UNIT - II
LINEAR ALGEBRA - II

Topic Learning Objectives

Upon Completion of this unit, students will be able to:

- Study the orthogonal and orthonormal properties of vectors.
- Use Gram-Schmidt process to factorize a given matrix as a product of an orthogonal matrix(Q) and an upper triangular invertible matrix(R).
- Diagonalize symmetric matrices using eigenvalues and eigenvectors.
- Decompose a given matrix into product of an orthogonal matrix (U), a diagonal matrix (Σ) and an orthogonal matrix (V^T).

Introduction

This section deals with the study of orthogonal and orthonormal vectors which forms the basis for the construction of an orthogonal basis for a vector space. The Gram-Schmidt process is applied to construct an orthogonal basis for the column space of a given matrix and further to decompose a given matrix to the form A = QR, where Q has orthonormal column vectors and R is an upper triangular invertible matrix with positive entries along the diagonal. This section also deals with finding the Eigenvalues and Eigenvectors of a square matrix, which is applied to diagonalize a square matrix A as $D = P^{-1}AP$. Further the singular value decomposition is studied wherein, a given matrix is resolved as a product of an orthogonal matrix (U), a diagonal matrix(Σ) and an orthogonal matrix V^T.

Definition. *Two vectors u and v in $\mathbb{R}^n$ are **orthogonal** to each other if $u \cdot v = 0$.*

Example. In $\mathbb{R}^2$, the vectors $u = (1, 2)$ and $v = (6, -3)$ are orthogonal as $u \cdot v = (1, 2) \cdot (6, -3) = 0$.

Definition. *A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $u_i \cdot u_j = 0$ whenever $i \neq j$*

Example. Let $u_1 = (3, 1, 1), u_2 = (-1, 2, 1), u_3 = (-\frac{1}{2}, -2, \frac{7}{2})$. Then $u_1 \cdot u_2 = (3, 1, 1) \cdot (-1, 2, 1) = -3 + 2 + 1 = 0$, $u_1 \cdot u_3 = (3, 1, 1) \cdot (-\frac{1}{2}, -2, \frac{7}{2}) = -\frac{3}{2} - 2 + \frac{7}{2} = 0$, $u_2 \cdot u_3 = (-1, 2, 1) \cdot (-\frac{1}{2}, -2, \frac{7}{2}) = \frac{1}{2} - 4 + \frac{7}{2} = 0$. Each pair of distinct vectors is orthogonal and so $\{u_1, u_2, u_3\}$ is an orthogonal set.

Definition. *A set $\{u_1, u_2, \ldots, u_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.*

Example. The standard basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{R}^n$, is an orthonormal set. Any non-empty subset of $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set.

Definition. *An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.*

Example. The set $S = \{(3, 1, 1), (-1, 2, 1), (-\frac{1}{2}, -2, \frac{7}{2})$ is an orthogonal basis for $\mathbb{R}^3$ as (i) S is an orthogonal set and (ii) S forms a basis for $\mathbb{R}^3$.

$$\begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ -\frac{1}{2} & -2 & \frac{7}{2} \end{vmatrix} = 3(7 + 2) - 1(-\frac{7}{2} + \frac{1}{2}) + 1(2 + 1) = 27 + 3 + 3 = 33 \neq 0$$

Definition. *An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.*

Example. *Show that $\{(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}), (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}), (-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}})\}$ is an orthonormal basis for $\mathbb{R}^3$, where*

Solution: Let $v_1 = (\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}})$, $v_2 = (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}), v_3 = (-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}})$.

$$v_1 \cdot v_2 = -\frac{3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$v_1 \cdot v_3 = -\frac{3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$v_2 \cdot v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal set.

$$v_1 \cdot v_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = 1$$

$$v_2 \cdot v_2 = \frac{2}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

$$v_3 \cdot v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$$

which shows that v_1, v_2, v_3 are unit vectors. Thus $\{v_1, v_2, v_3\}$ is an orthonormal set. Since the set is linearly independent and consists of three vectors form a basis for $\mathbb{R}^3$.

Definition. *A square matrix P with real entries and satisfying the condition $P^{-1} = P^T$ is called an **orthogonal matrix**. That is $PP^T = I$.*

Example. Let $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Clearly $P^{-1} = P^T$. Therefore P is an orthogonal matrix.

