...erties of integers: Mathematical Induction

Given any two distinct integers x, y, we know that we must have either $x < y$ or $y < x$. However, this is also true if x and y are rational numbers or real numbers.

What makes Z special in this situation?

Suppose we try to express the subset Z^+ of Z, using inequality symbol $>$ and $\geq$.

We can define the set of positive elements of Z as

$$Z^+ = \{x \in Z \mid x > 0\} \quad \text{or}$$

$$Z^+ = \{x \in Z \mid x \geq 1\}.$$

However, if we try to do likewise for rational numbers or real numbers, we find that

$Q^+ = \{x \in Q \mid x > 0\}$ and $R^+ = \{x \in R \mid x > 0\}$,

But we can't express Q^+ and R^+ using $\geq$ as in the case of Z^+.

The set Z^+ is different from the sets Q^+ & R^+ in such a way that every nonempty subset X of Z^+ contains an integer $a \leq x$, for all $x \in X$. That is X contains a smallest integer. This is not the case in either Q^+ or R^+.

The-Well Ordering Principle:

"Every non empty subset of Z^+ contains a smallest element" (Z^+ is well ordered).

This principle differentiates Z^+ from Q^+ or R^+.

Does this principle is usefull in anywhere in mathematics?

YES: Usefull in mathematical induction.

Finite induction principle or principle of Mathem. induction:

Let $S(n)$ denotes an open mathematical st. (or set of such open statements) that involves one or more occurences of the variable n, which represente a positive integer.

a) If $S(1)$ is true; and

b) If whenever $S(k)$ is true, where $k \in Z^+$, then $S(k+1)$ is true, then $S(n)$ is true for all $n \in Z^+$.

Proof: Let $S(n)$ be such an open statement satisfying conditions (a) and (b), and let $F = \{ t \in Z^+ \mid S(t) \text{ is false} \}$. We need to prove that $F = \phi$, so to obtain a contradiction we assume that $F \neq \phi$.

By the principle of Well-ordering, F has a least element s. Since $S(1)$ is true, it follows that $s \neq 1$, so $s > 1$, and consequently $s-1 \in Z^+$. With $s-1 \notin F$, we have $S(s-1)$ true. By condition (b) it follows that $S((s-1)+1) = S(s)$ is true, which contradicts that $s \in F$. This contradiction arises from the assumption that $F \neq \phi$. Consequently $F = \phi$. //

In the above statement, the condition (a) is referred to as the basis step. The condition (b) is called the inductive step.

Ex(1): For all $n \in Z^+$, $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Proof: For $n=1$, the open statement

$$S(n): \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2} \text{ becomes}$$

$$S(1): \sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

$\therefore$ $S(1)$ is true and we have our basis step.

Assume the result is true for $n=k$ (for some $k \in Z^+$).
We want to establish our inductive step by showing how the truth of $S(k)$ "forces" us to accept the truth of $S(k+1)$.
To establish the truth of $S(k+1)$, we need to show that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

we proceede as follows

$$\sum_{i=1}^{k+1} i = 1+2+\cdots+k+(k+1) = \sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Ex(2): A wheel of fortune has the numbers 1 to 36 painted on it in a random manner. Show that regardless of how the numbers are situated, three consequitive numbers total 55 or above.

Let x_1 be any nuber on the wheel, counting clockwise from x_1, label the other numbers $x_2, x_3 \cdots x_{36}$.
For the result to be false, we must have
$x_1+x_2+x_3 < 55$, $x_2+x_3+x_4 < 55, \ldots, x_{34}+x_{35}+x_{36} < 55$,
$x_{35}+x_{36}+x_1 < 55$, and $x_{36}+x_1+x_2 < 55$.
There are 36 inequalities each of terms $x_1, x_2 \cdots x_{36}$ appears (exactly) 3 times, so, each integers $1, 2, \cdots 36$ appears exactly 3 times. Adding all 36 inequalities

$$3\sum_{i=1}^{36} x_i = 3\sum_{i=1}^{36} i < 36(55) = 1980. \text{ But } \sum_{i=1}^{36} i = \frac{36(37)}{2} = 666$$

$1998 = 3(666) < 1980$ // (3)

Ex(3): prove that for each $n \in Z^+$, $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

proof: Let the open statement is $S(n)$

$$S(n): \sum_{i=1}^{n} i^2 = (n)(n+1)(2n+1)/6.$$

Basis: $n=1$

$$S(1): \sum_{i=1}^{1} i^2 = 1^2 = (1)(1+1)(2(1)+1)/6 = 6/6 = 1$$

so $S(1)$ is true.

Inductive step: Assume the truth of $S(k)$, for some $k \in Z^+$:

$$\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6 \quad \text{is true statement}$$

From the deduce the truth of

$$S(k+1): \sum_{i=1}^{k+1} i^2 = (k+1)((k+1)+1)(2(k+1)+1)/6$$

$$= (k+1)(k+2)(2k+3)/6.$$

Using the inductive hypothesis $S(k)$, we find that

$$\sum_{i=1}^{k+1} i^2 = 1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= [k(k+1)(2k+1)/6] + (k+1)^2$$

$$= (k+1)[[k](2k+1)/6 + k+1] = (k+1)[(2k^2+7k+6)/6]$$

$$= (k+1)(k+2)(2k+3)/6.$$

hence the general result follows from the principle of mathematical induction.

