ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Show that the mapping $t : R^2 \to R^3$ given by*

$$t(a,b) = (a+b, a-b, b) \quad \textit{for all } (a,b) \in R^2,$$

is a linear transformation. Find the range, rank, kernel and nullity of t.

SOLUTION For any $x=(a,b)$, $y=(c,d)\in R^2$ and any $\lambda,\mu\in R$, we have

$$t(\lambda x+\mu y)=t(\lambda a+\mu c,\lambda b+\mu d)$$
$$\Rightarrow\quad t(\lambda x+\mu y)=(\lambda a+\mu c+\lambda b+\mu d,\lambda a+\mu c-\lambda b-\mu d,\lambda b+\mu d)$$
$$\Rightarrow\quad t(\lambda x+\mu y)=(\lambda a+\lambda b+\mu c+\mu d,\lambda a-\lambda b+\mu c-\mu d,\lambda b+\mu d)$$
$$\Rightarrow\quad t(\lambda x+\mu y)=(\lambda a+\lambda b,\lambda a-\lambda b,\lambda b)+(\mu c+\mu d,\mu c-\mu d,\mu d)$$
$$\Rightarrow\quad t(\lambda x+\mu y)=\lambda(a+b,a-b,b)+\mu(c+d,c-d,d)$$
$$\Rightarrow\quad t(\lambda x+\mu y)=\lambda t(x)+\mu t(y)$$

$\therefore\quad t:R^2\to R^3$ is a linear transformation.

Since $B=\{(1,0),(0,1)\}$ is a basis for R^2 and $t(1,0)=(1,1,0), t(0,1)=(1,-1,1)$. Therefore, $B'=\{t(1,0),t(0,1)\}=\{(1,1,0),(1,-1,1)\}$ spans range $=I_m(t)$ (See Theorem 3 on page 249). Moreover, B' is a linearly independent set, because

$$\lambda(1,1,0)+\mu(1,-1,1)=(0,0,0)\Rightarrow\lambda=\mu=0$$

Hence, B' forms a basis for range(t).

$\therefore\qquad \text{rank}(t)=\dim(I_m(t))=\dim(\text{range}(t))=2$

and, $\qquad \text{nullity}(t)=\dim R^2-\text{rank}(t)=2-2=0.$

Thus, $\qquad \text{Ker}(t)=\text{Null Space}=\{(0,0)\}.$

EXAMPLE-2 *Let C be the field of complex numbers, and let $t:C^3\to C^3$ be a mapping given by*

$$t(a,b,c)=(a-b+2c,2a+b-c,-a-2b).$$

Show that t is a linear transformation and find its kernel.

SOLUTION It can be easily checked that

$t[\lambda(a,b,c)+\mu\,(x,y,z)]=\lambda t(a,b,c)+\mu t(x,y,z)$ for all $(a,b,c),(x,y,z)\in C^3$ and all $\lambda,\mu\in C$.

Therefore, $t:C^3\to C^3$ is a linear transformation.

$$\text{Ker}(t)=\{(a,b,c):\ t(a,b,c)=(0,0,0)\}$$
$$\therefore\quad (a,b,c)\in\text{Ker}(t)$$
$$\Leftrightarrow\quad t(a,b,c)=(0,0,0)$$
$$\Leftrightarrow\quad (a-b+2c,2a+b-c,-a-2b)=(0,0,0)$$
$$\Leftrightarrow\quad a-b+2c=0,2a+b-c=0,-a-2b=0$$

This is a homogeneous system of equations whose solution space is the kernel of t.

The coefficient matrix A given by

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & -3 & 2 \end{bmatrix} \text{ Applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & -3 \end{bmatrix} \text{ Applying } R_3 \to R_3 + R_2$$

Clearly, $\text{rank}(A) = 3 =$ Number of unknowns

So, it has only the trivial solution $a = b = c = 0$.

Hence, $\text{Ker}(t) = \{(0,0,0)\} =$ Null space.