Example. The matrix $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is an orthogonal matrix, since

$A^TA = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The row vector of A, namely $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$, $(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$ and $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ are orthonormal. So are the column vectors of A.

Note:

Let P be an $n \times n$ matrix with real entries. Then P is orthogonal if and only if the row (column) vectors of A form an orthonormal basis for $\mathbb{R}^n$.

Orthogonal Projections

Given a non-zero vector $\vec{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\vec{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\vec{u}$ and the other orthogonal to $\vec{u}$. We wish to write

$$\vec{y} = \hat{y} + \vec{z} \tag{1}$$

where $\hat{y} = \alpha\vec{u}$, for some scalar α and $\vec{z}$ is some vector orthogonal to $\vec{u}$.
Given any scalar α, let $\vec{z} = \vec{y} - \alpha\vec{u}$, so that 1 is satisfied. Then $\vec{y} - \hat{y}$ is orthogonal to $\vec{u}$ iff

$$0 = (\vec{y} - \alpha\vec{u}).\vec{u} = \vec{y}\cdot\vec{u} - (\alpha\vec{u})\cdot\vec{u}$$

$$= \vec{y}\cdot\vec{u} - \alpha(\vec{u}\cdot\vec{u})$$

That is, 1 is satisfied with $\vec{z}$ orthogonal to $\vec{u}$ iff

$$\alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} \quad \text{and} \quad \hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

The vector $\hat{y}$ is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$, and the vector $\vec{z}$ is called the component of $\vec{y}$ orthogonal to $\vec{u}$.

Example. Let $\vec{y} = (7,6)$ and $\vec{u} = (4,2)$. The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is given by,

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = 2(4,2) = (8,4)$$

Note:

1. The orthogonal projection of $\vec{y}$ onto a subspace W spanned by orthogonal vectors $\{u_1, u_2\}$ is given by $\hat{y} = \dfrac{\vec{y}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\vec{u_1} + \dfrac{\vec{y}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}}\vec{u_2}$.
2. The distance from a point $\vec{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\vec{y}$ to the nearest point in W.

Example. The distance from $\vec{y}$ to $W = \text{Span}\{u_1, u_2\}$, where $\vec{y} = (-1,-5,10)$, $u_1 = (5,-2,1)$, $u_2 = (1,2,-1)$. is given by

$$\begin{aligned}
\hat{y} &= \frac{(-1,-5,10)\cdot(5,-2,1)}{(5,-2,1)\cdot(5,-2,1)}(5,-2,1) + \frac{(-1,-5,10)\cdot(1,2,-1)}{(1,2,-1)\cdot(1,2,-1)}(1,2,-1) \\
&= (-1,-8,4) \\
\vec{y} - \hat{y} &= (-1,-5,10) - (-1,-8,4) = (0,3,6)
\end{aligned}$$

The distance from $\vec{y}$ to W is $\sqrt{0 + 3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$.

Exercise:

1. Determine which set of vectors are orthogonal.

   (a) $u_1 = (-1,4,-3), u_2 = (5,2,1), u_3 = (3,-4,-7)$,

   (b) $u_1 = (5,-4,0,3), u_2 = (-4,1,-3,8), u_3 = (3,3,5,-1)$.

2. Show that $\{(2,-3),(6,4)\}$ forms an orthogonal basis for $\mathbb{R}^2$.
3. Show that $\{(1,0,1),(-1,4,1),(2,1,-2)\}$ forms an orthogonal basis for $\mathbb{R}^3$.
4. Show that the matrix $U = \begin{bmatrix} \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -\frac{4}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & -\frac{7}{\sqrt{66}} \end{bmatrix}$ is an orthogonal matrix.
5. Find the orthogonal projection of $y=(2,6)$ onto $u=(7,1)$.
6. Let $u_1=(2,5,-1), u_2=(-2,1,1)$ and $y=(1,2,3)$. Find the orthogonal projection of y onto Span$\{u_1,u_2\}$.

Gram-Schmidt Orthogonalization

The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$. The construction converts a skewed set of axes into a perpendicular set.

