Lecture 22 (MA4020) LINEAR ALGEBRA Nov 23, 2021.

Lectures

23^{rd} (Tuesday)

26^{th} (Friday)

27^{th} (Saturday morning) (10:00 - 11:30 am)

(Dean academic email: shift 3^{rd} Dec Lecture ⟷ 27^{th} Nov)

I. Lecture 16 recording : (Included in Lecture 17 recording) (Lecture 17 was revision of Lecture 16 plus few topics)

III. EXAM : 30^{th} November

IV. Syllabus : All topics from Linear transformation onwards (including linear transformation)

Problem from previous Lecture:

Q. Define an inner product on $\mathbb{R}^2$ such that

$\langle e_1, e_1 \rangle = 2$, $\langle e_1, e_2 \rangle = -1$, $\langle e_2, e_2 \rangle = 3$

$$\langle -,- \rangle : \underset{\Downarrow}{\mathbb{R}^2} \times \underset{\Downarrow}{\mathbb{R}^2} \longrightarrow \underset{\text{scalar}}{\mathbb{R}}$$

$$\Big\langle (x_1, x_2) \quad (y_1, y_2) \Big\rangle \longrightarrow \underbrace{\qquad\qquad\qquad}_{\text{expression}}$$

<u>Exercise</u>

POLYNOMIALS AND MATRICES

Let K be a field.

[Notation K, we used F before

Additional Notes for fields
(shared before)]

By a polynomial over K, we mean a formal expression $f(t) = a_n t^n + \cdots + a_0$

where $a_i \in K$, $i = 0, 1, \ldots, n$

& t is a "variable".

SUM and PRODUCT:

$$f(t) = a_n t^n + \cdots + a_0 ; \qquad a_i \in K$$

$$g(t) = b_m t^m + \cdots + b_0 , \qquad b_j \in K$$

If $n \geq m$

re-write $g(t) = 0 t^n + \cdots + b_m t^m + \cdots + b_0$,

and then write the sum $f+g$ as

$$(f+g)(t) = (a_n + b_n) t^n + \cdots + (a_0 + b_0)$$

"Polynomial" where $b_j = 0$ if $j > m$.

If $c \in K$, then

$$(cf)(t) = ca_n t^n + \dots + ca_0 ;$$

Hence, (cf) is also a polynomial.

Thus, polynomial form a vector space over K

PRODUCT :

$$(fg)(t) = (a_n b_m) t^{n+m} + \dots + a_0 b_0$$

In general,

$$(fg)(t) = c_{n+m} t^{n+m} + \dots + c_0$$

where

$$c_K = \sum_{i=0}^{K} a_i \cdot b_{K-i} = a_0 b_K + a_1 b_{K-1} + \dots + a_K b_0$$

In the space of polynomials, we have product structure too, forming a polynomial algebra structure.

non-zero

Remark. Let f, g be polynomials with co-efficients in K. Then

$$\deg(fg) = \deg f + \deg g$$

" Proof is easy ".

NOTATION:

$K[t] :=$ set of all polynomials over K.

Theorem.

(i) Let $f \in \mathbb{C}[t]$, with $\deg f \geq 1$, then

f has a root in $\mathbb{C}$ [i.e. $\exists\ \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$]

How to prove this ?

(ii) $f \in \mathbb{C}[t]$ with $\deg f = n \geq 1$ and leading co-efficient is 1, then there exists complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$f(t) = (t - \alpha_1) \cdots (t - \alpha_n).$$

Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the distinct roots of the polynomial $f \in \mathbb{C}[t]$, then

$$f(t) = (t-\alpha_1)^{m_1} \cdots (t-\alpha_r)^{m_r}$$

with integers $m_1, \dots, m_r > 0$, and m_i is the multiplicity of root α_i.

Discussion.

Let A be a square matrix with co-efficients in K. Let $f \in K[t]$, and write

$$f(t) = a_n t^n + \dots + a_0 \; ; \quad \text{with } a_i \in K.$$

We define

$$f(A) = a_n A^n + \dots + a_0 I.$$

Matrix $\quad T : V \longrightarrow V$

A

$$f(T) = a_n T^n + \dots + a_0 \cdot I$$

Operator $\quad I : V \longrightarrow V$

Example. Let $f(t) = 3t^2 - 2t + 5$. Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}.$$

Then,

$$f(A) = 3 \cdot \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^2 - 2\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(A) = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$$

Theorem. Let $f, g \in K[t]$. Let A be a square matrix with co-efficients in K. Then

$$(f+g)(A) = f(A) + g(A)$$

$$(fg)(A) = f(A)g(A).$$

If $c \in K$, then $(cf)(A) = c f(A)$.

