Epidemics on networks

SI-model

2 states $\to$ susceptible, infected

fully mixed model = mass-action approx.

= model on complete network

$S(t)$ – # susceptible individuals at time t

$X(t)$ – # infected —//—

β – the probability of meeting betw. S and I

n inds in the population

$\frac{S}{n}$ – probability to meet S

$$\boxed{X \cdot \frac{S}{n} \cdot \beta}$$

$$X + S = n$$

$$\begin{cases} \frac{dX}{dt} = \beta \frac{XS}{n} \\ \frac{dS}{dt} = -\beta \frac{XS}{n} \end{cases}$$

$$s = \frac{S}{n} \qquad x = \frac{X}{n}$$

$$\begin{cases} \frac{dx}{dt} = \beta s x \\ \frac{ds}{dt} = -\beta s x \end{cases}$$

$$s + x = 1$$

$$S = 1 - x$$

$$\frac{dx}{dt} = \beta(1-x)x \quad (\text{logistic grow equation})$$

$$\frac{dx}{(1-x)x} = \beta\, dt$$

$$\int \frac{dx}{x} + \int \frac{dx}{1-x} = \int \beta\, dt$$

$$\ln|x| - \ln|1-x| = \beta t + C \quad \Big| \begin{array}{l} x > 0 \\ x < 1 \end{array}$$

