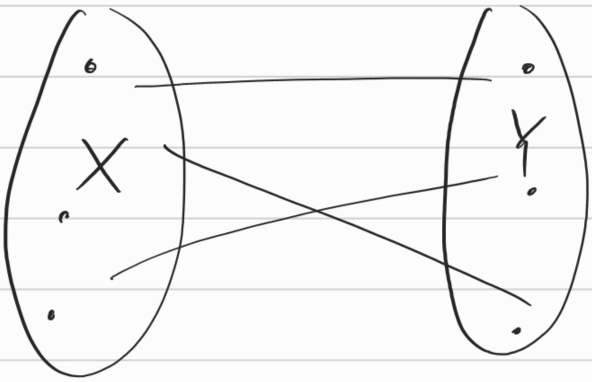
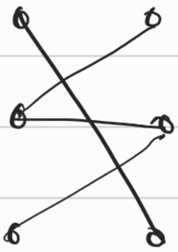
13/04/2022

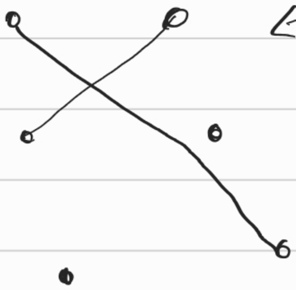
Given a bipartite graph G

$G = (X \cup Y, E)$



Defn: Matching is a subset of edges s.t. every vertex has at most one incident edge.



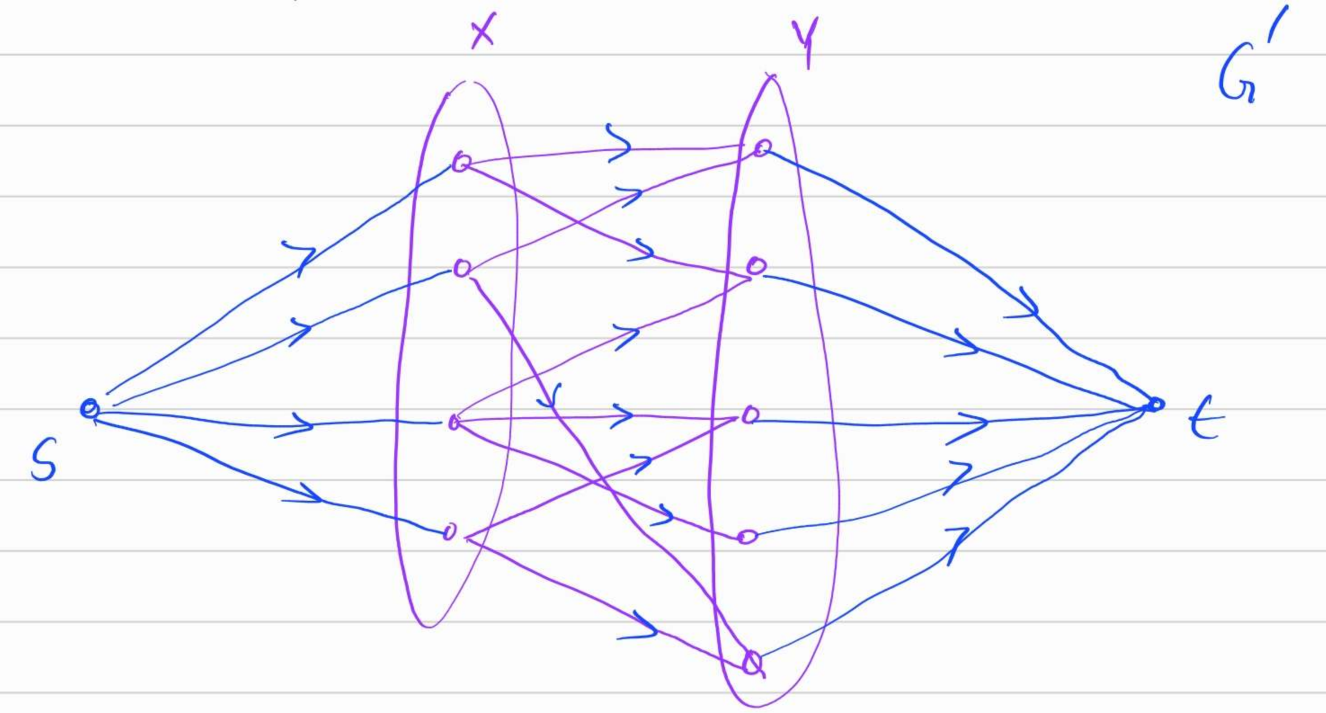


← matching of size 2.