Recursive definitions :-

Consider the integer sequence $b_0, b_1, b_2, b_3, \ldots$, where $b_n = 2n$ for all $n \in N$. Here we find that $b_0 = 2 \cdot 0 = 0$, $b_1 = 2 \cdot 1 = 2$, $b_2 = 2 \cdot 2 = 4$ and $b_3 = 2 \cdot 3 = 6$. If, for instance, we need to determine b_6, we simply calculate $b_6 = 2 \cdot 6 = 12$ without the need to calculate other b_n for any $n \in N$.

We can perform each calculations, because we have an explicit formula, $b_n = 2n$.

Consider an another integer sequence $a_0, a_1, a_2, a_3, \ldots$, where

$a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$; for all $n \in Z^+$ where $n \geq 3$

In this case, we don't have explicit formula in terms of n for all $n \in N$ that determines a_n for example if we want calculate a_6, we need to know the values a_5, a_4 and a_3 and these values require that we also know the values of a_2, a_1 and a_0.

$$a_6 = a_5 + a_4 + a_3$$

$$= (a_4 + a_3 + a_2) + (a_3 + a_2 + a_1) + (a_2 + a_1 + a_0)$$

$$= [(a_3 + a_2 + a_1) + (a_2 + a_1 + a_0) + a_2] + [(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0)$$

$$= [[(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0) + a_2] + [(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0)$$

$$= [[(3+2+1) + 3 + 2] + (3+2+1) + 3] + [(3+2+1) + 3 + 2] + (3+2+1)$$

$$= 37$$

Ex(4):

$$S(n): \sum_{i=1}^{n} 2(i-1) = n^2 \quad \text{for all } n \in Z^+$$

Ex(5): For every well defined arithmetic expressions the number of left brackets is equal to the number of right brackets.

Ex(6): Prove each of the following by mathematical induction.

(i) $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = (n)(2n-1)(2n+1)/3$

(ii) $1\cdot3 + 2\cdot4 + 3\cdot5 + \dots + n(n+1) = (n)(n+1)(2n+7)/6$

Ex(7): Consider the following program segment

```
for i := 1 to 123 do
   for j := 1 to i do
        print i*j
```

(i) How many times is the print statement of the third line executed? $\boxed{7626}$

(ii) Replace i in the second loop by i^2, how many times the print statement executed? $\boxed{627,874}$

$$\frac{123\,(123+1)}{2} = 123 * 62 = \boxed{7626}$$

$$\frac{123\,(124)\,(246+1)}{6} = \boxed{627,874}$$

This can be computed in another easier manner,

$a_3 = a_2 + a_1 + a_0 = 3+2+1 = 6$
$a_4 = a_3 + a_2 + a_1 = 6+3+2 = 11$
$a_5 = a_4 + a_3 + a_2 = 11+6+3 = 20$
$a_6 = a_5 + a_4 + a_3 = 20+11+6 = 37$

No matter how we arrive at a_6, we realize that the two integers sequences – $b_0, b_1, b_2, b_3, \dots,$ and $a_0, a_1, a_2, \dots,$ – are more than that they are numerically distinct.

The integers $0, 2, 4, 6, \dots$ are readily listed because we have explicit formula, $b_n = 2 \cdot n$ for all $n \in \mathbb{N}$.

But to list the integers in the sequence $a_0, a_1, a_2, \dots$ we might find it difficult, because we don't have explicit formula.

What is happening here for a sequence of integers can also occur for other mathematical concepts – such as sets and binary operations.

Some times it is difficult to define a mathematical concept in an explicit manner.

But for the sequence $a_0, a_1, a_2, \dots$ we may be able to define what we want interms of similar prior results. This concept is called recursively defined.

Therefore, although we don't have explict formula for the sequence $a_0, a_1, a_2, a_3, \dots,$ we do have a way of defining the integers a_n, for $n \in \mathbb{N}$, by recursion. The assignments

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 3$$

provides a base for recursion

the equation

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \text{ for } n \in \mathbb{Z}^+ \text{ where } n \geq 3$$

provides the recursive process.

The first order linear Recurrence Relation

A geometric progression is an infinite sequence of numbers, such as 5, 15, 45, 135, ... where the division of each term, other than the first, by its immediate predessor is a Constant, called the common ratio.
In the above sequence the common ratio is 3.
$15 = 3(5)$, $45 = 3(15)$, and so on.

If $a_0, a_1, a_2, \ldots\ldots$ is a geometric progression
then $\frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n} = \cdots = r$,
the common ratio.

