06/04/2022

**Claim:** $f'$ is a flow in $G$.

**Proof:-** verify two things

(1) Capacity constraints

(2) Conservation constraints.

→ **Capacity constraints**

Consider an edge $e=(u,v)$ on the path $P$.

**Case 1:** $e=(u,v)$ is a forward edge

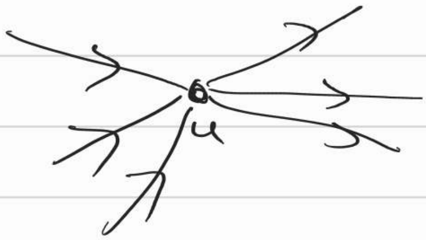
$$0 < f'(e) := f(e) + b \leq f(e) + (c(e) - f(e))$$
$$\leq c(e)$$

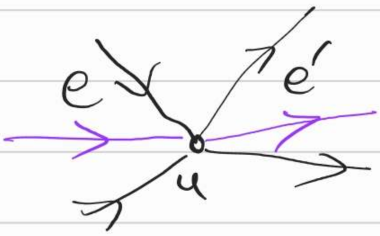
**Case 2:-** $e=(u,v)$ is a backward edge

$$f'(e) := f(e) - b \geq f(e) - f(e) = 0$$

## → Conservation Constraints.

Let $u$ be an internal vertex in $P$.





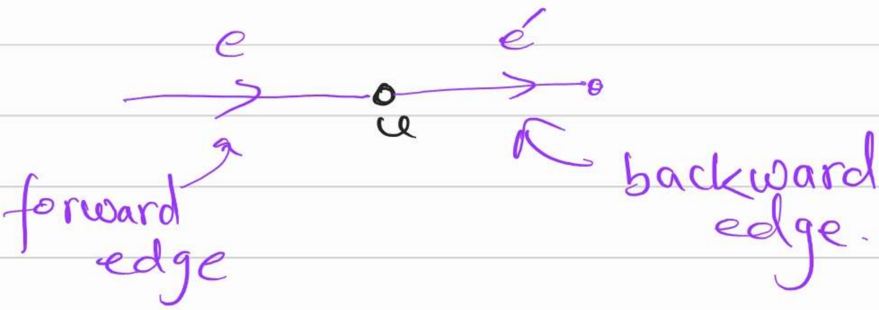

Cases 1 :- incoming edge is forward edge
outgoing edge is forward edge

$$f'(e) = f(e) + b$$

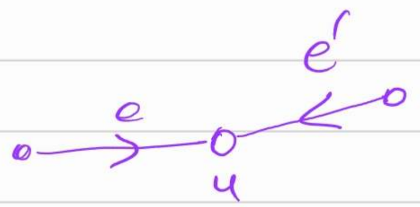
$$f'(e') = f(e') + b$$

$$\sum_{e \text{ into } u} f'(e) = \sum_{e \text{ out of } u} f'(e).$$

Case 2 :- incoming edge is a forward edge
outgoing edge is a backward edge




in $G_f$



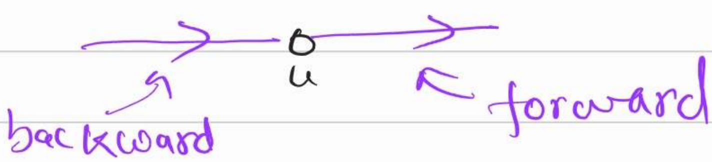

in $G$

$$f'(e) = f(e) + b$$

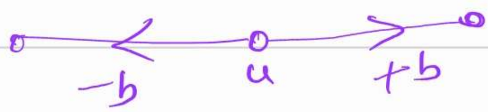
$$f'(e') = f(e) - b$$

Case 3 :- incoming edge is a backward edge

outgoing edge — forward edge.




in $G_f$




in $G$

Case 4 :- both edges are backward edges.

**Defn:-** A $s \rightsquigarrow t$ simple path in the residual graph $G_f$ is called an "Augmenting Path".

**claim:-** $v(f') := v(f) + b$

value of $f'$ — $v(f')$

$b$ — bottleneck$(P, f)$

$\Rightarrow v(f') > v(f)$.

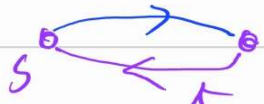
**Proof:-** $v(f) = \sum_{e \text{ out of } s} f(e)$

$v(f') = \sum_{e \text{ out of } s} f'(e)$


in $G_f$

The edge coming out of $s$ cann't be a backward edge.

suppose this is a backward edge




this edge exists in $G$

But this a contradiction to the fact that source vertex, $s$, has only outgoing edges.


$$\nu(f') = \sum_{e \text{ out of } s} f'(e) = \left( \sum_{\substack{e \text{ out of } s \\ \text{and } e \text{ is} \\ \text{not part of } P}} f'(e) \right) + \underset{\substack{\uparrow \\ e \text{ is the} \\ \text{starting} \\ \text{edge of } P}}{f'(e)}$$

$$= \underbrace{\sum_{e \text{ out of } s} f(e) + f(e)}\; + b$$

$$= \nu(f) + b$$

∎

## Max-flow (Ford-Fulkerson Algorithm)

Initialize $f(e) = 0$ $\forall e$ in $G$.

While there is a s-t path in $G_f$

let P be a simple s-t path in $G_f$

$f' := \text{Augment}(P, f)$

Update $f$ to be $f'$

Update $G_f$ to be $G_{f'}$

End while.

Return ($f$).

## Termination of Ford-Fulkerson

- Assume all capacities are integers.

Claim 3: At every stage of Ford-Fulkerson Algorithm the flow values as well as the residual capacities are integers.

Proof: proof by induction.

Claim 4 :- The no. of iterations of while loop is at most $C$.

$$\underset{\uparrow \text{ of any flow}}{v(f)} \leq \sum_{e \text{ out of } s} c(e) := C$$

Proof :- because $b \geq 1$.

Claim 5 : The Ford-Fulkerson algo. can be implemented to run in $O(C \cdot m)$ time, where $m$ is the no. of edges in $G$.

Proof :-

→ Finding an $s \rightsquigarrow t$ path.

$$O(m+n) = O(m)$$

because underlying undirected graph is connected, which implies $m \geq n-1$.

→ Augment using the $s \rightsquigarrow t$ path found.

$$O(n)$$

→ Construct residual graph $G_{f'}$ with respect to $f'$.

$$O(m).$$

maintain residual graph using adjacency list.