$$\ln \frac{x}{1-x} = \beta t + C$$

$$\frac{x}{1-x} = A e^{\beta t} \qquad \frac{1-x}{x} = A e^{-\beta t}$$

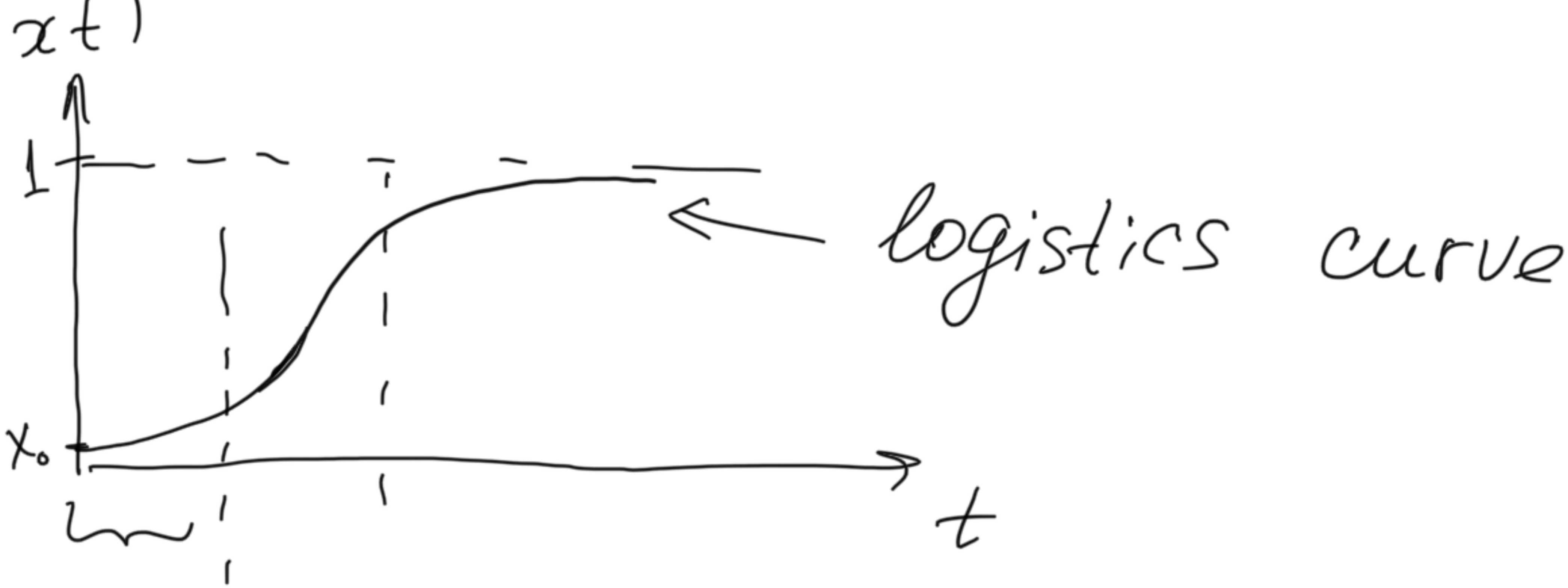
$$\frac{1}{} = 1 + A e^{-\beta t}$$

$$\frac{\ }{x} = 1 + Ae$$

$$x = \frac{1}{1 + Ae^{-\beta t}} = \frac{e^{\beta t}}{e^{\beta t} + A}$$

$$x(0) = x_0 \Rightarrow A = \frac{1}{x_0} - 1$$

$$\boxed{x = \frac{x_0 e^{\beta t}}{1 - x_0 + x_0 e^{\beta t}}}$$



SIR- model

S – — " —

I – — " —

R – recovery (+ immunity)

β – the probability of contact between S and I

γ – the speed of recovery

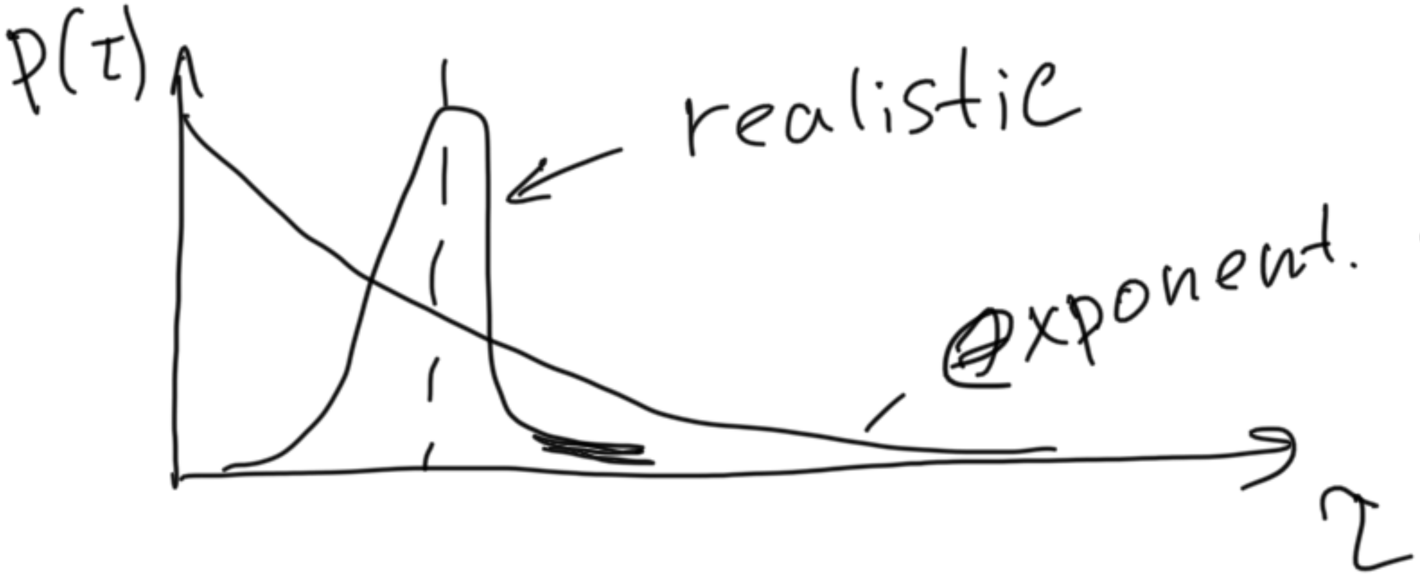
$\gamma d\tau$ – the probability to recover in $d\tau$

$$\lim_{\delta\tau \to 0} (1 - \gamma \delta\tau)^{\frac{\tau}{\delta\tau}} = e^{-\gamma\tau}$$ – the probability to stay infected at τ