The geometric progression 5, 15, 45, 135, ..., we have

$\boxed{a_{n+1} = 3a_n, \quad n \geq 0}$ → is the recurrence relation

This relation does not represent a unique geometric progression. The progression 7, 21, 63, 189, ... also satisfies the relation $a_{n+1} = 3a_n$, $n \geq 0$.

To pinpoint a particular sequence described by $a_{n+1} = 3a_n$ we need to know one of the terms of that sequence.

Hence, $\boxed{a_{n+1} = 3a_n, \quad n \geq 0 \quad a_0 = 5}$

This relation uniquely defines a sequence 5, 15, 45 ...

The relation $a_{n+1} = 3a_n$, $n \geq 0$ $a_1 = 21$ identifies 7, 21, 63, ...

The equation $a_{n+1} = 3a_n, n \geq 0$ is called recurrence relation because the value a_{n+1} dependent on a_n.

If the value of a_{n+1} depends only on its immediate predessor a_n, the relation is called <u>first order</u>.

In particular, this is a <u>"first-order linear homogeneous recurrance relation with constant coefficients"</u>.

The general form is $a_{n+1} = d\,a_n, n \geq 0$ where d is constant. The values such as a_0 or a_1 are called boundary conditions. The expression $a_0 = A$ is reffered as initial condition.

<u>EX</u>: $a_{n+1} = 3a_n, \; n \geq 0, \; a_0 = 5$

First few terms in the sequence are

$$a_0 = 5$$
$$a_1 = 3a_0 = 3(5)$$
$$a_2 = 3a_1 = 3(3a_0) = 3^2(5)$$
$$a_3 = 3a_2 = 3(3^2(5)) = 3^3(5)$$

For each $n \geq 0$ $\quad a_n = 5(3^n) \Longrightarrow$ This is called a general solution.

Now if we want to know $a_{10} = 5(3^{10}) = 295,245$ no need to start at a_0 and build up to a_9.

In general if $a_{n+1} = d\,a_n, \; n \geq 0$ & $a_0 = A$ the unique solution is

$$\boxed{a_n = A \cdot d^n, \quad n \geq 0}$$

Ex ①: Solve the recurrence $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$

This is just an alternative form of $a_{n+1} = 7a_n$ for $n \geq 0$ and $a_2 = 98$.

Therefor the general solution is

$a_n = a_0(7^n)$. Since $a_2 = 98 = a_0(7^2) = 2(7^2)$.

$\therefore a_0 = 2$, and $a_n = 2 \cdot (7^n)$, $n \geq 0$ is the unique solution.

Non-linear homogeneous recurrence relation

The recurrence relation $a_{n+1} - d a_n = 0$ is called linear because each subscripted term appears to the first power (as do variables x and y in line equations). In a linear relations there are no terms like $a_{n+1} a_n$, which appears in nonlinear recurrence relations.

Ex: Find a_{12} if $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ for $n \geq 0$, and $a_0 = 2$.

Although this is not linear in a_n, if we let $b_n = a_n^2$, then the new relation becomes

$b_{n+1} = 5 b_n$ where $n \geq 0$ & $b_0 = (2)^2 = 4$.

is a linear recurrence relation.

The solution is

$b_n = 4 \cdot (5)^n$

$\therefore a_n = \sqrt{4 \cdot (5)^n} = 2 \cdot (\sqrt{5})^n$ for $n \geq 0$ and $a_{12} = 2 \cdot (\sqrt{5})^{12}$
$= \boxed{31,250}$

First order Linear nonhomogeneous recurrence relations

Ex: Bubble-sort:

$$a_n = a_{n-1} + (n-1), \quad n \geq 2 \quad a_1 = 0$$

$a_1 = 0$

$a_2 = a_1 + (2-1) = 0 + 1 = 1$

$a_3 = a_2 + (3-1) = 1 + 2$

$a_4 = a_3 + (4-1) = 1 + 2 + 3$

$\vdots$

$\ldots \ldots \ldots$

$a_n = 1 + 2 + 3 + \cdots + (n-1) = (n-1)n/2 = \frac{n^2 - n}{2}$

$f: Z^+ \to R \quad \therefore f(n) = a_n = (n^2 - n)/2$

$$\boxed{f \in O(n^2)}$$

Ex (2): Consider the sequence
0, 2, 6, 12, 20, 30, 42, ...

⇒ Here
$a_0 = 0, a_1 = 2, a_2 = 6, a_3 = 12, a_4 = 20, a_5 = 30, a_6 = 42$

and $a_1 - a_0 = 2$, $a_3 - a_2 = 6$, $a_5 - a_4 = 10$
$a_2 - a_1 = 4$, $a_4 - a_3 = 8$, $a_6 - a_5 = 12$

These calculations suggest the recurrence relation

$$\boxed{a_n - a_{n-1} = 2n, \quad n \geq 1 \text{ and } a_0 = 0}$$

Consider the following n-equations

$a_1 - a_0 = 2$
$a_2 - a_1 = 4$
$a_3 - a_2 = 6$
⋮
$a_n - a_{n-1} = 2n$

Add; these equations

$$a_n - a_0 = 2 + 4 + 6 + \cancel{8} + 10 + \cdots + 2n$$
$$= 2[1 + 2 + 3 + 4 + 5 + \cdots + n]$$
$$= 2\left(\frac{n(n+1)}{2}\right) = n^2 + n$$

$\therefore a_n = a_0 + n^2 + n$ since $a_0 = 0$

$\boxed{a_n = n^2 + n}$ for $n \in N$

Ex (3): $a_n = n \cdot a_{n-1}$, where $n \geq 1$ & $a_0 = 1$.

