# Calculus 2: Integral Calculus

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The purpose of this note is to give a summary of the material that is covered under the bucket of Integral Calculus. We shall recall the definitions of the notions that are introduced during the period. The statements of all important results are mentioned and exercises, meant to serve the roles of assignments, are also added.

## 1. Riemann Integrals

Our foremost goal is to understand the integrability of a bounded real valued function on a closed interval. To this end, let $f : [a, b] \to \mathbb{R}$ be a bounded function. A **partition** $\mathcal{P}$ of the interval $[a, b]$ is a finite ordered subset $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. Note that, such a partition $\mathcal{P}$ divides $[a, b]$ into $n$ subintervals, namely $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.

Definition 1.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define,

$$m(f) = \inf\{f(x) : x \in [a, b]\} \quad \text{and} \quad M(f) = \sup\{f(x) : x \in [a, b]\}.$$

Given a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we define, for $i = 1, \ldots, n$, the quantities,

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We can easily deduce that $m(f) \le m_i(f) \le M_i(f) \le M(f)$ for all $i = 1, \ldots, n$. In this setting, We define the **lower sum** $L(\mathcal{P}, f)$ and the **upper sum** $U(\mathcal{P}, f)$ corresponding to $f$ and $\mathcal{P}$ as:

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \quad \text{and} \quad U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

Let us now take the opportunity to state an important inequality concerning the upper sums and lower sums that we have introduced.

Proposition 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P}$ be a partition of $[a, b]$. We have,*

$$m(f)(b - a) \le L(P, f) \le U(P, f) \le M(f)(b - a).$$

A partition $\mathcal{P}^*$ of $[a, b]$ is said to be a **refinement** of a partition $\mathcal{P}$ of $[a, b]$ if $\mathcal{P} \subset \mathcal{P}^*$. The following proposition shows that as we keep refining our partitions, the lower sums increase and the upper sums decrease.

Proposition 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P}, \mathcal{P}^*$ be two partitions of $[a, b]$. If $\mathcal{P}^*$ is a refinement of $\mathcal{P}$, then*

$$L(\mathcal{P}, f) \le L(\mathcal{P}^*, f) \le U(\mathcal{P}^*, f) \le U(\mathcal{P}, f).$$

*As a consequence, given any two partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a, b]$, we have $L(\mathcal{P}_1, f) \le U(\mathcal{P}_2, f)$.*

A few more definitions will put us in position to define the integrability and then the definite integral of a function (of course, if it is integrable!).

Let $\mathscr{P}([a,b])$ denote the set of all partitions of the interval $[a,b]$. We define the **lower Riemann integral** and the **upper Riemann integral** of $f$, denoted by $L(f)$ and $U(f)$ respectively, as follows:

$$L(f) := \sup\{L(\mathcal{P}, f) : \mathcal{P} \in \mathscr{P}([a,b]) \quad \textbf{and} \quad U(f) := \inf\{U(\mathcal{P}, f) : \mathcal{P} \in \mathscr{P}([a,b])\}.$$

It can be shown easily that $L(f) \leq U(f)$. A bounded function $f : [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** if and only if $L(f) = U(f)$. In such a case, we define

$$\int_a^b f(x)dx := U(f) = L(f).$$

EXAMPLE 1.4. Here are a couple of examples that we have already seen in our class:

(a) (Constant Function) The function $f : [a,b] \to \mathbb{R}$ given by $f(x) = c$ for all $x \in [a,b]$ is integrable and $\displaystyle\int_a^b f(x)dx = c(b-a)$.

(b) (Identity function) The function $f : [a,b] \to \mathbb{R}$ given by $f(x) = x$ for all $x \in [a,b]$ is integrable and $\displaystyle\int_a^b f(x)dx = (b^2 - a^2)/2$.

(c) (Dirichlet function) The function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

is NOT integrable.

EXERCISE 1.5. Let $f : [a,b] \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} 1 + x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

Is $f$ integrable? (Hint: You may not be able to compute $U(f)$, but you will find out that it is easy to determine a lower bound for $U(f)$ that is strictly bigger than $L(f)$ (this is easy to compute!)).

The following Theorem gives an alternative way of checking whether a given function is Riemann integrable.

THEOREM 1.6 (Riemann's condition). *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable iff for any $\epsilon > 0$, there exists a partition $\mathcal{P}_\epsilon$ such that*

$$U(\mathcal{P}_\epsilon, f) - L(\mathcal{P}_\epsilon, f) < \epsilon.$$

EXERCISE 1.7. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1) \\ 0 & \text{if} \quad x = 1. \end{cases}$$

Use Riemann's condition to show that $f$ is integrable on $[0,1]$. (Namely, suppose that I have given you a positive real number $\epsilon$. Find a partition $\mathcal{P}_\epsilon$ of $[0,1]$ such that $U(\mathcal{P}_\epsilon, f) - L(\mathcal{P}_\epsilon, f) < \epsilon$. Look for the easiest one!)