Proof. Let

$$f(t) = a_n t^n + \cdots + a_0$$

$$g(t) = b_m t^m + \cdots + b_0$$

Then

$$(fg)(t) = c_{m+n} t^{m+n} + \cdots + c_0$$

where $c_K = \sum_{i=0}^{K} a_i b_{K-i}$

By definition,

$$(fg)(A) = c_{m+n} A^{m+n} + \cdots + c_0 I$$

On the other hand,

$$f(A) = a_n A^n + \cdots + a_0 I$$

$$g(A) = b_m A^n + \cdots + b_0 I$$

Hence

$$f(A)\, g(A) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \cdot A^i \; b_j \cdot A^j$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \cdot b_j \cdot A^{i+j}$$

$$= \sum_{k=0}^{m+n} c_k A^k .$$

Thus $$f(A)\, g(A) = (fg)(A) .$$

The proofs for other cases: $(f+g)(A) = f(A) + g(A)$

& $(cf)(A) = c f(A)$

are also easy.

Examples. Let $f(t) = (t-1)(t+3)$

$$= t^2 + 2t - 3.$$

Then

$$f(A) = A^2 + 2A - 3I$$

$$= (A - I)(A + 3I)$$

$$(A^2 - IA + 3AI - 3I^2)$$

$$\|$$

$$A^2 + 2A - 3I$$

Examples. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be numbers. Let

$$f(t) = (t - \alpha_1) \cdots (t - \alpha_n).$$

Then

$$f(A) = (A - \alpha_1 I) \cdots (A - \alpha_n I).$$

Discussion.

V : vector space over K.

$A : V \longrightarrow V$ be a linear map

then $A^2 = A \circ A = A \cdot A$.

In general, $A^n = A \circ \cdots \circ A$ (n times), $A^0 = I$ identity

In general, $A^{m+n} = A^m A^n$ for all integers $m, n \geq 0$.

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{R}) = V$

$f(t)$ s.t. $f(A) = 0$

$\dim_{\mathbb{R}} V = 4$

Basis of $V = \left\{ e_{ij} \,\middle|\, \begin{matrix} i = 1,2 \\ j = 1,2 \end{matrix} \right\}$

$f(t) = a_n t^n + \dots + a_0 = 0$

choose

$a_n A^n + \dots + a_0 I = 0$

$I \cdot a_0 + \dots + a_n A^n = 0$

$\{A^n, \dots, A, I\} \in M_2(\mathbb{R})$

$\{I, A, A^2, \dots, A^n\} \in M_2(\mathbb{R})$ $n > 4$

lin. dependent

Theorem. Let A be an $n \times n$ matrix in a field K.

Then there exists a non-zero polynomial

$f \in K[t]$ such that $f(A) = 0$.

Proof.

$V = M_n(K)$

$\dim_K V = n^2$

Consider the sequence of vectors in V

$\underbrace{I, A, A^2, \dots, A^N}_{\text{Linearly dependent}}$ (where $N > n^2$)

$\Rightarrow a_N A^N + \dots + a_0 I = 0$, set $f(t) = a_N t^n + \dots + a_0$

choose $f(t) = a_N t^N + \dots + a_0$

THEOREM OF HAMILTON - CAYLEY.

Discussion.

V : f.d. vector space over a field K.

$A : V \longrightarrow V$ be a linear map.

Assume that V has a basis consisting of eigenvectors of A, say $\{v_1, \ldots, v_n\}$

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the corresponding eigenvalues.

Then the characteric polynomial of A is

$$p(t) = (t - \lambda_1) \cdots (t - \lambda_n)$$

$$p(A) = (A - \lambda_1 I) \cdots (A - \lambda_n I).$$

Observe that

$$p(A)(v_i) = 0 \qquad Av_i = \lambda_i v_i$$

$$p(A) v_i = (A - \lambda_1 I) \cdots (A - \lambda_n I) \underline{v_i}$$

$\neq 0$

$$= 0 \qquad \text{Since } Av_i = \lambda_i v_i$$

$p(A) = 0$

In general, we cannot find basis consisting of eigen vectors.

[Cayley - Hamilton Theorem]

Theorem. Let V be a f.d. v.s. over $\mathbb{C}$ with $\dim V \geq 1$, and let $A : V \longrightarrow V$ be a linear map. Let p be its characteristic polynomial. Then

$$p(A) = 0$$

Proof.

Note that matrix of $A : V \longrightarrow V$ is upper triangular.