Gram-Schmidt process

Given a basis $\{x_1,x_2,\ldots,x_p\}$ for a subspace W of $\mathbb{R}^n$. Define,

$$\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \frac{x_2\cdot v_1}{v_1\cdot v_1}v_1 \\
v_3 &= x_3 - \frac{x_3\cdot v_1}{v_1\cdot v_1}v_1 - \frac{x_3\cdot v_2}{v_2\cdot v_2}v_2 \\
&\vdots \\
v_p &= v_p - \frac{x_p\cdot v_1}{v_1\cdot v_1}v_1 - \frac{x_p\cdot v_2}{v_2\cdot v_2}v_2 - \cdots - \frac{x_p\cdot v_{p-1}}{v_{p-1}\cdot v_{p-1}}v_{p-1}
\end{aligned}$$

Then $\{v_1,v_2,\ldots,v_p\}$ is an orthogonal basis for W. In addition Span$\{v_1,v_2,\ldots,v_k\}$ = Span$\{x_1,x_2,\ldots,x_k\}$ for $1\le k\le p$.

Example. *Let $W = Span\{x_1,x_2\}$ where $x_1=(3,6,0)$ and $x_2=(1,2,2)$. Construct an orthogonal basis $\{v_1,v_2\}$ for W.*

Solution: Let $v_1=x_1$ and $v_2 = x_2 - \frac{x_2\cdot x_1}{x_1\cdot x_1}x_1 = (1,2,2) - \frac{(1,2,2)\cdot(3,6,0)}{(3,6,0)\cdot(3,6,0)}(3,6,0) = (0,0,2)$. Then $\{v_1,v_2\}$ is an orthogonal set of non-zero vectors in W. Since $\dim(W)=2$, the set $\{v_1,v_2\}$ is a basis in W.

Example. *Let $W = Span\{v_1,v_2,v_3\}$, where $v_1=(0,1,2), v_2=(1,1,2), v_3=(1,0,1)$. Construct an orthogonal basis $\{u_1,u_2,u_3\}$ for W.*

Solution: Let $u_1=v_1$.

$$u_2 = v_2 - \frac{v_2\cdot u_1}{u_1\cdot u_1}u_1 = (1,1,2) - \frac{(1,1,2)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2) = (1,0,0)$$

$$
\begin{aligned}
u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2, u_2} u_2 \\
&= (1,0,1) - \frac{(1,0,1)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2) - \frac{(1,0,1)\cdot(1,0,0)}{(1,0,0)\cdot(1,0,0)}(1,0,0) \\
&= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0) = \left(0, -\frac{2}{5}, \frac{1}{5}\right).
\end{aligned}
$$

QR Factorization

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $Col(A)$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Example. *Find a QR factorization of* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution: The columns of A are the vectors $\{x_1, x_2, x_3\}$. Let $v_1 = x_1 = (1,1,1,1)$

$$
\begin{aligned}
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (0,1,1,1) - \frac{(0,1,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) \\
&= (0,1,1,1) - \frac{3}{4}(1,1,1,1) = \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\
&= (0,0,1,1) - \frac{(0,0,1,1)\cdot(1,1,1,1)}{(1,1,1,1)\cdot(1,1,1,1)}(1,1,1,1) - \frac{(0,0,1,1)\cdot\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)}{\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\cdot\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)}\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
&= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{2}{3}\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)
\end{aligned}
$$

Therefore $\left\{(1,1,1,1), \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(0, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)\right\}$ forms an orthogonal basis for $Col(A)$ and $\left\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}), (0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})\right\}$ forms an orthonormal basis for $Col(A)$.
Therefore

$$
Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}
$$

We have $A = QR \implies Q^T A = Q^T QR \implies Q^T A = IR \implies Q^T A = R$ i.e., $R = Q^T A$.

