ALGORITHM

Step I *Form a matrix A whose columns are given vectors.*

Step II *Reduce the matrix in step-I to echelon form.*

Step III *See whether all columns have pivot elements or not. If all columns have pivo elements, then given vectors are linearly independent. If there is a column not havin a pivot element, then the corresponding vector is a linear combination of the preced ing vectors and hence linearly dependent.*

In Example 2, the matrix A whose columns are v_1, v_2, v_3 is

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\therefore \qquad A \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 - R_1$$

$$\Rightarrow \qquad A \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_3 \to R_3 - R_2$$

Pivots in the echelon form of matrix A have been encircled. We observe that the third col does not have a pivot. So, the third vector v_3 is a linear combination of the first two vecto and v_2. Thus, the vectors v_1, v_2, v_3 are linearly dependent.

EXAMPLE-3 *Show that the vectors* $v_1 = (1,2,3), v_2 = (2,5,7), v_3 = (1,3,5)$ *are linearly i pendent in* $R^3(R)$.

SOLUTION Let x, y, z be scalars such that

$$xv_1 + yv_2 + zv_3 = 0$$

$$\Rightarrow \quad x(1,2,3) + y(2,5,7) + z(1,3,5) = (0,0,0)$$

$$\Rightarrow \quad (x + 2y + z, 2x + 5y + 3z, 3x + 7y + 5z) = (0,0,0)$$

$$\Rightarrow \quad x + 2y + z = 0,\ 2x + 5y + 3z = 0,\ 3x + 7y + 5z = 0$$

The determinant of the coefficient matrix A of the above homogeneous system of equations is given by

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{vmatrix} = 1 \neq 0$$

So, the above system of equations has trivial solution only, i.e. $x = y = z = 0$.

Hence, the given vectors are linearly independent in $R^3(R)$.

EXAMPLE-4 *Show that the vectors* $v_1 = (1,1,2,4), v_2 = (2,-1,-5,2), v_3 = (1,-1,-4,0)$ *and* $v_4 = (2,1,1,6)$ *are linearly dependent in* $R^4(R)$.

SOLUTION Let x,y,z,t be scalars in R such that

$$xv_1 + yv_2 + zv_3 + tv_4 = 0$$

$$\Rightarrow \quad x(1,1,2,4) + y(2,-1,-5,2) + z(1,-1,-4,0) + t(2,1,1,6) = 0$$

$$\Rightarrow \quad (x+2y+z+2t, x-y-z+t, 2x-5y-4z+t, 4x+2y+0z+6t) = (0,0,0,0)$$

$$\Rightarrow \quad x+2y+z+2t = 0,\ x-y-z+t = 0,\ 2x-5y-4z+t = 0,\ 4x+2y+0z+6t = 0$$

The coefficient matrix A of the above homogeneous system of equations is

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 4R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - 3R_2,\ R_4 \to R_4 - 2R_2$$

Clearly, rank of $A = 2 <$ Number of unknowns. So, the above system has non-trivial solutions. Hence, given vectors are linearly dependent in $R^4(R)$.

Aliter The matrix A whose columns are v_1, v_2, v_3, v_4 is

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$$

The echelon form of matrix A is

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe that the third and fourth columns does not have pivots. So, third and fourth vectors are linear combinations of first two vectors v_1 and v_2. Thus, the vectors are linearly dependent.

EXAMPLE-5 *Determine whether the vectors* $f(x) = 2x^3 + x^2 + x + 1, g(x) = x^3 + 3x^2 + x - 2$ *and* $h(x) = x^3 + 2x^2 - x + 3$ *in the vector space* $R[x]$ *of all polynomials over the real number field are linearly independent or not.*

SOLUTION Let a, b, c be real numbers such that

$$af(x) + bg(x) + ch(x) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad a(2x^3 + x^2 + x + 1) + b(x^3 + 3x^2 + x - 2) + c(x^3 + 2x^2 - x + 3) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad (2a + b + c)x^3 + (a + 3b + 2c)x^2 + (a + b - c)x + (a - 2b + 3c) = 0 \quad \text{for all } x$$

$$\Rightarrow \quad 2a + b + c = 0,\ a + 3b + 2c = 0,\ a + b - c = 0,\ a - 2b + 3c = 0$$

The coefficient matrix A of the above system of equations is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix} \text{ Applying } R_1 \leftrightarrow R_3$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 3 \\ 0 & -3 & 4 \end{bmatrix} \text{ Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 9 \\ 0 & -1 & 3 \\ 0 & 0 & -5 \end{bmatrix} \text{ Applying } R_2 \to R_2 + 2R_3,\ R_4 \to R_4 - 3R_3$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 9 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ Applying } R_4 \to R_4 + \frac{5}{9} R_2$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix} \text{ Applying } R_2 \leftrightarrow R_3$$

Clearly, rank of A is equal to 3 which is equal to the number of unknowns a, b, c. So, the system has trivial solution only, i.e. $a = b = c = 0$.

So, $f(x), g(x), h(x)$ are linearly independent in $R[x]$.

EXAMPLE-6 *Let V be the vector space of functions from R into R. Show that the functions $f(t) = \sin t, g(t) = e^t, h(t) = t^2$ are linearly independent in V.*

SOLUTION Let x, y, z be scalars such that

$$x f(t) + y\, g(t) + z\, h(t) = 0 \text{ for all } t \in R$$

Putting $t = 0, \pi$ and $\frac{\pi}{2}$, respectively we get

$$x f(0) + y\, g(0) + z h(0) = 0$$
$$x f(\pi) + y\, g(\pi) + z\, h(\pi) = 0$$
$$x f\left(\frac{\pi}{2}\right) + y\, g\left(\frac{\pi}{2}\right) + z\, h\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad y = 0,\ y\, e^{\pi} + z\, \pi^2 = 0, x + y\, e^{\frac{\pi}{2}} + h\, \frac{\pi^2}{4} = 0$$

$$\Rightarrow \quad y = 0, z = 0, x = 0$$

Thus,

$$x f(t) + y g(t) + z h(t) = 0 \Rightarrow x = y = z = 0$$

Hence, $f(t), g(t), h(t)$ are linearly independent.