EXPONENTIAL DISTRIBUTION

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, If its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , 0 \le x < \infty \\ 0 & , \text{otherwise} \end{cases}$$

MEAN:

$$\text{Mean} = \mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x(\lambda e^{-\lambda x})\, dx$$

$$= \lambda \int_{0}^{\infty} \underset{u}{x}\, \underset{v}{e^{-\lambda x}}\, dx$$

Bernoulli's Rule

$\int uv = uv_1 - u'v_2 + u''v_3 - \cdots$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$$= \lambda \left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (1)\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty}$$

$$= \lambda \left[-\frac{1}{\lambda}\{0 - 0\} - \frac{1}{\lambda^2}\{\cancel{e^{-\infty}}^{0} - e^0\}\right]$$

$$= \lambda \left[\frac{1}{\lambda^2}\right] = \frac{1}{\lambda}$$

$$\therefore \boxed{\text{Mean} = \mu = \frac{1}{\lambda}}$$

VARIANCE:

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

→

$$= \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^{\infty} \underset{u}{\left(x - \frac{1}{\lambda}\right)^2} \underset{v}{e^{-\lambda x}} \, dx$$

$$= \lambda \left[\left(x - \frac{1}{\lambda}\right)^2 \left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2\left(x - \frac{1}{\lambda}\right)\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + (2)\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) \right]_0^{\infty}$$

$$= \lambda \left[-\frac{1}{\lambda}\left\{ \overset{e^{-\infty}}{0} - \left(-\frac{1}{\lambda}\right)^2 e^0 \right\} - \frac{2}{\lambda^2}\left\{ \overset{e^{-\infty}}{0} - \left(-\frac{1}{\lambda}\right) e^0 \right\} - \frac{2}{\lambda^3}\left\{ e^{-\infty} - e^0 \right\} \right]$$

$$= \lambda \left[+ \frac{1}{\lambda^3} - \cancel{\frac{2}{\lambda^3}} + \cancel{\frac{2}{\lambda^3}} \right] = \cancel{\lambda}\left(\frac{1}{\lambda^{\cancel{3}}}\right)$$

$$\boxed{\sigma^2 = \frac{1}{\lambda^2}}$$

Standard deviation $= SD(X) = \sigma = \sqrt{V(X)} = \frac{1}{\lambda}$

PROBLEMS

1. In a certain town the duration of a shower is exponentially distributed with mean equal to 5 minutes. What is the probability that shower will last for (i) less than 10 minutes ?
(ii) 10 minutes (or) More ?

Sol: Given Mean $= \frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$

Let X be the exponential variate, Then we have

$$f(x) = \frac{1}{5} e^{-\frac{1}{5}x}, \quad x > 0$$

(i) P [less than 10 min] $= P(x < 10)$

$$= \int_0^{10} f(x)\, dx$$

$$= \frac{1}{5} \int_0^{10} e^{-\frac{1}{5}x} dx$$

$$= 0.8647$$

$$\int_0^{10} e^{-1/5 x} dx = \left[\frac{e^{-1/5 x}}{-1/5}\right]_0^{10} = -5\left[e^{-1/5 x}\right]_0^{10} = -5\left[e^{-2} - e^{0}\right]$$

(ii) P [10 Minutes or More] $= P(x \geq 10)$

$P(A) = 1 - P(\bar{A})$

$$= 1 - P(x < 10) \quad \text{(OR)} \quad \int_{10}^{\infty} f(x)\, dx$$

$$= 1 - 0.8647$$

$$= 0.1353$$

[2] The length of a telephone conversation has a exponential distribution with a Mean of 3 min. Find the probability that a call (i) Ends in less than 3 min (ii) takes btw 3 and 5 minutes.

Sol: Given Mean $= \frac{1}{\lambda} = 3 \Rightarrow \lambda = \frac{1}{3}$

Let X be the exponential variate, Then we have

$$f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \quad x > 0$$

(i) P(Less than 3 min) $= P(x<3)$

$$= \int_0^3 f(x)\,dx$$

$$= \frac{1}{3}\int_0^3 e^{-\frac{1}{3}x}\,dx$$

$$= 0.6321.$$

(ii) P (btw 3 and 5 Min) $= P(3<x<5)$

$$= \int_3^5 f(x)\,dx$$

$$= \frac{1}{3}\int_3^5 e^{-\frac{1}{3}x}\,dx$$

$$= 0.1790$$

13 The daily turn over in a medical shop is exponentially distributed with Rs. 6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs. 500 on a randomly choosen day.

Sol: Let X be the exponential variate denoting the turn over per day.

Given Mean $= \frac{1}{\lambda} = 6000 \Rightarrow \lambda = \frac{1}{6000}$.

Let A be the turn over for which net profit is Rs. 500.

