Fundamental Theorem of Calculus

Fundamental Theorem of calculus (FTC).

Suppose $f:[a,b]\to\mathbb{R}$ be an integrable function.

Define $F:[a,b]\to\mathbb{R}$ by $F(x)=\boxed{\int_a^x f(t)\,dt}$ where $x\in[a,b]$

The function F is continuous.

In fact, F satisfies "a Lipschitz condition":

there exists an $\alpha\in\mathbb{R}$ $(\alpha\geq 0)$ such that

for $c\in[a,b]$ we have $|F(x)-F(c)|\leq\alpha|x-c|$

Part ① If f is continuous at $c\in[a,b]$, then F is differentiable at c. Also, $F'(c)=f(c)$.

Part ② If f is differentiable and f' is integrable in $[a,b]$, then

$$\int_a^b f'(t)\,dt=f(b)-f(a).$$

Proposition:

(Domain additivity of Riemann integrals)

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $c\in[a,b]$.

We have, f is integrable on $[a,b]$

$\iff$ f is integrable on $[a,c]$ and $[c,b]$.

In such a case, we have

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Examples

① $f:[a,b]\to\mathbb{R}$

$f(x)=[x]$ greatest integer function

$f:[-1,1]\longrightarrow\mathbb{R}.$

$x\in[-1,0)$ $\quad x=-\frac{1}{2}$

$[x]=-1$

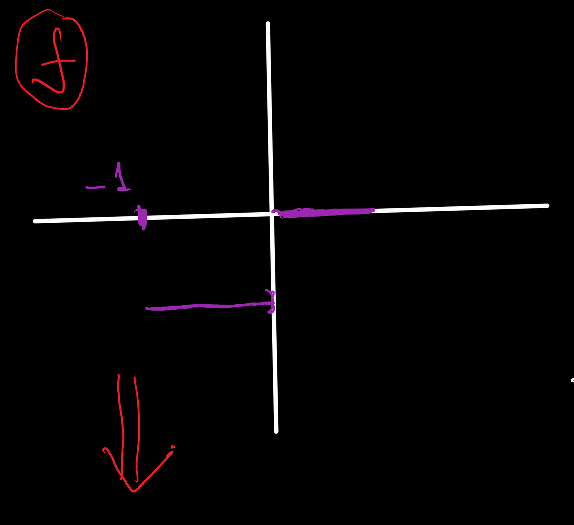
$x\in[0,1]$

f is not continuous

Is f integrable?

Yes (monotone)

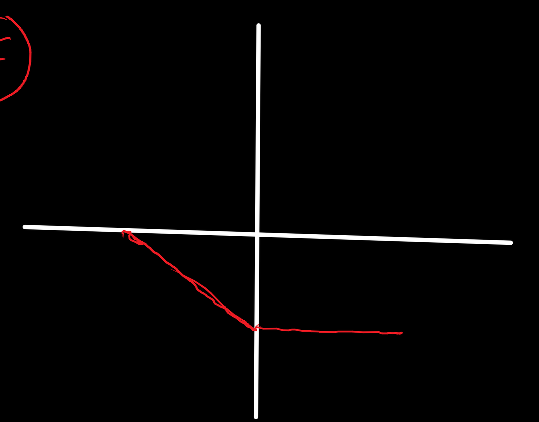
f



By FTC, the function $F(x)=\int_{-1}^{x}f(t)\,dt$ is continuous

$$F(x)=\begin{cases}-1-x & \text{if } x\in[-1,0]\\ -1 & \text{if } x\in(0,1].\end{cases}$$

F



Example ②

You should not use indefinite integrals.

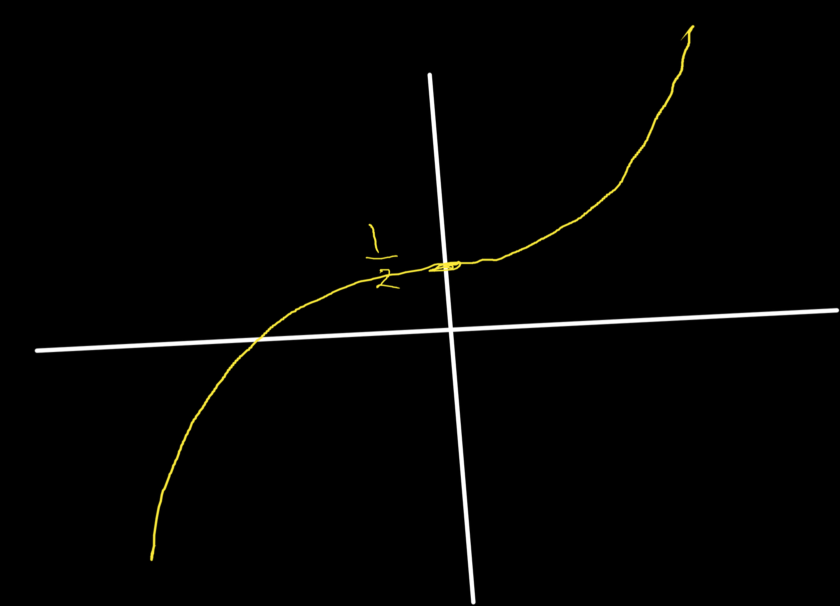
$f:[-1,1]\to\mathbb{R}.$

$f(x)=|x|.$ continuous $\Rightarrow$ integrable

Exercises

$$F(x)=\int_{-1}^{x}f(t)\,dt=\begin{cases}\frac{(1-x^2)}{2} & \text{if } x\in[-1,0]\\ \frac{(1+x^2)}{2} & \text{if } x\in(0,1]\end{cases}$$

$F(x)$ is differentiable.



Example ③

$g:[0,1]\to\mathbb{R}$ $\qquad g(x)=\int_0^x\frac{dt}{t^6+1}$

ⓐ Evaluate: $g'\left(\frac{1}{2}\right)$

ⓑ Evaluate: $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\frac{dt}{t^6+1}.$

ⓐ $f(t)=\frac{1}{t^6+1}$ $\qquad [0,1].$ This function is integrable.

$$g(x)=\int_0^x\frac{dt}{t^6+1}\xRightarrow{\text{FTC 1.}}g'\left(\tfrac{1}{2}\right)=f\left(\tfrac{1}{2}\right)=\frac{1}{\left(\frac{1}{2}\right)^6+1}$$

ⓑ
$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\frac{dt}{t^6+1}=\lim_{h\to 0}\frac{1}{h}\left(\int_0^{x+h}\frac{dt}{t^6+1}-\int_0^x\frac{dt}{t^6+1}\right)\leftarrow \text{domain additivity of integrable functions}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(g(x+h)-g(x)\right)$$

$$=g'(x)$$

by FTC ① $=f(x)=\frac{1}{x^6+1}$