# CS 1010 Discrete Structures Lecture 5: Sequences & Summations, Cardinality of Sets

Maria Francis

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# Sequences

- Ordered lists of elements.
- Used in counting and an important data structure.
- $1, 2, 3, 5, 8$ is a sequence of 5 terms and $1, 3, 9, 27, 81, \ldots, 3^n, \ldots$ is an infinite sequence.
- How to define them formally?
- A sequence is a function from a subset of the set of integers (usually either $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$ to a set $S$.
- $a_n$ : image of integer $n$, called the term of a sequence.
- $\{a_n\}$ : denotes the entire sequence. But this notation looks like a set notation, context decides which one we are using.
- $a$ is arbitrary, it could be any letter. Typically lowercase letters.

# Sequences

- Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$. The list of terms in this sequence $a_1, a_2, \ldots,$ are $1, \frac{1}{2}, \frac{1}{3}, \ldots$.
- Another notation for sequences : listing elements between paranthesis (), $(a, b, c)$ : sequence of length 3.
- Elements of a sequence need not be distinct, unlike sets. Also sequences are ordered and sets are not.
- How to link sets and sequences? Cartesian Product of sets $S_1 \times S_2 \times \cdots \times S_n$. This is a new set of all sequences where the first component is from $S_1$, second from $S_2$, etc.
- A product of $n$ copies of a set $S$ is denoted as $S^n$. Example:

$$\begin{aligned}\{0,1\}^3 = &\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ &(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}\end{aligned}$$

# Geometric Progression

- Geometric progression is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

  where $a \in \mathbb{R}$ is the initial term and $r \in \mathbb{R}$ is the common ratio.
- Example:
  - Let $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$ and $d_n = 6 \cdot (1/3)^n$ be three geometric progressions
  - With initial terms and common ratios as follows: 1 and $-1$; 2 and 5; and 6 and $\frac{1}{3}$ resply for $n = 0$.
  - The sequences look like this:

$$1, -1, 1, -1, 1, \ldots,$$
$$2, 10, 50, 250, 1250, \ldots,$$
$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots,$$

# Arithmetic Progression

- A sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$, where $a \in \mathbb{R}$ is the initial term and $d \in \mathbb{R}$ is the common difference.
- Example: Sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ with the initial terms and common differences: $-1$ and 4, 7 and $-3$ resply starting with $n = 0$.

$$-1, 3, 7, 11, \ldots,$$

$$7, 4, 1, -2, \ldots,$$

- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$ and an arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$, where $d, a$ are real numbers.

# Strings

- Finite sequences of the form $a_1, a_2, \ldots, a_n$ occur often in computer science – they are called strings denoted as $a_1 a_2 \ldots a_n$.
- We have already seen bit strings.
- The length of a string is the number of terms in this string, empty string, $\lambda$ is the string with no terms and has length zero.
- $\lambda$ is also used for an empty sequence. There are other notations too.

# Recurrence Relation

- Specifying sequences : either a single formula or provide one or more initial terms together with a rule of determining subsequent terms from those that precede them.
- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$ $\forall$ $n$ with $n \geq n_0$ where $n_0 \in \mathbb{N}$.
- A sequence that satisfies this recurrence relation is a solution.
- Recursive relations recursively define a sequence.
- Example: $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and let $a_0 = 2$.
  - $a_1 = a_0 + 3 = 2 + 3 = 5$, $a_2 = 5 + 3 = 8$, $a_3 = 8 + 3 = 11$.
- Initial conditions : specify the values of the terms that precede the first term where the recursive relation takes effect.

# Fibonacci Sequence

- Named after the Italian mathematician Fibonacci born in the 12th century.
- This sequence we will keep seeing in this course as well as in many other areas within and outside of computer science.
- $\{f_n\}$: Initial conditions $f_0 = 0, f_1 = 1$ and recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

  for $n = 2, 3, 4, \ldots$.
- How does the sequence look like: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

# Factorials as a Sequence

- Consider $a_n = n!$, we have $n! = n(n-1)(n-2)\ldots 2 \cdot 1 = n(n-1)! = na_{n-1}$.
- Thus sequence of factorials is a solution to $a_n = na_{n-1}$, with initial condition $a_1 = 1$.
- Solving recurrence relation with initial conditions means finding a formula, closed formula for terms of the sequence.
- Given $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ .
  - Is $a_n = 3n$ (the closed formula) a solution? Yes since $2(3(n-1)) - 3(n-2) = 3n = a_n$.
  - What about $a_n = 2^n$? No since $a_0 = 1, a_1 = 2, a_2 = 4$ and $2a_1 - a_0 = 3 \neq a_2$.
  - What about $a_n = 5$? $2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$ and so $a_n = 5$ is a solution.