Size of a matching equals the no. of edges it contains.

← matching of size 3.

Goal :- Find a maximum matching given a bipartite graph G.



Construct a directed graph G' as shown above from G.

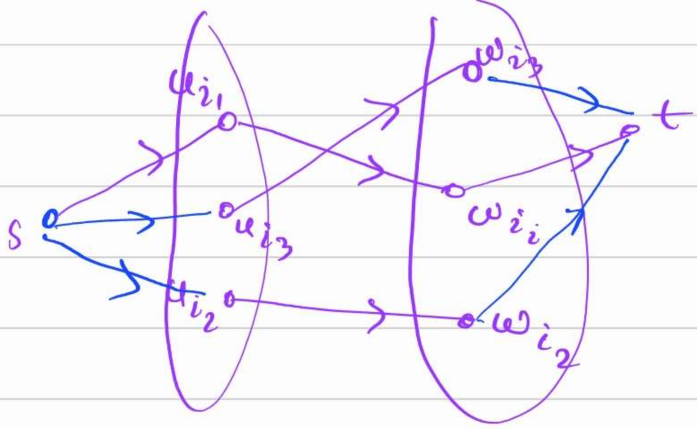
$c(e) = 1$ iff $e \in G'$

$c(e) \in \{1, 0\}$

Claim: Value of a Maximum flow in G' equals the size of maximum matching in G.

Proof:-

Given a matching $M = \{ (u_{i_1}, w_{i_1}), (u_{i_2}, w_{i_2}), (v_{i_3}, w_{i_3}) \}$



$s \to u_{i_1} \to w_{i_1} \to t$

$s \to u_{i_2} \to w_{i_2} \to t$

$s \to u_{i_3} \to w_{i_3} \to t$

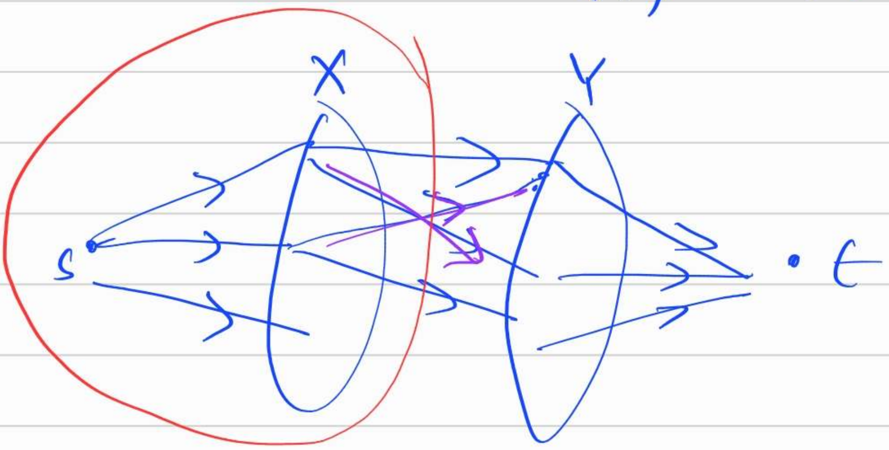
We get a flow f s.t. $v(f) = |M|$.

<u>Reverse direction:</u>

Given a flow f,

Construct a matching M s.t.