$p(\tau)d\tau$ – the probability that ind., who was infected at, will recover at time period $\tau - \tau + d\tau$

$$e^{-\gamma\tau}\gamma\, d\tau = p(\tau)\, d\tau$$

↳ exponential distr.



$$\begin{cases} \dfrac{ds}{dt} = -\beta s x \\[2ex] \dfrac{dx}{dt} = \beta s x - \gamma x \\[2ex] \dfrac{dr}{dt} = \gamma x \end{cases}$$

$s + x + r = 1$

$$\frac{dS}{dt} = -\beta S \frac{1}{\gamma} \frac{dr}{dt}$$

$$\frac{dS}{dt} \cdot \frac{1}{S} = -\frac{\beta}{\gamma} \cdot \frac{dr}{dt}$$

$$\underbrace{\frac{S'}{S}}_{\circ} = -\frac{\beta}{\gamma} r'$$

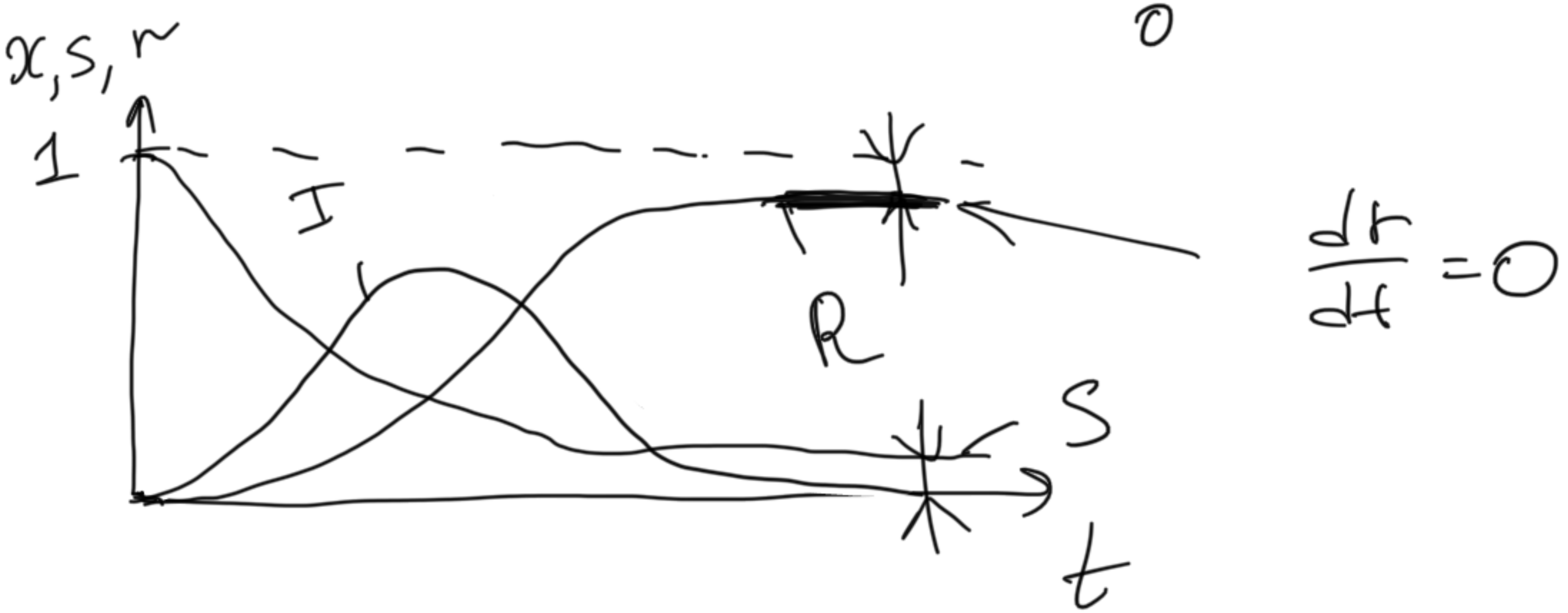
$$(\ln S)' = -\frac{\beta}{\gamma} r' \Rightarrow \ln S = -\frac{\beta}{\gamma} r + C$$

