2.6 LINEAR COMBINATIONS

LINEAR COMBINATION. *Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n$ be n vectors in V and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n scalars in F. Then the vector $\lambda_1 v_1 + \cdots + \lambda_n v_n$ (or $\sum_{i=1}^{n} \lambda_i v_i$) is called a linear combination of $v_1, v_2, \ldots, v_n$. It is also called a linear combination of the set $S = \{v_1, v_2, \ldots, v_n\}$. Since there are finite number of vectors in S, it is also called a finite linear combination of S.*

If S is an infinite subset of V, then a linear combination of a finite subset of S is called a finite linear combination of S.

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Express $v = (-2, 3)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (1, 1)$ and $v_2 = (1, 2)$.*

SOLUTION Let x, y be scalars such that

$$
\begin{aligned}
& v = xv_1 + yv_2 \\
\Rightarrow\quad & (-2,3) = x(1,1) + y(1,2) \\
\Rightarrow\quad & (-2,3) = (x+y, x+2y) \\
\Rightarrow\quad & x+y = -2 \text{ and } x+2y = 3 \\
\Rightarrow\quad & x = -7, y = 5
\end{aligned}
$$

Hence, $v = -7v_1 + 5v_2$.

EXAMPLE-2 *Express $v = (-2, 5)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (-1, 1)$ and $v_2 = (2, -2)$.*

SOLUTION Let x, y be scalars such that

$$
\begin{aligned}
& v = xv_1 + yv_2 \\
\Rightarrow\quad & (-2,5) = x(-1,1) + y(2,-2) \\
\Rightarrow\quad & (-2,5) = (-x+2y, x-2y) \\
\Rightarrow\quad & -x+2y = -2 \text{ and } x-2y = 5
\end{aligned}
$$

This is an inconsistent system of equations and so it has no solution.

Hence, v cannot be written as the linear combination of v_1 and v_2.

EXAMPLE-3 *Express $v = (1, -2, 5)$ in R^3 as a linear combination of the following vectors:*

$$v_1 = (1,1,1), v_2 = (1,2,3), v_3 = (2,-1,1)$$

ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Express* $v = (-2,3)$ *in* $R^2(R)$ *as a linear combination of the vectors* $v_1 = (1,1)$ *and* $v_2 = (1,2)$.

SOLUTION Let x, y be scalars such that

$$
\begin{aligned}
& v = xv_1 + yv_2 \\
\Rightarrow \quad & (-2,3) = x(1,1) + y(1,2) \\
\Rightarrow \quad & (-2,3) = (x+y, x+2y) \\
\Rightarrow \quad & x+y = -2 \text{ and } x+2y = 3 \\
\Rightarrow \quad & x = -7, y = 5
\end{aligned}
$$

Hence, $v = -7v_1 + 5v_2$.

EXAMPLE-2 *Express* $v = (-2,5)$ *in* $R^2(R)$ *as a linear combination of the vectors* $v_1 = (-1,1)$ *and* $v_2 = (2,-2)$.

SOLUTION Let x, y be scalars such that

$$
\begin{aligned}
& v = xv_1 + yv_2 \\
\Rightarrow \quad & (-2,5) = x(-1,1) + y(2,-2) \\
\Rightarrow \quad & (-2,5) = (-x+2y, x-2y) \\
\Rightarrow \quad & -x+2y = -2 \text{ and } x-2y = 5
\end{aligned}
$$

This is an inconsistent system of equations and so it has no solution.

SOLUTION Let x, y, z be scalars in R such that

$$v = xv_1 + yv_2 + zv_3$$

$$\Rightarrow \quad (1,-2,5) = x(1,1,1) + y(1,2,3) + z(2,-1,1)$$

$$\Rightarrow \quad (1,-2,5) = (x+y+2z,\ x+2y-z,\ x+3y+z)$$

$$\Rightarrow \quad x+y+2z = 1, x+2y-z = -2, x+3y+z = 5$$

This system of equations can be written in matrix as follows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \text{ Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

or,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix} \text{ Applying } R_3 \to R_3 - 2R_2$$

This is equivalent to

$$x+y+2z = 1,\ y-3z = -3,\ 5z = 10$$

$$\Rightarrow \quad x = -6,\ y = 3,\ z = 2$$

Hence, $v = -6v_1 + 3v_2 + 2v_3$.

EXAMPLE-4 *Express* $v = (2,-5,3)$ *in* $R^3(R)$ *as a linear combination of the vectors* $v_1 = (1,-3,2), v_2 = (2,-4,-1), v_3 = (1,-5,7)$.

SOLUTION Let x, y, z be scalars such that

$$v = xv_1 + yv_2 + zv_3$$

$$\Rightarrow \quad (2,-5,3) = x(1,-3,2) + y(2,-4,-1) + z(1,-5,7)$$

$$\Rightarrow \quad (2,-5,3) = (x+2y+z,\ -3x-4y-5z,\ 2x-y+7z)$$

$$\Rightarrow \quad x+2y+z = 2,\ -3x-4y-5z = -5,\ 2x-y+7z = 3$$

This system of equations, in matrix form, can be written as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ Applying } R_2 \to R_2 + 3R_1, R_3 \to R_3 + (-2)R_1$$

$$\text{or, } \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix} \text{ Applying } R_3 \to R_3 + \frac{5}{2}R_2$$

This is an inconsistent system of equations and so has no solution. Hence, v cannot be written as a linear combination of v_1, v_2 and v_3.