$a_1 = 1 \cdot a_0 = 1$
$a_2 = 2 \cdot a_1 = 1 \cdot 2$
$a_3 = 3 \cdot a_2 = 1 \cdot 2 \cdot 3$
$a_4 = 4 \cdot a_3 = 1 \cdot 2 \cdot 3 \cdot 4$
⋮
$a_n = n(1 \cdot 2 \cdot 3 \cdot \ldots (n-1)) = n!$

The Second-order Linear Homogeneous Recurrence Relation With constant Co-efficient.

Let $K \in Z^+$ and $C_n, C_{n-1}, C_{n-2}, \cdots, C_{n-K} (\neq 0)$ are real numbers. If a_n for $n \geq 0$ is a discrete function, then

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + \cdots + C_{n-K} a_{n-K} = f(n) \quad \text{--- (1)} \quad n \geq K$$

is called a linear recurrence relation with constant Coefficients of order K.

When $f(n) = 0$ for all $n \geq 0$, the relation is called homogeneous; otherwise it is called non-homogeneous.

Consider a homogeneous relation of order 2.

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \quad \text{--- (2)} \quad n \geq 2$$

We need a general solution in the form $\boxed{a_n = C r^n}$ where $C \neq 0$ & $r \neq 0$.

Substitute $a_n = Cr^n$ in eqn (2), we get

$$C_n cr^n + C_{n-1} cr^{n-1} + C_{n-2} cr^{n-2} = 0$$

With C & $r \neq 0$, this eqn becomes

$$C_n r^n + C_{n-1} r^{n-1} + C_{n-2} r^{n-2} = 0 \quad n \geq 2$$

$n=2$

$$C_n r^2 + C_{n-1} r^1 + C_{n-2} r^0 = 0$$

$$\boxed{C_2 r^2 + C_1 r + C_0 = 0} \rightarrow \text{quadratic eqn which is called a characteristic equation.}$$

The unique soln to this reccurrence relation is found as follows.

(1) Find the roots of the characteristic eqn; let they are r_1 & r_2.

Three cases.

(i) r_1, r_2 are real & distinct

(ii) r_1, r_2 are complex

(iii) r_1, r_2 are real & equal.

<u>Real and Distinct roots</u> :-

<u>Gen soln</u>: $\boxed{a_n = c_1(r_1)^n + c_2(r_2)^n}$ $n \geq 0$ & c_1 & c_2 are constants. — (3)

<u>Ex (1)</u>: Solve the reccurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0$ where $n \geq 2$ and

$a_0 = -1$

$a_1 = 8$

<u>Soln</u>: If $a_n = cr^n$ with c & $r \neq 0$

$$cr^n + cr^{n-1} - 6cr^{n-2} = 0$$

The characteristic eqn is

$$r^2 + r - 6 = 0$$

Find the roots of this eqn.

$r_1 = 2$ & $r_2 = -3$

$$\boxed{\therefore a_n = c_1(2)^n + c_2(-3)^n} \quad \text{— (4)}$$

$a_0 = -1, a_1 = 8$

$a_0 = -1 = c_1(2^0) + c_2(-3)^0 = c_1 + c_2$

$8 = a_1 = c_1(2^1) + c_2(-3)^1 = 2c_1 - 3c_2$

$c_1 = 1, \; c_2 = -2,$

$\boxed{\therefore a_n = 2^n - 2(-3)^n}$ $n \geq 0$ is a unique soln of the given reccurrence relation

Ex(7): Solve the recurrence relation

$F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0, F_1 = 1$.

Sol^n: This relation we can write as

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \text{ with } F_0 = 0, F_1 = 1.$$

The characteristic eq^n is

$$r^2 - r - 1 = 0$$

The characteristic roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

The general solution is

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solve for A & B.