$$v(f) = |M|.$$

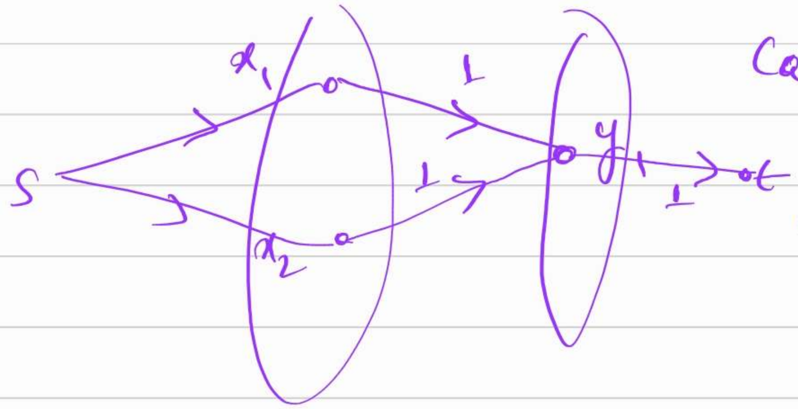


$f(e) \in \{0, 1\}$

Define $M = \{ e \mid e$ crosses from X to Y and $f(e) = 1 \}$

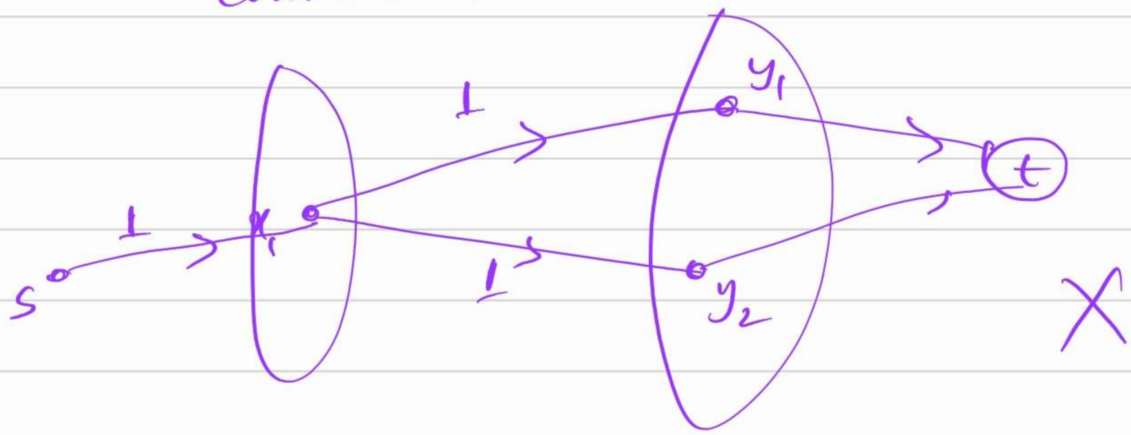
<u>Claim:</u> M is a matching.

<u>Proof:-</u>



Can such a scenario occur in a flow ?

Cann't happen because of conservation condition.



The two things together implies that M is a matching.

Claim :- $|M| = v(f)$

Proof :- Let (A, B) be an s-t-cut.

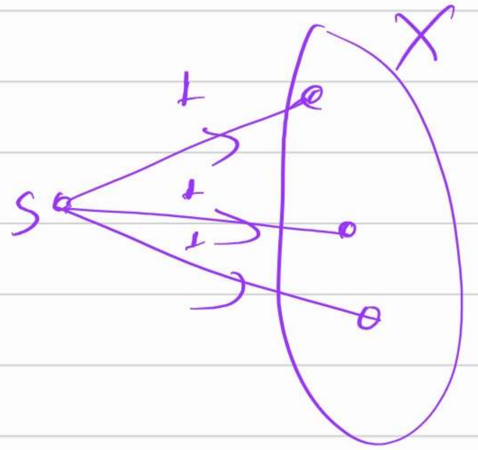
$$v(f) = f^{out}(A) - f^{in}(A)$$

Consider $(\{s\} \cup X, \; Y \cup \{t\})$ s-t-cut.

