16/03/2022

Correctness of Dijkstra's Algo:

At any point of time we are maintaining a set $T \subseteq V$ of explored vertices.

Lemma: For any vertex $u \in T$, the path from s to u in T is a shortest length path from s to u.

Proof: By induction on $|T|$. Initially $T = \{s\}$

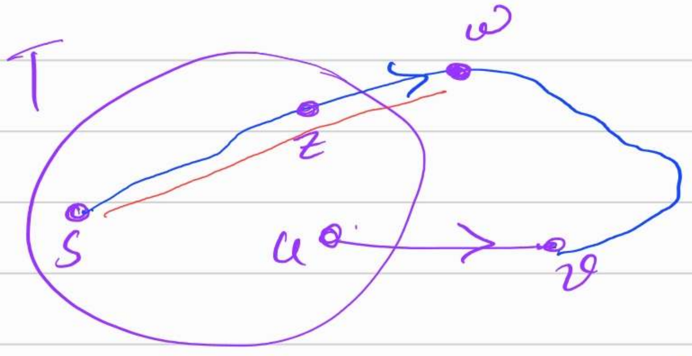
Base case: $|T| = 1$: $T = \{s\}$; $d(s) = 0$

This holds trivially.

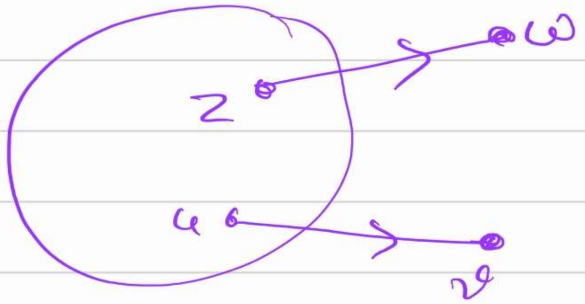
Induction Hypothesis: Suppose the claim is true until $|T| = k$.

Induction Step: consider the step when $|T| = k+1$.

Let v be the vertex that's added to T to grow the size from k to $k+1$.



v is being added to T in this step.



At this point w is also being considered but the algorithm chose v over w.

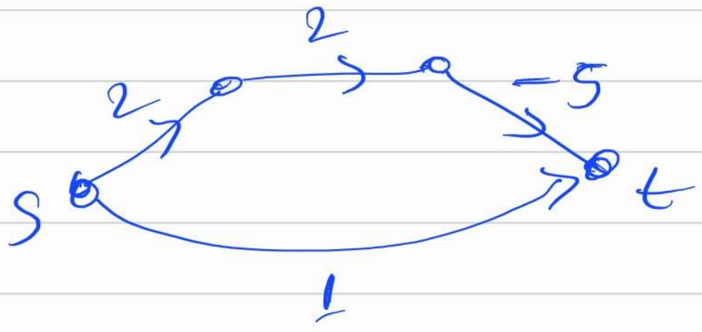
$d(w) \geq d(v)$.

Since you chose an arbitrary path from s to v. excluding the edge (u,v)..

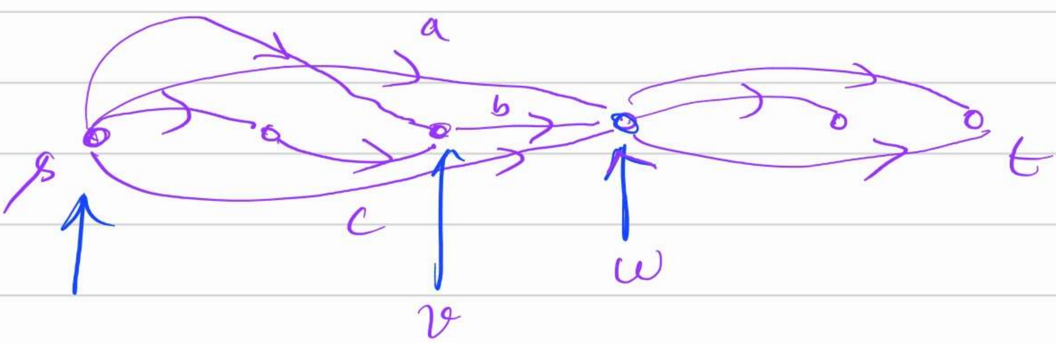
we obtain s to v path via u is a shortest length path.