$n=0$

$$F_0 = 0 = A + B \qquad \therefore B = -A$$

$n=1$ $$F_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\therefore 1 = A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right)$$

$$2 = A(1+\sqrt{5}) - A(1-\sqrt{5})$$

$$2 = A[\cancel{1} + \sqrt{5} - \cancel{1} + \sqrt{5}] = 2A\sqrt{5}$$

$$\therefore A = \frac{\cancel{2}}{\cancel{2}\cdot\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$B = -1/\sqrt{5}$$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0$$

Repeated real roots:

Let $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \cdots + c_{n-k} a_{n-k} = 0$ with $c_n (\neq 0), c_{n-1}, \cdots, c_{n-k} (\neq 0)$ real constants, and r the characteristic root of multiplicity m, where $2 \le m \le K$, then the general solution that involves r has the form

$$A_0 r^n + A_1 n r^n + A_2 \cdot n^2 \cdot r^n + A_3 \cdot n^3 \cdot r^n + \cdots + A_{m-1} n^{m-1} r^n$$

$$= r^n \left[A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \cdots + A_{m-1} n^{m-1} \right]$$

Where $A_0, A_1, A_2, \cdots, A_{m-1}$ are arbitrary constants.

Ex: if $K = 2$, a quadratic eqn with equal roots.

$$\boxed{a_n = r^n [A_0 + A_1 n]} \quad , n \geq 0$$

Ex: solve the recurrence relation

$a_{n+2} - 4a_{n-1} + 4a_n = 0$, where $n \geq 0$

or $a_n - 4a_{n-1} + 4a_{n-2} = 0$, where $n \geq 0$ $\quad a_0 = 1, a_1 = 3$

The characteristic eqn is

$r^2 - 4r + 4 = 0$, The roots are $r = r_1 = r_2 = 2$.
r is a root of multiplicity 2.

$\therefore$ The general solution is

$a_n = 2^n [c_0 + c_1 \cdot n]$, $n \geq 0$

$n=0$ $\quad a_0 = 1 = 1[c_0 + 0] \therefore \quad 1 = c_0$

$n=1$ $\quad a_1 = 3 = 2[c_0 + 1 \cdot c_1] \therefore \quad c_0 + c_1 = 3/2$

$c_1 = \frac{3}{2} - 1 = \cancel{} = 1/2$

$\cancel{}$

$\therefore a_n = (2)^n \left[1 + (\tfrac{1}{2}) n \right] = \boxed{2^n + \tfrac{1}{2} \cdot n (2^n)}$ for $n \geq 0$

Second order linear homogeneous recurrence relation with constant co-efficients.

Complex roots:

DeMoivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n \geq 0 \quad \text{—(1)}$$

If $z = x+iy \in \mathbb{C}$, $z \neq 0$, we can write

$$z = r(\cos\theta + i\sin\theta), \text{ where } r = \sqrt{x^2+y^2}$$

and $\tan\theta = y/x$, for $x \neq 0$.

If $x = 0$, then

case (1) $y > 0$

$$z = yi = yi\sin\left(\frac{\pi}{2}\right) = y\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

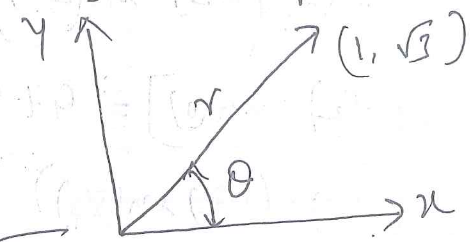
case (2) $y < 0$

$$z = yi = |y|\, i\sin\left(\frac{3\pi}{2}\right) = |y|\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$$

In all the cases

$$z^n = (x+iy)^n = r^n(\cos n\theta + i\sin n\theta), \text{ for } n \geq 0$$

Example: $(1+\sqrt{3}i)^{10}$ compute.



Here $r = \sqrt{1+(\sqrt{3})^2} = \sqrt{4} = 2$ & $\theta = \pi/3$

$\therefore$ $1+\sqrt{3}i = 2\left(\cos(\pi/3) + i\sin(\pi/3)\right)$

$$(1+\sqrt{3}i)^{10} = 2^{10}\left(\cos(10 \cdot \pi/3) + i\sin\left(10\frac{\pi}{3}\right)\right) = 2^{10}\left(\cos(4\pi/3) + i\sin(4\pi/3)\right)$$

$$= 2^{10}\left(\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)i\right) = \boxed{(-2^9)(1+\sqrt{3}i)}$$

1 Solve the following recurrence relation :-

$a_n = 2(a_{n-1} - a_{n-2})$, where $n \geq 2$ & $a_0 = 1$, $a_1 = 2$

$\Rightarrow \quad a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic eqn is $r^2 - 2r + 2 = 0$

The roots of this eqn are

$r_1 = 1+i \;, \; r_2 = 1-i.$

The general solutions is

$a_n = c_1(r_1)^n + c_2(r_2)^n = c_1(1+i)^n + c_2(1-i)^n$

where c_1 and c_2 are arbitrary constant (complex).