COROLLARY 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable iff there is a sequence of partitions $(P_n)$ of $[a,b]$ such that*

$$U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f) \to 0 \quad as \quad n \to \infty.$$

PROPOSITION 1.9 (Domain additivity of Riemann integrals). *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in [a, b]$. Then $f$ is integrable on $[a, b]$ iff $f$ is integrable of $[a, c]$ and $[c, b]$. In such a case,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

As of now, whenever we have talked about an interval $[a, b]$, we have implicitly assumed that $a \leq b$. Let us emphasize that we have defined the notion of integrability of a function $f : [a, b] \to \mathbb{R}$ over the subset $\{x : a \leq x \leq b\}$ and that we have not associated any direction or orientation with this subset. Having said that, in order to obtain uniformity of presentation, we shall use the convention that

$$\int_b^a f(x)dx := -\int_a^b f(x)dx.$$

Let us also emphasize that the above equality is just a convention and that it does not follow from any of our definitions, except when $a = b$.

## 2. Integrable functions

Let us look at classes of functions that are known to be integrable. We shall omit the proofs of these results.

PROPOSITION 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(a) *If $f$ is continuous, then $f$ is integrable.*
(b) *More generally, if $f$ has only finitely many points of discontinuities in $[a, b]$, then $f$ is integrable.*
(c) *If $f$ is monotone, then $f$ is integrable.*

EXAMPLE 2.2. The above proposition opens up a huge class of integrable functions. We look at a few of them below. However, it will help if you try to cook up examples of functions that satisfies one of the conditions in the above proposition.

(a) Let $f : [a, b] \to \mathbb{R}$ be the function given by $f(x) := [x]$, the integer part of $x$. Then $f$ is (monotonic and hence) integrable.
(b) A polynomial function (continuous and hence) is integrable.
(c) The function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = |x|$ is (continuous and hence) integrable.
(d) The "infinite step" function

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{n} & \text{if} \quad \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases}$$

is (monotonic and hence) integrable.

EXERCISE 2.3. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that the set

$$\{x \in [a, b] : f(x) \neq g(x)\}$$

is finite. If $g$ is integrable, then show that $f$ is integrable and that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

EXERCISE 2.4. Show that the function $f : [0, 3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in [0, 1) \\ 2 & \text{if} \quad x \in [1.2) \\ 1 & \text{if} \quad x \in [2, 3] \end{cases}$$

is neither monotone, nor continuous but integrable. Compute $\displaystyle\int_0^3 f(x)dx$. (Hint: Use the above exercise).

### 3. Algebraic and order properties of integrable functions

PROPOSITION 3.1 (Algebraic properties). *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions and $c \in \mathbb{R}$ a constant. Then*

(a) *$f + g$ is integrable and* $\int_a^b (f+g)(x) = \displaystyle\int_a^b f(x)dx + \int_a^b g(x)dx$.

(b) *$cf$ is integrable and* $\displaystyle\int_a^b (cf)(x) = c\int_a^b f(x)dx$.

(c) *$fg$ is integrable.*

(d) *If there exists $\delta > 0$ such that $|f(x)| \geq \delta^1$ for all $x \in [a, b]$, then $1/f$ is integrable.*

COROLLARY 3.2. *If $f, g : [a, b] \to \mathbb{R}$ are integrable, then so are* $\max\{f, g\}$ *and* $\min\{f, g\}$.

EXERCISE 3.3. Give examples of two functions $f, g : [a, b] \to \mathbb{R}$ such that $f$ and $g$ are not integrable, but $f + g$ and $fg$ are integrable. Show that if $cf$ is integrable for some non-zero constant $c$, then so is $f$.

PROPOSITION 3.4 (Order properties). *Let $f, g : [a, b] \to \mathbb{R}$ be integrable.*

(a) *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then* $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(b) *The function $|f|$ is integrable and* $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

EXERCISE 3.5. Find an $f : [a, b] \to \mathbb{R}$ such that $f$ is not integrable, but $|f|$ is integrable.

EXERCISE 3.6. Suppose that $f : [a, b] \to \mathbb{R}$ is an integrable function such that $f(x) \geq 0$ for all $x \in [a, b]$.

(a) Show that $\displaystyle\int_a^b f(x)dx \geq 0$.

(b) If $f$ is continuous and $\displaystyle\int_a^b f(x)dx = 0$, then show that $f(x) = 0$ for all $x \in [a, b]$.

(c) Show that (b) is false if $f$ is not continuous.

### 4. Fundamental Theorem of Calculus

As you may have observed during the lectures, we have not made use of the theory of differential calculus while developing the theory of integral calculus. In this section, we state a wonderful result that shows that differentiation and integration are in some sense inverse to each other.

THEOREM 4.1 (Fundamental Theorem of Calculus (FTC)). Let $f : [a, b] \to \mathbb{R}$ be an integrable function.

(a) (FTC - Part 1)If $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)dt.$$

[1]this makes the function $1/f$ bounded in $[a, b]$.

Then $F$ is continuous. Moreover, if $f$ is continuous at $c \in [a,b]$, then $F$ is differentiable at $c$ and $F'(c) = f(c)$. Consequently, if $f$ in continuous on $[a,b]$, then $F$ is differentiable on $[a,b]$ and $F' = f$ on $[a,b]$.

(b) (FTC - Part 2) If $f$ is differentiable and $f'$ is integrable on $[a,b]$, then

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be a function. We say that $f$ has an **antiderivative** $F$ on $[a,b]$ if there exists a differentiable function $F : [a,b] \to \mathbb{R}$ such that $F' = f$. We have seen that if an antiderivate of a function exists, then it is unique upto addition of a constant.

