Chapter-1:

The rule of sum:

If a first task can be performed in m ways while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of m+n ways.

Examples

(1) A college library has 40 textbooks on DMS and 50 textbooks on DSC. By the rule of sum, a student at this college can select among 40 + 50 = 90 textbooks in order to learn more about one or the other of these two subjects.

(2) The rule can be extended to more than two tasks as long as no pair of tasks can occur simultaneously.
For example, a teacher can recomend any of the 20 books to student to learn programming language.

5 books on C++
5 books on Fortran
5 books on pascal
5 books on Java.

The Rule of product:

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in a designated order, in mn ways.

Example ③ : HOD, Committee A Committee B

5 7

Frame the syllabus Scrutinize the sylabus.

If HOD wants to speak with any one of these members he had $5+7=12$ outcomes.

If he wants to interact with a faculty in A on Monday and with a faculty in B on Tuesday then he had $5\times7=35$ ways.

This rule also can be extended behand two tasks.

Ex ①: How many license plates can be manufactured consisting of two letters followed by four digits

(i) If no letter or digit can be repeated.

$26\times25\times10\times9\times8\times7 = 3,276,000$ diffent plates.

(ii) If repeation of letter & digits is allowed.

$26\times26\times10\times10\times10\times10 = 6,760,000$ plates.

(iii) If repeatations are allowed, how many of the plates have only vowels (A, E, I, O, U) and even digits.

$5\times5\times\cancel{5\times5\times5\times5} = 15,625$ plates

$10\times10\times10\times10 = 25\times10000 = 25,0000$

(iv) Any one letter is vowel.

$(5\times21\times10\times10\times10\times10)*2 = ?$

Permutations:

Linear arrangement of objects (distinct) is Called permutation.
Counting such that how many such arrangements of n distinct objects is possible is basically a extension of the rule of product.

EX ① A class contains 10 students, 5 of these are to be choosen and ask them to sit in a single desk. How many arrangements are possible.

$$\underset{10}{\overset{1}{}} \times \underset{9}{\overset{2}{}} \times \underset{8}{\overset{3}{}} \times \underset{7}{\overset{4}{}} \times \underset{6}{\overset{5}{}} = 30,240.$$

Definition: For an integer $n \geq 0$, n factorial (denoted $n!$) is defined by

$$0! = 1$$

$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \quad \text{for } n \geq 1$$

$$(n+1)! = (n+1)(n!)$$

ex ①: $$10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{10!}{5!}$$

$$(n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots 3 \times 2 \times 1}{(n-r)(n-r-1)\cdots 3 \times 2 \times 1}$$

$$P(n,r) = \boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

Ex-② : Consider a word COMPUTER.

1) How many permutations of these letters possible? $8!$

2) Only 5 letters are chosen $p(8,5)$

$$= \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$$

3) If repetition is allowed how many words of length 10 can be formed.

$8 \times 8 \times \cdots \times 8$ (10 times) $= 8^{10}$ words.

Ex-③ How many permutations are possible for the word BALL.

Ans: 12 not $24 = 4!$ why?

$$\frac{4!}{2!} = \boxed{12}$$

they are.

ABLL	LABL
ALBL	LALB
ALLB	LBAL
BALL	LBLA
BLAL	LLAB
BLLA	LLBA

Ex-④ : MASSASAUGA

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25,200$$ possible arrangements

$$\frac{7!}{3!\,1!\,1!\,1!\,1!} = 840$$ arrangements All 4 A's are together.

Combination :-

In general, if we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n, denoted $C(n,r)$, where $0 \le r \le n$, satisfies $(r!) \times C(n,r) = P(n,r)$ and

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}, \quad 0 \le r \le n$$

Sometimes $C(n,r)$ is denoted as $\binom{n}{r}$.

Ex. ①:

a) A student taking a DMS examination is directed to answer any seven of 10 questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}\,3!} = \boxed{120 \text{ ways.}}$$

b) If the student must answer three questions from the first five and four questions from the last five.

3 questions can be selected from the first five in $\binom{5}{3} = 10$ ways.

4 questions can be selected from the last five in $\binom{5}{4} = 5$ ways.

$\therefore$ By the rule of product $10 \times 5 = \boxed{50 \text{ ways.}}$

c) Student must ~~answ~~ 7 out of 10 where at least 3 are selected from the first five.

There are 3 cases to consider.

i) 3 from first five $\binom{5}{3} = 10$
 4 from last five $\binom{5}{4} = 5$ $\therefore 10 \times 5 = 50$

ii) 4 from first five $\binom{5}{4} = 5$
 3 from last five $\binom{5}{3} = 10$ $\therefore 5 \times 10 = 50$

iii) 5 from first five $\binom{5}{5} = 1$ $\therefore 1 \times 10 = 10$
 2 from last five $\binom{5}{2} = 10$

By the rule of sum $50 + 50 + 10 = 110$ ways the student can answer the examination

Q-2: How many number of arrangements of the
(i) letters 'TALLAHASSEE' are there.

