Modern Physics Assignment 1

(Solutions)

January 11, 2021

Question 1.

Energy of the electron in H atom

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r}$$

Now, uncertainty principle gives

$$\Delta x \Delta p = \hbar$$
$$\implies \Delta p = \frac{\hbar}{\Delta x} = \frac{\hbar}{r}$$

Where we have defined the uncertainty in position as the radius. We can replace Δp with p for this purpose. So we get the energy equation as:

$$E = \frac{\hbar^2}{2mr} - \frac{e^2}{4\pi\epsilon_0 r}$$

For ground state, we minimize the energy and solve for r.

$$\frac{dE}{dr} = 0$$
$$\implies r = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$
$$\implies r = 0.53\text{Å}$$

Question 2.

Given, rest mass energy of the electron, $E_0 = m_e c^2 = 0.5$ MeV, and the wavelength range is $\lambda = 10^{-15}$ m. So, from de Broglie hypothesis, we have

$$pc = \frac{hc}{\lambda} \approx 124.08\text{MeV} >> E_0 \tag{1}$$

So, we need a relativistic calculation for this. The total energy required is then given by

$$E = \sqrt{p^2c^2 + E_0^2} = 124.081\text{MeV} \tag{2}$$

Question 3.

The wavefunction of a free particle,

$$\psi_k = e^{i(kx-\omega t)}$$

The energy expectation value is given by,

$$\langle E\rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2})\psi(x,t)dx$$

$$\implies \langle E\rangle = \int_{-\infty}^{+\infty} e^{-i(kx-\omega t)}(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2})e^{i(kx-\omega t)}dx$$

$$\implies \langle E\rangle = \frac{< p^2 >}{2m}$$

Question 4.

(a) The wavefunction is

$$\psi(x) = \frac{1}{(\pi\sigma_0^2)^{1/4}}\exp(\frac{-x^2}{2\sigma_0^2})\exp(\frac{ip_0x}{\hbar})$$

So, probability density is given by:

$$\begin{aligned}\psi^*\psi &= |\psi(x)|^2 \\ &= \frac{1}{\sigma_0\sqrt{\pi}}\exp(\frac{-x^2}{\sigma_0^2})\end{aligned}$$

The expectation value of x is

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x|\psi(x)|^2 dx \\ &= 0\end{aligned}$$

Since the integrand above was an odd function. Again, the expectation value of momentum is

$$\begin{aligned}\langle p\rangle &= \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar\frac{\partial}{\partial x})\psi(x) \\ &= \frac{-i\hbar}{\sigma_0\sqrt{\pi}}\int_{-\infty}^{\infty}(\frac{-x}{\sigma_0^2}+\frac{ip_0}{\hbar})\exp(\frac{-x^2}{\sigma_0^2})dx \\ &= 0 - \frac{-i\hbar}{\sigma_0\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{ip_0}{\hbar}\exp(\frac{-x^2}{\sigma_0^2})dx\end{aligned}$$

where the first part of the integrand is again an odd function, so it gives us zero. Using the standard integral for the form $\exp(-ax^2+bx)$ we have:

$$\begin{aligned}\langle p\rangle &= \frac{\hbar p_0}{\hbar\sigma_0\sqrt{\pi}}\sqrt{\frac{\pi}{1/\sigma_0^2}} \\ &= p_0\end{aligned}$$

(b) The momentum space wavefunction is given by

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{(\pi\sigma_0^2)^{1/4}}\exp(\frac{-x^2}{2\sigma_0^2})\exp(\frac{i(p_0-p)x}{\hbar})dx$$

The amplitude of $\phi(p)$ is when $p = p_0$, so

$$\begin{aligned}
\phi(p_0) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{1}{(\pi\sigma_0^2)^4} \exp(\frac{-x^2}{2\sigma_0^2}) dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(\pi\sigma_0^2)^{1/4}} \sqrt{2\pi\sigma_0^2} \\
&= (\frac{\sigma_0^2}{\pi\hbar^2})^{1/4}
\end{aligned}$$

Question 5.

Check uploads for the `Mathematica` notebook.

Question 6.

$\lambda_0 = 6800\text{Å}$
The work function for Sodium is given by,

$$W_0 = \frac{hc}{\lambda_0} = \left[\left(\frac{6.625 \times 10^{-34} \times 3 \times 10^8}{6800 \times 10^{-10}} \right) \frac{1}{1.6 \times 10^{-19}} \right] eV \tag{3}$$

$$\implies W_0 = 1.8eV \tag{4}$$

Question 7.

$\lambda_0 = 3000$ Å
$\lambda = 1200$ Å

$$K.E._{max} = \frac{hc}{\lambda} - \frac{hc}{\lambda_0} \tag{5}$$

$$\implies K.E._{max} = (6.625 \times 10^{-34} \times 3 \times 10^8) \left[\frac{1}{1200 \times 10^{-10}} - \frac{1}{3000 \times 10^{-10}} \right] J = 6.2eV \tag{6}$$