Lecture 13 MA4020 (LINEAR ALGEBRA)

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Revision:

Theorem. Let $T : V \longrightarrow W$ be a linear transformation and assume that V is finite-dimensional. Then

$$\boxed{\dim V = \dim(\ker T) + \dim(\operatorname{im} T)}$$

– Dimension Formula

Notations.

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

$$\begin{array}{ccc} \| & & \| \\ \dim(\operatorname{null} T) & + & \dim(\operatorname{range} T) \\ \| & & \| \\ \text{nullity of } T & & \text{rank of } T \end{array}$$

$$\boxed{\dim V = \text{rank} + \text{nullity}}$$

Rank-nullity formula

Discussion

(I.) Is it possible to define a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which is surjective (onto) ?

$$\dim(\mathbb{R}^2) = \dim(\ker T) + \underset{\substack{\| \text{ (If onto)} \\ \dim(\mathbb{R}^3)}}{\dim(\operatorname{im} T)}$$

$$2 = (\geq 0) + 3$$

(Not Possible)

(II.) Is it possible to define a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which is injective (one-one) ?

$$\dim(\mathbb{R}^3) = \dim(\overset{\substack{\{0\} \text{ (If one-one)} \\ \|}}{\ker T}) + \dim(\operatorname{im} T)$$

$$3 = 0 + (< 3)$$

(Not possible)

Corollary. If V and W are finite-dimensional vector spaces such that $\dim V < \dim W$, then

$$T : V \longrightarrow W$$ cannot be surjective (onto).

Proof.

Using dimension formula,

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

$\parallel$ (If onto)

$$\dim V \overset{\uparrow}{=} \dim(\ker T) + \dim W$$

Equality is not possible.

Corollary. If V and W are finite-dimensional vector spaces such that $\dim V > \dim W$, then

$$T : V \longrightarrow W$$ cannot be injective (one-one).

Case study.

Define
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$X \longmapsto T_A(X) := A \cdot X$$

where A is some fixed matrix of size $m \times n$,

$$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m \times n}$$

$$\ker T = \{ X \in \mathbb{R}^n \text{ s.t. } \underbrace{T_A(X) = 0}_{A \cdot X = 0} \}$$

$$\operatorname{im} T = \{ Y \in \mathbb{R}^m \text{ s.t. } Y = T(X) \text{ for some } X \in \mathbb{R}^n \}$$

Definition.

Rank of a matrix $A \overset{\text{notation}}{:=} \operatorname{rank}(A) \overset{\text{def.}}{=} \dim(\operatorname{im} T)$

Nullity of a matrix $A \overset{\text{notation}}{:=} \operatorname{nullity}(A) \overset{\text{def}}{=} \dim(\ker T)$

$$\boxed{\operatorname{rank}(A) + \operatorname{nullity}(A) = n = \dim(\mathbb{R}^n).}$$

Assume that $r = \text{rank}(A)$

$$\begin{aligned} r &= \text{rank}(A) \\ &\parallel \\ &\dim(\text{im}\, T) \\ &\parallel \\ &\dim\left(\{B \in \mathbb{R}^m \text{ s.t. } AX = B \text{ has a solution}\}\right) \end{aligned}$$

(i) Now, define a linear transformation

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$X \longmapsto T_A(X) := A \cdot X \quad , \text{ where } \det(A) \neq 0.$$

Note that $\det(A) \neq 0 \implies A$ is invertible.

$$\Downarrow$$

$AX = B$ has a unique solution for every $B \in \mathbb{R}^n$.

$\ker T = \{(0)\}$

$\dim(\ker T) = 0$

$\dim(\text{im}\, T) = n$

(ii). Assume that $\det(A) = 0$

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$X \longmapsto T_A(X) := A \cdot X$$

In general, not all equations $AX = B$ would have solutions. (refer to row-reduced echelon form discussion).

Those equations for which $AX = B$ has a solution, would have more than one solution.

- If X_1 and X_2 are solutions to $AX = 0$,

$$\Rightarrow A(X_1 + X_2) = 0$$

- Let Y be a solution to $AX = B$, (i.e. $A \cdot Y = B$)

then note that $Y + X_1$ and $Y + X_2$ are also a solution to $AX = B$.

$\dim(\ker T)$

$\dim(\operatorname{im} T) < n$

Remark. Every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by left multiplication by some $m \times n$ matrix.

Set.