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831,600.$$

(ii) In how many of these arrangements have no adjacent A's.

Discard A's, then

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040 \text{ ways.}$$

Consider one of 5040 arrangement as shown below

$\uparrow E \uparrow E \uparrow S \uparrow T \uparrow L \uparrow L \uparrow S \uparrow H \uparrow$

3 A's can be placed in 9 positions.

$\therefore \binom{9}{3} = 84$ ways.

$\therefore$ By the rule of product $5040 \times 84 = 423,360$

Binomial Theorem:

If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0$$

$$= \sum_{K=0}^{n} \binom{n}{K} x^K y^{n-K}$$

Special case: $n=4$, the co-efficient of x^2y^2 in the expansion of $(x+y)^4 = \underset{1^{st}}{(x+y)}\cdot\underset{2^{nd}}{(x+y)}\cdot\underset{3^{rd}}{(x+y)}\cdot\underset{4^{th}}{(x+y)}$

is the number of ways in which we can select two x's from the four x's one of which is available in each factor.

Also, we note that when we select two x's, we use two factors, leaving us with two other factors from which we can select the two y's that are needed. For ex, among the possibility we can select (1) x from the first two factors and y from the last two or (2) x from the first and third factors and y from the second and fourth.

	factor for x	factor for y
(1)	1, 2	3, 4
(2)	1, 3	2, 4
(3)	1, 4	2, 3
(4)	2, 3	1, 4
(5)	2, 4	1, 3

Consequently, the co-efficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2}=6$, the number of ways to select two distinct objects from a collection of four distinct objects.

<u>Proof</u>: In the expansion of $(x+y)^n$

$$\underset{1^{st}}{(x+y)}\cdot\underset{2^{nd}}{(x+y)}\cdots\cdots\underset{n^{th}}{(x+y)}$$

the co-efficient of $\boxed{x^k y^{n-k}}$ where $0 \le k \le n$ is the number of ways in which we can select k x's and (($n-k$) no. of y's) from n available factors

$\therefore$ The total number of such selections of size k from a collection of size n is $C(n,k)=\binom{n}{k}$, and from this binomial theorem follows.

<u>Ex ①</u>: (a) The co-efficient of x^5y^2 in the expansion $(x+y)^7$ is $\binom{7}{5}=\binom{7}{2}=21$.

* (b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$ Replace $2a$ by x and $-3b$ by y. $(x+y)^7$. x^5y^2 coefficient is 21. OR $\binom{7}{5}x^5y^2=\binom{7}{5}(2a)^5(-3b)^2$

$$=21(2)^5a^5(-3)^2b^2=(21*32*9)\,a^5b^2=\boxed{6048\,a^5b^2}$$

$$\frac{n!}{\quad}=\frac{7!}{\quad}=\frac{7\times6\times5!}{\quad}=21$$

Multinomial theorem:

For +ve integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \dots + x_t)^n$ is $\dfrac{n!}{n_1!\, n_2!\, n_3!\, n_4! \dots n_t!}$

where $0 \le n_i \le n$ & $1 \le i \le t$. & $n_1 + n_2 + \dots + n_t = n$.

$$\binom{n}{n_1, n_2, n_3 \dots n_t}$$

Ex: In the expansion of $(x+y+z)^7$

① the co-efficient of $\boxed{x^2 y^2 z^3}$ is $\binom{7}{2,2,3}$

$= \dfrac{7!}{2!\,2!\,3!} = 210$, while the co-efficient of

$\boxed{xyz^5}$ is $\binom{7}{1,1,5} = \dfrac{7!}{5!} = 42$ and the

co-efficient of $\boxed{x^3 z^4}$ is $\binom{7}{3,0,4} = \dfrac{7!}{3!\,0!\,4!} = 35$

② co-efficient of $a^2 b^3 c^2 d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$.

$v = a$, $w = 2b$, $x = -3c$, $y = 2d$ & $z = 5$

$(v + w + x + y + z)^{16}$ in this determine co-efficient of $v^2 w^3 x^2 y^5 z^4 = \binom{16}{2,3,2,5,4} = 302,702,400$

$$\binom{16}{2,3,2,5,4} (a)^2 (2b)^3 (-3c)^2 (2d)^5 (5)^4 = \boxed{435,891,456,000,000\, a^2 b^3 c^2 d^5}$$

Combination with repeatation :-

Choosing r objects from n collections, there are $\binom{n}{r}$ combinations are possible. If repetation is allowed, then there are n^r ways for an integer $r \geq 0$.