$x+iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2+y^2}$

$\therefore 1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \qquad r = \sqrt{1^2+1^2} = \sqrt{2}$

$1-i = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\pi/4\right)\right)$

$= \sqrt{2}\left(\cos(\pi/4) - i\sin(\pi/4)\right)$

This yields

$a_n = c_1(1+i)^n + c_2(1-i)^n$

$= c_1\left[\sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)\right]^n + c_2\left[\sqrt{2}\left(\cos(\pi/4) - i\sin(\pi/4)\right)\right]^n$

$= c_1\left[(\sqrt{2})^n\left(\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right)\right] + c_2\left[(\sqrt{2})^n\left(\cos(n\pi/4) - i\sin\left(n\pi/4\right)\right)\right]$

$a_n = (c_1+c_2)\left((\sqrt{2})^n\cos\left(\frac{n\pi}{4}\right)\right) + (c_1-c_2)i\left((\sqrt{2})^n\sin\left(\frac{n\pi}{4}\right)\right)$

$n=0$, $a_0 = 1 = \left[(c_1+c_2)\cos(0) + (c_1-c_2)i\sin(0)\right] = c_1+c_2$

$n=1$, $a_1 = 2 = \left((c_1+c_2)(\sqrt{2})\cos\left(\frac{\pi}{4}\right) + (c_1-c_2)i\left((\sqrt{2})\sin(\pi/4)\right)\right)$

$2 = \sqrt{2}\cos\left(\frac{\pi}{4}\right) + (c_1-c_2)i(\sqrt{2})(\sin(\pi/4))$

$2 = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}\right) + (c_1-c_2)i\,\sqrt{2}\cdot(1/\sqrt{2})$

$2 = 1 + (c_1-c_2)i \qquad \therefore (c_1-c_2)i = 2-1 = 1$

Third order Linear homogeneous relations

(1). Solve the recurrence relation

$2a_{n+3} - a_{n+2} - 2a_{n+1} + a_n = 0$ for $n \geq 0$

with $a_0 = 0,\ a_1 = 1,\ a_2 = 2$.

$\Rightarrow$ We can rewrite this eqn as

$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0$ for $n \geq 3$

The characteristic eqn is $2r^3 - r^2 - 2r + 1 = 0$

i-e $(2r-1)(r^2-1) = 0$

$\therefore 2r-1 = 0,\ 2r = 1,\ r_1 = 1/2$

& $r^2 - 1 = 0$ gives us $r_2 = 1$ & $r_3 = -1$.

All the three roots are real distinct.

$\therefore$ The general soln is

$$a_n = C_1(r_1)^n + C_2(r_2)^n + C_3(r_3)^n$$

Let $n = 0$,

$a_0 = 0 = C_1 + C_2 + C_3$

$n=1$ $\quad a_1 = 1 = C_1\left(\frac{1}{2}\right) + C_2 + C_3(-1)$

$n=2$ $\quad a_2 = 2 = C_1\left(\frac{1}{2}\right)^2 + C_2(1)^2 + C_3(-1)^2$

$\therefore\ C_1 + C_2 + C_3 = 0$

$\frac{1}{2}C_1 + C_2 - C_3 = 1$

$\frac{1}{4}C_1 + C_2 + C_3 = 2$

Solving these 3 simultaneous eqns we get

$C_1 = -8/3,\quad C_2 = 5/2$ & $C_3 = 1/6$

$\therefore a_n = \left(\frac{-8}{3}\right)\left(\frac{1}{2}\right)^n + \left(\frac{5}{2}\right)(1)^n + \frac{1}{6}(-1)^n$ is the required solution. //

(2) Find the general solution of the relation

$a_n - 7a_{n-2} + 10_{n-4} = 0, \quad n \geq 4$

We can rewrite this as

$a_n + 0a_{n-1} - 7a_{n-2} + 0a_{n-3} + 10a_{n-4} = 0$

$\therefore$ the characteristic eqn is

$r^4 + 0r^3 - 7r^2 + 0r + 10 = 0$

i-e $r^4 - 7r^2 + 10 = 0$

This yields

$r^2 = \frac{7 \pm \sqrt{49-40}}{2} = \frac{1}{2}(7 \pm 3)$

$\therefore r_1^2 = 5$ & $r_2^2 = 2$

$\therefore r_1 = \pm\sqrt{5}$ & $r_2 = \pm\sqrt{2}$

$\therefore \sqrt{5}, -\sqrt{5}, \sqrt{2}$ & $-\sqrt{2}$ are the distinct real roots of the eqn.

$\therefore a_n = c_1(\sqrt{5})^n + c_2(-\sqrt{5})^n + c_3(\sqrt{2})^n + c_4(-\sqrt{2})^n$

where c_1, c_2, c_3 & c_4 are constants.

Nonhomogeneous Linear Relations of second & higher order :-

Let
$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + \cdots + C_{n-k} a_{n-k} = f(n)$ for

$n \geq k \geq 2$ — (1)

where $C_n, C_{n-1}, \ldots, C_{n-k}$ are real constants with $C_n \neq 0$
and $f(n)$ is a given real-valued function of n.

The general solution of the recurrence relation (1)
is given by

$$a_n = a_n^{(h)} + a_n^{(P)} \quad \text{— (2)}$$

where $a_n^{(h)}$ is the general solution of the homogeneous part of the eqn (1) i.e $f(n) = 0$.

$a_n^{(P)}$ is any particular solution of the relation (1).