$Av_1 = a_{11}v_1$

$$A = \begin{pmatrix} \boxed{a_{11}} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ \vdots & & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$$

Let $v_1, v_2, \ldots, v_n$ be those basis elements such that

$$A v_1 = a_{11} v_1$$

$$A v_2 = a_{12} v_1 + a_{22} v_2$$

$$\vdots$$

$$A v_n = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{nn} v_n$$

Set $V_1 = \{v_1\}$

$V_2 = \{v_1, v_2\}$

$\vdots$

$V_n = \{v_1, v_2, \dots, v_n\}$

Thus,

$$A v_i = a_{ii} v_i + \text{an element of } V_{i-1}.$$

or

$$(A - a_{ii} I) v_i \in V_{i-1}$$

The characteristic polynomial of A is

$$p(t) = (t - a_{11}) \cdots (t - a_{nn})$$

& hence

$$p(A) = (A - a_{11} I) \cdots (A - a_{nn} I)$$

We shall prove by induction that

$$(A - a_{11} I) \cdots (A - a_{ii} I) v = 0 \quad \text{for all } v \in V_i. \quad i = 1, \dots, n.$$

When $i = n$, we get our theorem.

Let $i = 1$, then $(A - a_{11} I) v_1 = A v_1 - a_{11} v_1$

$= 0$ ✓

Let $i > 1$, and assume that statement holds for $i-1$.

$$\left[\begin{array}{r} (A - a_{11} I) \cdots (A - a_{i-1,i-1} I)\, v = 0 \\ \text{for all } v \in V_i\,;\ i = 1, 2, \ldots, i-1 \end{array} \right]$$

Let $v \in V_i$, then

$$v = v' + c\, v_i \quad \text{with } v' \in V_{i-1} \text{ \& some scalar } c \in K.$$

Note that $(A - a_{ii} I)\, v' \in V_{i-1}$

$$\Downarrow \qquad [\because A V_{i-1} \subset V_{i-1}]$$

$$a_{ii} v' \in V_{i-1}$$

By induction,

$$(A - a_{11} I) \cdots (A - a_{i-1,i-1} I)(A - a_{ii} I)\, v' = 0$$

On the other hand,

$(A - a_{ii} I)\, c\, v_i \in V_{i-1}$, and hence by induction,

$$(A - a_{11} I) \cdots (A - a_{i-1,i-1} I)(A - a_{ii} I) c v_i = 0.$$

Hence for all v in V_i, we have

$$(A - a_{11} I) \cdots (A - a_{ii} I) v = 0$$

$$\Rightarrow \quad p(A) v = 0$$

$\neq 0$

$$\Rightarrow \quad \boxed{p(A) = 0}$$

Corollary. $A \in M_n(\mathbb{C})$ and p be its characteristic polynomial. Then

Caution: //

$$p(A) = \det(t \cdot I - A)^{(A)} \neq 0$$

$$\begin{array}{c} \| \\ 0 \end{array} \qquad t = A$$

Do not substitute directly A for t.

[Cayley-Hamilton Theorem]

Corollary. Let V be a finite dim. vec. space over the field K, and let $A : V \longrightarrow V$ be a linear map. Let P be the characteristic polynomial of A. Then

$$P(A) = 0.$$

Euclidean Algorithm.

Theorem. Let f, g be polynomials over the field K, assume that $\deg g \geq 0$. Then there exist polynomial q, r in $K[t]$ such that

$$\boxed{f(t) = q(t)\, g(t) + r(t),}$$

$g(t) \overline{)\, f(t)\,} \left(q(t) \right.$ and $\deg r < \deg g$. The polynomials q, r are uniquely

$r(t)$ determined by these condition.

Corollary. Let $f \in K[t]$, a non-zero polynomial.

Let $\alpha \in K$ be such that $f(\alpha) = 0$.

Then there exists a polynomial $q(t) \in K[t]$ such that

$$f(t) = (t-\alpha)\, q(t).$$

Proof.

define
$$f(t) = q(t)(t-\alpha) + r(t)$$

$$\underset{\substack{\uparrow \\ \text{constant},}}{\deg r} < \underset{\substack{\| \\ 1}}{\deg (t-\alpha)}$$

So

$$0 = f(\alpha) = q(\alpha)(\alpha-\alpha) + r(\alpha) = 0$$

$$\Downarrow$$

$$r(\alpha) = 0$$

$$\Rightarrow \quad r = 0.$$

Corollary. Suppose $f \in K[t]$ has ~~a~~ ~~roots~~ every root is in K, then there exists elements $\alpha_1, \dots, \alpha_n \in K$ and $c \in K$ such that

$$f(t) = c(t-\alpha_1)\cdots(t-\alpha_n).$$

Corollary. Let f be a polynomial of degree n in $K[t]$.