EXAMPLE-3 *Let $t : R^4 \to R^3$ be the linear transformation defined by*

$$t(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$$

Find a basis and dimension of (i) the image of t (ii) the kernel of t.

SOLUTION (i) We know that if vectors $v_1, v_2, \ldots, v_n$ span a vector space $V(F)$ and $t : V \to V'$ is a linear transformation, then $t(v_1),\ t(v_2), \ldots, t(v_n)$ span $I_m(t)$. Consider vectors $e_1^{(4)} = (1,0,0,0)$, $e_2^{(4)} = (0,1,0,0)$, $e_3^{(4)} = (0,0,1,0)$ and $e_4^{(4)} = (0,0,0,1)$ forming standard basis of R^4. Then, $t(e_1^{(4)})$, $t(e_2^{(4)})$, $t(e_3^{(4)})$, $t(e_4^{(4)})$ span $I_m(t)$.

We have, $t(e_1^4) = (1,1,1)$, $t(e_2^{(4)}) = (-1,0,1)$, $t(e_3^{(4)}) = (1,2,3)$, $t(e_4^{(4)}) = (1,-1,-3)$

In order to find the basis and dimension of $I_m(t)$, let us form a matrix A whose rows are images of $e_1^{(4)}, e_2^{(4)}, e_3^{(4)}, e_4^{(4)}$.

We have,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

Let us now reduce A to echelon form by using elementary row operations:

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \text{ Applying } R_2 \to R_2 + R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - R_2, R_4 \to R_4 + 2R_2$$

Clearly, there are two non-zero rows. So, $v_1 = (1,1,1)$, $v_2 = (0,1,2)$ form a basis of $I_m(t)$, hence $\dim(I_m(t)) = 2$.

(ii) Let $v = (x,y,z,t)$ be an arbitrary element in $\text{Ker}(t)$. Then,

$t(v) = (0,0,0)$

$\Rightarrow \quad (x-y+z+t,\ x+2z-t,\ x+y+3z-3t) = (0,0,0)$

$\Rightarrow \quad x-y+z+t = 0,\ x+2z-t = 0,\ x+y+3z-3t = 0$

$\text{Ker}(t)$ is the solution space of the above homogeneous system of equations which can be written in matrix form as follows:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1.$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 + (-2)R_2$$

We observe that there are two non-zero rows in the coefficient matrix, so its rank is 2. So, there are two (= Number of variables – rank) free variables.

The homogeneous system is equivalent to

$$x-y+z+t = 0,\ y+z-2t = 0$$

Taking z and t as free variables, we obtain the following solutions:
(i) $z = -1,\ t = 0,\ y = 1,\ x = 2$ (ii) $z = 0,\ t = 1,\ y = 2,\ x = 1$

Thus, $(2,1,-1,0)$ and $(1,2,0,1)$ form a basis of $\text{Ker}(t)$

EXAMPLE-4 *Let $t : R^3 \to R^3$ be the linear transformation defined by*

$$t(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z)$$

Find a basis and the dimension of (i) the image of t (ii) the kernel of t.

SOLUTION (i) The standard basis of R^3 is $\{e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\}$, where $e_1^{(3)} = (1,0,0)$, $e_2^{(3)} = (0,1,0)$ and $e_3^{(3)} = (0,0,1)$.

$$\therefore \quad t\left(e_1^{(3)}\right) = (1,0,1),\ t\left(e_2^{(3)}\right) = (2,1,1),\ t\left(e_3^{(3)}\right) = (-1,1,-2)$$

Since $t\left(e_1^{(3)}\right), t\left(e_2^{(3)}\right), t\left(e_3^{(3)}\right)$ span $I_m(t)$. So, to find a basis and the dimension of $I_m(t)$, we form a matrix A whose rows are $t\left(e_1^{(3)}\right), t\left(e_2^{(3)}\right), t\left(e_3^{(3)}\right)$ as given below.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

Now we reduce A to echelon form as shown below:

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - R_2$$

Thus, $v_1 = (1,0,1)$ and $v_2 = (0,1,-1)$ form a basis of $I_m(t)$ and hence $\dim(I_m(t)) = 2$.