Then $\frac{8}{100} \times A = 500$ $[\because 8\%$ profit $]$

$$\Rightarrow \boxed{A = 6250}$$

Thus the probability that the Net profit exceeds Rs. 500 is given by

$$P(\text{Net profit} > 500) = P(X > 6250)$$

$$= 1 - P(X \le 6250)$$

$$= 1 - \int_0^{6250} f(x)\, dx$$

$$= 1 - \int_0^{6250} \frac{1}{6000} e^{-\frac{1}{6000}x} dx \qquad f(x) = \lambda e^{-\lambda x}$$

$$= 0.353$$

4. After the appointment of a new sales manager, the sales in a two-wheeler show room is exponentially distributed with mean equal to 4. If two days are selected at random, what is the probability that

(i) ~~on~~ both days the sales is over 5 units. ?

(ii) The sales is over 5 units on atleast one of the two days.?

Soln: Given Mean $= \frac{1}{\lambda} = 4 \Rightarrow \lambda = \frac{1}{4}$

Let X be a exponential variate, then

$$f(x) = \frac{1}{4} e^{-\frac{1}{4}x}, \quad x > 0$$

(i) $P(\text{over 5 units}) = P(x > 5)$ (OR) $= 1 - P(x \le 5)$

$$= \int_5^\infty f(x)\,dx \qquad = 1 - \int_0^5 f(x)\,dx$$

$$= \int_5^\infty \frac{1}{4} e^{-\frac{1}{4}x}\,dx \qquad = 0.2865$$

$$= e^{-5/4} = 0.2865$$

∴ The probability that the sales is over 5 units on both days. Here $n = 2$, $P = 0.2865$, $q = 1 - p = 1 - 0.2865$

By Binomial distribution, $P(x=2) = {}^2C_2 (0.2865)^2 (1-0.2865)^0 = 0.082$

(ii) The probability that the sales is over 5 units in atleast one of the two days $= P(x=1) + P(x=2)$

$$= {}^2C_1 (0.2865)^1 (1-0.2865)^1 + 0.082$$

$$= 0.4909$$

Assignments :

1 For the exponential variate X with mean value 5. Evaluate (i) $P(0 < x < 1)$ (ii) $P(-\infty < x < 10)$

2 The average turnout in a departmental store is Rs 10,000/= and the net profit is 8%. If the turnout has an exponential distribution, find the probability that the net profit will exceed Rs. 3000/= each on two consecutive days choosen at random

(Ans: $e^{-7.5}$, refer problem 3)

5 At a certain city bus stop, ~~thate~~ three buses arrive per hour, on an average. Assuming that the time between successive arrivals is exponentially distributed, find the probability that the time between the arrival of successive buses is (i) Less than 10 minutes and (ii) at least 30 minutes.

Sol: 3 buses in 1 hr. $\Rightarrow$ 1 bus in 20 min.

$\therefore$ Mean $= \frac{1}{\lambda} = 20 \Rightarrow \lambda = \frac{1}{20}$

(i) P (Less than 10 min) $= P(x < 10) = \int_0^{10} \frac{1}{20} e^{-\frac{1}{20}x} dx = 0.3935$

(ii) P (atleast 30 min) $= P(x \geq 30) = 1 - P(x < 30)$

$= 1 - \int_0^{30} \frac{1}{20} e^{-\frac{1}{20}x} dx$

$= 0.2231$

[6] If the Life time of a certain type of Electric bulbs is distributed as an exponential variate with mean of 1000 hours, what is the probability that a bulb will Last for more than 1500 hours? If two bulbs are selected at random, find the probability that (i) both the bulbs (ii) atleast one bulb will last for more than 1500 hours.

Sol: Given $\frac{1}{\lambda} = 1000 \Rightarrow \lambda = \frac{1}{1000}$

Let X be a exponential variate, then

$$f(x) = \frac{1}{1000} e^{-\frac{1}{1000}x} \quad x > 0$$

(i) P (More than 1500 hrs) $= P(x > 1500)$

$$= 1 - P(x \leq 1500)$$

$$= 1 - \int_0^{1500} \frac{1}{1000} e^{-\frac{1}{1000}x} dx$$

$$= e^{-3/2}.$$

(ii) If two bulbs are selected at random, here $n = 2$

$p = e^{-3/2}$, $q = 1 - p = 1 - e^{-3/2}$

$$\therefore P(x=2) = {}^2C_2 (e^{-3/2})^2 (1 - e^{-3/2})^0 = e^{-3}$$

$$P(\text{atleast one of the bulb}) = P(x=1) + P(x=2)$$

$$= {}^2C_1 (e^{-3/2})^1 (1 - e^{-3/2})^1 + e^{-3}$$

$$= 2 \times e^{-3/2}(1 - e^{-3/2}) + e^{-3} = 0.3965$$