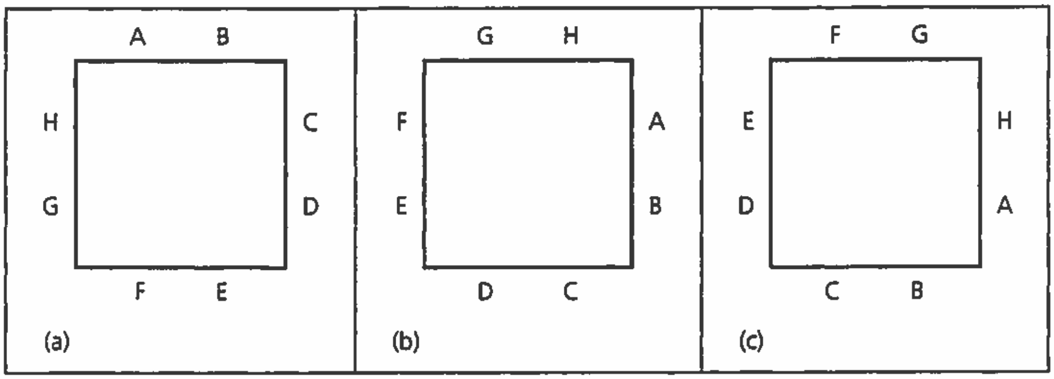

Figure 1.6

31. Determine the number of six-digit integers (no leading zeros) in which (a) no digit may be repeated; (b) digits may be repeated. Answer parts (a) and (b) with the extra condition that the six-digit integer is (i) even; (ii) divisible by 5; (iii) divisible by 4.

32. a) Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $n!/(3!)^k$ is an integer.

b) Generalize the result of part (a).

33. a) In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part (a) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

34. How many distinct four-digit integers can one make from the digits 1, 3, 3, 7, 7, and 8?

35. a) In how many ways can seven people be arranged about a circular table?

b) If two of the people insist on sitting next to each other, how many arrangements are possible?

36. a) In how many ways can eight people, denoted A, B, . . . , H be seated about the square table shown in Fig. 1.6, where Figs. 1.6(a) and 1.6(b) are considered the same but are distinct from Fig. 1.6(c)?

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other?

37. Sixteen people are to be seated at two circular tables, one of which seats 10 while the other seats six. How many different seating arrangements are possible?

38. A committee of 15—nine women and six men—is to be seated at a circular table (with 15 seats). In how many ways can the seats be assigned so that no two men are seated next to each other?

39. Write a computer program (or develop an algorithm) to determine whether there is a three-digit integer abc $(= 100a + 10b + c)$ where $abc = a! + b! + c!$.

1.3 Combinations: The Binomial Theorem

The standard deck of playing cards consists of 52 cards comprising four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards: ace, 2, 3, . . . , 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine of clubs), KD (king of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer important, then the six permutations AH–9C–KD, AH–KD–9C, 9C–AH–KD, 9C–KD–AH, KD–9C–AH, and KD–AH–9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, *with no reference to order*, corresponds to 3! permutations of three cards. In equation form this translates into

$(3!) \times$ (Number of selections of size 3 from a deck of 52)

$$= \text{Number of permutations of size 3 for the 52 cards}$$
$$= P(52, 3) = \frac{52!}{49!}.$$

Consequently, three cards can be drawn, without replacement, from a standard deck in $52!/(3!\,49!) = 22{,}100$ ways.

If we start with n distinct objects, each *selection*, or *combination*, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \le r \le n.$$

In addition to $C(n, r)$ the symbol $\binom{n}{r}$ is also frequently used. Both $C(n, r)$ and $\binom{n}{r}$ are sometimes read "n choose r." Note that for all $n \ge 0$, $C(n, 0) = C(n, n) = 1$. Further, for all $n \ge 1$, $C(n, 1) = C(n, n-1) = n$. When $0 \le n < r$, then $C(n, r) = \binom{n}{r} = 0$.

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem. When order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

EXAMPLE 1.18

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in $C(20, 11) = \binom{20}{11} = 20!/(11!\,9!) = 167{,}960$ ways. However, once the 11 arrive, how she arranges them around her rectangular dining table is an arrangement problem. Unfortunately, no part of the theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

EXAMPLE 1.19

Lynn and Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the five numbers (between 1 and 49 inclusive). This she can do in $\binom{49}{5}$ ways (since matching does *not* involve order). Meanwhile Patti selects the powerball—here there are $\binom{42}{1}$ possibilities. Consequently, by the rule of product, Lynn and Patti can select the six numbers for their PowerBall ticket in $\binom{49}{5}\binom{42}{1} = 80{,}089{,}128$ ways.

