

Assignment 4

Suraj - CS20BTECH11050

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<https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment%204/Python%20codes>

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1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is

$$f(x) = \begin{cases} \frac{7}{32} x^6 (2 - x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

then k equals _____

2 SOLUTION

Let $X \in [0, 2]$ be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x) dx = 1 \quad (2.0.1)$$

$$\Pr(x) = \frac{1}{2} \quad (\because \text{Uniform order}) \quad (2.0.2)$$

The PDF for X is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.3)$$

The CDF for X is

$$F(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.4)$$

Definition 1. k^{th} order statistic : If the sample is arranged in an ascending order, the k^{th} order statistic is the k^{th} element from the left. $X_{(k)}$ is called k^{th} vector of order statistic For a sample $\{X_1, X_2, \dots, X_n\}$ of size n :

$$X_{(1)} = \min \{X_1, X_2, \dots, X_n\} \quad (2.0.5)$$

$$X_{(n)} = \max \{X_1, X_2, \dots, X_n\} \quad (2.0.6)$$

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(k)} \leq \dots \leq X_{(n)} \quad (2.0.7)$$

Let $X_{(k)}$ be vector of order statistic of (X_1, X_2, \dots, X_n)

Lemma 2.1. Marginal probability density (PDF) for a k_{th} order statistic of a random sample of size n given CDF= $F(x)$ and PDF= $f(x)$ is given by

$$f_{(k,n)}(x) = n^{n-1} C_{k-1} (1 - F(x))^{n-k} F(x)^{k-1} f(x) \quad (2.0.8)$$

Proof. The CDF of the k^{th} order statistic from a sample of size n is:

$$F_{(k,n)}(x) = \Pr(X_{(k)} \leq x) \quad (2.0.9)$$

$$= \sum_{j=k}^n {}^n C_j (1 - F(x))^{n-j} F(x)^j \quad (2.0.10)$$

Deriving PDF of k^{th} order statistic from a sample of size n :

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx} \left(\sum_{j=k}^n {}^n C_j (1 - F(x))^{n-j} F(x)^j \right) \quad (2.0.11)$$

$$\begin{aligned} f_{(k,n)}(x) &= \sum_{j=k}^n {}^n C_j (j) (1 - F(x))^{n-j} F(x)^{j-1} f(x) \\ &\quad - \sum_{j=k}^n {}^n C_j (n - j) (1 - F(x))^{n-j-1} F(x)^j f(x) \end{aligned} \quad (2.0.12)$$

$$S_1 = \sum_{j=k}^n \frac{n!}{(n-j)!(j-1)!} (1-F(x))^{n-j} F(x)^{j-1} f(x) \quad (2.0.13)$$

$$S_2 = \sum_{j=k}^n \frac{n!}{(n-j-1)!j!} (1-F(x))^{n-j-1} F(x)^j f(x) \quad (2.0.14)$$

let $i = j + 1$ change the limits for the summation in equation (2.0.14)

$$S_2 = \sum_{i=k+1}^n \frac{n!}{(n-i)!(i-1)!} (1-F(x))^{n-i} F(x)^{i-1} f(x) \quad (2.0.15)$$

$$S_1 - S_2 = \frac{n!}{(n-k)!(k-1)!} (1-F(x))^{n-k} F(x)^{k-1} f(x) \quad (2.0.16)$$

$$f_{(k,n)}(x) = n^{n-1} C_{k-1} (1-F(x))^{n-k} F(x)^{k-1} f(x) \quad (2.0.17)$$

□

Using Lemma (2.1) given $n = 8$ substituting n , equation (2.0.3) and equation (2.0.4) in above equation (2.0.17) we get:

$$f_{(k,n)}(x) = 8^7 C_{k-1} \left(1 - \frac{x}{2}\right)^{8-k} \left(\frac{x}{2}\right)^{k-1} \frac{1}{2} \quad (2.0.18)$$

$$f_{(k,n)}(x) = \frac{1}{32} {}^7C_{k-1} (2-x)^{8-k} x^{k-1} \quad (2.0.19)$$

Comparing the marginal probability density function obtained in equation (2.0.19) with the equation given in question

$$\frac{1}{32} {}^7C_{k-1} (2-x)^{8-k} x^{k-1} = \frac{7}{32} (2-x) x^6 \quad (2.0.20)$$

$$\therefore k = 7 \quad (2.0.21)$$

Hence the marginal probability density in the given problem is 7th order uniform statistic and **the value of k is 7**

