Lecture 4. MA4020 - Linear Algebra

August 31, 2021.

Determinants.

$$\begin{bmatrix} & \\ & \end{bmatrix}_{n\times n} \xrightarrow{\text{function}} \text{expression}$$

example.

$$f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$

$$A = [\ \because\ ]_{n\times n} \longmapsto \det(A)$$

$$A \longrightarrow \underset{d_f(A)}{\overset{\parallel}{}}$$

Let A be an $n\times n$ matrix $[a_{ij}]$. Let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and the j^{th} column of A. Then

$$\underset{\underset{d_f(A)}{\parallel}}{\det(A)} = a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots \pm a_{n1} \det A_{n1}.$$

[Expansion by minors along first column]

Note that the definition above is recursive in nature.

Moreover one may define $\det(A)$ using expansion by minors on other rows and on columns.

Denote "d_f" for det.

(1). Observe that $d_f(I_n) = 1$.

(2). The function d_f is linear in the rows of the matrix.

Let R_i denote the row vector, i^{th} row of the matrix.

Then

$$A = \begin{bmatrix} - R_1 - \\ \vdots \\ - R_n - \end{bmatrix} \qquad \text{(in terms of row vector)}$$

d_f is linear means

$$d_f \begin{bmatrix} \vdots \\ - R+S - \\ \vdots \end{bmatrix}$$

(Here R and S are i^{th} row vectors of A, for some i.)

$$= d_f \begin{bmatrix} \vdots \\ - R - \\ \vdots \end{bmatrix} + d_f \begin{bmatrix} \vdots \\ - S - \\ \vdots \end{bmatrix},$$

and

$$d_f \begin{bmatrix} \vdots \\ - cR - \\ \vdots \end{bmatrix} = c \, d_f \begin{bmatrix} \vdots \\ - R - \\ \vdots \end{bmatrix}.$$

d_f linear function

③. If two adjacent rows of a matrix A are equal, then $d_f(A) = 0$.

Any $n \times n$ matrix.

Proof.

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ d & e & f \end{bmatrix}$$

$$\det(A) = a \cdot \cancel{\det\begin{pmatrix} b & c \\ e & f \end{pmatrix}} - a \cdot \cancel{\det\begin{pmatrix} b & c \\ e & f \end{pmatrix}} + d \cdot \det\begin{pmatrix} b & c \\ b & c \end{pmatrix}$$

$$bc - bc = 0$$

Using Induction

We prove this by induction on n.

Assume that rows j and $j+1$ are equal.

Then A_{i1} (matrix obtained as minor by removing 1^{st} column and i^{th} row)

"Proof by induction on n".

Let $n = 2$. $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix}$

Two adjacent rows of A are equal.

$$d_f(A) = a_{11} \cdot \det[a_{12}] - a_{11} \cdot \det[a_{12}]$$

$$= a_{11} \cdot a_{12} - a_{11} \cdot a_{12}$$

$$= 0$$

Let us try for $n = 3$.

Assume that $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{equal}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\}$$

$$d_f(A) = a_{11} \cdot d_f \begin{bmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{bmatrix} - \cancel{a_{21} \cdot d_f \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}}$$

$$+ \cancel{a_{21} \cdot d_f \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}}$$

$$= 0$$

In other words,

$$d_f(A) = a_{11} \cdot d_f(A_{11}) - a_{21} \cdot d_f(A_{21}) + a_{21} \cdot d_f(A_{31})$$

"formal Proof" by induction.

Step 1. (Base case). Let $n = 2$, then $d_f(A) = 0$.

Step 2. (Induction hypothesis).

Assume that for all square matrices of size less than or equal to K, whenever two adjacent rows are equal, then $d_f(A) = 0$

$A = [a_{ij}]_{(K+1) \times (K+1)}$, claim. $d_f(A) = 0$

Suppose that rows j and $j+1$ are equal.

Consider the matrices A_{i1} ; $i = 1, 2, \dots, K+1$.

Note that A_{i1} will have two equal rows except when $i = j$ or $i = j+1$.

by induction hypothesis

Thus $\boxed{d_f(A_{i1}) = 0}$, whenever $i \neq j$, $i \neq j+1$.

Also observe that for $i = j$ and $i = j+1$, we have $A_{j1} = A_{j+1,1}$ and $a_{j1} = a_{j+1,1}$.

Thus

same

$$d_f(A) = \underbrace{\cdots\cdots}\ \pm a_{j1}\, d_f(A_{j1}) \mp \cancel{a}\, a_{j+1,1}\, d_f(A_{j+1,1}) \pm \cdots .$$

same

$$= \pm a_{j1}\, d_f(A_{j1}) \mp a_{j1}\, d_f(A_{j1})$$

$$= 0$$

Hence if two adjacent rows of any square matrix A are equal, then $d_f(A) = 0$.

④ If a multiple of one row is added to an adjacent row, the determinant is unchanged.

$$d_f \begin{bmatrix} \vdots \\ - R - \\ - S + cR - \\ \vdots \end{bmatrix} = d_f \begin{bmatrix} \vdots \\ - R - \\ - S - \\ \vdots \end{bmatrix} + c\, \underbrace{d_f \begin{bmatrix} - R - \\ - R - \end{bmatrix}}_{= 0}$$

$$= d_f \begin{bmatrix} \vdots \\ - R - \\ - S - \\ \vdots \end{bmatrix}$$

⑤. If two adjacent rows are interchanged, then determinant is multiplied by -1.