Ex:

$3^2 = 9$ ways A B C

AB, AC, AA,
BA, BC, BB,
CC, CA, CB

$nC_r = 3C_2 = \frac{3!}{2!} = 3$

$\begin{Bmatrix} AB \\ BC \\ AC \end{Bmatrix}$

$$\frac{(n+r-1)!}{r!\,(n-1)!} = \binom{n+r-1}{r} = C(n+r-1, r).$$

Ex ①: In shop 20 items are available for shopping. Each one has 12 flavours. You need to shop for 12 items.

$C(20+12-1, 12) = C(31, 12) = 141,120,525$ ways.

Ex ②: In how many ways can we distribute 7 apples and 6 oranges among 4 children such that each child receives at least one apple.

$C(4+3-1, 3) = 20$ ways to distribute 3 apples to students

$C(4+6-1, 6) = 84$ ways to distribute 6 oranges to children

$20 \times 84 = 1680$ ways.

Ex 3 : In how many ways can one distribute 10 identical marbles to six distinct containers.

⇒ It is same as that of finding the no. of non-negative integer solutions to the equations $x_1+x_2+\cdots+x_6=10$.

This is $C(6+10-1, 10) = 3003$.

Ex 4 : $x_1+x_2+\cdots+x_6 < 10$? How many sols.

$x_1+x_2+\cdots+x_6+x_7=10$

$\Downarrow$

$y_1+y_2+\cdots+y_6+y_7=9$ ✓

$x_i = y_i \quad 1 \le i \le 6$

$y_7 = x_7 - 1$

$C(7+9-1, 9) = 5005$.

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Ex 5 :

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for i ← 1 to 20 do
    for j ← 1 to i do
        for k ← 1 to j do
            print (i×j+k)
```

How many times print is executed.

$\binom{3+10-1}{10} = \binom{12}{10}$ ~~$\binom{30}{10}$~~ $\binom{20+3-1}{3} = \binom{22}{3} = 1540$

3 - loops :

r - loops $\binom{20+r-1}{r}$ times executed.

Theorem: There are $C(n+r-1, r)$ r-combination from a set with n-elements when repetition of elements is allowed.

Proof: Each r-combination of a set with n-element when repition is allowed can be represented by a list of $n-1$ bars and r stars.

The $n-1$ bars are used to mark off n different cells, with the i^{th} cell containing a star for each time the i^{th} element of the set occur in the combination.

For ex, a 6-combination of a set with 4 elements is represented with 3 bars and 6 stars.

(i) $**|*||***$ or (ii) $*|*|**|**$ Etc.

(i) represents the combination containing exactly two of the first elmt, one of the second elmt, none of the 3^{rd} elmt and three of the 4^{th} elmt.

$\therefore$ The list have $n-1+r$ positions, in these we have r positions to place r-no of stars $\therefore C(n-1+r, r)$ //.

Q: $\boxed{A's}$ $\boxed{B's}$ $\boxed{C's}$ set of 3 box's consisting of A's, B's & C's
choose 2 letters. $n=3$ A B C

$\quad$ 1st 2nd 3rd. $\quad$ 3-1=2 bars & 2 stars

$C(n+r-1, r)$

1) $*||* \Rightarrow AC$
2) $**|| \Rightarrow AA$
3) $|*|* \Rightarrow BC$
4) $||** \Rightarrow CC$
5) $*|*| \Rightarrow AB$
6) $|**| \Rightarrow BB$

Six combinations with repetition allowed.

$\therefore n=3 \quad C(3-1+2, 2)$

$r=2 \quad C(4,2) = \frac{4!}{2!\,2!}$

$= \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2!}}{\cancel{2!} \times \cancel{2!}}$

$= 6.$

H.W OR Tutorial questions:-

1) Given a positive integer n and a non-negative integer not exceeding n, find the number of r-combinations and r-permutations of a set with n-elements.
2) Given positive integers n and r, find the number of r-permutation and r-combination of a set with n-elements when repetition is allowed.
3) Given an equation $x_1 + x_2 + \dots + x_n = C$, where C is a constant, and $x_1, x_2, \dots x_n$ are non-negative integers, list all the solutions.
4) Given a positive integer n, list all the permutations of the set $\{1, 2, 3, \dots, n\}$ in lexicographic order.

5. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-combinations of the set $\{1,2,3,\ldots,n\}$ in lexicographic order.

6. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-permutations of the set $\{1,2,3,\ldots,n\}$ in lexicographic order.

7. Given a positive integer n, list all the combinations of the set $\{1,2,3,\ldots,n\}$

8. Given positive integers n and r, list all the r-combinations with repetition allowed, of the set $\{1,2,\ldots,n\}$