$\boxtimes$

▷ Handling negative edges.
(without negative cycles in a graph)



DAG:



$$d(w) = \min \{ d(s)+a,\ d(s)+c,\ d(v)+b \}$$

Consider the vertices in a topological order.

let $d(v)$:= distance of a shortest length path from s to v.

$$d(v) = \begin{cases} 0 & \text{if } v = s \\ \min\limits_{e:(u,v)\in E} \left\{ d(u) + \omega((u,v)) \right\} \end{cases}$$

Correctness of the algo follows from the correctnes of the recurrence, which can be proved via induction.

<u>Running time</u>: $O(m+n)$ time

<u>Directed graphs</u>: (Bellman-Ford Algorithm)

Problem with the previous recurrence is that it will go into infinite loop over a cycle.

Consider a directed graph G. Suppose it has no negative directed cycle.

Then how many edges a shortest length path from s to v could have?

A shortest length path will not have a cycle.

Hence at most $n-1$ edges

On a shortest length path.

$dist[i, v] :=$ the length of a shortest length path from s to v using at most i edges

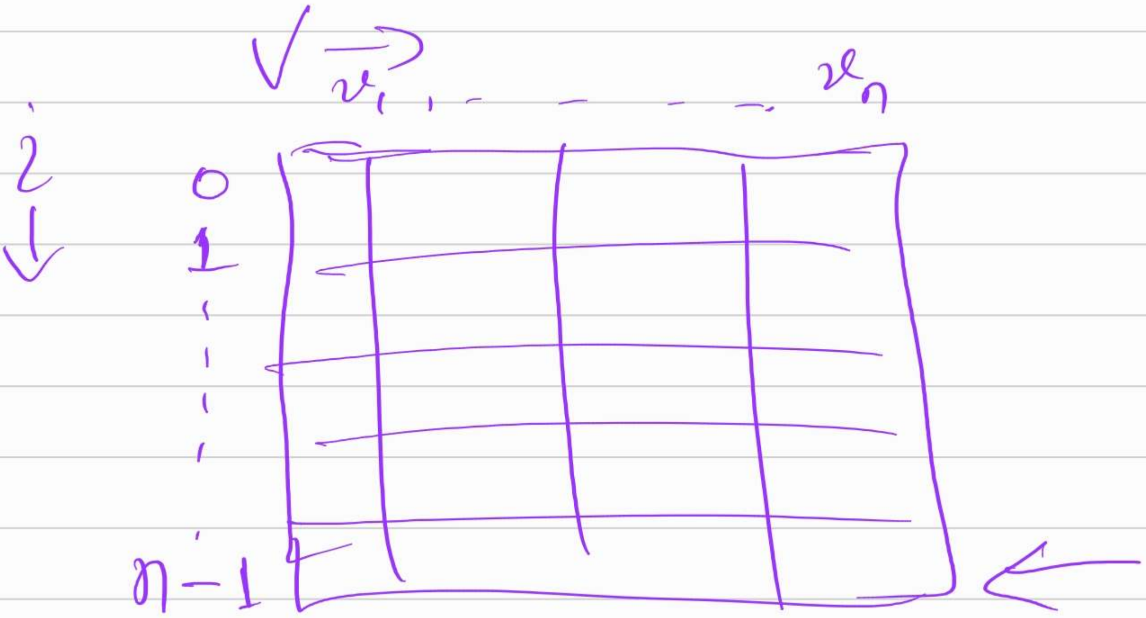
$$dist[i,v] = \begin{cases} 0 & \text{if } i=0 \text{ and } v=s. \\ \infty & \text{if } i=0 \text{ and } v \neq s \\ \min\left\{ dist[i-1,v], \min\limits_{\substack{e:(u,v) \\ u \to v}} \left\{ dist[i-1,u] + \omega((u,v)) \right\} \right\} \end{cases}$$

$$\text{dist}[i,v] = \min\left[\text{dist}[i-1,v],\ \min_{\substack{(u,v)\in E\\ u\to v}}\left\{\text{dist}[i-1,u] + \omega((u,v))\right\}\right]$$

$\text{pred}[i,v] \leftarrow u$

Correctness follows from induction.

$\text{dist}[i,v]$ in an array of size $n \times n$.



$\text{dist}[n-1, v]$ = length of a shortest path from s to v.

Initialize an array dist[,] of size $n \times n$.

$dist[0, v] = \infty$ if $v \neq s$.

$dist[0, s] = 0$

For $i = 1$ to $n-1$

For $v \in V$

Compute $dist[i, v]$ using the above recurrence.

(update $pred[i, v]$ when $dist[i, v]$ changes due to u.).

End For

End For

Output $dist[n-1, v]$ for all $v \in V$.

time : $\leq O(n^3)$.

For every vertex the minimum is computed over the incoming edges.

Hence the total time is $O(n \cdot m)$.