(ii) Let $v = (x,y,z) \in \text{Ker}(t)$. Then,

$$t(v) = (0,\ 0,\ 0)$$

$$\Rightarrow \quad (x + 2y - z,\ y + z,\ x + y - 2z) = (0,\ 0,\ 0)$$

$$\Rightarrow \quad x + 2y - z = 0,\ y + z = 0,\ x + y - 2z = 0$$

Kernel of t is the solution space of this homogeneous system of equations which can be written in matrix form as follows:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - R_1$$

or, $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\quad x+2y-z=0,\ y+z=0$

Clearly, rank of the coefficient matrix is 2. So, there is one free variable and hence $\dim(\mathrm{Ker}(t)) = 1$. Taking z as the free variable and setting $z=1$, we get $y=-1$ and $x=3$. Hence, $(3,-1,1)$ forms a basis of $\mathrm{Ker}(t)$.

EXAMPLE-5 *Let $t: R^4 \to R^3$ be a linear transformation defined by*

$$t(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d).$$

Then find the basis and dimension of the (i) $I_m(t)$, (ii) $\mathrm{Ker}(t)$.

SOLUTION (i) By definition, $I_m(t) = \{t(v) : v \in R^4\}$. Since the set $\{e_1^{(4)}, e_2^{(4)}, e_3^{(4)}, e_4^{(4)}\}$, being standard basis, spans R^4. Therefore, $t(e_1^{(4)})=(1,1,1)$, $t(e_2^{(4)})=(-1,0,1)$, $t(e_3^{(4)})=(1,2,3)$ and $t(e_4^{(4)})=(1,-1,-3)$ span $I_m(t)$.

To find the basis of $I_m(t)$, we will now find linearly independent vectors from the set $\{t(e_1^{(4)}), t(e_2^{(4)}), t(e_3^{(4)}), t(e_4^{(4)})\}$, which spans $I_m(t)$. For this, form a matrix A whose rows are $t(e_1^{(4)}), t(e_2^{(4)}), t(e_3^{(4)}), t(e_4^{(4)})$, i.e.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}.$$

Applying elementary transformations this matrix reduces to the following echelon matrix

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The non-zero rows of the above echelon matrix form a basis of $I_m(t)$. Hence, $\{(1,1,1), (0,1,2)\}$ is a basis of $I_m(t)$. Hence, $\dim(I_m(t)) = \mathrm{rank}(t) = 2$.

(ii) By definition, $\mathrm{Ker}(t) = \{v \in R^4 : t(v) = 0 \in R^3\}$.

If $v \in \mathrm{Ker}(t) = 0 \in R^3$ for all $v \in R^4$.

$\therefore \quad t(a,b,c,d) = (0,0,0)$ for all $a,b,c,d \in R$

$\Rightarrow \quad a-b+c+d=0,\ a+2c-d=0,\ a+b+3c-3d=0.$ (i)

Forming the matrix of coefficients of the above system of equations and reducing it to echelon form, we obtain

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the system of equations given in (i) is equivalent to the system

$$\left.\begin{array}{r} a-b+c+d=0 \\ b+c+d=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} b=2d-c \\ a=-2c+d \end{array}\right\} \qquad \text{(ii)}$$

From (ii), we observe that the values of a and b depend upon c and d and hence c and d are two free variables. Therefore,

$$\operatorname{Ker}(t)=\{(-2c+d,\ 2d-c,\ c,d): c,d\in R\}$$

$$\Rightarrow \text{Nullity}(t)=\dim(\operatorname{Ker}(t))=\text{Number of free variables}=2.$$

To find a basis of $\operatorname{Ker}(t)$, we give arbitrary values to c and d. For example, for $c=0,\ d=1$, we have $a=1,\ b=2$ and for $c=1,\ d=0$, we have $a=-2,\ b=-1$. Therefore, $\{(1,2,0,1), (-2,-1,1,0)\}$ is a basis for $\operatorname{Ker}(t)$.