$\mathcal{B} = (e_1, \ldots, e_n)$ standard ordered basis for $\mathbb{R}^n$

(column vectors)

$\mathcal{B}' = (e_1', \ldots, e_m')$ standard ordered basis for $\mathbb{R}^m$

(column vectors)

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$e_j \longmapsto T(e_j) \in \mathbb{R}^m \quad \text{for all } j = 1, \ldots, n.$$

$$T(e_1) = a_{11} e_1' + a_{21} e_2' + \cdots + a_{m1} e_m'$$

$$T(e_2) = a_{12} e_1' + a_{22} e_2' + \cdots + a_{m2} e_m' \qquad \left(\begin{array}{c} a_{ij} \in \mathbb{R} \\ \text{some scalars} \end{array} \right)$$

$$\vdots$$

$$T(e_n) = a_{1n} e_1' + a_{2n} e_2' + \cdots + a_{mn} e_m'$$

$X \in \mathbb{R}^n$

$$X = e_1 x_1 + \cdots + e_n x_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{column vector}$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto T(X)$$

$$\begin{aligned}
&\parallel \\
&T(e_1 x_1 + e_2 x_2 + \cdots + e_n x_n) \\
&\parallel \\
&T(e_1)\, x_1 + T(e_2)\cdot x_2 + \cdots + T(e_n)\cdot x_n \\
&\parallel \\
&\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\end{aligned}$$

(Block-multiplication)

$$T(X) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot X$$

$$\uparrow$$

$T_{\mathcal{B}, \mathcal{B}'}$ Matrix of linear transformation w.r.t. basis $\mathcal{B}$ and $\mathcal{B}'$.

Details.

$$T(x) = T(e_1)x_1 + T(e_2)x_2 + \cdots + T(e_n)x_n$$

$$= (a_{11}e_1' + a_{21}e_2' + \cdots + a_{m1}e_m')x_1$$

$$+ (a_{12}e_1' + a_{22}e_2' + \cdots + a_{m2}e_m')x_2$$

$$+ \cdots$$

$$+ (a_{1n}e_1' + a_{2n}e_2' + \cdots + a_{mn}e_m')x_n$$

$$= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)e_1'$$

$$+ (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)e_2'$$

$$+ \cdots$$

$$+ (a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)e_m'$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A \cdot X$$

Discussion.

$T : V \longrightarrow W$ linear transformation

$\mathcal{B}_V = (v_1, \dots, v_n)$ basis for V

$\mathcal{B}_W = (w_1, w_2, \dots, w_m)$ basis for W

$$v_1 \longmapsto T(v_1) \in W$$
$$\vdots$$
$$v_n \longmapsto T(v_n) \in W$$

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$\vdots$$
$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

$$T_{\mathcal{B}_V, \mathcal{B}_W} = \begin{array}{c} \begin{array}{cccc} T(v_1)\downarrow & T(v_2)\downarrow & \cdots & T(v_n)\downarrow \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{l} \leftarrow w_1 \\ \leftarrow w_2 \\ \\ \leftarrow w_m \end{array}$$

$m \times n$ matrix

Matrix of linear transformation w.r.t. bases $\mathcal{B}_V$ and $\mathcal{B}_W$.

Examples.

1. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (x+y, x-y)$

$\mathcal{B} = (\underset{(1,0)}{e_1}, \underset{(0,1)}{e_2})$ standard basis for $\mathbb{R}^2$ (ordered basis)

$$T(e_1) = T(1,0) = (1,1) = 1\cdot e_1 + 1\cdot e_2$$

$$T(e_2) = T(0,1) = (1,-1) = 1\cdot e_1 + (-1)\cdot e_2$$

$$T_{\mathcal{B},\mathcal{B}} = \begin{array}{cc} T(e_1) & T(e_2) \\ \left[\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right. & \left.\right] \end{array} \begin{array}{c} e_1 \\ e_2 \end{array}$$

Now we will change the basis:

$$\mathcal{B}_d = (\underset{v_1}{e_1+e_2}, \underset{v_2}{e_1-e_2})$$

$v_1 = e_1 + e_2 = (1,1)$

$v_2 = e_1 - e_2 = (1,-1)$

$$\mathcal{B}_c = (\underset{\substack{w_1 \\ (2,0)}}{2e_1}, \underset{\substack{w_2 \\ (1,3)}}{e_1 + 3e_2})$$

$$T_{\mathcal{B}_d,\mathcal{B}_c} = \begin{array}{cc} T(v_1) & T(v_2) \end{array}$$
$$\begin{bmatrix} 1 & -1/3 \\ 0 & 2/3 \end{bmatrix} \begin{array}{c} w_1 \\ w_2 \end{array}$$

$$T(v_1) = T(1,1) = (2,0) = 1\cdot w_1 + 0\cdot w_2$$

$$T(v_2) = T(1,-1) = (0,2) = \left(-\tfrac{1}{3}\right)\cdot w_1 + \tfrac{2}{3}\cdot w_2$$

2.