$$
\therefore R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.
$$

Example. *Find a QR factorization of* $A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution: Let $\{x_1, x_2, x_3\}$ be the columns of the matrix A. Let $v_1 = x_1 = (1, -1, -1, 1, 1)$

$$\begin{aligned}
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (2, 1, 4, -4, 2) - \frac{(2, 1, 4, -4, 2) \cdot (1, -1, -1, 1, 1)}{(1, -1, -1, 1, 1) \cdot (1, -1, -1, 1, 1)} (1, -1, -1, 1, 1) \\
&= (2, 1, 4, -4, 2) - \frac{-5}{5} (1, -1, -1, 1, 1) = (3, 0, 3, -3, 3) \\
v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\
&= (5, -4, -3, 7, 1) - \frac{(5, -4, -3, 7, 1) \cdot (1, -1, -1, 1, 1)}{(1, -1, -1, 1, 1) \cdot (1, -1, -1, 1, 1)} (1, -1, -1, 1, 1) \\
&\qquad - \frac{(5, -4, -3, 7, 1) \cdot (3, 0, 3, -3, 3)}{(3, 0, 3, -3, 3) \cdot (3, 0, 3, -3, 3)} (3, 0, 3, -3, 3) \\
&= (5, -4, -3, 7, 1) - \frac{20}{5} (1, -1, -1, 1, 1) - \frac{-12}{36} (3, 0, 3, -3, 3) = (2, 0, 2, 2, -2)
\end{aligned}$$

Therefore $\{(1, -1, -1, 1, 1), (3, 0, 3, -3, 3), (2, 0, 2, 2, -2)\}$ forms an orthogonal basis for $Col(A)$ and $\{(1/\sqrt{5}, -1/\sqrt{5}, -1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}), (1/2, 0, 1/2, -1/2, 1/2), (1/2, 0, 1/2, 1/2, -1/2)\}$ forms an orthonormal basis for $Col(A)$. Therefore

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

Therefore $R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

Example. *Find the orthogonal basis for the column space of the matrix* $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$.

Solution: Let $\{x_1, x_2, x_3\}$ be the columns of A, where $x_1 = (3, 1, -1, 3), x_2 = (-5, 1, 5, -7), x_3 = (1, 1, -2, 8)$.

Let $v_1 = (3,1,-1,3)$,

$$\begin{aligned}
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (-5,1,5,-7) - \frac{(-5,1,5,-7)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3) \\
&= (-5,1,5,-7) - \frac{-40}{20}(3,1,-1,3) = (1,3,3,-1) \\
v_3 &= \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (1,1,-2,8) - \frac{(1,1,-2,8)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3) \\
&\quad - \frac{(1,1,-2,8)\cdot(1,3,3,-1)}{(1,3,3,-1)\cdot(1,3,3,-1)}(1,3,3,-1) \\
&= (1,1,-2,8) - \frac{30}{20}(3,1,-1,3) - \frac{-10}{20}(1,3,3,-1) = (-3,1,1,3)
\end{aligned}$$

Therefore $\{(3,1,-1,3),(1,3,3,-1),(-3,1,1,3)\}$ is an orthogonal basis for the column space of the given matrix.

Example. *Find the orthogonal basis for the column space of the matrix* $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

Solution: Let $\{x_1, x_2, x_3\}$ be the columns of A, where $x_1 = (-1,3,1,1)$, $x_2 = (6,-8,-2,-4)$, $x_3 = (6,3,6,-3)$. Let $v_1 = (-1,3,1,1)$.

$$\begin{aligned}
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (6,-8,-2,-4) - \frac{(6,-8,-2,-4)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1) \\
&= (6,-8,-2,-4) - \frac{-36}{12}(-1,3,1,1) = (3,1,1,-1) \\
v_3 &= \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (6,3,6,-3) - \frac{(6,3,6,-3)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1) \\
&\quad - \frac{(6,3,6,-3)\cdot(3,1,1,-1)}{(3,1,1,-1)\cdot(3,1,1,-1)}(3,1,1,-1) \\
&= (6,3,6,-3) - \frac{6}{12}(-1,3,1,1) - \frac{30}{12}(3,1,1,-1) = (-1,-1,3,-1)
\end{aligned}$$

Therefore $\{(-1,3,1,1),(3,1,1,-1),(-1,-1,3,-1)\}$ is an orthogonal basis for the column space of the given matrix.

Exercise

1. Let $W = \text{Span}\{v_1, v_2\}$, where $v_1 = (1,1)$ and $v_2 = (2,-1)$. Construct an orthogonal basis $\{u_1, u_2\}$ for W.