EXAMPLE 1.20

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways.}$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3} = 10$ ways, and the other four questions can be selected in $\binom{5}{4} = 5$ ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

i) The student answers three of the first five questions and four of the last five: By the rule of product this can happen in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways, as in part (b).

ii) Four of the first five questions and three of the last five questions are selected by the student: This can come about in $\binom{5}{4}\binom{5}{3} = 5 \times 10 = 50$ ways—again by the rule of product.

iii) The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that this last case can occur in $\binom{5}{5}\binom{5}{2} = 1 \times 10 = 10$ ways.

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$ selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

EXAMPLE 1.21

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in $\binom{53}{9} = 4{,}431{,}613{,}550$ ways.

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in $\binom{50}{6} = 15{,}890{,}700$ ways.

c) For a certain tournament the team must comprise four juniors and five seniors. The teacher can select the four juniors in $\binom{28}{4}$ ways. For each of these selections she has $\binom{25}{5}$ ways to choose the five seniors. Consequently, by the rule of product, she can select her team in $\binom{28}{4}\binom{25}{5} = 1{,}087{,}836{,}750$ ways for this particular tournament.

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following example demonstrates this.

EXAMPLE 1.22

The gym teacher of Example 1.21 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in $\binom{36}{9}$ ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right)$$

$$= \frac{36!}{9!\,9!\,9!\,9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

b) For an alternative solution, consider the 36 students lined up as follows:

1st student	2nd student	3rd student	...	35th student	36th student

To select the four teams, we must distribute nine A's, nine B's, nine C's, and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!\,9!\,9!\,9!}, \quad \text{as in part (a).}$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

EXAMPLE 1.23

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow E \uparrow E \uparrow S \uparrow T \uparrow L \uparrow L \uparrow S \uparrow H \uparrow$$

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of a list of $n+1$ terms like $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called the *Sigma notation* because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i.$$

Here, the letter i is called the *index* of the summation, and this index accounts for all integers starting with the *lower limit* m and continuing on up to (and including) the *upper limit* $m+n$.

We may use this notation as follows.

1) $\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j$, for there is nothing special about the letter i.

2) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2$, because $0^2 = 0$.

3) $\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + \cdots + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$.

4) $\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34) = 2(7+8+9+10) = 2\sum_{i=7}^{10} i$.

5) $\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$.

6) $\sum_{i=1}^{5} a = a + a + a + a + a = 5a$.

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 1.20 as

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = \sum_{i=3}^{5} \binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4} \binom{5}{7-j}\binom{5}{j}.$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this book.

EXAMPLE 1.24

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called *strings*, made up from a prescribed *alphabet* of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of *length* 2. Among the 27 strings of length 3 are 000, 012, 202, and 110.

In general, if n is any positive integer, then by the rule of product there are 3^n strings of length n for the alphabet 0, 1, and 2. If $x = x_1x_2x_3 \cdots x_n$ is one of these strings, we define the *weight* of x, denoted $\text{wt}(x)$, by $\text{wt}(x) = x_1 + x_2 + x_3 + \cdots + x_n$. For example, $\text{wt}(12) = 3$ and $\text{wt}(22) = 4$ for the case where $n = 2$; $\text{wt}(101) = 2$, $\text{wt}(210) = 3$, and $\text{wt}(222) = 6$ for $n = 3$.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string x contains no 1's, then each of the 10 locations in x can be filled with either 0 or 2, and by the rule of product there are 2^{10} such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}2^8$ strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table 1.2.

Table 1.2

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is $2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10} = \sum_{n=0}^{5} \binom{10}{2n}2^{10-2n}$.

Often we must be careful of *overcounting* — a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

EXAMPLE 1.25

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

   i) ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.
   ii) five of spades, seven of spades, ten of spades, seven of diamonds, and the king of diamonds.
   iii) two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

   If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in the deck that are not clubs. Consequently, she can make her selection in $\binom{39}{5}$ ways.

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were *not* counted in part (a). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

   of all five-card hands contain at least one club.

c) Can we obtain the result in part (b) in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1}\binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

   Something here is definitely *wrong*! This answer is larger than that in part (b) by more than one million hands. Did we make a mistake in part (b)? Or is something wrong with our present reasoning?

