ with just one term removed, x^2.
(h) $0, 0, 0, 1, 2, 3, 4, \ldots$
So after the third term its the same as $1/(1-x)^2$, to account for that we can multiply with x^3 to get $x^3/(1-x)^2$.

(3) Use generating functions to find the number of ways to choose a dozen bagels from three varietiesegg, salty, and plainif at least two bagels of each kind but no more than three salty bagels are chosen.

At least two bagels of each variety is to be selected and therefore each variety is represented by the series, $x^2 + x^3 + \cdots$. But no more than three salty bagels can be chosen, which implies for salty bagels the series have to be cut short after x^3. That is, $x^2 + x^3$. Thus the OGF we are looking at is:

$$(x^2 + x^3)(x^2 + x^3 + x^4 + \cdots)^2$$

and we need to determine the coefficient of x^{12} here. Which is the same as finding the coefficient of x^6 in this equation,

$$(1 + x)(1 + x + x^2 + \cdots)^2.$$

From the table we have this is equal to $\frac{(1+x)}{(1-x)^2}$.

By splitting the fraction, we see that what we have to compute is the coefficient of x^6 in $\frac{1}{(1-x)^2}$ and x^5 in $\frac{1}{(1-x)^2}$.

The first one from table we have that the coefficient of x^6 is $k+1=7$ and the coefficient of x^5 is 6. Therefore the answer is 13.

(4) Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$. Also, $x^2G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2} x^k$. Thus,

$$\begin{aligned} G(x) - 5xG(x) + 6x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 5a_{k-1}x^k + \sum_{k=2}^{\infty} 6a_{k-2}x^k \\ &= a_0 + a_1 x - 5a_0 x + \sum_{k=2}^{\infty} 0 \cdot x^k = 6. \end{aligned}$$

Thus $G(x)(1-5x+6x^2) = 6$, $G(x) = \frac{6}{(1-5x+6x^2)}$.
To use standard formulae from the table we use partial fractions to get,

$$\frac{6}{(1-5x+6x^2)} = \frac{A}{1-3x} + \frac{B}{1-2x}.$$

Solving for A and B, we get

$$G(x) = \frac{18}{1-3x} + \frac{-12}{1-2x}.$$

We can use the table now to get that $G(x) = \sum_{k=0}^{\infty}(18 \cdot 3^k - 12 \cdot 2^k)x^k$. Again from the table we can get, $a_k = 18 \cdot 3^k - 12 \cdot 2^k$.

We can use CAS to solve partial fractions.

```
from sympy import *
x, y, z = symbols('x y z')
expr = (6)/(1-5*x+6*x**2)
apart(expr)
```

(5) Use generating functuns to prove Pascal's identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when n and r are positive integers with $r < n$. Hint : use the identity,

$$(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}.$$

As per the identity we have,

$$\begin{aligned}(1+x)^{n-1} + x(1+x)^{n-1} &= \sum_{k=0}^{\infty} C(n-1,k)x^k + \sum_{k=0}^{\infty} C(n-1,k)x^{k+1} \\ &= 1 + \sum_{k=1}^{\infty} C(n-1,k)x^k + \sum_{k=1}^{\infty} C(n-1,k-1)x^k \\ &= 1 + \sum_{k=1}^{\infty}(C(n-1,k) + C(n-1,k-1))x^k\end{aligned}$$

But we have

$$\begin{aligned}(1+x)^n &= (1+x)^{n-1} + x(1+x)^{n-1} \\ \sum_{k=0}^{\infty} C(n,k)x^k &= 1 + \sum_{k=1}^{\infty}(C(n-1,k) + C(n-1,k-1))x^k \\ 1 + \sum_{k=1}^{\infty} C(n,k)x^k &= 1 + \sum_{k=1}^{\infty} C(n-1,k)x^k + \sum_{k=1}^{\infty} C(n-1,k-1)x^k \\ \sum_{k=1}^{\infty} C(n,k)x^k &= \sum_{k=1}^{\infty}(C(n-1,k)C(n-1,k-1))x^k\end{aligned}$$

Matching coefficients we get and replacing $k = r$ we get required identity.

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