$$r(0) = 0$$

$$S = S_0 e^{-\frac{\beta}{\gamma} r}, \quad S_0 = S(0)$$

$$\underline{x = 1 - S - r} \Rightarrow \boxed{\frac{dr}{dt} = \gamma\left(1 - r - S_0 e^{-\frac{\beta}{\gamma} r}\right)}$$

$$t = \frac{1}{\gamma}\int_0^{r} \frac{du}{1-u-s_0 e^{-\frac{\beta}{\gamma}u}}$$



$$\frac{dr}{dt} = 0 \quad : \quad r(t) = \bar{r}$$

$$1 - \bar{r} - s_0 e^{-\frac{\beta}{\gamma}\bar{r}} = 0,$$

$\bar{r}$ - total fraction of recovered

$$x_0 = \frac{c}{\quad}$$

n $\qquad c \approx 0$

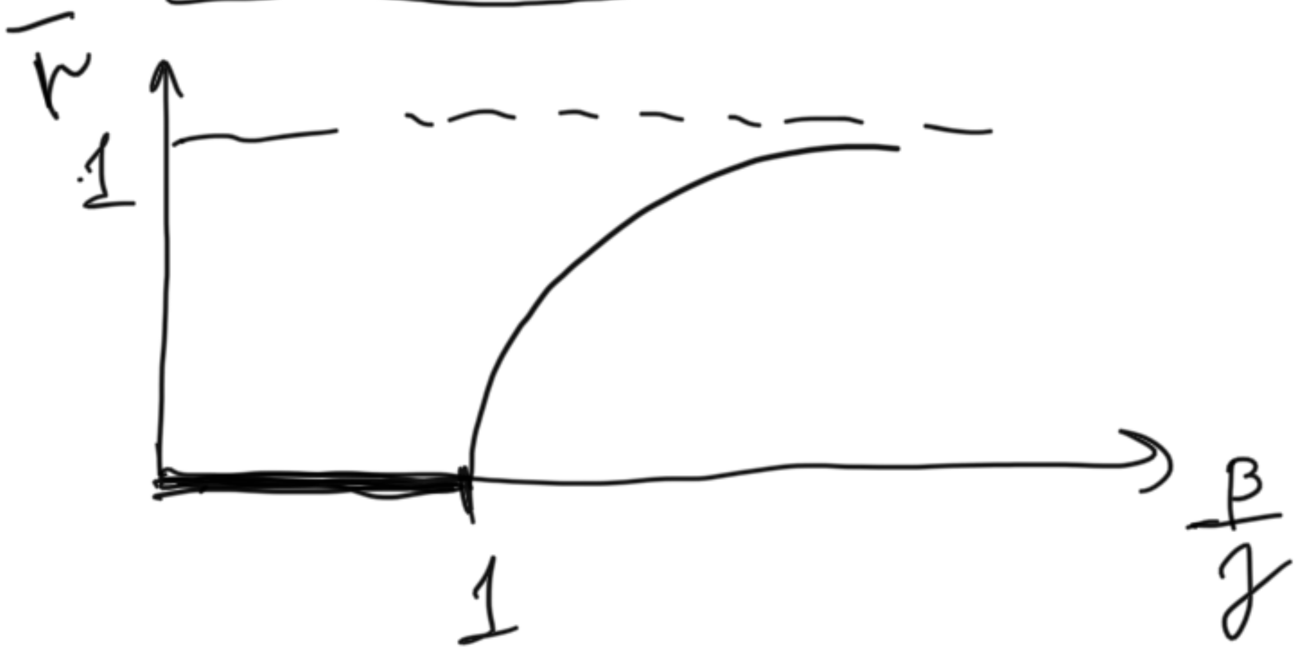
