

# Assignment 4

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<https://github.com/Suraj11050/Assignments-AI1103/tree/main/Assignment%204/Python%20codes>

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1 GATE 2021 (ST), Q.17 (STATISTICS SECTION)

If the marginal probability density function of the  $k^{th}$  order statistic of a random sample of size 8 from a uniform distribution on  $[0, 2]$  is

$$f(x) = \begin{cases} \frac{7}{32} x^6 (2 - x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

then  $k$  equals \_\_\_\_\_

## 2 SOLUTION

Let  $X \in [0, 2]$  be a random variable of uniform order statistic distribution of sample size 8 then

$$\int_0^2 \Pr(x) dx = 1 \quad (2.0.1)$$

$$\Pr(x) = \frac{1}{2} \quad (\because \text{Uniform order}) \quad (2.0.2)$$

The PDF for  $X$  is

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.3)$$

The CDF for  $X$  is

$$F(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases} \quad (2.0.4)$$

*Definition 1.*  $k^{th}$  order statistic : If the sample is arranged in an ascending order, the  $k^{th}$  order statistic is the  $k^{th}$  element from the left.  $X_{(k)}$  is called  $k^{th}$  vector of order statistic For a sample  $\{X_1, X_2, \dots, X_n\}$  of size  $n$ :

$$X_{(1)} = \min \{X_1, X_2, \dots, X_n\}$$

$$X_{(n)} = \max \{X_1, X_2, \dots, X_n\}$$

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(k)} \leq \dots \leq X_{(n)}$$

Let  $X_{(k)}$  be vector of order statistic of  $(X_1, X_2, \dots, X_n)$

*Lemma 2.1.* Marginal probability density (PDF) for a  $k_{th}$  order statistic of a random sample of size  $n$  given CDF=  $F(x)$  and PDF=  $f(x)$  is given by

$$f_{(k,n)}(x) = n^{n-1} C_{k-1} (1 - F(x))^{n-k} F(x)^{k-1} f(x)$$

*Proof.* The CDF of the  $k^{th}$  order statistic from a sample of size  $n$  is:

$$F_{(k,n)}(x) = \Pr(X_{(k)} \leq x)$$

$$= \sum_{j=k}^n {}^n C_j (1 - F(x))^{n-j} F(x)^j \quad (2.0.5)$$

Deriving PDF of  $k^{th}$  order statistic from a sample of size  $n$ :

$$\frac{d}{dx} F_{(k,n)}(x) = \frac{d}{dx} \left( \sum_{j=k}^n {}^n C_j (1 - F(x))^{n-j} F(x)^j \right)$$

$$f_{(k,n)}(x) = \sum_{j=k}^n {}^n C_j (j) (1 - F(x))^{n-j} F(x)^{j-1} f(x) - \sum_{j=k}^n {}^n C_j (n - j) (1 - F(x))^{n-j-1} F(x)^j f(x)$$

$$S_1 = \sum_{j=k}^n \frac{n!}{(n-j)!(j-1)!} (1-F(x))^{n-j} F(x)^{j-1} f(x) \quad (2.0.6)$$

$$S_2 = \sum_{j=k}^n \frac{n!}{(n-j-1)!j!} (1-F(x))^{n-j-1} F(x)^j f(x) \quad (2.0.7)$$

let  $i = j + 1$  change the limits for the summation in equation (2.0.7)

$$S_2 = \sum_{i=k+1}^n \frac{n!}{(n-i)!(i-1)!} (1-F(x))^{n-i} F(x)^{i-1} f(x)$$

$$S_1 - S_2 = \frac{n!}{(n-k)!(k-1)!} (1-F(x))^{n-k} F(x)^{k-1} f(x) \quad (2.0.8)$$

$$f_{(k,n)}(x) = n^{n-1} C_{k-1} (1-F(x))^{n-k} F(x)^{k-1} f(x) \quad (2.0.9)$$

□

Using Lemma (2.1) given  $n = 8$  substituting  $n$ , equation (2.0.3) and equation (2.0.4) in above equation (2.0.9) we get:

$$f_{(k,n)}(x) = 8^7 C_{k-1} \left(1 - \frac{x}{2}\right)^{8-k} \left(\frac{x}{2}\right)^{k-1} \frac{1}{2}$$

$$f_{(k,n)}(x) = \frac{1}{32} {}^7C_{k-1} (2-x)^{8-k} x^{k-1} \quad (2.0.10)$$

Comparing the marginal probability density function obtained in equation (2.0.10) with the equation given in question

$$\frac{1}{32} {}^7C_{k-1} (2-x)^{8-k} x^{k-1} = \frac{7}{32} (2-x) x^6$$

$$\therefore k = 7 \quad (2.0.11)$$

Hence the marginal probability density in the given problem is 7<sup>th</sup> order uniform statistic and **the value of k is 7**