Remark 4.2. If $f : [a,b] \to \mathbb{R}$ is continuous, then we can see from the FTC-Part 1, that $f$ has an antiderivative. However, there exist integrable functions $f : [a,b] \to \mathbb{R}$ such that $f$ has no antiderivatives. You may have seen in Calculus 1, that if $F : [a,b] \to \mathbb{R}$ is a differentiable function, then $F'$ has the intermediate value property. Note that, the integer part function does not satisfy intermediate value property if $b - a > 1$ and as a consequence, if can not be the derivative of any function on $[a,b]$. However, as we have seen, this function is integrable.

Exercise 4.3. Define $g : [0,1] \to \mathbb{R}$ by $g(x) := \displaystyle\int_0^x \frac{1}{t^6+1} dt$.

(a) Evaluate $g'(\frac{1}{2})$.
(b) Compute

$$\lim_{h \to 0} \left( \frac{1}{h} \int_x^{x+h} \frac{1}{t^6+1} dt \right).$$

Exercise 4.4. Determine $H'(x)$, where $H(x) = \displaystyle\int_0^{x^2} \frac{\sin \pi t}{1+t^2} dt$.

Exercise 4.5. Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Define $G : [a,b] \to \mathbb{R}$ by

$$G(x) := \int_x^b f(t)dt.$$

Show that $G$ is continuous on $[a,b]$. Further, if $f$ is continuous at $c \in [a,b]$, then $G$ is differentiable at $c$ and $G'(c) = -f(c)$. (Hint: Make judicial use of domain additivity and Fundamental theorem of calculus, your solution should NOT take more than a couple of sentences!).

The following proposition helps us in determining the Riemann integral of the product of two functions:

Proposition 4.6 (Integration by parts). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable*[2]. *Assume that $g : [a,b] \to \mathbb{R}$ is integrable and has an antiderivative $G$ on $[a,b]$. Then*

$$\int_a^b f(x)g(x)dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx.$$

Next, we consider the method of substitution for evaluating a Riemann integral.

[2] It is possible to think that the derivative of a differentiable function is always integrable. However, it is not true. The function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is differentiable in $[-1,1]$ but its derivative is unbounded in $[-1,1]$. Consequently, $f'$ is not integrable

PROPOSITION 4.7 (Integration by substitution). *Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi([\alpha, \beta]) = [a, b]$. If $\phi$ is differentiable and $\phi'$ is integrable on $[a, b]$, then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable. Furthermore,*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} (f \circ \phi)(t)\phi'(t)dt.$$

EXERCISE 4.8. Evaluate the following integrals:

$$(i) \int_0^{1/4} \frac{x}{\sqrt{1-4x^2}}dx, \qquad (ii) \int_1^8 x^{1/3}(x^{4/3}-1)^{1/2}dx.$$

EXERCISE 4.9. Let $f$ be a continuous function on the interval $[-a, a]$ for some real number $a > 0$. Show that

(i) If $f$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

(ii) If $f$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.

## 5. Riemann Sums

In this section, we introduce the notion of Riemann sums corresponding to a definite integral. This helps us in estimating a definite integral which is difficult to compute using the techniques that we have developed till now. To this end, let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We define a **Riemann sum** $S(\mathcal{P}, f)$ corresponding to the function $f$ and the partition $\mathcal{P}$ as:

$$S(\mathcal{P}, f) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

where $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. It is easy to see that $L(\mathcal{P}, f) \leq S(\mathcal{P}, f) \leq U(\mathcal{P}, f)$. The set $\mathcal{T} := \{t_1, \ldots, t_n\}$ is called a **tag set** associated to $\mathcal{P}$. Strictly speaking, the Riemann sum $S(\mathcal{P}, f)$ depends upon the tag set and consequently, it is often denoted by $S(\mathcal{P}, \mathcal{T}, f)$.

For a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$, the **mesh of** $\mathcal{P}$, denoted $\mu(\mathcal{P})$ is defined as:

$$\mu(\mathcal{P}) := \max\{x_i - x_{i-1} : i = 1, \ldots, n\}.$$

We state an important result that improves upon the Riemann's condition:

THEOREM 5.1. *Let $f$ be integrable on $[a, b]$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \epsilon,$$

*for every partition $\mathcal{P}$ satisfying $\mu(\mathcal{P}) < \delta$.*

COROLLARY 5.2. *Let $f$ be integrable on $[a, b]$, and let $(\mathcal{P}_n)$ be a sequence of partitions of $[a, b]$ such that $\mu(\mathcal{P}_n) \to 0$. Then*

$$U(\mathcal{P}_n, f) - L(\mathcal{P}_n, f) \to 0$$

*as $n \to \infty$. Furthermore, if $S(\mathcal{P}_n, f)$ is a Riemann sum corresponding to $\mathcal{P}_n$ and $f$, then*