$S_0 = 1 - x_0 = 1 - \frac{c}{n}$

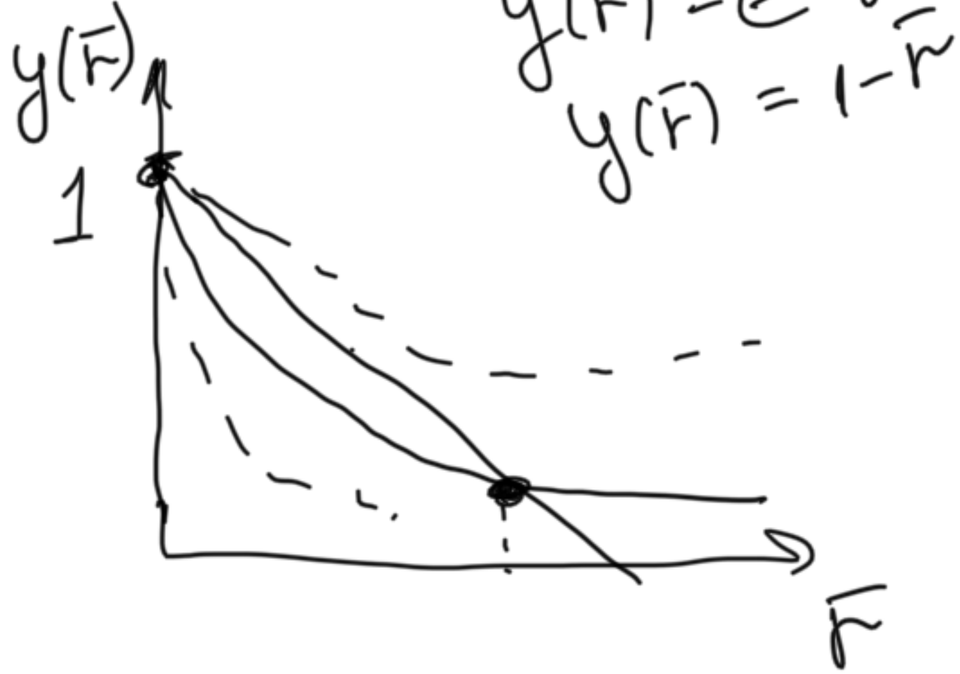
$r_0 = 0$

$n \to \infty, \quad c \approx 0 \Rightarrow \boxed{\begin{array}{l} S_0 = 1 \\ x_0 = 0 \end{array}}$

$\bar{r}$ – "size" of epidemics

$$\boxed{1 - \bar{r} = e^{-\frac{\beta}{\gamma}\bar{r}}}$$



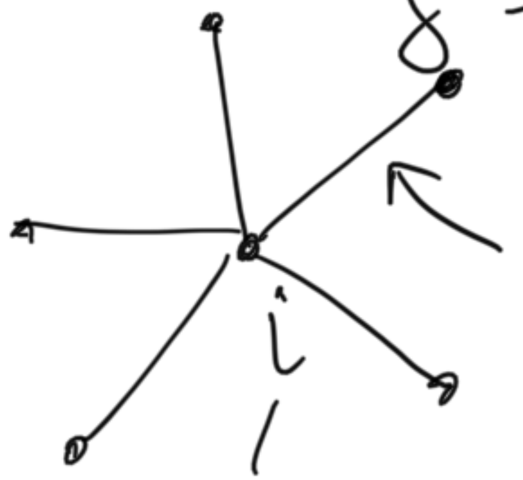


$$\boxed{\frac{\beta}{\gamma} = 1}$$

phase transition

$\frac{\beta}{\gamma} \leq 1$: no epidemics spread

Models on networks



j - infected

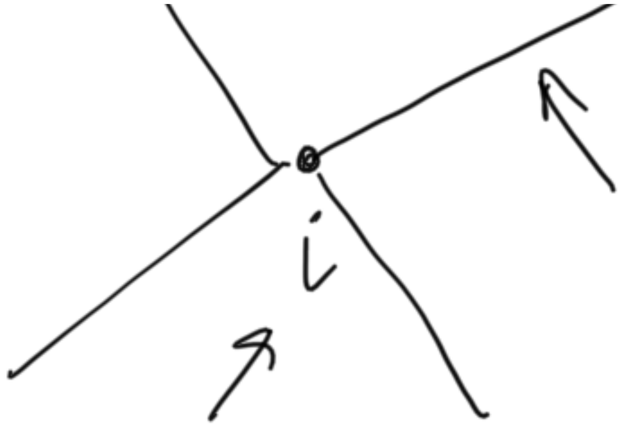
β - probability to become infected

Suscept.

