POISSON DISTRIBUTION

A probability distribution which has the following probability function is called poisson distribution

$$p(x) = \frac{e^{-m} m^x}{x!} \quad ; \; x = 0, 1, 2, \ldots \quad \text{and } m > 0.$$

(or) $= \frac{e^{-\lambda} \lambda^x}{x!}$

Here the variate X is discrete and it is called poisson variate, m is the parameter of P.D.

NOTE:

poisson distribution may be treated as a Limiting form of B.D. when

(i) The probability of success p is very small ($p \to 0$) so that $np = m$ is finite.

(ii) The Number of repetitions n is very large ($n \to \infty$)

MEAN and VARIANCE of a poisson variate

MEAN:

Let X be a poisson variate with parameter m, Then

$$p(x) = \frac{e^{-m} m^x}{x!} \quad ; \; x = 0, 1, 2, 3, \ldots$$

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} x \, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-m} m^x}{x!}$$

$$= 0 + \sum_{x=1}^{\infty} \frac{e^{-m} m^x}{(x-1)!} \qquad \Big| \; \because \frac{x}{x!} = \frac{1}{(x-1)!}$$

$$= e^{-m} m \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= e^{-m} \cdot m \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right]$$

$$= m e^{-m} \cdot e^{m} \qquad \left| \; e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \sim \right.$$

$$\boxed{\text{Mean} = E(x) = m.}$$

VARIANCE :

$$V(X) = E(X^2) - [E(X)]^2 \quad \text{—— II}$$

Consider $E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) p(x)$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} m^x}{x!}$$

$$= 0 + 0 + \sum_{x=2}^{\infty} \frac{e^{-m} \cdot m^x}{(x-2)!} \qquad \left| \; \because \frac{x(x-1)}{x!} = \frac{1}{(x-2)!} \right.$$

$$= e^{-m} \cdot m^2 \cdot \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!}$$

$$= e^{-m} \cdot m^2 \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right]$$

$$= e^{-m} \cdot m^2 \cdot e^{+m}$$

$$= m^2.$$

$$\therefore E(X^2) = E[X(X-1)] + E(X) \qquad \left| \; \begin{aligned} &E[X(X-1)] = \\ &E(X^2 - X) \\ &= E(X^2) - E(X) \\ &\therefore E(X^2) = E(X^2) - E(X) + E(X) \end{aligned} \right.$$

$$= m^2 + m.$$

Using II,

$$V(X) = m^2 + m - (m)^2$$

$$\boxed{V(X) = m.}$$

Thus for a poisson variate Mean & Variance are Equal

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Examples of poisson variate

1. Number of deaths occuring in a city in a day
2. Number of vehicles crossing a junction in one hour
3. Number of Defects in a Manufacturing article.

Relation between Binomial and poisson Distribution

In a B.D. if n is Large, then the probability p of occurence of an event is close to <u>ZERO</u> so that $q = 1-p$ is close to <u>ONE</u>, the event is called a <u>RARE</u> event. For such cases the B.D. is very closely approximated by the poisson distribution with Mean $= \mu$ (or) $m = np$.

In fact the P.D. gives an acceptable approximation to specific B.D. when $p \leq 0.1$ & $n \geq 10$.

To fit Poisson distribution

If N is the total frequency, Then frequency of x is given by

$$T_x = N \times P(x)$$

$$\boxed{T_x = N \times \frac{e^{-m} m^x}{x!}} \quad ; x = 0, 1, 2, \ldots$$

This formula is used to Estimate Theoritical frequency distribution.

