CS:1010 DISCRETE STRUCTURES

PRACTICE QUESTIONS
LECTURE 3

Instructions

- Try these questions before class. Do not submit!

(1) Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

(2) P.T. if x is irrational, then $1/x$ is irrational.

(3) Are these steps for finding solutions of $\sqrt{x+3} = 3 - x$ correct?
  (a) $\sqrt{x+3} = 3 - x$ is given
  (b) $x + 3 = x^2 - 6x + 9$ obtained by squaring both sides of (1),
  (c) $0 = x^2 - 7x + 6$ obtained by subtracting $x + 3$ from both sides of (2),
  (d) $0 = (x - 1)(x - 6)$ obtained by factoring the RHS of (3),
  (e) $x = 1$ or $x = 6$ which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.

(4) P.T. at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

(5) P.T. $log_4 6$ is irrational.

(6) Let the coefficient of the polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.
  (a) Explain why the lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.
  (b) Carefully prove the lemma.
You may find it helpful to appeal to:
Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer.

(7) Consider a different proof that $\sqrt{2}$ is irrational, taken from American Mathematical Monthly :
Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer $q > 0$ s.t. $(\sqrt{2}-1)q$ is a nonnegative integer. Let $q^{'} := (\sqrt{2}-1)q$. Clearly $0 < q^{'} < q$. But an easy computation shows that $(\sqrt{2}-1)q^{'}$ is a nonnegative integer, contradicting the minimality of q.
   (a) This proof was written for an audience of college teachers, and at this point it is a little more concise than desirable. Write out a more complete version which includes an explanation of each step.
   (b) Now that you have justified the steps in this proof, do you have a preference for one of these proofs over the other? Why?

(8) Bogus Proof: It is a fact that arithmetic mean is at least as large as the geometric mean,
$$\frac{a+b}{2} \geq \sqrt{ab}$$
for all nonnegative real numbers a and b. But the following proof has something objectionable about it. Can you identify it? If
$$\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab}, \\ \Rightarrow a+b &\geq 2\sqrt{ab}, \\ \Rightarrow a^2+2ab+b^2 &\geq 4ab, \\ \Rightarrow a^2-2ab+b^2 &\geq 0, \\ \Rightarrow (a-b)^2 &\geq 0 \end{aligned}$$
which we know is true (since a and b are real numbers, $(a-b)$ is a real number and the square is always positive or zero) so the proof is true.

(9) Bogus Proof: $1/8 > 1/4$
$$\begin{gathered} 3 > 2 \\ 3log_{10}(1/2) > 2log_{10}(1/2) \\ log_{10}(1/2)^3 > log_{10}(1/2)^2 \\ (1/2)^3 > (1/2)^2. \end{gathered}$$