$$S(\mathcal{P}_n, f) \to \int_a^b f(x)dx.$$

The above corollary equips us with a rather strong method of computing the integral $\int_a^b f(x)dx$ if it is known that $f$ is integrable. Choose a sequence of partitions $(\mathcal{P}_n)$ of $[a,b]$ with (the only requirement) $\mu(\mathcal{P}_n) \to 0$. Pick a tag set corresponding to $\mathcal{P}_n$ and $f$. We emphasize that the partitions and points in the tag sets could be chosen in a way that makes the computation of the summation is convenient. This enables us to find approximations of the Riemann integral of $f$ even when we are not able to evaluate it exactly. For instance,if $f$ does not have an antiderivative in $[a,b]$ or if we are not able to think of an antiderivative of $f$, then the Part 2 of the FTC can not be applied to compute the definite integral. However, we may resort to calculating it approximately. In practice, it will generally be convenient to use the partitions $\mathcal{P}_n$ that divide $[a,b]$ into $n$ equal parts and choose the tag sets to be the ones consisting of either the right end points or the left end points of the subintervals. The following simple exercise is essentially a reformulation of the above statement.

EXERCISE 5.3. Assuming that $f$ is integrable on $[0,1]$, show that

$$\lim_{n\to\infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right) = \int_0^1 f(x)dx.$$

EXERCISE 5.4. Consider the sequence whose $n$-th term is the following. In each case, determine the limit of the sequence by expressing the $n$-th term as a Riemann sum for a suitable function:

(a) $\frac{1}{n^{17}}\sum_{i=1}^{n} i^{16}$.

(b) $\frac{1}{n^{5/2}}\sum_{i=1}^{n} i^{3/2}$.

(c) $\sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$.

(d) $S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{\frac{3}{2}} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}$.

EXERCISE 5.5. Does $\lim_{n\to\infty}\frac{1}{n^{18}}\sum_{i=1}^{n} i^{16}$ exist? If yes, then determine the limit.

## 6. Improper Integrals

Having worked with integrals of *bounded* real valued functions on *closed intervals* $[a,b]$, it is natural to wonder whether it is possible to integrate unbounded functions. Also what about integrating functions on unbounded intervals such as $(-\infty,a], [a,\infty)$ and even over $(-\infty,\infty)$? In this section, we address these issues with the notion of improper integrals.

**6.1. Improper integral of first kind.** Integrals on unbounded intervals are known as improper integrals of first kind. Here are the three possible cases.

(i) Suppose that $a \in \mathbb{R}$ and for every $x \in \mathbb{R}$ with $a \le x$ the function $f : [a,x] \to \mathbb{R}$ is integrable. We say that the integral

$$\int_a^\infty f(t)dt$$

is *convergent* if the limit

$$\lim_{x\to\infty}\int_a^x f(t)dt$$

exists. In such a case, we say that the *value* of the integral $\int_a^\infty f(t)dt$ is $\lim_{x\to\infty}\int_a^x f(t)dt$.

(ii) Suppose that $b \in \mathbb{R}$ and for every $x \in \mathbb{R}$ with $x \le b$ the function $f : [x,b] \to \mathbb{R}$ is integrable. We say that the integral

$$\int_{-\infty}^b f(t)dt$$

is *convergent* if the limit

$$\lim_{x\to-\infty}\int_x^b f(t)dt$$

exists. In such a case, we say that the *value* of the integral $\int_{-\infty}^b f(t)dt$ is $\lim_{x\to-\infty}\int_x^b f(t)dt$.

(iii) Suppose that for every pair of real numbers $a$ and $b$, the function $f$ is integrable on $[a,b]$. We say that the integral

$$\int_{-\infty}^\infty f(t)dt$$

is *convergent* if both the improper integrals

$$\int_0^\infty f(t)dt \quad \text{and} \quad \int_{-\infty}^0 f(t)dt$$

are convergent. In such a case, the value of $\int_{-\infty}^\infty f(t)dt$ is $\int_{-\infty}^0 f(t)dt + \int_0^\infty f(t)dt$.

If $\lim_{x\to\infty}\int_{-x}^x f(t)dt$ exists, then it is called the *Cauchy principal value* for the integral $\int_{-\infty}^\infty f(t)dt$. It is readily verified that the convergence of the integral guarantees the existence of its Cauchy principal value. However, as the following exercise shows, the converse is not true.

Exercise 6.1. Show that $\int_0^\infty \frac{2x}{x^2+1}dx$ is divergent and conclude that $\int_{-\infty}^\infty \frac{2x}{x^2+1}dx$ is also divergent. Prove that $\lim_{b\to\infty}\int_{-b}^b \frac{2x}{x^2+1}dx = 0$.

Example 6.2 (Worked out in class).

(i) $\int_1^\infty \frac{1}{t}dt$ *is divergent.*

(ii) $\int_1^\infty \frac{1}{t^2}dt$ *is convergent.*

(iii) $\int_{-\infty}^0 te^t dt$ *is convergent.*

(iv) $\int_1^\infty \frac{1}{t^p}dt$ *is convergent if* $p > 1$ *and divergent otherwise.*

Exercise 6.3. Determine whether the following integrals are convergent and in such cases evaluate them:

(a) $\int_1^\infty e^{-x}dx$.

(b) $\int_0^\infty \frac{1}{1+x}dx$.