2. Find the orthonormal basis for the subspace spanned by the vectors $u_1 = (1,-4,0,1)$, $u_2 = (7,-7,-4,1)$.

3. Find the QR factorization of the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$

Answer

1. $v_1 = (1,1), v_2 = (\frac{3}{2}, -\frac{3}{2})$
2. $v_1 = (1,-4,0,1), v_2 = (5,1,-4,-1)$

Eigenvalues and Eigenvectors

If A is a square matrix of order n, we can find the matrix $A - \lambda I$, where I is unit matrix of order n. The determinant of this matrix equated to zero, i.e,

$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$ is called the characteristic equation of A.

On expanding the determinant, the characteristic equation takes the form
$(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0$,
where k_is are expressible in terms of the elements a_{ij}. The roots of this equation are called the characteristic roots or latent roots or eigenvalues of the matrix A.

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then the linear transformation

$$Y = AX \tag{2}$$

carries the column vector X into the column vector Y by means of the square matrix A. In practice it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves. Let X be such a vector which transforms into λX by means of the transformation 2. Then, $\lambda X = AX$ or $AX - \lambda IX = 0$ or

$$[A - \lambda I]X = 0 \tag{3}$$

The equation (3) represents a system of n homogeneous linear equations,

$$\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 &+ \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 &+ \dots + a_{2n}x_n = 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots &+ (a_{nn}-\lambda)x_n = 0 \end{aligned} \tag{4}$$

which will have a non-trivial solution only if the coefficient matrix is singular. i.e, $|A - \lambda I| = 0$. This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A. It has n roots and corresponding to each root, the equation (3)(or equation (4)) will have a non-zero solution, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, which is known as the Eigenvector or latent vector.

Observation 1: Corresponding to n distinct Eigenvalues, we get n independent Eigenvectors. But when two or more Eigenvalues are equal, it may or may not be possible to get

linearly independent Eigenvectors corresponding to the repeated roots.
Observation 2: If x_i is a solution for a Eigenvalues λ_i then it follows from (3) that cx_i is also a solution, where c ia an arbitrary constant. Thus the Eigenvector corresponding to an Eigenvalues is not unique, but may be any one of the vectors cx_i.

Example. *Find the Eigenvalues and Eigenvectors of the matrix* $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

Solution: $|A - \lambda I| = 0 \implies \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} \implies \lambda^2 - \lambda = 0 \implies \lambda = 0, \lambda = 1.$

For $\lambda = 1$, $(A - \lambda I)X = 0 \implies \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + x_2 = 0 \implies x_2 = x_1$

Letting $x_1 = 1 \implies x_2 = 1 \therefore X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = 0$, $(A - \lambda I)X = 0 \implies \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_2 = -x_1$

Letting $x_1 = 1 \implies x_2 = -1 \therefore X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Example. *Find the Eigenvalues and Eigenvectors of the matrix* $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Solution: $|A - \lambda I| = 0 \implies \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \implies \lambda^2 + 1 = 0 \implies \lambda = +i, \lambda = -i.$

For $\lambda = i$, $(A - \lambda I)X = 0 \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -ix_1 - x_2 = 0 \implies x_2 = -ix_1$

Letting $x_1 = 1 \implies x_2 = -i \therefore X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

For $\lambda = -i$, $(A - \lambda I)X = 0 \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies ix_1 - x_2 = 0 \implies x_2 = ix_1$

Letting $x_1 = 1 \implies x_2 = i \therefore X = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Example. *Find the Eigenvalues and Eigenvectors of the matrix* $A = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$.

Solution: $|A - \lambda I| = 0 \implies \begin{vmatrix} 3 - \lambda & 4 & 2 \\ 3 & 5 - \lambda & 4 \\ 0 & 1 & 2 - \lambda \end{vmatrix} = 0 \implies \lambda^3 - 10\lambda^2 + 15\lambda = 0$

$\implies \lambda = 5 + \sqrt{10},\ 5 - \sqrt{10},\ 0$

For $\lambda = 5 + \sqrt{10}$, $|A - \lambda I| = 0 \implies \begin{bmatrix} -2 - \sqrt{10} & 4 & 2 \\ 3 & -\sqrt{10} & 4 \\ 0 & 1 & -3 - \sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\implies \frac{x_1}{3\sqrt{10} + 6} = \frac{-x_2}{-9 - 3\sqrt{10}} = \frac{x_3}{3} \implies \frac{x_1}{2 + \sqrt{10}} = \frac{x_2}{3 + \sqrt{10}} = \frac{x_3}{1}$$