PROBLEMS

III Prove that Poisson distribution is a Limiting form of a Binomial distribution when $n \to \infty$, & $p \to 0$ so that $np = m$ is finite

PROOF: By Binomial distributions we have

$$b(x; n, p) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad ; \quad x = 0, 1, 2, \dots n$$

Using Mean $= m = np$ we have $p = \frac{m}{n}$ & $q = 1 - \frac{m}{n}$

$$\therefore b(x; n, p) = \frac{n!}{(n-x)!\, x!} \left(\frac{m}{n}\right)^{x} \left(1 - \frac{m}{n}\right)^{n-x}$$

$$= \frac{n(n-1) \dots [n-(x-1)]\cancel{(n-x)!}}{\cancel{(n-x)!}\, x!} \frac{m^{x}}{n^{x}} \left(1 - \frac{m}{n}\right)^{n-x}$$

$$= \frac{\cancel{n^{x}}\left[(1-1/n)(1-2/n) \dots \left\{1 - \frac{(x-1)}{n}\right\}\right]}{\cancel{n^{x}}\; x!} \cdot m^{x}\left(1 - \frac{m}{n}\right)^{n-x}$$

$$= \frac{m^{x}}{x!} \left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \dots \left(1 - \frac{x-1}{n}\right)\right] \frac{\left(1 - \frac{m}{n}\right)^{n}}{\left(1 - \frac{m}{n}\right)^{x}}$$

Taking limit as $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} b(x; n, p) = \lim_{n\to\infty}\left[\frac{m^{x}}{x!}\left\{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \dots \left(1-\frac{x-1}{n}\right) \frac{\left(1 - \frac{m}{n}\right)^{n}}{\left(1 - \frac{m}{n}\right)^{x}}\right\}\right]$$

$$= \frac{m^{x}}{x!} \lim_{n\to\infty} \left(1 - \frac{m}{n}\right)^{n} \qquad \Big|\ \because \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0. \ \ \& \ \lim_{n\to\infty}\left(1 - \frac{m}{n}\right)^{x} = 1^{x} = 0.$$

$$= \frac{m^{x}}{x!} \lim_{n\to\infty} \left[\left(1 - \frac{m}{n}\right)^{-\frac{n}{m}}\right]^{-m}$$

$$= \frac{m^{x}}{x!} e^{-m} \qquad \Big|\ \because \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{n} = e^{x}.$$

$$\therefore \lim b(x; n, p) = \frac{e^{-m} m^{x}}{x!} \quad ; \quad x = 0, 1, 2, 3, \dots$$

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2. There is a chance that 5 %. of the pages of book contain typographical errors. If 100 pages of the book are choosen at random, find the probability that 2 of these pages contain typoglaphical errors using (i) Binomial distribution (ii) poisson distribution.

Sol: (i) Let X be Binomial variate

$n \to 100$

$p = 5\% = 0.05$, $q = 1-p = 0.95$

$\therefore P(2 \text{ pages}) = P(x=2) = {}^{100}C_2 (0.05)^2 (0.95)^{98}$ | $P(x) = {}^nC_x p^x q^{n-x}$

$= 0.081$

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(ii) Let X be a Poisson variate

Mean $= \mu = np$

$= 100 \times 0.05 = \underline{\underline{5}}$

$\therefore P(x) = \dfrac{e^{-\mu} \mu^x}{x!}$ [μ (or) m or λ]

$\Rightarrow P(x=2) = \dfrac{e^{-5} 5^2}{2!} = 0.084$

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$\boxed{3}$ Alpha particles are emitted by a radioactive source at an average rate of 5 in a 20 minute interval. Using the poisson distribution, find the probability that there will be (i) Exactly two emissions
(ii) at least two emissions.

Sol: Let X be a poisson variate

$n \to 20$

$\mu = 5$

$\therefore P(x) = \frac{e^{-5} 5^{x}}{x!}$; $x = 0, 1, 2, \ldots$

(i) P(Exactly 2 emissions) $= P(x=2)$

$= \frac{e^{-5} 5^{2}}{2!} = 0.0842$

(ii) P (atleast 2 emissions) $= P(x \geq 2)$

$= 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - \left[\frac{e^{-5} 5^{0}}{0!} + \frac{e^{-5} 5^{1}}{1!}\right]$

$= 0.9596$

A car hire firm has 2 cars, which it hires out day by day. The demand for a car on each day is distributed as a P.D. with mean 1.5. Calculate the probability that a randomly choosen day (i) neither car is used (ii) some demand is refused.