(c) $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2-x}} dx.$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6} dx.$

Let us now look at some methods for determining whether a given improper integral is convergent:

PROPOSITION 6.4 (Direct comparison test). *Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ be functions that are integrable on $[a, x]$ for all $x \geq a$. If $0 \leq f(t) \leq g(t)$ for all $t \geq a$, then*

$$\int_a^\infty g(t)dt \text{ is convergent } \Longrightarrow \int_a^\infty f(t)dt \text{ is convergent}$$

*and*

$$\int_a^\infty f(t)dt \text{ is divergent } \Longrightarrow \int_a^\infty g(t)dt \text{ is divergent.}$$

EXAMPLE 6.5 (Worked out in class).

(i) $\displaystyle\int_1^\infty \frac{\sin x}{x} dx$ *converges.*

(ii) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}$ *diverges.*

PROPOSITION 6.6 (Limit comparison test). *Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ be functions that are integrable on $[a, x]$ for all $x \geq a$. If*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = c, \quad \text{where} \quad c \neq 0, \infty$$

*then the integrals*

$$\int_a^\infty f(t)dt \quad \text{and} \quad \int_a^\infty g(t)dt$$

*both converge or both diverge.*

EXAMPLE 6.7 (Worked out in class).

(i) $\displaystyle\int_1^\infty \frac{1}{1+x^2} dx$ *converges.*

(ii) $\displaystyle\int_1^\infty \frac{3}{5+e^x}$ *converges.*

EXERCISE 6.8. Determine whether the following integrals are convergent:

(a) $\displaystyle\int_0^\infty \frac{1}{1+x} dx.$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^6+1}} dx.$

(c) $\displaystyle\int_1^\infty \frac{e^x}{x} dx.$

(d) $\displaystyle\int_4^\infty \frac{2}{t^{3/2}-1} dt.$

(e) $\displaystyle\int_2^\infty \frac{2}{\sqrt{x-1}} dx.$

**6.2. Improper integral of second kind.** Roughly speaking, integrals of functions that are unbounded on a bounded interval are called the improper integrals of second kind. As before, let us look at four cases that may arise here:

(i) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : (a, b] \to \mathbb{R}$ be a function such that $f$ is unbounded on $(a, b]$ but $f$ is integrable on $[x, b]$ for all $x \in (a, b]$. We say that the integral $\displaystyle\int_{a<t\leq b} f(t)dt$ is *convergent*, if the limit
$$\lim_{x\to a^+}\int_x^b f(t)dt$$
exists. In such a case, we say that the value of the integral $\displaystyle\int_{a<t\leq b} f(t)dt$ is given by $\displaystyle\lim_{x\to a^+}\int_x^b f(t)dt$. The integral is called *divergent* if it is not convergent.

(ii) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b) \to \mathbb{R}$ be a function such that $f$ is unbounded on $[a, b)$ but $f$ is integrable on $[a, x]$ for all $x \in [a, b)$. We say that the integral $\displaystyle\int_{a\leq t<b} f(t)dt$ is *convergent*, if the limit
$$\lim_{x\to b^-}\int_a^x f(t)dt$$
exists. In such a case, we say that the value of the integral $\displaystyle\int_{a\leq t<b} f(t)dt$ is given by $\displaystyle\lim_{x\to b^-}\int_a^x f(t)dt$. The integral is called *divergent* if it is not convergent.

(iii) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : (a, b) \to \mathbb{R}$ be a function such that $f$ is unbounded on $(a, b)$ but $f$ is integrable on $[x, y]$ for all $x, y \in (a, b)$. We say that the integral $\displaystyle\int_{a<t<b} f(t)dt$ is *convergent*, if both the improper integrals
$$\int_{a<t\leq c} f(t)dt \quad\text{and}\quad \int_{c\leq t<b} f(t)dt$$
exist for some $c \in (a, b)$. In such a case, we say that the value of the integral $\displaystyle\int_{a<t<b} f(t)dt$ is given by $\displaystyle\int_{a<t\leq c} f(t)dt + \int_{c\leq t<b} f(t)dt$. The integral is called *divergent* if it is not convergent.

(iv) Let $a, b \in \mathbb{R}$ with $a < b$ and $c \in (a, b)$. Let $f$ be unbounded on $[a, c)$ and $(c, b]$ but $f$ is integrable on $[a, x]$ and $[y, b]$ for all $x, y \in (a, b)$ satisfying $a \leq x < c < y \leq b$. We say that the integral $\displaystyle\int_a^b f(t)dt$ is *convergent*, if both the improper integrals
$$\int_{a\leq t<c} f(t)dt \quad\text{and}\quad \int_{c<t\leq b} f(t)dt$$
exist. In such a case, we say that the value of the integral $\displaystyle\int_a^b f(t)dt$ is given by $\displaystyle\int_{a\leq t<c} f(t)dt + \int_{c<t\leq b} f(t)dt$. The integral is called *divergent* if it is not convergent.

Example 6.9 (Worked out in class).

(i) $\displaystyle\int_0^1 \frac{1}{1-x}dx$ *is divergent.*

(ii) $\displaystyle\int_0^3 \frac{1}{(t-1)^{2/3}}$ *is convergent and its value is* $3 + 3.2^{1/3}$.

Exercise 6.10. Determine whether the following integrals are convergent. Also, determine the value of the integrals that are convergent.