Finding $a_n^{(P)}$ for arbitrary $f(n)$ is a tedious task.

In some special cases that we can find $a_n^{(P)}$ in straight forward way.

Following are the some special cases.

(1) Suppose $f(n)$ is a polynomial of degree q and 1 is not a root of the characteristic equation of the homogeneous part of the relation (1).

In this case, $a_n^{(P)}$ is taken in the form

$$\boxed{a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q} \quad \text{— (3)}$$

where $A_0, A_1 \ldots A_q$ are constants to be evaluated using the fact that $a_n = a_n^{(P)}$

(2) Suppose $f(n)$ is a polynomial of degree q & 1 is the root of multiplicity (m) of the homogeneous part then

$$\boxed{a_n^{(P)} = n^m \{A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q\}}$$

(3) Suppose $f(n) = \alpha b^n$, where α is a constant & b is not a root then

$$\boxed{a_n^{(P)} = A_0 b^n}$$

(4) Suppose $f(n) = \alpha b^n$, where α is a constant & b is a root of multiplicity m of the C.E.

$$\boxed{a_n^{(P)} = A_0 n^m b^n}$$

Ex (1): Solve the recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8, \quad \text{for } n \geq 2$$

and $a_0 = 1$, $a_1 = 2$.

$\Rightarrow$ The homogeneous part of the given relation is $K^2 + 4K + 4 = 0$

or $(K+2)^2 = 0$ $\therefore$ The roots are $-2, -2$.

Hence $a_n^{(h)} = (A + Bn)(-2)^n$ where A & B are constants.

Keeping RHS of the relation in mind, we seek $a_n^{(P)}$ in the form

$$a_n^{(P)} = A_0$$

Substitute $a_n^{(P)}$ for a_n in given relation

$$A_0 + 4A_0 + 4A_0 = 8$$

$$\therefore A_0 = 8/9$$

The general solution is

$$a_n = a_n^{(h)} + a_n^{(P)} = (A + Bn)(-2)^n + 8/9$$

$n=0$: $a_0 = 1 = A + 8/9$, $n=1$: $a_1 = 2 = (A+B)(-2) + 8/9$

These gives $A = 1/9$ & $B = -2/3$.

Hence $\boxed{a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + 8/9}$ //

Ex ⑦: Solve the recurrence relation

$a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - n, \ n \geq 0$

$\Rightarrow$ The characteristic eqn of the given relation is

$K^2 - 10K + 21 = 0$ whose roots are 3 & 7.

$\therefore a_n^{(h)} = A(3)^n + B(7)^n$

The RHS i.e $f(n) = 3n^2 - 2$ is a polynomial of degree 2.

$\therefore a_n^{(P)} = A_0 + A_1 n + A_2 n^2$

Substitute $a_n^{(P)}$ in the relation

$\{A_0 + A_1(n+2) + A_2(n+2)^2\} - 10\{A_0 + A_1(n+1) + A_2(n+1)^2\}$
$+ 21\{A_0 + A_1 n + A_2 n^2\} = 3n^2 - 2$

$\{A_0 + A_1 n + 2A_1 + A_2 n^2 + 4A_2 + 4 \cdot n \cdot A_2\}$
$-10\{A_0 + A_1 n + A_1 + A_2 n^2 + A_2 + 2 \cdot n \cdot A_2\}$
$+21\{A_0 + A_1 n + A_2 n^2\} = 3n^2 - 2$

Equating the corresponding terms

$12A_2 = 3, \quad -16A_2 + 12A_1 = 0, \quad -6A_2 - 8A_1 + 12A_0 = -2$

These gives
$A_2 = 4, \ A_1 = 16/3, \ A_0 = 47/9$

$\therefore a_n^{(P)} = 4n^2 + \frac{16}{3}n + \frac{47}{9}$

$\therefore \boxed{a_n = A3^n + B7^n + \left(4n^2 + \frac{16}{3}n + \frac{47}{9}\right)}$ //.

Solve the recurrence relation $a_n - 3a_{n-1} = n,\ n \geq 1,\ a_0 = 1$

This relation represents an infinite set of eqns:

$n=1 \qquad a_1 - 3a_0 = 1$

$n=2 \qquad a_2 - 3a_1 = 2$

$n=3 \qquad a_3 - 3a_2 = 3$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

Multiply the first eqn by x^1; second eqn by x^2 and third eqn by x^3 and so on, we obtain

$n=1 \qquad a_1 x^1 - 3a_0 x^1 = 1x^1$

$n=2 \qquad a_2 x^2 - 3a_1 x^2 = 2x^2$

$n=3 \qquad a_3 x^3 - 3a_2 x^3 = 3x^3$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

Adding the above equations, we find that

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n \qquad \text{--- } (1)$$

We want to solve for a_n in terms of n. To accomplish this, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the ordinary generating function for the sequence $a_0, a_1, a_2, a_3, \ldots\ldots$

The equation (1) can be rewritten as

$$\left(f(x) - a_0\right) - 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} n x^n \left(= \sum_{n=0}^{\infty} n x^n\right) \qquad \text{--- } (2)$$