There are at most n roots of f in K.

Proof.

Suppose f has more than n roots, say m,

$\alpha_1, \alpha_2, \dots, \alpha_m$ be distinct roots of f in K,

then $f(t) = (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_m)g(t)$

for some polynomial $g(t)$

[Contradiction $\deg f \geq m$]

Ideal. By an ideal of $K[t]$, we mean a subset I of $K[t]$ satisfying the following conditions:

- $0 \in K[t]$ is in I
- $f, g \in I \Rightarrow f+g \in I$
- $f \in I$ and $g \in K[t]$, then $f \cdot g \in I$

$\langle n \rangle$ $\mathbb{Z}$

Theorem. Let I be an ideal of $K[t]$. Then there exists a polynomial g which is a generator of I.

$$I = \langle g \rangle$$

Single polynomial

Theorem. $I = \langle f_1, f_2 \rangle \subseteq K[t]$. Then

$$I = \langle g \rangle, \quad g = GCD(f_1, f_2)$$

greatest common divisor of f_1 and f_2

$\langle 2, 5 \rangle = \langle \gcd(2,5) \rangle$

- $K[t]$ Unique Factorization Domain

field

PID

$x^2+1 = a(x)\,b(x) \in \mathbb{R}[x]$

- $f \in K[t]$ is irreducible (over K) if its degree ≥ 1 and if given a factorization $f = h_1 h_2$ with $h_1, h_2 \in K[t]$, then $\deg h_1$ or $\deg h_2 = 0$ (i.e. one of them is a constant).

M.Sc

$\mathbb{Z}$ m, n o p a b

Lemma. Let $f \in K[t]$ be irreducible polynomial.

$h_1, h_2 \in K[t]$ and assume f divides $h_1 h_2$.

Then $f \mid h_1$ or $f \mid h_2$.

$p \mid ab$ then $p \mid a$ or $p \mid b$

Corollary. $f \in K[t]$, $\deg f \geq 1$, then

$$f = c \cdot f_1 \cdot f_2 \cdot \cdots \cdot f_s$$

where $f_1, f_2, \ldots, f_s$ are irreducible polynomials with leading co-efficients 1.

Corollary.

$f \in \mathbb{C}[t]$, $\deg f \geq 1$, then

$$f = c \cdot (t - \alpha_1) \cdots (t - \alpha_n).$$

where $\alpha_i \in \mathbb{C}$ & $c \in \mathbb{C}$.

Theorem. Let $f(t) \in K[t]$ and suppose that $f = f_1 \cdot f_2$ and $GCD(f_1, f_2) = 1$.

Let $T : V \longrightarrow V$ be a linear map. Assume that $f(T) = 0$. Let

$W_1 = $ kernel of $f_1(T)$

$W_2 = $ kernel of $f_2(T)$.

Then $V = W_1 \oplus W_2$.

Theorem. V : v.s. over $\mathbb{C}$, $T : V \longrightarrow V$

$p(t)$ be a polynomial such that $p(T) = 0$,

Let $p(t) = (t - \alpha_1)^{m_1} \cdots (t - \alpha_r)^{m_r}$.

Let $W_i = $ kernel of $(A - \alpha_i \cdot I)^{m_i}$. Then

$V = W_1 \oplus \cdots \oplus W_r$

The minimal polynomial.

Recall: V f.d. v.s. over the field.

$T : V \longrightarrow V$ linear operator.

$p(t) = \det(t \cdot I - A)$ characteristic equation

By Cayley-Hamilton theorem,

$$p(A) = 0 \quad \left(\text{or} \quad p(T) = 0 \right)$$

$$I = \{ f \in K[t] \text{ s.t. } f(A) = 0 \}$$
$$\left(\text{or } f(T) = 0 \right)$$

collection of all polynomials s.t. $f(A) = 0$

Algebra

I is an ideal in $K[t]$
$\uparrow$
PID

$I = \langle m \rangle$ $\quad m(T) = 0$ or $m(A) = 0$

unique generator (monic polynomial)

"The minimal Polynomial for T"

~~The monic polyno~~

The (monic) minimal polynomial is

- monic $\quad m(t) = 1 \cdot t^n + a_{n-1} t^{n-1} + \cdots + a_0$
- $m(T) = 0 \quad m(A) = 0$
- $\deg m \leqslant \deg f$, where f is such that $f(T) = 0$

"minimal" in the sense of degree.

Theorem. The zeros/roots of characteristic polynomial and the minimal polynomial are same.