# How do we solve these recurrences?

- Simplest method: iteration via examples. Many more powerful methods are there but we will discuss them later.
- Solve: $a_n = a_{n-1} + 3$.
- Keep successively applying the recurrence starting with $a_1 = 2$, $a_2 = 2 + 3 = 5$, $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2, \cdots$
- $a_n = a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1)$. (closed formula)
- What we did above is called forward substitution, starting with initial condition and then reaching $a_n$.

# How do we solve these recurrences?

- We can also work backwards until we reach $a_1 = 2$. Backward substitution:

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&\ \vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1).
\end{aligned}
$$

- During each iteration we obtain the next term by adding 3 to the previous term.
- We obtain $n$th term after $n-1$ iterations $\rightarrow$ we add $3(n-1)$ to the initial term $a_0 = 2$ to get $a_n$.
- $a_n = 2 + 3(n-1)$ - the required closed form!

# Application of Recurrences

- Calculating compound interest:
  - A person deposits 10,000 rupees in a savings bank that yields 11 percent/year with interest compounded annually.
  - What will the amount be after 30 years?
  - $P_n$ – amount after $n$ years, but this is the amount after $n-1$ years + interest for the $n$th year.
  - $P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}$.

# Application of Recurrences

$$\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3 P_0 \\
&\vdots \\
P_n &= (1.11)P_{n-1} = (1.11)^n P_0.
\end{aligned}$$

$P_0 = 10,000$, $n = 30$, $P_n = (1.11)^n 10000$ and after 30 years, amount is 228922.97.

# Special Integer Sequences

- Finding a closed formula, a recurrence relation or some other rule for constructing the terms of a sequence is a common research problem.
- Some cases only a few terms of a sequence is known, how does one identify the entire sequence.
- Note: There are infinite sequences that start with any finite set of initial terms.
- But knowing the first few terms may help us make an educated conjecture about the identity of the sequence and then verify.
- Some questions you can ask to get to the formula or relation:
  - Do the same value occur many times in a row?
  - Do add same amount to the previous term or does the amount change with the position in the sequence?
  - Is it multiplying the value?
  - Are there cycles among terms?

# How to find closed formulas?

- $1, 3, 5, 7, 9$ : Adding 2 to each term we get the next term. $a_n = 2n + 1$ seems to be a possible conjecture.
- $1, -1, 1, -1, 1$: Each term looks like is getting multiplied with $-1$. A possible solution: $(-1)^n$, $n = 0, 1, 2, \ldots$, a geometric progression with $a = 1$ and $r = -1$.
- What about $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$? A reasonable rule is : the integer $n$ appears exactly $n$ times.
- What happens when we change initial conditions of a recurrence relation?
  - Consider the sequence whose first 10 terms are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123$.
  - Starting with the 3rd term here each term is the sum of the two previous terms, i.e. a recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1, L_2 = 3$.
  - This is called a Lucas sequence and the recurrence relation is the same as Fibonacci sequence.

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# How to find closed formulas?

- Finding a rule sometimes is easier by comparing with well-known integer sequences.
- See below the first 10 terms of some sequences:

| *n*th Term | First 10 Terms |
|---|---|
| $n^2$ | $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$ |
| $n^3$ | $1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots$ |
| $n^4$ | $1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, \ldots$ |
| $2^n$ | $2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$ |
| $3^n$ | $3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, \ldots$ |
| $n!$ | $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, \ldots$ |
| $f_n$ | $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ |

# How to find closed formulas?

- Conjecture a formula for $\{a_n\}$ whose first 10 terms are $1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047$.
- Unlike before very hard to find a pattern.
- Looking at ratio of consecutive terms we see that the ratio is not a constant but close to 3.
- Therefore the formula could have $3^n$. Compare with the sequence $\{3^n\}$.
- The $n$th term is 2 less than the power of 3, $a_n = 3^n - 2$ is a possible formula.