   For example, suppose that Ellen first selects

   the three of clubs

   and then selects

   the five of clubs,
   king of clubs,
   seven of hearts, and
   jack of spades.

If, however, she first selects

the five of clubs

and then selects

the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,

is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first selects

the king of clubs

and then follows this by selecting

the three of clubs,
five of clubs,
seven of hearts, and
jack of spades

is not different from the other two selections mentioned earlier.

Consequently, this approach is *wrong* because we are overcounting — by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 1.3. From the results in Table 1.3 we see, for example, that there are $\binom{13}{2}\binom{39}{3}$ five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are $\binom{13}{3}\binom{39}{2}$ such hands.

Table 1.3

Number of Clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Nonclubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

Since no two of the cases in Table 1.3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}$$

$$=\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$

$$=(13)(82{,}251)+(78)(9139)+(286)(741)+(715)(39)+(1287)(1)$$

$$=2{,}023{,}203.$$

We shall close this section with three results related to the concept of combinations.

First we note that for integers n, r, with $n \geq r \geq 0$, $\binom{n}{r} = \binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size r from a collection of n distinct objects, the selection process leaves behind $n-r$ objects. Consequently, $\binom{n}{r} = \binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size r (objects chosen) and the selections of size $n-r$ (objects left behind). An example of this correspondence is shown in Table 1.4, where $n=5$, $r=2$, and the distinct objects are 1, 2, 3, 4, and 5. This type of correspondence will be more formally defined in Chapter 5 and used in other counting situations.

Table 1.4

Selections of Size $r=2$ (Objects Chosen)		Selections of Size $n-r=3$ (Objects Left Behind)	
1. 1, 2	6. 2, 4	1. 3, 4, 5	6. 1, 3, 5
2. 1, 3	7. 2, 5	2. 2, 4, 5	7. 1, 3, 4
3. 1, 4	8. 3, 4	3. 2, 3, 5	8. 1, 2, 5
4. 1, 5	9. 3, 5	4. 2, 3, 4	9. 1, 2, 4
5. 2, 3	10. 4, 5	5. 1, 4, 5	10. 1, 2, 3

Our second result is a theorem from our past experience in algebra.

THEOREM 1.1 *The Binomial Theorem.* If x and y are variables and n is a positive integer, then

$$(x+y)^n=\binom{n}{0}x^0y^n+\binom{n}{1}x^1y^{n-1}+\binom{n}{2}x^2y^{n-2}+\cdots$$
$$+\binom{n}{n-1}x^{n-1}y^1+\binom{n}{n}x^ny^0=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Before considering the general proof, we examine a special case. If $n=4$, the coefficient of x^2y^2 in the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)}\ \underset{\text{4th factor}}{(x+y)}$$

is the number of ways in which we can select two x's from the four x's, one of which is available in each factor. (Although the x's are the same in appearance, we distinguish them as the x in the first factor, the x in the second factor, ..., and the x in the fourth factor. Also, we note that when we select two x's, we use two factors, leaving us with two other factors from which we can select the two y's that are needed.) For example, among the possibilities, we can select (1) x from the first two factors and y from the last two or (2) x from the first and third factors and y from the second and fourth. Table 1.5 summarizes the six possible selections.

Table 1.5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2

Consequently, the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2} = 6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Now we turn to the proof of the general case.

Proof: In the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)} \cdots \underset{n\text{th factor}}{(x+y)}$$

the coefficient of $x^k y^{n-k}$, where $0 \le k \le n$, is the number of different ways in which we can select k x's [and consequently $(n-k)$ y's] from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n, k) = \binom{n}{k}$, and from this the binomial theorem follows.

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient*. Notice that it is also possible to express the result of Theorem 1.1 as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}.$$

EXAMPLE 1.26

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and $\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$.

COROLLARY 1.1 For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and
b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part (a) follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part (b) results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem.*

THEOREM 1.2 For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, ..., and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.22, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\cdots-n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this product is equal to

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.* (When $t = 2$ this reduces to a binomial coefficient.)

EXAMPLE 1.27

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = 35$.
b) Suppose we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$. If we replace a by v, $2b$ by w, $-3c$ by x, $2d$ by y, and 5 by z, then we can apply the multinomial theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But $\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4(a^2b^3c^2d^5) = 435{,}891{,}456{,}000{,}000\,a^2b^3c^2d^5$.