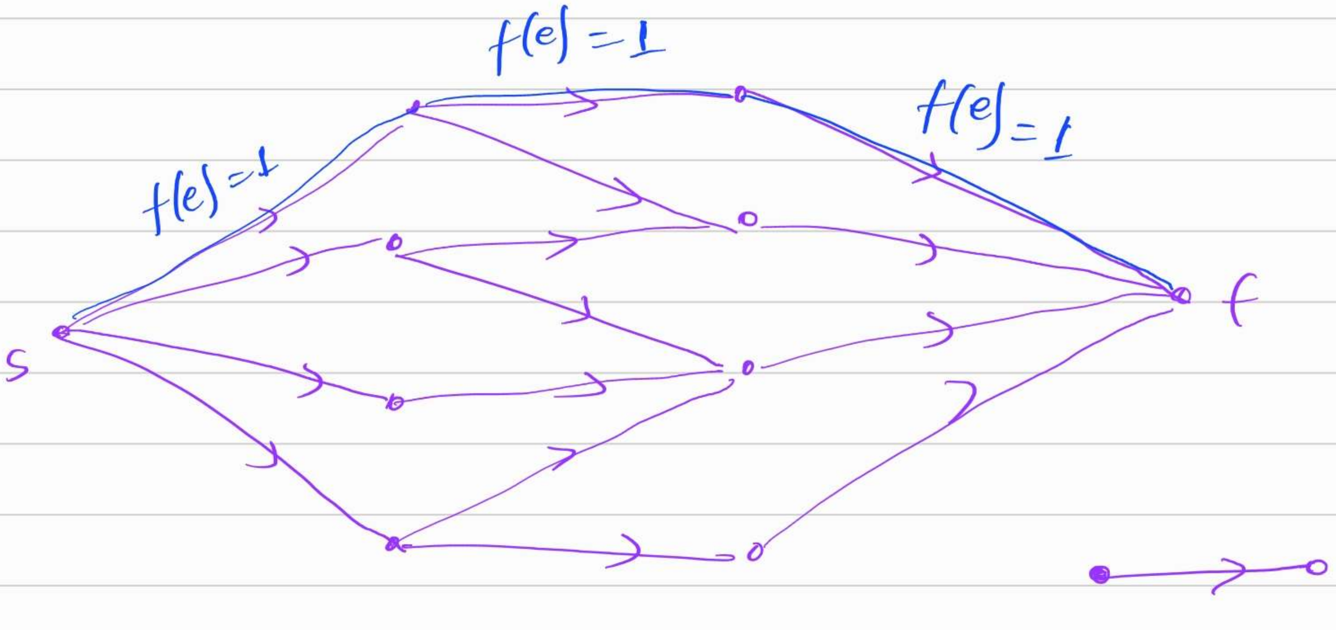
$$v(f) = f^{out}(\{s\} \cup X) - f^{in}(\{s\} \cup X)$$
$$= f^{out}(\{s\} \cup X) - 0$$
$$= f^{out}(\{s\} \cup X) = |M|$$