# Integer Sequences

- Integer sequences come up in many places : prime number sequences, number of moves to solve the Towers of Hanoi problem with $n$ disks, etc.
- You can see $200,000$ integer sequences in the Online Encyclopedia of Integer Sequences started by Neil Sloane.
- Check it out! Just enter a few initial terms and it will find sequences for you that match those terms.
- Journal of Integer Sequences is a good journal in this area and a possible source for your write up topics. It is an all-volunteer magazine that is fully free. It shows you how mathematicians and most academic community publish work with virtually no remuneration and still thrive.

# Summations

- What happens when we add terms of a sequence?
- Sum of terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$ is denoted as $\sum_{j=m}^{n} a_j$ or $\sum_{m \leq j \leq n} a_j$.
- $j$ is the called index of summation and $j$ is an arbitrary choice of variable ( $i$ or $k$ are also common choices).
- i.e. $\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$ .
- Lower limit is $m$ and upper limit is $n$.
- Usual laws of summation hold such as $\sum_{j=1}^{n}(ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$, where $x_i$s and $y_i$s are real numbers. Proof needs mathematical induction.
- Sum of first 100 terms of $\{a_j\}$ where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$ is denoted as $\sum_{j=1}^{100} \frac{1}{j}$.

# Value of Summations

- Value of $\sum_{j=1}^{5} j^2$? Expanding we get $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ which is equal to 55.
- Value of $\sum_{k=4}^{8} (-1)^k$? Expanding we get $(-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$ which is equal to 1.
- Sometimes it is useful to shift the index of summation likewhen we want to add two sums but their indices do not match. It is important to make the appropriate changes or else there will be errors.
- Eg: In $\sum_{j=1}^{5} j^2$ if we want summation to run between 0 and 4 then we do the following change:
  - $k = j - 1$ since $k = 0$ when $j = 1$ and $k = 4$ when $j = 5$ and therefore $j^2$ is $(k+1)^2$.

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2.$$

# Geometric Series

- Formula for sums of terms of geometric progression.
- If $a, r \in \mathbb{R}$ and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} \text{ if } r \neq 1, \\ (n+1)a \text{ if } r = 1 \end{cases}$$

- Let $S_n = \sum_{j=0}^{n} ar^j$.

$$\begin{aligned} rS_n &= r\sum_{j=0}^{n} ar^j \text{ Perturbation Method} \\ &= \sum_{j=0}^{n} ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \text{ shifting index of summation} \end{aligned}$$

# Geometric Series

$$rS_n = \sum_{k=0}^{n} ar^k + (ar^{n+1} - a)$$

removing $k = n + 1$ term and adding $k = 0$ term .

$$= S_n + (ar^{n+1} - a)$$

- We have $rS_n = S_n + (ar^{n+1} - a)$.
- Solve for $S_n$ for the two cases $r = 1$ and $r \neq 1$. For $r = 1$, $S_n = \sum_{j=0}^{n} ar^j$, which is $S_n = (n + 1)a$.
- Perturbation method: Perturb the sum so that we can combine the original sum with the perturbation (subtract in this context) to get something simpler.

# Different Indices

- Indices need not always be 1 to something, you can run the indices over all values in a set.

$$\sum_{s \in S} f(s),$$

  sum of all values of $f(s)$ for all members $s$ of $S$.
- Double (multiple) summations arise when you need to analyze multiple things like nested loops in computer programs, i.e. evaluate sums of sums.
- Consider $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$. Expand the inner sum, replace with closed form and then continue with the outer one.
- $\sum_{i=1}^{4} (i + 2i + 3i) = \sum_{i=1}^{4} 6i = 60.$

# Order of Summations

- If no obvious closed form for inner sum. Try exchanging order.
- $n$th Harmonic Number $H_n$: $\sum_{i=1}^{n} \frac{1}{i}$. There is no known closed form expression for harmonic numbers.
- The below theorem gives close upper and lower bounds on $H_n$:

$$\text{Since } \int_1^n \frac{1}{x} dx = ln(x)\Big|_1^n = ln(n)$$

$$ln(n) + \frac{1}{n} \leq H_n \leq ln(n) + 1.$$

  We used the result : Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. Let $S := \sum_{i=1}^{n} f(i)$ and $I := \int_1^n f(x)dx$. Then, $I + f(1) \leq S \leq I + f(n)$.
- $n$th harmonic number is very close to $ln(n)$. There are better approximations than this. (Writeup topic!)