EXERCISES 1.3

1. Calculate $\binom{6}{2}$ and check your answer by listing all the selections of size 2 that can be made from the letters a, b, c, d, e, and f.

2. Facing a four-hour bus trip back to college, Diane decides to take along five magazines from the 12 that her sister Ann Marie has recently acquired. In how many ways can Diane make her selection?

3. Evaluate each of the following.

a) $C(10, 4)$ b) $\binom{12}{7}$ c) $C(14, 12)$ d) $\binom{15}{10}$

4. In the Braille system a symbol, such as a lowercase letter, punctuation mark, suffix, and so on, is given by raising at least one of the dots in the six-dot arrangement shown in part (a) of Fig. 1.7. (The six Braille positions are labeled in this part of the figure.) For example, in part (b) of the figure the dots in positions 1 and 4 are raised and this six-dot arrangement represents the letter c. In parts (c) and (d) of the figure we have the representations for the letters m and t, respectively. The definite article "the" is shown in part (e) of the figure, while part (f) contains the form for the suffix "ow." Finally, the semicolon, ;, is given by the six-dot arrangement in part (g), where the dots at positions 2 and 3 are raised.

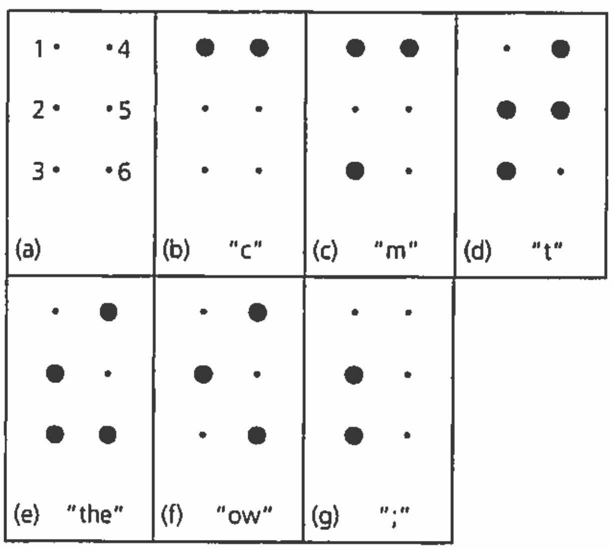

Figure 1.7

a) How many different symbols can we represent in the Braille system?

b) How many symbols have exactly three raised dots?

c) How many symbols have an even number of raised dots?

5. a) How many *permutations* of size 3 can one produce with the letters m, r, a, f, and t?

b) List all the *combinations* of size 3 that result for the letters m, r, a, f, and t.

6. If n is a positive integer and $n > 1$, prove that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

7. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions? (b) there must be six men and six women? (c) there must be an even number of women? (d) there must be more women than men? (e) there must be at least eight men?

8. In how many ways can a gambler draw five cards from a standard deck and get (a) a flush (five cards of the same suit)? (b) four aces? (c) four of a kind? (d) three aces and two jacks? (e) three aces and a pair? (f) a full house (three of a kind and a pair)? (g) three of a kind? (h) two pairs?

9. How many bytes contain (a) exactly two 1's; (b) exactly four 1's; (c) exactly six 1's; (d) at least six 1's?

10. How many ways are there to pick a five-person basketball team from 12 possible players? How many selections include the weakest and the strongest players?

11. A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if (a) there are no restrictions? (b) he must answer the first two questions? (c) he must answer at least four of the first six questions?

12. In how many ways can 12 different books be distributed among four children so that (a) each child gets three books? (b) the two oldest children get four books each and the two youngest get two books each?

13. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

14. A gym coach must select 11 seniors to play on a football team. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

15. a) Fifteen points, no three of which are collinear, are given on a plane. How many lines do they determine?

b) Twenty-five points, no four of which are coplanar, are given in space. How many triangles do they determine? How many planes? How many tetrahedra (pyramidlike solids with four triangular faces)?

16. Determine the value of each of the following summations.

a) $\sum_{i=1}^{6}(i^2+1)$ b) $\sum_{j=-2}^{2}(j^3-1)$ c) $\sum_{i=0}^{10}[1+(-1)^i]$

d) $\sum_{k=n}^{2n}(-1)^k$, where n is an odd positive integer

e) $\sum_{i=1}^{6} i(-1)^i$

17. Express each of the following using the summation (or Sigma) notation. In parts (a), (d), and (e), n denotes a positive integer.

a) $\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$, $n \geq 2$

b) $1 + 4 + 9 + 16 + 25 + 36 + 49$

c) $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + 7^3$

d) $\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$

e) $n - \left(\frac{n+1}{2!}\right) + \left(\frac{n+2}{4!}\right) - \left(\frac{n+3}{6!}\right) + \cdots$
$+ (-1)^n \left(\frac{2n}{(2n)!}\right)$

18. For the strings of length 10 in Example 1.24, how many have (a) four 0's, three 1's, and three 2's; (b) at least eight 1's; (c) weight 4?