Since $\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=0}^{\infty} a_n x^n = f(x)$ and $a_0 = 1$, LHS of (2) becomes

$$\left(f(x) - 1\right) - 3x f(x)$$

Generating function for the sequence $0, 1, 2, 3, \ldots\ldots$ is

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots, \text{ so}$$

$$(f(x)-1) - 3x f(x) = \frac{x}{(1-x)^2}$$

&

$$f(x) = \frac{1}{(1-3x)} + \frac{x}{(1-x)^2(1-3x)}$$

Using partial fraction decomposition

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)}$$

OR

$$x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

By substituting $x = 1$

$$1 = A\underbrace{(1-1)}_{0}(1-3\cdot 1) + B(1-3\cdot 1) + C\underbrace{(1-1)^2}_{0}$$

$$1 = -2B \qquad \boxed{\therefore B = -1/2}$$

By substituting $x = 1/3$

$$\frac{1}{3} = A(1-1/3)\underbrace{(1-3\cdot\tfrac{1}{3})}_{0} + B\underbrace{(1-3\cdot\tfrac{1}{3})}_{0} + C(1-1/3)^2$$

$$\frac{1}{3} = C\left(\frac{2}{3}\right)^2 \qquad \boxed{\therefore C = 3/4}$$

By substituting $x = 0$

$$0 = A + B + C \qquad \therefore A = -B - C = -(B+C)$$

$$= -\left[-\frac{1}{2} + \frac{3}{4}\right] = -\left(\frac{-2+3}{4}\right)$$

$$\boxed{A = -1/4}$$

$$\therefore f(x) = \frac{1}{1-3x} + \frac{(-1/4)}{1-x} + \frac{(-1/2)}{(1-x)^2} + \frac{(3/4)}{(1-3x)}$$

$$f(x) = \frac{(7/4)}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2}$$

Find the co-efficient x^n, we can find a_n.

i) $(7/4)/(1-3x) = (7/4)[1/(1-3x)]$

$= (7/4)[1+3x+(3x)^2+(3x)^3+\cdots]$

$\therefore$ Co-efficient of x^n is $\boxed{(7/4)3^n}$

ii) $(-1/4)/(1-x) = (-1/4)[1+x+x^2+\cdots]$

$\therefore$ Co-efficient of x^n is $\boxed{(-1/4)}$

iii) $(-1/2)/(1-x)^2 = (-1/2)(1-x)^{-2}$

$= (-1/2)\left[\binom{-2}{0}+\binom{-2}{1}(-x)+\binom{-2}{2}(-x)^2+\binom{-2}{3}(-x)^3+\cdots\right]$

$\therefore$ Co-efficient of x^n is given by

$(-1/2)\binom{-2}{n}(-1)^n = (-1/2)(-1)^n\binom{2+n-1}{n}\cdot(-1)^n$

$= (-1/2)(n+1)$

Therefore $a_n = \left(\frac{7}{4}\right)3^n - (1/2)n - \frac{1}{2} - \frac{1}{4}$

$\boxed{a_n = (7/4)3^n - (1/2)n - (3/4)}$, $n \geq 0$

$n=1$ $\quad a_1 = \left(\frac{7}{4}\right)3 - \frac{1}{2} - \frac{3}{4}$

$= \frac{21}{4} - \frac{1}{2} - \frac{3}{4} = \frac{21-2-3}{4} = \frac{16}{4} = 4$

$a_0 = A(3)^0 + \frac{-3}{4} - \frac{1}{2}$

$a_n = A\cdot 3^n + Bn$

$\frac{-3-2}{4} = \frac{-5}{4} + A = 1$

$1 + \frac{3}{4} = A$

$7/4 = A$

$A = \frac{1+5/4}{1}$

$$a_{n+2} - 5a_{n+1} + 6a_n = 2$$

$a_0 = 3, a_1 = 7$

$a_n^{(P)} = A_0 \quad r \neq 1.$

$A_0 - 5A_0 + 6A_0 = 2$

$2A_0 = 2$

$\therefore A_0 = 2/2 = 1$

$x^2 - 5x + 6 = 0$

$r_1 = 3, \; r_2 = 2$

$a_n^{(h)} = A(3)^n + B(2)^n$

$\therefore a_n = a_n^{(h)} + a_n^{(P)}$

$a_n = A(3)^n + B(2)^n + 1$

$n = 0$

$A + B + 1 = 3$

$A + B = 3 - 1 = 2 \quad — (1)$

$n = 1$

$3A + 2B + 1 = 7$

$3A + 2B = 7 - 1 = 6 \quad — (2)$

$A + B = 2 \quad \times 3$

$\cancel{3A} + 3B = 6$

$\cancel{3A} + 2B = 6$

$B = 0$

$\therefore \cancel{A + B = 2}$

$A + 0 = 2$

$A = 2$

$$\boxed{a_n = 2(3)^n + 1}$$