# Order of Summations

- Back to our question of no obvious closed form for inner sum. Compute the sum of first $n$ harmonic numbers:

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k} \frac{1}{j}.$$

- Note from the previous result we can see that it is close to $nln(n) - n + 1$.

$$\int_1^n ln(x)dx = xln(x) - x\Big|_1^n$$
$$= nln(n) - n + 1.$$

What about the exact sum?

# Order of Summations

- If we think about the pairs $(k, j)$ over which we are summing they form a triangle given by the picture:

| | | $j$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | $\dots$ | $n$ |
| $k$ | 1 | 1 | | | | | | |
| | 2 | 1 | 1/2 | | | | | |
| | 3 | 1 | 1/2 | 1/3 | | | | |
| | 4 | 1 | 1/2 | 1/3 | 1/4 | | | |
| | | $\dots$ | | | | | | |
| | $n$ | 1 | 1/2 | | $\dots$ | | | $1/n$ |

- The above summation sums each row and then adds the row sums.
- What if we add instead the columns and then add the column sums?

# Order of Summations

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k}\frac{1}{j}$$

$$= \sum_{j=1}^{n}\sum_{k=j}^{n}\frac{1}{j} \text{ (Exchanged the order)}$$

$$= \sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n} 1$$

$$= \sum_{j=1}^{n}\frac{1}{j}(n-j+1) = \sum_{j=1}^{n}\frac{n+1}{j} - \sum_{j=1}^{n}\frac{j}{j}$$

$$= (n+1)\sum_{j=1}^{n}\frac{1}{j} - \sum_{j=1}^{n} 1 = (n+1)H_n - n \text{ (Closed form!)}.$$

# Standard Sums

| Sum | Closed Form |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r-1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$ | $\frac{1}{(1-x)^2}$ |

Once you know the formula proving using mathematical induction is always possible and we will see that later.
Sums involving infinite series need calculus typically and we will not cover it here but these sums are useful in discrete mathematics too.

# Standard Sums

- Knowing standard sums can be useful.
- Suppose we need to calculate $\sum_{k=50}^{100} k^2$.

$$\begin{aligned}\sum_{k=50}^{100} k^2 &= \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 \\ &= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} \\ &= 338350 - 40425 = 297925.\end{aligned}$$

- We have used the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ from the standard sums list.

# What about Products?

- Closed forms for products? We do not need to develop entirely new set of tools!
- Consider $n! = \prod_{i=1}^{n} i$. Take logarithm to get a sum!
- More generally, $P = \prod_{i=1}^{n} f(i)$ gives $ln(P) = \sum_{i=1}^{n} ln(f(i))$.
- Use summing tools for $ln(P)$ and then exponentiate to undo logarithm.

$$\begin{aligned} ln(n!) &= ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) \\ &= ln(1) + ln(2) + \cdots + ln(n-1) + ln(n) \\ &= \sum_{i=1}^{n} ln(i). \end{aligned}$$

No known closed form exists but there are some bounds.

# Bounds for $n!$

- Using previous results we get

$$nln(x) - n + 1 \leq \sum_{i=1}^{n} ln(i) \leq nln(n) - n + 1 + ln(n)$$

$$\frac{n^n}{e^n - 1} \leq n! \leq \frac{n^{n+1}}{e^{n-1}} \text{ (By exponentiating)}$$

- We used the result : Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. Let $S := \sum_{i=1}^{n} f(i)$ and $I := \int_1^n f(x)dx$. Then,

$$I + f(1) \leq S \leq I + f(n).$$

- Stirling's formula also gives a useful bound on $n!$. Writeup topic!

# Infinite Sets

- Any data on a computer is limited by the size of the computer's memory so why not only stick to finite sets?
- But we are using infinite sets all throughout, $\mathbb{N}, \mathbb{R}$, etc.
- It does not make sense to talk of only bounded number of bounded measurements that can be made in a bounded universe when it is more convincing and easier to accept theories for the larger, original sets.

# Infinite Sets

- That is, writing a program to add two nonnegative integers with up to as many digits as, say, the stars in the sky is no different from writing a program that would add any two integers, no matter how many digits they had.
- It is also a nice setting to practice proof methods and see how discrete techniques work over infinite sets.
- We will use these techniques in other places too like uncomputable functions. So we make a brief tour of the world of infinite elements.