(i) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}dx.$

(ii) $\displaystyle\int_0^2 \frac{1}{1-x^2}dx.$

(iii) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}dx.$

## 7. Area

**7.1. Area under a curve.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose, for the time being, that $f(x) \geq 0$ for all $x \in [a,b]$. Note that, in this case, the graph of $f$ always lies above the $x$-axis. We denote by $R_f$ the region bounded above by the graph of the equation $y = f(x)$, below by the $x$ axis, to the left by the line $x = a$ and to the right by the line $x = b$; i.e.

$$R_f = \{(x,y) : a \leq x \leq b, 0 \leq y \leq f(x)\}.$$

We say that the region $R_f$ has an **area** if $f$ is integrable on $[a,b]$ and in such a case, we say write

$$\text{Area}(R_f) := \int_a^b f(t)dt.$$

In this particular case, namely, when $f(x) \geq 0$ for all $x \in [a,b]$, it follows from the Order properties that $R_f \geq 0$. This is something nice that we would particularly wish for that area of a region is nonnegative.

Now what happens if we do not have the leisure of dealing with a function that takes nonnegative values. Namely, how do we define the area of the most obvious region in the case when the graph of $f$ does not lie above $x$ axis completely (or at all!)? What we shall do in this case is not farfetched. Simply put, we compute the area of the regions that lies above $x$-axis and below $x$-axis (we want the area of this region to be nonnegative as well!) and add them up. More concretely, we follow the procedure mentioned below.

Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions. We define the following two functions on the interval $[a,b]$:

$$\max(f,g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad \min(f,g)(x) := \min\{f(x), g(x)\}.$$

A simple computation, left for you to convince yourselves, shows that,

$$\max(f,g) = \frac{f+g+|f-g|}{2} \quad \text{and} \quad \min(f,g) = \frac{f+g-|f-g|}{2}.$$

Using the order properties, we see that the functions $\max(f,g)$ and $\min(f,g)$ are integrable.

Now we back to our original situation. We have an integrable function $f : [a,b] \to \mathbb{R}$. We define

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := -\min\{f, 0\}.$$

It follows easily that $f^+$ and $f^-$ are nonnegative functions and a simple calculation shows that

$$\int_a^b f(t)dt = \text{Area}(R_{f^+}) - \text{Area}(R_{f^-}).$$

However, in this case, simply computing the integral will not give you the area of the region bounded by the graph of $f$ and $x$-axis. In fact, what the integral gives us is known as **the sigend area**. The requisite area, also known as **the total area** of such a region is given by the formula:

$$\text{Area}(R_{f^+}) + \text{Area}(R_{f^-}).$$

This phenomenon is captured nicely in the following example:

EXAMPLE 7.1. Let $f : [0, 2\pi] \to \mathbb{R}$ be given by $f(x) = \sin x$. The signed area, as mentioned above, is given by

$$\int_0^{2\pi} \sin x dx = (-\cos x)|_{x=0}^{x=2\pi} = 0.$$

It is easy to see that,

$$f^+(x) = \begin{cases} \sin x & \text{if} \quad x \in [0, \pi] \\ 0 & \text{if} \quad x \in [\pi, 2\pi] \end{cases}$$

whereas

$$f^-(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \pi] \\ -\sin x & \text{if} \quad x \in [\pi, 2\pi] \end{cases}$$

Using the FTC along with the domain additivity of Riemann integrals one derives,

$$\text{Area}(R_{f^+}) = 2 \quad \text{and} \quad \text{Area}(R_{f^-}) = 2.$$

When we compute the integral $\int_0^{2\pi} f(x)dx$, we compute the difference between the above two areas and consequently they cancel each other. Finally, the total area of the region is obtained by adding the the two areas mentioned above. Consequently, the total area is 4.

EXERCISE 7.2. Determine the total area of the region bounded by the graph[3] of $f$ and $x$ axis as $x$ varies in the intervals as follows:

(i) $y = -x^2 - 3x, \quad -3 \le x \le 2.$
(ii) $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2.$
(iii) $y = x^{1/3}, \quad -1 \le x \le 8.$
(iv) $y = x^2 - 4x + 3. \quad 0 \le x \le 3.$

**7.2. Area between two curves.** Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1(x) \le f_2(x)$ for all $x \in [a, b]$. Then the **area** of the region between the curves given by $y = f_1(x)$ and $y = f_2(x)$ that is bounded by the vertical lines $x = a$ and $x = b$, i.e. the region

$$R := \{(x, y) \in \mathbb{R}^2 : a \le x \le b, \quad f_1(x) \le y \le f_2(x)\},$$

is given by

$$\text{Area}(R_{f_2 - f_1}) = \int_a^b (f_2(x) - f_1(x))\, dx.$$

However, if we are not in such a nice situation where one function dominates the other, then the procedure of solving such a problem is as follows: If we can subdivide the corresponding region into finitely many nonoverlapping subregions of the types considered above, then the area of $R$ is given by the sum of areas of these subregions. For example, suppose that there exists $c_1, \ldots, c_n \in [a, b]$ such that $f_1(x) \ge f_2(x)$ for all $x \in [a, c_1]$, $f_1(x) \le f_2(x)$ for all $x \in [c_1, c_2]$ etc.,

[3]To get a true feeling on how the corresponding region looks like, it will be better if you could draw the graph of the function. You are expected to know curve sketching by now. Just in case, you do not know, or you do not want to spend a lot of time sketching graphs of functions, I recommend using https://www.desmos.com/calculator

then we compute the area of $R$ by adding up the areas of all the regions such as $R_{f_1-f_2}$ on $[a, c_1]$, $R_{f_2-f_1}$ on $[c_1, c_2]$ etc. It turns out that

$$\text{Area}(R) = \int_a^b |f_1(x) - f_2(x)|dx.$$

We note that, if we take $f_1$ or $f_2$ to be the zero function, then we derive the material covered in the previous subsection. We finish this subsection, as always, by listing a few exercises.

