Numericals solution of ODE (first order)
↳ order differential Equation

* Taylor's Series method
* Runge Kutta Method

$$\frac{dy}{dx} + y = x$$

$\underbrace{y(x_0) = y_0}$ condition

ex:- $y(1) = 1.1$

x_0 - initial condition

$$\frac{dy}{dx} = x - y$$

First order equation is of the form $\frac{dy}{dx} = (x-y)$ ↳ $f(x,y)$

Taylor's Series expansion → infinite series ↳ of y about the point $x = x_0$ or in powers of $(x - x_0)$ is given by

$$y = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

1. Using Taylor series method find y at $x = 0.1, 0.2, 0.3$ considering terms upto 3rd degree given $\frac{dy}{dx} = x^2 + y^2$ & $y(0) = 1$

$y' = x^2 + y^2$ $\qquad x_0 = 0,\ y_0 = 1$

$y'(x_0) = y'(0) = 1$

$y'' = 2x + 2y \cdot y'$ $\qquad y''(0) = 2(0) + 2(1)(1) = 0 + 2 = \underline{2}$

$y''' = 2 + 2y \cdot y'' + 2y' \cdot y'$ $\qquad y'''(0) = 2 + 2(1)(2) + 2(1)(1)$

$= 2 + 4 + 2 = \underline{8}$

Taylor's equation in terms of x is given by

$$y = y(0) + x \cdot y'(1) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0)$$

$$= 1 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8)$$

$y(x) = 1 + x + \frac{2x^2}{2} + \frac{\cancel{8}^{4} x^3}{\cancel{6}_3} + \cdots$

$y(x) = 1 + x + x^2 + \frac{4x^3}{3}$

$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)}{3} = 1.\cancel{24}1123$

$y(0.2) = 1.2506$

$y(0.3) = 1.428$

② Using Taylor's Series method, solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking step size $h = 0.1$ given $y = 10$ at $x = 0$ initially considering upto 4th order degree.

$x_0 = 0, \; y_0 = 10$

$y' = x^2 + y \qquad y'(x_0) = 0 + 10 = 10$

$y'' = 2x + y' \qquad y''(x_0) = 0 + 10 = 10$

$y''' = 2 + y'' \qquad y'''(x_0) = 2 + 10 = 12$

$y^{iv} = y''' \qquad y^{iv}(x_0) = 2 + 10 = 12$

$y(x) = y(x_0) + x(y')(x_0) + \frac{x^2}{2!} y''(x_0) + \frac{x^3}{3!} y'''(x_0) + \frac{x^4}{4!} y^{iv}(x_0)$

$= 10 + 10x + \frac{x^2}{\cancel{2}_1} \times \cancel{10}^5 + \frac{x^3}{\cancel{6}_1} \times \cancel{12}^2 + \frac{x^4}{\cancel{24}_2} \cdot \cancel{12}^1$

$y(x) = 10 + 10x + 5x^2 + 2x^3 + \frac{x^4}{2}$

$y(0) = 10$

$y(0.1) = 11.0520$

$y(0.2) = 12.2165$

① Given find $5x\cdot y' + y^2 - 2 = 0$, $y(4)=1$, compute y at $x = 4.1, 4.2, 4.3$ by Taylor's series method.

$y(x) = y$.
$y(4) = 1$

$x=4$, $y=1$

$5(4)\cdot y' + 1 - 2 = 0$

$20y' - 1 = 0$

$y'(4) = 1/20$

$y' = 1/20$

differentiation

$5xy' + y^2 - 2 = 0$

$5xy'' + 5y' + 2y\cdot y' = 0$ — ①

$5(4)y'' + \cancel{5}\times\frac{1}{\cancel{20}_4} + \cancel{2}\cdot 1\times\frac{1}{\cancel{20}} = 0$

$20y'' = -\frac{1}{4} - \frac{1}{10}$

$20y'' = -\frac{52}{20}$

$y'' = -0.0175$

Diff eqn ①

$5xy''' + 5(-0.0175) + 2y\cdot y'' + 2y'\cdot y' = 0$

$5xy''' + 5(-0.0175) + 2(4)\cdot(-0.0175) + 2\,\cancel{(-0.0175)(-0.0175)}$

$5(4)y''' - 0.0875 + 2(4)\cdot(-0.0175)$

$20y''' - 0.0875 + 0.14 - 6.125$

$y'''(4) = -0.0105$

$5xy'' + 2yy' + 5y' = 0$

$5xy'' + 2(y')^2 + 2yy'' + 5y'' = 0$

$5xy''' + 5y'' + 2(y')^2 +$

Taylors series expansion of y in terms of $(x-4)$ is

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2!}y''(4) + \frac{(x-4)^3}{3!}y'''(4)$$

$$= 1 + (x-4)0.05 + \frac{(x-4)^2}{2!}(-0.0175) + \frac{(x-4)^3}{3!}(-0.105)$$

$y(4.1) = 1.0049$

$y(4.2) = 1.0097$

$y(4.3) = 1.0142$