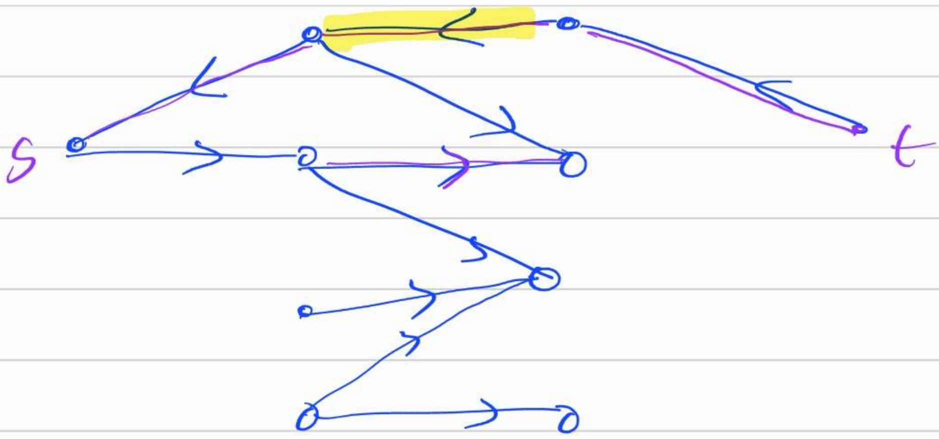
$\boxtimes$

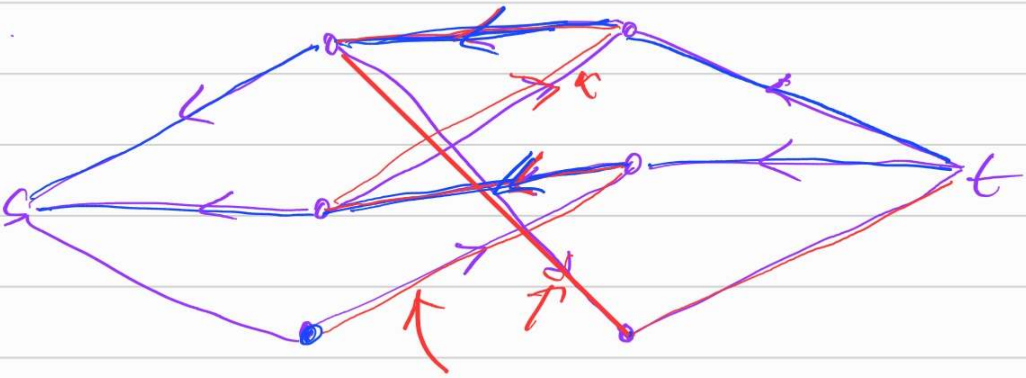
Thm :- One can find the max. matching in $O(mn)$ time.



$$\sum_{e \text{ going out of } s} c(e) \le n.$$

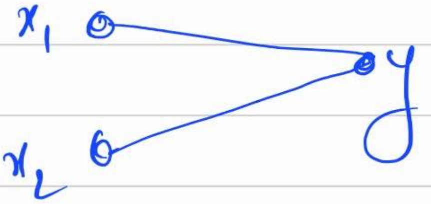






Augmenting paths are also called alternating paths in context with matching.

Defn :- Perfect matching is a matching where every vertex has exactly one edge incident on it.



minimum condition for a perfect matching in a bipartite graph is $|X| = |Y|$

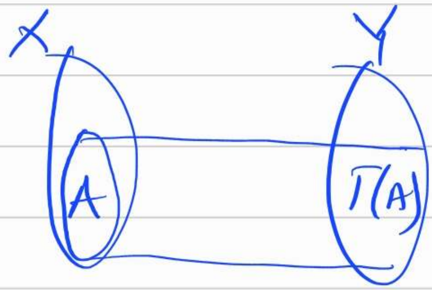
Find the max flow (= max. matching) and check max-flow = $|X| = |Y|$

Otherwise, max-flow $< |X| = |Y|$

<u>Define</u> :- $\Gamma(A)$ where $A \subseteq X$

$\Gamma(A) = \{ y \in Y \mid$ s.t. y is adjacent to some vertex in A }

(Hall's thm)



if G has perfect Matching

then $|\Gamma(A)| \geq |A|$

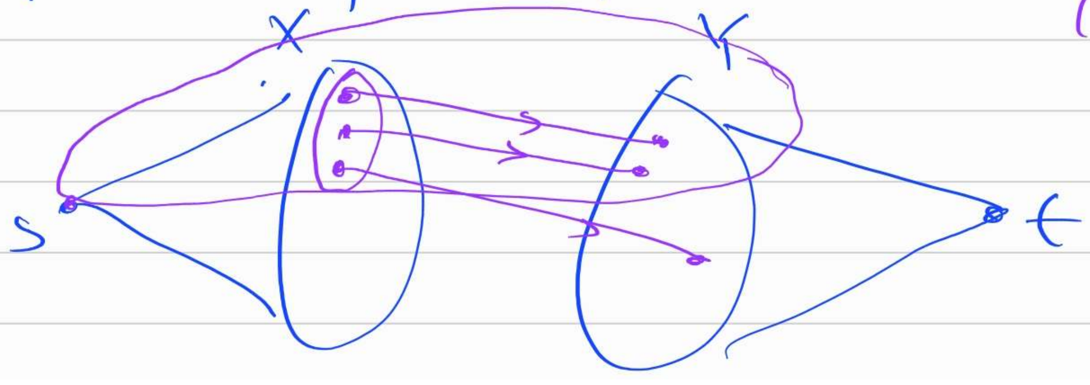
Thm :- Consider a bipartite graph $G = (X \cup Y, E)$

s.t. $|X| = |Y|$.

Then either G has a perfect matching

or $\exists\ A \subseteq X$ s.t. $|\Gamma(A)| < |A|$.

Proof :- from max-flow - find the min cut
(A', B')



$A' = \{s\} \cup$ some part of X $\cup$ some part of Y

$A := A' \cap X$

Claim: $|\Gamma(A)| < |A|$