EXAMPLE-5 *Express the polynomial $f(x) = x^2 + 4x - 3$ in the vector space V of all polynomials over R as a linear combination of the polynomials $g(x) = x^2 - 2x + 5, h(x) = 2x^2 - 3x$ and $\phi(x) = x + 3$.*

SOLUTION Let α, β, γ be scalars such that

$$f(x) = u\, g(x) + v\, h(x) + w\, \phi(x) \quad \text{for all } x \in R$$

$$\Rightarrow \quad x^2 + 4x - 3 = u(x^2 - 2x + 5) + v(2x^2 - 3x) + w(x + 3 \quad \text{for all } x \in R$$

$$\Rightarrow \quad x^2 + 4x - 3 = (u + 2v)x^2 + (-2u - 3v + w)x + (5u + 3w) \quad \text{for all } x \in R \qquad \text{(i)}$$

$$\Rightarrow \quad u + 2v = 1, -2u - 3v + w = 4, 5u + w = -3 \qquad \text{(ii)}$$

The matrix form of this system of equations is

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

This system of equations is equivalent to

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -10 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -8 \end{bmatrix} \text{ Applying } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 5R_1$$

$$\text{or, } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 52 \end{bmatrix} \text{ Applying } R_3 \to R_3 + 10R_2$$

Thus, the system of equations obtained in (i) is consistent and is equivalent to

$$u + 2v = 1, v + w = 6 \text{ and } 13w = 52 \Rightarrow \quad u = -3, v = 2, w = 4$$

Hence,
$$f(x) = -3g(x) + 2h(x) + 4\phi(x).$$

REMARK. *The equation (i) obtained in the above solution is an identity in x, that is it holds for any value of x. So, the values of u, v and w can be obtained by solving three equations which can be obtained by given any three values to variable x.*

EXAMPLE-6 *Let $V = R^{2\times 2}$ be the vector space of all 2×2 matrices (with entries in R) over field R and let*

$$M=\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}, A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } C=\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

be three matrices in V. Express matrix M as a linear combination of A, B and C.

SOLUTION Let x, y, z be three scalars in R such that

$$M = xA + yB + zC$$

$$\Rightarrow \quad \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + z\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} x+y+z & x+2y+z \\ x+3y+4z & x+4y+5z \end{bmatrix}$$

$$\Rightarrow \quad x+y+z=4,\ x+2y+z=7,\ x+3y+4z=7,\ x+4y+5z=9 \ldots \text{(i)}$$

The matrix form of this system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 7 \\ 9 \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 5 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1$$

or,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -3 \\ -4 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - 2R_2$$

Thus, the system of equations in (i) is equivalent to the following system of equations

$$x+y+z=4, y=3, 3z=-3, 4z=-4$$

$$\therefore \quad x=2,\ y=3,\ z=-1$$

Hence, $M = 2A + 3B - C$

EXAMPLE-7 *Consider the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$ in $R^3(R)$. Find k so that $u = (1,k,4)$ is a linear combination of v_1 and v_2.*

SOLUTION Let x, y be scalars such that

$$u = xv_1 + yv_2$$
$$\Rightarrow \quad (1,k,4) = x(1,2,3) + y(2,3,1)$$
$$\Rightarrow \quad (1,k,4) = (x+2y, 2x+3y, 3x+y)$$
$$\Rightarrow \quad x+2y = 1, 2x+3y = k \text{ and } 3x+y = 4$$

Solving $x+2y=1$ and $3x+y=4$, we get $x = \frac{7}{5}$ and $y = -\frac{1}{5}$

Substituting these values in $2x+3y=k$, we get $k = \frac{11}{5}$

EXAMPLE-8 *Consider the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$ in $R^3(R)$. Find conditions on a,b,c so that $u = (a,b,c)$ is a linear combination of v_1 and v_2.*

SOLUTION Let x, y be scalars in R such that

$$u = xv_1 + yv_2$$
$$\Rightarrow \quad (a,b,c) = x(1,2,3) + y(2,3,1)$$
$$\Rightarrow \quad (a,b,c) = (x+2y, 2x+3y, 3x+y)$$
$$\Rightarrow \quad x+2y = a, 2x+3y = b \text{ and } 3x+y = c$$

Solving first two equations, we get $x = -3a+2b$ and $y = 2a-b$.

Substituting these values in $3x+y=c$, we get

$3(-3a+2b)+(2a-b) = c$ or, $7a-5b+c=0$ as the required condition.

EXERCISE 2.6

1. Express $v = (3,-2)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (-1,1)$ and $v_2 = (2,1)$.
2. Express $v = (-2,5)$ in $R^2(R)$ as a linear combination of the vectors $v_1 = (2,-1)$ and $v_2 = (-4,2)$.
3. Express $v = (3,7,-4)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (1,2,3)$, $v_2 = (2,3,7)$ and $v_3 = (3,5,6)$.
4. Express $v = (2,-1,3)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (3,0,3)$, $v_2 = (-1,2,-5)$ and $v_3 = (-2,-1,0)$.
5. Let V be the vector space of all real polynomials over field R of all real numbers. Express the polynomial $p(x) = 3x^2+5x-5$ as a linear combination of the polynomials $f(x) = x^2+2x+1, g(x) = 2x^2+5x+4$ and $h(x) = x^2+3x+6$.
6. Let $V = P_2(t)$ be the vector space of all polynomials of degree less than or equal to 2 and t be the indeterminate. Write the polynomial $f(t) = at^2+bt+c$ as a linear combination of the polynomials $p_1(t) = (t-1)^2, p_2(t) = t-1$ and $p_3(t) = 1$.
7. Write the vectors $u = (1,3,8)$ in $R^3(R)$ as a linear combination of the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,1)$.