$\therefore X = \begin{bmatrix} 2 + \sqrt{10} \\ 3 + \sqrt{10} \\ 1 \end{bmatrix}$

For $\lambda = 5-\sqrt{10}$, $|A-\lambda I| = 0 \implies \begin{bmatrix} -2+\sqrt{10} & 4 & 2 \\ 3 & \sqrt{10} & 4 \\ 0 & 1 & -3+\sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{-3\sqrt{10}+6} = \dfrac{-x_2}{-9+3\sqrt{10}} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2-\sqrt{10}} = \dfrac{x_2}{3-\sqrt{10}} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2-\sqrt{10} \\ 3-\sqrt{10} \\ 1 \end{bmatrix}$

with $\lambda = 0$, $|A-\lambda I| = 0 \implies \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{6} = \dfrac{-x_2}{6} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2} = \dfrac{x_2}{-2} = \dfrac{x_3}{1}$

$\therefore X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Example. *Find the Eigenvalues and Eigenvectors of the matrix* $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Solution: $|A-\lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 + 3\lambda^2 - 4 = 0$

$\implies \lambda = 1, -2, -2$

For $\lambda = 1$, $|A-\lambda I| = 0 \implies \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{9} = \dfrac{-x_2}{9} = \dfrac{x_3}{9} \implies \dfrac{x_1}{1} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} \therefore X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = -2$, $|A-\lambda I| = 0 \implies \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + 3x_2 + 3x_3 = 0 \implies x_1 = -x_2 - x_3$

Letting $x_2 = k_1, x_3 = k_2 \implies x_1 = -k_1 - k_2 \therefore X = \begin{bmatrix} -k_1-k_2 \\ k_1 \\ k_2 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are the linearly independent Eigenvectors corresponding to $\lambda = -2$.

Diagonalization of a Matrix

Suppose the $n \times n$ matrix A has n linearly independent Eigenvectors. Let P be the matrix with its columns as n linearly independent vectors of A. Then $P^{-1}AP$ is a diagonal matrix D. The diagonal entries of D the Eigenvalues of A.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Note:

1. Any matrix with distinct Eigenvalues can be diagonalized.
2. The diagonalization matrix P is not unique.
3. Not all matrices posses n linearly independent Eigenvectors, so not all matrices are diagonalizable.
4. Diagonalizability of A depends on enough Eigenvectors.
5. Diagonalizability can fail only if there are repeated Eigenvalues.
6. The Eigenvalues of A^k are $\lambda_1^k, \lambda_2^k \ldots, \lambda_n^k$ and each Eigenvector of A is still an Eigenvector of A^k. $[D^k = DD \ldots D(k\,\text{times}) = (P^{-1}AP)(P^{-1}AP) \ldots (P^{-1}AP) = P^{-1}A^kP]$.

Example. *Diagonalize the matrix* $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution: $|A - \lambda I| = 0 \implies \begin{vmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 11\lambda + 24 = 0 \implies (\lambda-3)(\lambda-8) = 0 \implies \lambda = 3, 8.$

For $\lambda = 3, (A - 3I)X = 0 \implies \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x_1 + x_2 = 0$. Letting $x_1 = 1 \implies x_2 = -2$. Hence $X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda = 8, (A - 8I)X = 0 \implies \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + 2x_2 = 0$. Letting $x_2 = 1 \implies x_1 = 2$. Hence $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\implies P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}$

Example. *Diagonalize the matrix* $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$.

Solution: $|A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = 0 \implies (\lambda - 1)(\lambda + 3) = 0 \implies \lambda = -3, 1.$

For $\lambda = -3, (A + 3I)X = 0 \implies \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x_1 = 0 \implies x_1 = 0$. Let $x_2 = 1$. Hence $X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda = 1, (A - I)X = 0 \implies \begin{vmatrix} 0 & 0 \\ 0 & -4 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x_2 = 0 \implies x_2 = 0$. Let $x_1 = 1$. Hence $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\implies P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

Example. *Diagonalize the matrix* $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

Solution: $|A-\lambda I|=0 \implies \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$