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x, y, z) \longmapsto (x, y, z)$$

(A) $\mathcal{B}_d = (e_3, e_1, e_2)$ with $e_3 = v_1$, $e_1 = v_2$, $e_2 = v_3$

$\mathcal{B}_c = (e_1+e_2, e_1-e_2, e_3)$ with $e_1+e_2 = w_1$, $e_1-e_2 = w_2$, $e_3 = w_3$

$$T(v_1) = T(0,0,1) = (0,0,1) = 0 \cdot w_1 + 0 \cdot w_2 + 1 \cdot w_3$$

$$T(v_2) = T(1,0,0) = (1,0,0) = \frac{1}{2} w_1 + \frac{1}{2} w_2 + 0 \cdot w_3$$

$$T(v_3) = T(0,1,0) = (0,1,0) = \frac{1}{2} w_1 + \left(-\frac{1}{2}\right) w_2 + 0 \cdot w_3$$

$$T_{\mathcal{B}_d, \mathcal{B}_c} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 1 & 1 & 0 \end{bmatrix}$$

(B) $\mathcal{B}_d = (e_1, e_2, e_3)$ $\mathcal{B}_c = (e_1, e_2, e_3)$

$$T_{\mathcal{B}_d, \mathcal{B}_c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

$$T = \frac{d}{dx} : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$$

$$f \longmapsto f'$$

Set. Basis for $\mathcal{P}_3(\mathbb{R}) := \mathcal{B}_d = (1, x^2, x, x^3)$

Basis for $\mathcal{P}_2(\mathbb{R}) = \mathcal{B}_c = (x^2, 1, x)$

$$T(1) = 0 = 0 \cdot x^2 + 0 \cdot 1 + 0 \cdot x$$

$$T(x^2) = 2x = 0 \cdot x^2 + 0 \cdot 1 + 2 \cdot x$$

$$T(x) = 1 = 0 \cdot x^2 + 1 \cdot 1 + 0 \cdot x$$

$$T(x^3) = 3x^2 = 3 \cdot x^2 + 0 \cdot 1 + 0 \cdot x$$

$$T_{\mathcal{B}_d, \mathcal{B}_c} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Discussion.

V finite dimensional vector space ($\dim V = n$)

$$\begin{tikzcd} & V \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{B}_V \arrow[rr, leftrightarrow] & & \mathcal{B}'_V \end{tikzcd}$$

Basis of V $\rightarrow$ $\mathcal{B}_V$

How are two bases of the same vector space related ?

$\mathcal{B}'_V = \mathcal{B}_V \cdot P^{-1}$ where $P \in GL_n(\mathbb{R})$

W finite dimensional vector space

$$\begin{tikzcd} & W \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{B}_W \arrow[rr, leftrightarrow] & & \mathcal{B}'_W \end{tikzcd}$$

Basis of W $\rightarrow$ $\mathcal{B}_W$

$\mathcal{B}'_W = \mathcal{B}_W \cdot Q^{-1}$, where $Q \in GL_n(\mathbb{R})$

$$V \longrightarrow W$$

$$\mathcal{B}'_V = \mathcal{B}_V \cdot P^{-1} \qquad\qquad \mathcal{B}'_W = \mathcal{B}_W \cdot Q^{-1}$$

$$\begin{tikzcd}
\mathcal{B}_V \arrow[r, dash, dotted] \arrow[d, leftrightarrow] & T_{\mathcal{B}_V, \mathcal{B}_W} \arrow[r, dash, dotted] \arrow[d, leftrightarrow, "\text{connection}"] & \mathcal{B}_W \arrow[d, leftrightarrow] \\
\mathcal{B}'_V \arrow[r, dash, dotted] & T_{\mathcal{B}'_V, \mathcal{B}'_W} \arrow[r, dash, dotted] & \mathcal{B}'_W
\end{tikzcd}$$

$T_{\mathcal{B}_V, \mathcal{B}_W}$: Matrix of linear transformation w.r.t. bases $\mathcal{B}_V$ and $\mathcal{B}_W$

$T_{\mathcal{B}'_V, \mathcal{B}'_W}$: Matrix of linear transformation w.r.t. bases $\mathcal{B}'_V$ and $\mathcal{B}'_W$

$$\begin{tikzcd}
\mathcal{B}_V \arrow[r, dash, dotted] & A \arrow[r, dash, dotted] \arrow[d, leftrightarrow, "\text{connection}"] & \mathcal{B}_W \\
\mathcal{B}'_V \arrow[r, dash, dotted] & A' \arrow[r, dash, dotted] & \mathcal{B}'_W
\end{tikzcd}$$

"We shall discuss this in the next lecture".