SI- model on network

j

$x_i = 1 - s_i$



$\beta dt \quad (t \to t+dt)$

s_i:

↳ the probability to be suscept.

x_i — — // — to be infected

$s_i + x_i = 1$

$$\begin{cases} \dfrac{dx_i}{dt} = \beta s_i \sum_j A_{ij} x_j = \beta(1-x_i)\sum_j A_{ij} x_j \\ \dfrac{ds_i}{dt} = -\beta s_i \sum_j A_{ij} x_j = -\beta s_i \sum_j A_{ij}(1-s_j) \end{cases}$$

$$x_i(0) = \frac{\quad}{h}, \quad s_i = 1 - \frac{\quad}{h}$$

$$h \to \infty \implies \begin{matrix} s_i \approx 1 \\ x_i \approx 0 \end{matrix}$$

<u>Early</u> <u>limit</u> in time

$$\frac{dx_i}{dt} = \beta \underline{\underline{(1 - x_i)}} \sum A_{ij} x_j \overset{\text{early limit}}{\approx} \beta \sum A_{ij} x_j$$

$$\frac{dx_i}{dt} = \beta \sum A_{ij} x_j$$

$$\frac{d\vec{x}}{dt} = \beta A \vec{x}, \quad \vec{x} = \sum_{r=1}^{n} a_r(t) \cdot \vec{v}_r$$

eig. vector of A

$$\frac{d\vec{x}}{dt} = \sum_{r=1}^{n} \frac{da_r(t)}{dt}\vec{v}_r = \beta A \sum_{r=1}^{n} a_r(t) \underset{\text{eig. val.}}{k_r} \vec{v}_r$$

$$\frac{da_r(t)}{dt} = \beta k_r a_r(t)$$

$$a_r(t) = a_r(0)\, e^{\beta k_r t}$$

$$x(t) = \sum_{r=1}^{n} a_r(0) e^{\beta k_r t} \vec{v}_r$$

k_1 – largest eig. val. of A

$$x(t) \approx \underline{a_1(0)\, e^{\cdots}\, v_1}$$

SIR model on the graph

$$\begin{cases} \dfrac{ds_i}{dt} = -\beta s_i \sum A_{ij} x_j \\[2ex] \dfrac{dx_i}{dt} = \beta s_i \sum A_{ij} x_j - \gamma x_i \\[2ex] \dfrac{dr_i}{dt} = \gamma x_i \end{cases}$$

$s_i + x_i + r_i = 1$

$$x_i(0) = \frac{c}{n} \to 0 \quad (n \to \infty)$$

$$s_i(0) = 1 - \frac{c}{n} \to 1 \quad (n \to \infty)$$

$$r_i(0) = 0$$

<u>Early</u> <u>limit</u>

$$S_i \simeq 1$$

$$\frac{dx_i}{dt} = \beta \sum_j A_{ij} x_j - \gamma x_i = \sum_j x_j \underbrace{(\beta A_{ij} - \gamma \delta_{ij})}_{\beta M_{ij}}$$

$$M_{ij} = A_{ij} - \frac{\gamma}{\beta} \delta_{ij}$$

$$M = A - \frac{\gamma}{\beta} I$$

$$\frac{d\vec{x}}{dt} = \beta M \vec{x}$$

$\vec{v}_k$ - eig vect. of A

$$M\vec{v}_r = \left(A - \frac{\gamma}{\beta} I\right)\vec{v}_r = k_r \vec{v}_r - \frac{\gamma}{\beta}\vec{v}_r =$$

$$= \underline{\left(k_r - \frac{\gamma}{\beta}\right)} \vec{v}_r$$

$$x(t) \approx a_1(0)\, e^{\underbrace{(\beta k_1 - \gamma)}_{} t} \, \vec{v}_1$$

$\beta k_1 - \gamma = 0$: phase transition

$$\frac{\beta}{\gamma} = \frac{1}{k_1}$$