$\implies \lambda = 3,\ 6,\ 8$

For $\lambda = 3$, $(A-\lambda I)X = 0 \implies \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{5} = \dfrac{-x_2}{-5} = \dfrac{x_3}{5} \implies \dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{1} \therefore X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 6$, $(A-\lambda I)X \implies \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{-1} = \dfrac{-x_2}{1} = \dfrac{x_3}{2} \implies \dfrac{x_1}{-1} = \dfrac{x_2}{-1} = \dfrac{x_3}{2} \therefore X = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

For $\lambda = 8$, $(A-\lambda I)X \implies \begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{5} = \dfrac{-x_2}{5} = \dfrac{x_3}{0} \implies \dfrac{x_1}{1} = \dfrac{x_2}{-1} = \dfrac{x_3}{2} \therefore X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/6 & -1/6 & 1/3 \\ 1/2 & -1/2 & 0 \end{bmatrix}$.

Example. *Diagonalize the matrix* $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

Solution: $|A-\lambda I|=0 \implies \begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 12\lambda^2 - 21\lambda + 98 = 0$

$\implies \lambda = -2,\ 7,\ 7$

For $\lambda = -2$, $(A-\lambda I)X \implies \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{36} = \dfrac{-x_2}{-18} = \dfrac{x_3}{-36} \implies \dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{-2} \therefore X_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 7$, $(A-\lambda I)X = 0 \implies \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

As the second and third row are dependent on the first row, we get only one equation in three unknowns. i.e., $-4x_1 - 2x_2 + 4x_3 = 0$. Letting x_1 and x_3 as arbitrary implies $x_2 = -2x_1 + 2x_3$. With $x_1 = 1, x_3 = 2$ we get $x_2 = 2$. With $x_1 = 2, x_3 = 1$ we get $x_2 = -2$.

$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Exercise:

Diagonalize the matrices

1. $\begin{bmatrix} -2 & -36 & 0 \\ -36 & -23 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$ 3. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

Singular Value Decomposition

Definition. *Let A be any matrix. The square root of non-zero eigenvalues of AA^T (or A^TA) are known as the singular values of A.*

Any $m \times n$ matrix A can be factored into $A = U\Sigma V^T$, where U and V are orthogonal matrices and Σ is a diagonal matrix. The columns of $U(m \times m)$ are Eigenvectors of AA^T, and the columns of $V(n \times n)$ are Eigenvectors of A^TA. The r singular values on the diagonal of $\Sigma(m \times n)$ are the square roots of the non-zero Eigenvalues of both AA^T and A^TA.

Note:

The diagonal(but rectangular) matrix Σ has Eigenvalues from A^TA. These positive entries(also called sigma) will be $\sigma_1, \sigma_2, \ldots, \sigma_r$, such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. They are the singular values of A. When A multiplies a column v_j of V, it produces σ_j times a column of $U(A = U\Sigma V^T \implies AV = U\Sigma)$.

Example. *Decompose* $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ *as $U\Sigma V^T$, where U and V are orthogonal matrices.*

Solution: $AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $|AA^T - \lambda I| = 0$,

$$\implies \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 4-\lambda & 4 \\ -2 & 4 & 4-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 9\lambda^2 = 0 \implies \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 9$$

For $\lambda = 9, [AA^T - \lambda I]X = 0 \implies \begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -8x_1 - 2x_2 - 2x_3 = 0,$

$-18x_2 + 18x_3 = 0 \implies x_1 = -(1/2)x_3, x_2 = x_3 \implies X_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 0, [AA^T - \lambda I]X = 0 \implies \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 2x_2 + 2x_3$

$\implies X_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Hence $U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$.

$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$, $|A^TA - \lambda I| = 0 \implies |9 - \lambda| = 0 \implies \lambda = 9$

Then $(A^TA - \lambda I)X = 0 \implies \begin{bmatrix}0\end{bmatrix}\begin{bmatrix}x_1\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}$. Let $x_1 = 1$. Then $X = \begin{bmatrix}1\end{bmatrix}$. Hence $V = \begin{bmatrix}1\end{bmatrix}$ or $V^T = \begin{bmatrix}1\end{bmatrix}$. 9 is an Eigenvalues of both AA^T and A^TA. The rank of $A = \begin{bmatrix}-1\\2\\2\end{bmatrix}$ is 1.