EXERCISE 7.3. Determine the areas of the regions enclosed by two curves in each of the following cases:

(i) $y = x^2 - 2$, $y = 2$.
(ii) $x = y^3$, $x = y^2$.
(iii) $y = 2x^2$, $y = y = x^4 - 2x^2$.
(iv) $y = -x^3 + 3x$, $y = 2x^3 - x^2 - 5x$.
(v) $y = \frac{x^3}{3} - x$, $y = \frac{x}{3}$.

## 8. Volume of a solid

**8.1. Slice method.** Let $D$ be a bounded subset $\mathbb{R}^3$. A cross-section of $D$ obtained by cutting $D$ by a plane in $\mathbb{R}^3$ is called a **slice** of $D$. Let $a < b$ and suppose that $D$ lies in between the planes $x = a$ and $x = b$ that are perpendicular to $x$-axis. For $s \in [a, b]$, consider the slice of $D$ by the plane $x = s$, namely the region

$$D_s = \{(x, y, z) \in D : x = s\}.$$

Suppose that $D_s$ has an area $A(s)$. This gives us an area function:

$$A : [a, b] \to \mathbb{R}$$

given by

$$A(s) = \text{Area}(D_s).$$

We define the **volume** of $D$ by

$$\text{Vol}(D) = \int_a^b A(x)dx, \tag{1}$$

provided the area function is integrable. We look at some particular cases, namely, the solids of revolutions. If a subset $D \subset \mathbb{R}^3$ is obtained by revolving a planar region about an axis, then $D$ is known as a **solid of revolution**.

EXAMPLE 8.1. Let us consider two more particular instances:

(a) Let $f : [a, b] \to \mathbb{R}$ be an integrable function and assume that $f \geq 0$. Let $R_f$ denote the region is given in the previous section, namely,

$$R_f = \{(x, y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

Let $D \subset \mathbb{R}^3$ be obtained by revolving $R_f$ about $x$-axis. Note that, for every $s \in [a, b]$, the slice $D_s$ is a disc of radius $f(s)$. Consequently, the area function is given by

$$A(x) = \pi f(x)^2.$$

Since $f$ is integrable, so is $A(x)$. Consequently, the equation (1) applies, and we obtain

$$\text{Vol}(D) = \int_a^b \pi f(x)^2 dx.$$

(b) (**Washer Method**) Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions with $f_1 \leq f_2$ on $[a, b]$. Let

$$R = \{(x, y) : a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}.$$

Suppose that $D$ is a solid that is obtained by revolving $R$ about $x$-axis. Then the slice $D_s$ is a *washer shaped region* with inner radius $f_1(s)$ and outer radius $f_2(s)$. Consequently, the area function is given by

$$A(x) = \pi \left((f_2(x))^2 - (f_1(x))^2\right).$$

Since $f_1, f_2$ are integrable, so is the function $A(x)$. We derive from (1) that

$$\mathrm{Vol(D)} = \int_a^b \pi \left((f_2(x))^2 - (f_1(x))^2\right) dx.$$

Similar formulae can be obtained for computing the volumes of solids if the roles of $x$ and $y$ are interchanged. Here are the exercises for the current subsection.

Exercise 8.2. Find the volume of the solid $D$ enclosed by the cyllinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Exercise 8.3. Determine the volume of the solids of revolution obtained by revolving the regions bounded by lines and curves about $x$-axis.

(i) $y = \sqrt{9 - x^2}$, $y = 0$.
(ii) $y = x^2 + 1$, $y = x + 3$.

Exercise 8.4 (Volume of a torus). Let $a, b \in \mathbb{R}$ with $0 < a < b$. The disk $x^2 + y^2 \leq a^2$ is rotated about a line $x = b$ to generate a torus. Find its volume. (Hint: You may use the fact that $\displaystyle\int_{-a}^{a} \sqrt{a^2 - y^2} dy = \frac{\pi a^2}{2}$, since this is the area of a semicirle of radius $a$).

**8.2. Shell method.** Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be such that $f_1 \leq f_2$ where $0 \leq a < b$. Let $D$ denote the solid generated by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$ and $x = b$ about $y$-axis. If $f_1, f_2$ are integrable on $[a, b]$, then we define,

$$\mathrm{Vol}(D) := 2\pi \int_a^b x(f_2(x) - f_1(x)).$$

As in the case with slice methods, the above goes through if we interchange the roles of $x$ and $y$.

Exercise 8.5. In each of the following exercises, determine the volume of the solids generated by revolving the regions about $y$-axis:

(i) The region bounded by $y = \sqrt{x^2 + 1}$, $x = 0$, $x = \sqrt{3}$ and $x$-axis.
(ii) $y = 2x - 1$, $y = \sqrt{x}$, and $x = 0$.
(iii) $y = \frac{9x}{\sqrt{x^3+1}}$, $x$-axis, $x = 0$ and $x = 3$.