$$d_f \begin{bmatrix} \vdots \\ - R - \\ - S - \\ \vdots \end{bmatrix} = d_f \begin{bmatrix} \vdots \\ - R - \\ - (S - R) - \\ \vdots \end{bmatrix}$$

$$= d_f \begin{array}{l} \\ i^{th}\text{ row} \\ i{+}1^{th}\text{ row} \\ \\ \end{array} \begin{bmatrix} & \vdots & \\ — & R + (S-R) & — \\ — & (S-R) & — \\ & \vdots & \end{bmatrix}$$

$$= d_f \begin{bmatrix} & \vdots & \\ — & S & — \\ — & (S-R) & — \\ & \vdots & \end{bmatrix}$$

$$= d_f \begin{bmatrix} & \vdots & \\ — & S & — \\ — & (-R) & — \\ & \vdots & \end{bmatrix}$$

$$= - d_f \begin{bmatrix} & \vdots & \\ — & S & — \\ — & R & — \\ & \vdots & \end{bmatrix}$$

6. If two rows of a matrix A are equal, then $d_f(A) = 0$.

$$A \xrightarrow[\text{rows finitely many times.}]{\text{interchanging}} A' \quad (\text{adjacent rows are equal})$$

$$d_f(A) = \pm\, d_f(A') = 0$$

Conclusions.

1. If a multiple of one row is added to another row, the determinant is unchanged.

$$d_f\left(E_{ij}(\lambda) \cdot A \right) = d_f(A)$$

2. If two rows are interchanged, the determinant is multiplied by -1.

$$d_f\left(P_{ij} \cdot A \right) = -\,d_f(A)$$

3. By linearity property of d_f, we have

$$d_f\left(E_i(\lambda) \cdot A \right) = \lambda \cdot d_f(A)$$

Special cases: $A = I$

(a) $d_f\left(E_{ij}(\lambda) \right) = 1$

(b) $d_f\left(P_{ij} \right) = -1$

(c) $d_f\left(E_i(\lambda) \right) = \lambda \qquad (\lambda \neq 0)$

(4). Let E be an elementary matrix and let A be arbitrary square matrix. Then

$$\boxed{d_f(E \cdot A) = d_f(E) \cdot d_f(A).}$$

Question. How to compute determinant $d_f(A)$?

Recall that A can be reduced by elementary row operations to a matrix A' which is either the identity matrix I or else its bottom row is zero.

square matrix

$$A \sim E_1 A \sim \cdots \quad E_K \cdots E_1 A = A'$$

$$A' = E_K E_{K-1} \cdots E_2 \cdot E_1 \cdot A$$

$$d_f(A') = d_f\big(\underbrace{E_K \cdots E_2 \cdot E_1 A}_{d_f(E_1)\, d(A)}\big)$$

|| Inductively

$$= d_f(E_K) \cdots d_f(E_1) \cdot d_f(A)$$

Thus

$$\underset{\det(A)}{d_f(A)} = \frac{1}{\underset{\neq 0}{d_f(E_K)} \cdots \underset{\neq 0}{d_f(E_1)}} \cdot d_f(A').$$

I

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Theorem. (Axiomatic Characterization of the Determinant):

The determinant function (given by expansion by minors on the first column,

$$\det A = a_{11} \det A_{11} - a_{21} \det A_{21} + , \cdots , \pm a_{n1} \det A_{n1})$$

is the only one satisfying rules $*_1$, $*_2$, and $*_3$.

$d_f := \det$

$*_1$: $\det(I) = 1$

$*_2$: $\det(A)$ is linear in the rows of the matrix

$*_3$ If two adjacent rows of A are equal, then $\det(A) = 0$.

Proof.

Recall,

$$d_f(A') = d_f(E_k) \cdots d_f(E_1) \cdot d_f(A)$$

Using $*_1$, $*_2$ and $*_3$, we have determined $d_f(E_k), \ldots, d_f(E_1)$. & thus $d_f(A')$.

$\square$

Notation. $d_f(A) := \det(A)$

Corollary. A square matrix A is invertible if and only if $\det A \neq 0$.

$$\underbrace{E_K \cdots E_1}A = A' \quad I \text{ or } \begin{bmatrix} & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\neq 0}{\det(E_K)} \cdots \underset{\neq 0}{\det(E_K)} \cdot \underset{\neq 0}{\det(A)} = \underset{\neq 0}{\det(A')} \quad \Leftrightarrow$$

Theorem. Let A, B be any two $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B).$$

Proof. Recall, $\boxed{\det(E \cdot B) = \det(E)\det(B)}$ where E is elementary matrix.

We divide the proof in two cases:

Case 1. Assume that A is invertible.

then $A = E_K \cdots E_1 \cdot I$ (product of elementary matrices)

Thus
$$\boxed{\begin{aligned} \det(A) &= \det(E_K \cdots E_1) \\ &= \det(E_K) \cdots \det(E_1) \end{aligned}}$$

Now,

$$\det(AB) = \det(\underbrace{E_K \cdots E_1}_{A} \cdot B)$$

$$= \underbrace{\det(E_K) \cdots \det(E_1)}_{\det(A)} \cdot (\det B)$$

$$= \det(A) \cdot \det B$$

Case 2. Assume that A is not invertible, then $\det(A) = 0$

Enough to prove that $\det(AB) = 0$.

$$A \rightsquigarrow E_1 A \rightsquigarrow \cdots \rightsquigarrow E_K E_{K-1} \cdots E_1 A = A'$$

$$\underset{0}{\underset{\parallel}{\det(A'B)}} = \det(\underbrace{E_K \cdots E_1}_{} \cdot AB)$$

$$= \underset{\neq 0}{\det(E_K)} \cdots \underset{\neq 0}{\det(E_1)} \cdot \underset{0}{\underset{\parallel}{(\det(AB))}}.$$

Since $\det(E_i) \neq 0$ for any i

$$\Rightarrow \det(AB) = 0.$$

Corollary. If A is invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

$A \cdot A^{-1} = I = A^{-1}A.$

Discussion on Problems, Please see the recording.