Therefore Σ has only one non-zero entry $\sigma_1 = \sqrt{9} = 3$. That is $\Sigma = \begin{bmatrix}3\\0\\0\end{bmatrix}$

Therefore the SVD of $A = \begin{bmatrix}-1\\2\\2\end{bmatrix}$ is $\begin{bmatrix}-1/3 & 2/3 & 2/3\\2/3 & -1/3 & 2/3\\2/3 & 2/3 & -1/3\end{bmatrix}\begin{bmatrix}3\\0\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix}$.

Example. *Obtain the SVD of* $A = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$.

Solution: $AA^T = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix}2-\lambda & 1\\1 & 1-\lambda\end{vmatrix} = 0 \implies \lambda^2 - 3\lambda + 1 = 0 \implies \lambda_1 = \dfrac{3-\sqrt{5}}{2}$, $\lambda_2 = \dfrac{3+\sqrt{5}}{2}$

For $\lambda = \dfrac{3-\sqrt{5}}{2}$, $(AA^T - \lambda I)X = 0 \implies \begin{bmatrix}\dfrac{1+\sqrt{5}}{2} & 1\\1 & \dfrac{-1+\sqrt{5}}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\implies \dfrac{1+\sqrt{5}}{2}x_1 + x_2 = 0$. Letting $x_1 = -1$, then $x_2 = \dfrac{1+\sqrt{5}}{2}$

Therefore $X = \begin{bmatrix}-1\\\dfrac{1+\sqrt{5}}{2}\end{bmatrix} = \begin{bmatrix}-1\\\alpha\end{bmatrix}$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

For $\lambda = \dfrac{3+\sqrt{5}}{2}$, $(AA^T - \lambda I)X = 0 \implies \begin{bmatrix}\dfrac{1-\sqrt{5}}{2} & 1\\1 & \dfrac{-1-\sqrt{5}}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\implies \dfrac{1-\sqrt{5}}{2}x_1 + x_2 = 0$. Letting $x_1 = -1$, then $x_2 = \dfrac{1-\sqrt{5}}{2}$

Therefore $X = \begin{bmatrix}-1\\\dfrac{1-\sqrt{5}}{2}\end{bmatrix} = \begin{bmatrix}-1\\\beta\end{bmatrix}$, where $\beta = \dfrac{1-\sqrt{5}}{2}$.

Hence $U = \begin{bmatrix}\dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{-1}{\sqrt{1+\beta^2}}\\\dfrac{\alpha}{\sqrt{1+\alpha^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}}\end{bmatrix}$

As $A^TA = AA^T$ $\quad V^T = \begin{bmatrix}\dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{\alpha}{\sqrt{1+\alpha^2}}\\\dfrac{-1}{\sqrt{1+\beta^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}}\end{bmatrix}$ and $\Sigma = \begin{bmatrix}\sqrt{\lambda_1} & 0\\0 & \sqrt{\lambda_2}\end{bmatrix}$.

Example. *Obtain the SVD of* $A = \begin{bmatrix}-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$.

Solution: $AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 4\lambda + 3 = 0$

$\implies \lambda_1 = 1, \lambda_2 = 3$

For $\lambda = 3$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_1 = -x_2$

Letting $x_2 = 1 \implies x_1 = -1 \therefore x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 1$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 - x_2 = 0 \implies x_1 = x_2$

Letting $x_2 = 1 \implies x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$|A^TA - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 4\lambda^2 + 3\lambda = 0$

$\implies \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

For $\lambda = 0$ $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies x_1 - x_2 = 0, x_2 - x_3 = 0 \implies x_1 = x_2, x_2 = x_3$

Letting $x_3 = 1 \implies x_2 = 1, x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 1$ $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies -x_1 + x_2 - x_3 = 0, x_2 = 0 \implies x_1 = -x_3, x_2 = 0$

Letting $x_3 = 1 \implies x_2 = 0, x_1 = -1 \therefore x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$, $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies -2x_1 - x_2 = 0, x_2 + 2x_3 = 0 \implies 2x_1 = x_2, x_2 = -2x_3$

Letting $x_3 = 1 \implies x_2 = -2, x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence $U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

$V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

Exercise:

Obtain the SVD of the following matrices

1. $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$,
2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$,
3. $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$