Exercise 8.6. Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \in (0, \pi] \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that $xf(x) = \sin x$ for all $x \in [0, \pi]$. Hence compute the area of the solid obtained by rotating the region bounded by $y = f(x)$, $x = 0$, $x = \pi$ and $x$-axis.

## 9. Arc length

Let $C \subset \mathbb{R}^2$ be a **parametrized curve** given by $(x(t), y(t))$ where $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuously diferentiable[4]) functions on $[\alpha, \beta]$. The **arc length** of such a curve, denoted by $\ell(C)$, is given by the integral

$$\ell(C) := \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

In particular, we come across two special cases:

(a) Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to \mathbb{R}$ be continuously differentiable, and let a *smooth curve* $C$ be given by $y = f(x)$, $x \in [a, b]$. Then the arc length of $C$ is equal to

$$\ell(C) = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

(b) Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to \mathbb{R}$ be continuously differentiable, and let a *smooth curve* $C$ be given by $x = g(y)$, $y \in [a, b]$. Then the arc length of $C$ is equal to

$$\ell(C) = \int_a^b \sqrt{1 + g'(y)^2}\ dy.$$

Exercise 9.1.

(a) Determine the lengths of the parametrized curves:
   (i) $x = 1 - t$, $y = 2 + 3t$, $t \in \left[-\frac{2}{3}, 1\right]$.
   (ii) $x = t^3$, $y = 3t^2/2$, $t \in [0, \sqrt{3}]$.
   (iii) $x = \frac{(2t+3)^{3/2}}{3}$, $y = t + \frac{t^2}{2}$, $0 \le t \le 3$.

(b) Determine the lengths of the following curves:
   (i) $x = \frac{y^4}{4} + \frac{1}{8y^2}$, $y \in [1, 2]$.
   (ii) $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$, $1 \le x \le 8$.
   (iii) $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}dt$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. (Hint: Use FTC).

## 10. Surface of revolution

Let $C \subset \mathbb{R}^2$ be a smooth curve[5] parametrically given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Suppose that $L \subset \mathbb{R}^2$ be a line given by $ax + by + c = 0$ such that $C$ does not cross $L$. Then the **area of** $S$, the **surface of revolution** obtained by rotating $C$ around $L$, is given by

$$\text{Area}(S) = 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\ dt,$$

where $\rho(t)$ is the distance of $(x(t), y(t))$ from the line $L$. We know from our high school geometry that

$$\rho(t) = \frac{|ax(t) + by(t) + c|}{\sqrt{a^2 + b^2}}.$$

Putting all the information together, we obtain

$$\text{Area}(S) = 2\pi \int_\alpha^\beta \frac{|ax(t) + by(t) + c|}{\sqrt{a^2 + b^2}}\sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

As in the case with arc length, we have the following two particular situations:

[4] a function is continuously differentiable if its derivative is continuou

[5] This means that $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuously differentiable

(a) Let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to \mathbb{R}$ be continuously differentiable, and let a *smooth curve* $C$ be given by $y = f(x)$, $x \in [a, b]$. Let $L$ be the $x$-axis and the curve $C$ does not cross $L$. Then the area of the surface of revolution $S$ obtained by rotating $C$ about $L$ is given by

$$\text{Area}(S) = 2\pi \int_a^b |f(x)|\sqrt{1 + f'(x)^2}\ dx.$$

(b) Let $a, b \in \mathbb{R}$ with $a < b$, $g : [a, b] \to \mathbb{R}$ be continuously differentiable, and let a *smooth curve* $C$ be given by $x = g(y)$, $y \in [a, b]$. Let $L$ be the $y$-axis and the curve $C$ does not cross $L$. Then the area of the surface of revolution $S$ obtained by rotating $C$ about $L$ is given by

$$\ell(C) = 2\pi \int_a^b |g(y)|\sqrt{1 + g'(y)^2}\ dy.$$

EXERCISE 10.1. Determine the surface area of the cone frustum generated by revolving the line segment

$$y = \frac{x}{2} + \frac{1}{2}, \qquad 1 \le x \le 3$$

about $x$ axis. Check your result with the geometry formula that a frustum surface area is given by $\pi(d_1 + d_2) \times$ slant height.

EXERCISE 10.2. Find the areas of the surfaces of revolution generated by revolving the curve $C$ about the line $L$ is each of the following cases:

(a) $C : y = \sqrt{x + 1}$, $\quad 1 \le x \le 5$ and $L : x$- axis.
(b) $C : x = \sqrt{2y - 1}$, $\quad \frac{5}{8} \le x \le 1$ and $L : y$-axis.

EXERCISE 10.3. Suppose that the semicircle $y = \sqrt{r^2 - x^2}$, $x \in [-r, r]$ is rotated about $x$-axis to generate a sphere of radius $r$. Let $AB$ be an arc of the semicircle that lies above an interval of length $h$ on $x$-axis. Show that the area swept out by $AB$ does not depend on the location of the interval. Namely, if $AB$ lies above the interval $[a, a + h]$, then show that the area of the surface of revolution thus generated is independent of $a$.

(**An interesting consequence:** If you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust).

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