Sol: Let X be a poisson variate

Given Mean $= \mu = 1.5$

$n = 2.$

$$\therefore \quad p(x) = \frac{e^{-1.5}(1.5)^x}{x!}$$

(i) P(Neither car is used) $= P(x=0)$

$$= \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231$$

(ii) P(Some demand is Refused) $= P(x > 2)$

$$= 1 - P(x \leq 2)$$

$$= 1 - \{P(x=0) + P(1) + P(2)\}$$

$$= 1 - \left[e^{-1.5} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 0.1913$$

5 The Number of accidents occuring in a city in a day is a poisson variate with mean 0.8. Find the probability that on a randomly selected day
(i) There are no accidents
(ii) There are accidents.

Sol: Let X denote the No. of accidents per day, Then X is a poisson variate with $m = 0.8$.

$$\therefore \quad P(x) = \frac{e^{-0.8}(0.8)^x}{x!} \quad ; \; x = 0, 1, 2, \ldots$$

(i) $P[\text{No accidents}] = P[x=0] = \frac{e^{-0.8}(0.8)^0}{0!} = e^{-0.8} = 0.449$

(ii) $P[\text{accidents occur}] = 1 - P(\text{No accidents}) = 1 - P(x=0) = 0.551$.

6 Given that 2% of the fuses manufactured by a firm are defective, find by using poisson distribution, the probability that a Box containing 200 fuses has
(i) No defective fuses (ii) at least one defective fuse
(iii) Exactly 3-defective fuse.
(iv) 3 (or) more defective fuses.

Sol: Probability that Fuse is defective $= p = 2\% = 0.02$.

$n = 200$.

$\therefore \quad m = np \;\Rightarrow\; m = 0.02 \times 200 = 4$.

$\left(\because \text{Also } p = 0.02 \text{ (very small) } \& \; n = 200 \text{ very large.}\right)$

Let X be a poisson variate, Then

$$P(x) = \frac{e^{-4}\,4^x}{x!} \quad ; \; x = 0, 1, 2, \ldots$$

(i) $P[\text{No defective fuse}] = P(X=0) = \frac{e^{-4}4^0}{0!} = e^{-4} = 0.0183$

(ii) $P[\text{atleast one defective}] = P[X \geq 1] = 1 - P(X<1) = 1 - P(0) = 0.9817$

(iii) $P[\text{Exactly 3-defective}] = P[X=3] = \frac{e^{-4}4^3}{3!} = 0.1952$

(iv) $P[\text{3 (or) More}] = P(X \geq 3) = 1 - P(X<3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 0.762.$

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7 The Number of accidents in a city by taxi drivers is distributed as poisson variate with mean 3. If there are 500 taxi drivers in the city. Find the Number of taxi drivers with (i) No accidents (ii) Some accidents (iii) 3-accidents.

Sol: Let X is a poisson variate with parameter $m = 3$ and $N = 500$.

$$\therefore P(x) = \frac{m^x e^{-x}}{x!} = \frac{3^x e^{-3}}{x!} \; ; x = 0, 1, 2, \ldots$$

(i) $P[\text{No accidents}] = P[X=0] = \frac{e^{-3} \cdot 3^0}{0!} = 0.0498.$

$\therefore$ The Number of drivers with No accidents $= 500 \times 0.0498 = 24.9 \simeq 25.$

(ii) $P[\text{Some accidents}] = 1 - P(\text{No accidents}) = 1 - P(X=0) = 0.9502.$

$\therefore$ The Number of drivers with Some accidents $= 500 \times 0.9502 = 475.$

(iii) $P[\text{3-accidents}] = P(X=3) = \frac{e^{-3}3^3}{3!} = 0.2241$

$\therefore$ The Number of drivers with 3-accidents $= 500 \times 0.2241 \doteq 112$

[8] The probability that a Razor Blades manufactured by a firm is defective is $\frac{1}{500}$. Blades are supplied in packets of 10 each. Use Poisson distribution to calculate the approximate number of packets containing

(i) No defective (ii) Exactly one defective (ii) 2-defective. in a Lot of 10,000 each.

