Assignment 4

Suraj - CS20BTECH11050

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```
https://github.com/Suraj11050/Assignments-
    AI1103/tree/main/Assignment%204/Python
    %20codes
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    AI1103/blob/main/Assignment%204/
    Assignment%204.tex
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1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f(x) = \begin{cases} \frac{7}{32} x^6 (2-x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

then k equals ______________

2 SOLUTION

Let $X \in [0, 2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x)\ dx = 1 \tag{2.0.1}$$

$$\Pr(x) = \frac{1}{2} \ (\because \text{Uniform order}) \tag{2.0.2}$$

The PDF for X is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.0.3}$$

The CDF for X is

$$F(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.0.4}$$

Definition 1. k^{th} order statistic : If the sample is arranged in an ascending order, the k^{th} order statistic is the k^{th} element from the left. $X_{(k)}$ is called k^{th} vector of order statistic For a sample $\{X_1, X_2, \cdots X_n\}$ of size n:

$$X_{(1)} = \min\{X_1, X_2, \cdots X_n\} \tag{2.0.5}$$

$$X_{(n)} = \max\{X_1, X_2, \cdots X_n\} \tag{2.0.6}$$

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(k)} \le \cdots \le X_{(n)} \tag{2.0.7}$$

Let $X_{(k)}$ be vector of order statistic of $(X_1, X_2 \cdots, X_n)$

Lemma 2.1. Marginal probability density (PDF) for a k_{th} order statistic of a random sample of size n given CDF= $F(x)$ and PDF= $f(x)$ is given by

$$f_{(k,n)}(x) = n\, {}^{n-1}\mathrm{C}_{k-1}\ (1 - F(x))^{n-k}\, F(x)^{k-1} f(x) \tag{2.0.8}$$

Proof. The CDF of the k^{th} order statistic from a sample of size n is:

$$F_{(k,n)}(x) = \Pr(X_{(k)} \le x) \tag{2.0.9}$$

$$= \sum_{j=k}^{n} {}^{n}\mathrm{C}_j\ (1 - F(x))^{n-j}\, F(x)^j \tag{2.0.10}$$

Deriving PDF of k^{th} order statistic from a sample of size n:

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx}\left(\sum_{j=k}^{n} {}^{n}\mathrm{C}_j\ (1 - F(x))^{n-j}\, F(x)^j\right) \tag{2.0.11}$$

$$f_{(k,n)}(x) = \sum_{j=k}^{n} {}^{n}\mathrm{C}_j\ (j)(1 - F(x))^{n-j}\, F(x)^{j-1}\, f(x) - \sum_{j=k}^{n} {}^{n}\mathrm{C}_j\ (n-j)(1 - F(x))^{n-j-1}\, F(x)^j\, f(x) \tag{2.0.12}$$

$$S_1 = \sum_{j=k}^{n} \frac{n!}{(n-j)!\,(j-1)!}(1-F(x))^{n-j}F(x)^{j-1}f(x) \tag{2.0.13}$$

$$S_2 = \sum_{j=k}^{n} \frac{n!}{(n-j-1)!\,j!}(1-F(x))^{n-j-1}F(x)^{j}f(x) \tag{2.0.14}$$

let $i = j+1$ change the limits for the summation in equation (2.0.14)

$$S_2 = \sum_{i=k+1}^{n} \frac{n!}{(n-i)!\,(i-1)!}(1-F(x))^{n-i}F(x)^{i-1}f(x) \tag{2.0.15}$$

$$S_1 - S_2 = \frac{n!}{(n-k)!\,(k-1)!}(1-F(x))^{n-k}F(x)^{k-1}f(x) \tag{2.0.16}$$

$$f_{(k,n)}(x) = n\,{}^{n-1}\mathrm{C}_{k-1}\ (1-F(x))^{n-k}F(x)^{k-1}f(x) \tag{2.0.17}$$

□

Using Lemma (2.1) given $n = 8$ substituting n, equation (2.0.3) and equation (2.0.4) in above equation (2.0.17) we get:

$$f_{(k,n)}(x) = 8\,{}^{7}\mathrm{C}_{k-1}\left(1-\frac{x}{2}\right)^{8-k}\left(\frac{x}{2}\right)^{k-1}\frac{1}{2} \tag{2.0.18}$$

$$f_{(k,n)}(x) = \frac{1}{32}\,{}^{7}\mathrm{C}_{k-1}\ (2-x)^{8-k}\ x^{k-1} \tag{2.0.19}$$

Comparing the marginal probability density function obtained in equation (2.0.19) with the equation given in question

$$\frac{1}{32}\,{}^{7}\mathrm{C}_{k-1}\ (2-x)^{8-k}\ x^{k-1} = \frac{7}{32}\ (2-x)\ x^{6} \tag{2.0.20}$$

$$\therefore k = 7 \tag{2.0.21}$$

Hence the marginal probability density in the given problem is 7^{th} order uniform statistic and **the value of k is 7**