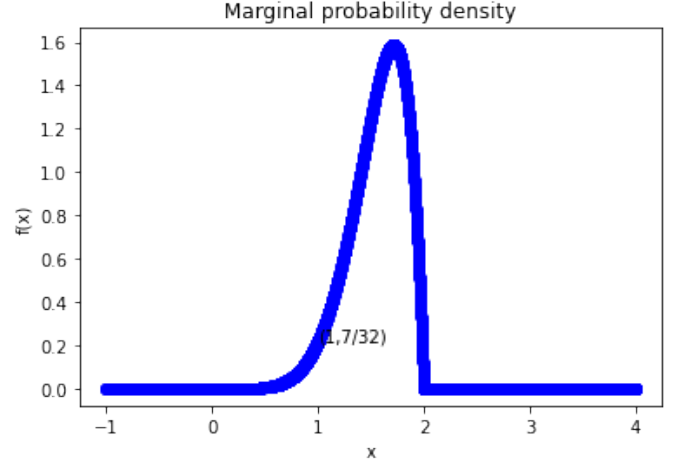


Fig. 1: PDF for order statistics at k=7

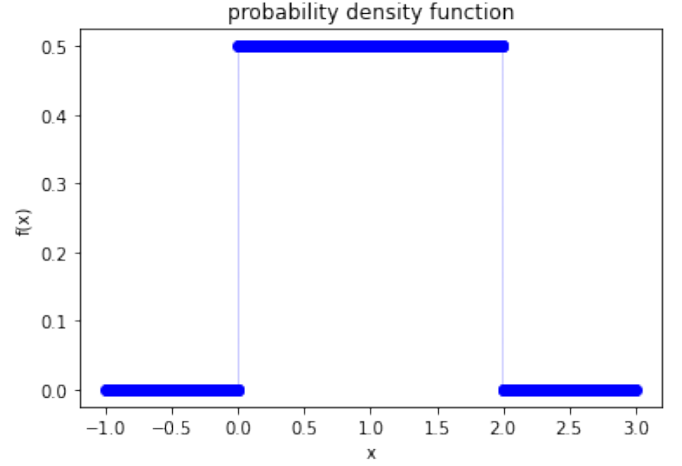


Fig. 2: PDF for uniform distribution

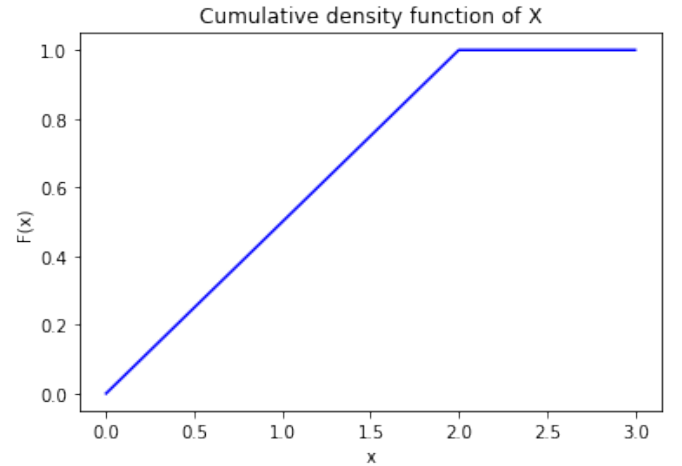


Fig. 3: CDF for uniform distribution