19. Consider the collection of all strings of length 10 made up from the alphabet 0, 1, 2, and 3. How many of these strings have weight 3? How many have weight 4? How many have even weight?

20. In the three parts of Fig. 1.8, eight points are equally spaced and marked on the circumference of a given circle.

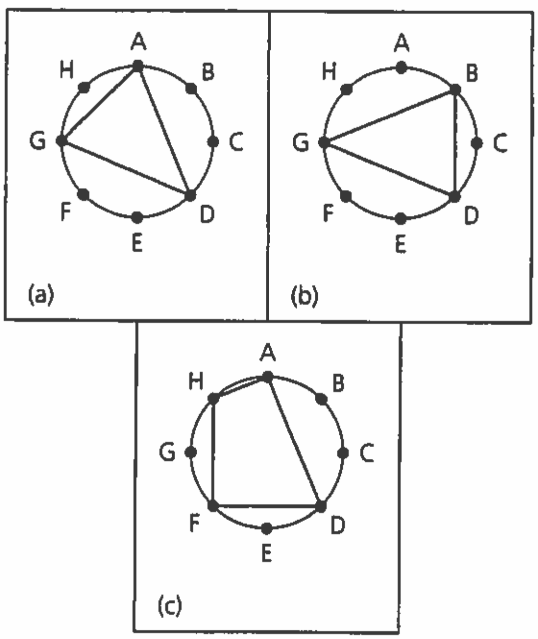

Figure 1.8

a) For parts (a) and (b) of Fig. 1.8 we have two different (though congruent) triangles. These two triangles (distinguished by their vertices) result from two selections of size 3 from the vertices A, B, C, D, E, F, G, H. How many different (whether congruent or not) triangles can we inscribe in the circle in this way?

b) How many different quadrilaterals can we inscribe in the circle, using the marked vertices? [One such quadrilateral appears in part (c) of Fig. 1.8.]

c) How many different polygons of three or more sides can we inscribe in the given circle by using three or more of the marked vertices?

21. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

22. a) In the complete expansion of $(a+b+c+d) \cdot (e+f+g+h)(u+v+w+x+y+z)$ one obtains the sum of terms such as agw, cfx, and dgv. How many such terms appear in this complete expansion?

b) Which of the following terms do *not* appear in the complete expansion from part (a)?

i) afx ii) bvx iii) chz
iv) cgw v) egu vi) dfz

23. Determine the coefficient of x^9y^3 in the expansions of (a) $(x+y)^{12}$, (b) $(x+2y)^{12}$, and (c) $(2x-3y)^{12}$.

24. Complete the details in the proof of the multinomial theorem.

25. Determine the coefficient of

a) xyz^2 in $(x+y+z)^4$

b) xyz^2 in $(w+x+y+z)^4$

c) xyz^2 in $(2x-y-z)^4$

d) xyz^{-2} in $(x-2y+3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

26. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of (a) $(w+x+y+z+1)^{10}$, (b) $(2w-x+3y+z-2)^{12}$, and (c) $(v+w-2x+y+5z+3)^{12}$.

27. Determine the sum of all the coefficients in the expansions of

a) $(x+y)^3$ b) $(x+y)^{10}$ c) $(x+y+z)^{10}$

d) $(w+x+y+z)^5$

e) $(2s-3t+5u+6v-11w+3x+2y)^{10}$

28. For any positive integer n determine

a) $\sum_{i=0}^{n} \frac{1}{i!(n-i)!}$ b) $\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!}$

29. Show that for all positive integers m and n,

$$n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}.$$

30. With n a positive integer, evaluate the sum

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n}.$$

31. For x a real number and n a positive integer, show that

a) $1 = (1+x)^n - \binom{n}{1}x^1(1+x)^{n-1}$
$+ \binom{n}{2}x^2(1+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$

b) $1 = (2+x)^n - \binom{n}{1}(x+1)(2+x)^{n-1}$
$+ \binom{n}{2}(x+1)^2(2+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}(x+1)^n$