# Infinite Cardinality

- A bit of history: Georg Cantor was studying the convergence of Fourier series and found some series which converge most of the time but there were infinite number of points where they did not converge.
- He realised he needed to compare infinite sets.
- His definition: The sets $A$ and $B$ have the same cardinality iff there is a one-to-one correspondence from $A$ to $B$. When $A$ and $B$ have the same cardinality we write $|A| = |B|$.
- As big as : infinite set $A$ is as big as set $B$ if there is a surjection from $A$ to $B$ or equivalently an injection from $B$ to $A$.
- $A$ is strictly smaller than $B$ or $A$ strict $B$ when $A$ is not as big as $B$.

# Infinite Cardinality

- Cantor got diverted from Fourier series and became involved full-time in this study of infinite sets and his ideas have a lot of impact on computer science and mathematics.
- But his own time he was considered to live in an irrelevant Cantor's paradise. Read Logicomix!
- Note: we are comparing sizes not defining what a size is. This needs definition of ordinals.
- Most of the as big as and same size properties carry over from finite sets to infinite sets. But some important ones do not!

# Properties

1. There is a surjection from $A$ to $B$ iff there is an injection from $B$ to $A$.
2. If there is a surjection from $A$ to $B$ and from $B$ to $C$ then there is a surjection from $A$ to $C$.
3. If there is a bijection from $A$ to $B$ and from $B$ to $C$ then there is a bijection from $A$ to $C$.
4. There is a bijection from $A$ to $B$ iff there is a bijection from $B$ to $A$.

Schröder-Bernstein: For any sets $A, B$ if there is a surjection from $A$ to $B$ and $B$ to $A$ then there is a bijection from $A$ to $B$.

# Infinite Cardinality

- Intuitively it says if $A$ is at least as big as $B$ and $B$ is at least as big as $A$ then $A$ is the same as $B$.
- It feels obvious but for infinite sets its actually a technical proof! ( A possible writeup topic !)
- Schröder's proof was flawed and Bernstein, student of Cantor, gave a proof in 1897. But later it was seen that Dedekind in his notes had given a proof in 1887.
- S.T. $|(0,1)| = |(0,1]|$:
  - Not very obvious to get a bijective mapping between the two sets.
  - Since $(0,1) \subset (0,1]$ an inclusion map is a one-to-one mapping.
  - $g(x) = x/2$ is one-to-one and maps $(0,1]$ to $(0,1/2] \subset (0,1)$. Many other one-one maps from $(0,1]$ to $(0,1)$ can also be found.
  - From Schröder-Bernstein theorem we know that one-one maps are enough, we have the result.

# Properties

- For all sets $A, B$ either there is a surjection from $A$ to $B$ or from $B$ to $A$. Very technical set theory proof!
- $A$ strict $B$ and $B$ strict $C$ implies $A$ strict $C$ for all sets $A, B, C$.
  - Assume that $\neg(A$ strict $C)$ holds which implies there is a surjection from $A$ to $C$.
  - Now since $B$ strict $C$ from above result we can conclude there is a surjection from $C$ to $B$.
  - This means we have a surjective function from $A$ to $B$ contradiction to the fact that $A$ strict $B$.

# Properties that do not hold

- One property that does not carry over to finite sets is that adding something new makes a set bigger.
- Let $A$ be a set and $b \notin A$. Then $A$ is infinite iff there is a bijection from $A$ to $A \cup \{b\}$.
- Since $A$ is not the same size as the $A \cup \{b\}$ when $A$ is finite, we show that they are the same size when $A$ is infinite.
  - Since $A$ is infinite it has at least one element, $a_0$.
  - Since $A$ is infinite it has at least two elements, $a_1, a_0$ s.t. $a_1 \neq a_0$.
  - Since $A$ is infinite it has at least three elements, $a_2, a_1, a_0$ s.t. $a_2 \neq a_1 \neq a_0$.
  - Continue this way, we get an infinite sequence: $a_0, a_1, a_2, \ldots, a_n, \ldots$ of different elements of $A$.

# Properties that do not hold

- What is the bijection? $e : A \cup \{b\} \to A$:

$$\begin{aligned} e(b) &:= a_0, \\ e(a_n) &:= a_{n+1} \text{ for } n \in \mathbb{N}, \\ e(a) &:= a \text{ for } a \in A \setminus \{b, a_0, a_1, \ldots\}. \end{aligned}$$

- Related to the Hilbert's Grand Hotel problem. A paradox that cannot happen with finite sets. We will see that when we see countable sets in the next lecture.
- We continue our discussion on types of infinite sets – countable sets and uncountable sets – and show some surprising results associated with them (($\mathbb{Q}$ is countable!) in the next lecture.