Sol: Let X is a poisson variate, Then.

$$P(x) = \frac{m^x e^{-m}}{x!} \quad ; \; x = 0, 1, 2, \ldots$$

Given $n = 10$, $P = 1/500 = 0.002$ & $N = 10,000$

$\therefore \quad m = np \Rightarrow m = 10 \times 0.002 = 0.02$

(i) $P[\text{No. defective}] = P(X=0) = 0.9802$

$\therefore$ The Number of packets with No ~~accidents~~ defective $= 10,000 \times 0.9802 = 9802$.

(ii) $P[\text{one defective}] = P(X=1) = 0.0196$

$\therefore$ The Number of packets with one ~~accident~~ defective $= 10,000 \times 0.0196 = 196$

(iii) $P[\text{Two defective}] = P(X=2) = -$

$\therefore$ The Number of packets with 2 defective $= 10,000 \times -$

$=$

9 The Probability that an Individual suffers a bad reaction from a certain injection is 0.001. Using poisson distribution, determine the probability that out of 2000 Individuals (a) Exactly 3 (b) More than 2 will suffer a bad reaction.

Sol: Given $p = 0.001$ & $n = 2000$.

$\therefore$ Mean $= m = np = 2000 \times 0.001 = 2.$

Let x be a poisson variate which denotes the Number of persons who suffers a bad reaction, Then $P(x) = \frac{e^{-2} 2^x}{x!}$.

(i) $P[\text{exactly } 3] = P[x=3] = \frac{e^{-2} 2^3}{3!} = 0.1804.$

(ii) $P[\text{More than } 2] = P[x > 2] = 1 - P(x \leq 2) = 1 - [P(0) + P(1) + P(2)]$

$= 0.32333.$

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10 Assignment

The Average No of telephone calls booked at an Exchange btw 10.00 am & 10.10 am is 4. Find the probability that on a randomly selected day 2 (or) More calls are booked btw 10.00 am and 10.10 am. On How many days a year, would you except booking of 2 (or) More calls during that time gap?

Sol: Here $m = 4$, $P(x) = \frac{e^{-4} 4^x}{x!}$; $x = 0, 1, 2 \ldots$

$P[2 \text{ (or) More calls}] = 1 - P[\text{Less than 2 calls}]$

$= 1 - [P(0) + P(1)] = 0.9085.$

The Estimation for 365 days $= 365 \times 0.9085 = 332.$

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(5)

III Fit a poisson distribution to the following data. Find the corresponding Theoritical estimates for f_i.

x :	0	1	2	3	4	5
f :	22	13	5	5	3	2

Sol: Mean $= m = \frac{\Sigma f x}{\Sigma f} = 1.2 = \frac{\Sigma f x}{N} = \frac{(\ )}{50}$

The probability function in poisson distribution is given by

$$P(x) = \frac{e^{-1.2}(1.2)^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

The Theoritical (Expected) Frequency distribution is given by

$$T_x = N \times P(x) = 50 \times \frac{e^{-1.2}(1.2)^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

$T_0 = 50 \times \frac{e^{-1.2}(1.2)^0}{0!} = 15.06 \simeq 15$

$T_1 = \quad = 18.07$

$T_2 = \quad = 10.84$

$T_3 = \quad = 4.336$

$T_4 = \quad = 1.301$

$T_5 = \quad = 0.3122$

Thus the observed and Theoritical distributions are

x :	0	1	2	3	4	5
observed f :	22	13	5	5	3	2
Theoritical f (T_x) :	15	18	11	4	1	0

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