ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Apply the Gram-Schmidt orthogonalization process to the basis* $B = \{(1, 0, 1), (1, 0, -1), (0, 3, 4)\}$ *of the inner product space* R^3 *to find an orthogonal and an orhtonormal basis of* R^3.

SOLUTION Let $v_1 = (1, 0, 1)$, $v_2 = (1, 0, -1)$ and $v_3 = (0, 3, 4)$. Further, let $w_1 = v_1 = (1, 0, 1)$.

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - 0 = v_2 = (1, 0, -1) \qquad [\because \langle v_2, w_1 \rangle = 0]$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\Rightarrow \quad w_3 = v_3 - \frac{4}{2} v_1 + \frac{4}{2} v_2$$

$$\Rightarrow \quad w_3 = v_3 - 2v_1 + 2v_2$$

$$\Rightarrow \quad w_3 = (0, 3, 4) + (-2, 0, -2) + (2, 0, -2) = (0, 3, 0)$$

Thus, $\{w_1, w_2, w_3\}$ is an orthogonal basis of R^3.

In order to obtain an orthonormal basis of R^3, let us normalize w_1, w_2, w_3

We have,

$$||w_1||^2 = 2, \ ||w_2||^2 = 2 \quad \text{and} \quad ||w_3||^2 = 9$$

Let $\quad u_i = \dfrac{w_i}{||w_i||}; \quad i = 1,2,3.$

Then, $\quad u_1 = \left(\dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}}\right), \ u_2 = \left(\dfrac{1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}\right), \ u_3 = (0,1,0)$

form an orthonormal basis for R^3.

EXAMPLE-2 *Let $V = P_3(t)$ be the vector space of all polynomials $f(t)$ of degree less than of equal to 3 with inner product defined by* $< f,g >= \int_{-1}^{1} f(t)\, g(t)dt.$

Apply Gram-Schmidt orthogonalization process to find an orthogonal basis with integral coefficients and the an orthonormal basis from the basis $\{1, t, t^2, t^3\}$.

PROOF. Let $f_0 = 1, f_1 = t, f_2 = t^2, f_3 = t^3$ form the given basis. Then,

$$\langle f_0, f_0\rangle = \int_{-1}^{1} 1dt = 2, \ \langle f_1, f_1\rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3}, \ \langle f_2, f_2\rangle = \int_{-1}^{1} t^4 dt = \frac{2}{5}, \ \langle f_3, f_3\rangle = \int_{-1}^{1} t^6 dt = \frac{2}{7}$$

$$\langle f_0, f_1\rangle = \int_{-1}^{1} tdt = 0, \langle f_0, f_2\rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3}, \langle f_0, f_3\rangle = \int_{-1}^{1} t^3 dt = 0$$

$$\langle f_1, f_2\rangle = \int_{-1}^{1} t^3 dt = 0, \langle f_1, f_3\rangle = \int_{-1}^{1} t^4 dt = \frac{2}{5}, \ \langle f_2, f_3\rangle = \int_{-1}^{1} t^5 dt = 0$$

Let $g_0 = f_0 = 1$,

$$g_1 = f_1 - \frac{\langle f_1, f_0\rangle}{\langle f_0, f_0\rangle} f_0 = t - 0 = t$$

$$g_2 = f_2 - \frac{\langle f_2, f_0\rangle}{\langle f_0, f_0\rangle} f_0 - \frac{\langle f_2, f_1\rangle}{\langle f_1, f_1\rangle} f_1 = t^2 - \frac{2}{3} \times \frac{1}{2} = t^2 - \frac{1}{3}$$

$$g_3 = f_3 - \frac{\langle f_3, f_0\rangle}{\langle f_0, f_0\rangle} f_0 - \frac{\langle f_3, f_1\rangle}{\langle f_1, f_1\rangle} f_1 - \frac{\langle f_3, f_2\rangle}{\langle f_2, f_2\rangle} f_2 = t^3 - \frac{2}{5} \times \frac{3}{2} \times t - 0 = t^3 - \frac{3}{5}t$$

Thus, $\left\{g_0 = 1, \ g_1 = t, \ g_2 = t^2 - \dfrac{1}{3}, \ g_3 = t^3 - \dfrac{3}{5}t\right\}$ is an orthogonal basis V. Multiplying g_2 by 3 and g_3 by 5, we obtain $\{\phi_0(t) = 1, \ \phi_1(t) = t, \ \phi_2(t) = 3t^2 - 1, \ \phi_3(t) = 5t^3 - 3t\}$ as an orthogonal basis with integral coefficients.

Now,

$$||\phi_0(t)||^2 = < 1, 1 > = \int_{-1}^{1} 1 dt = 2 \quad \Rightarrow \quad ||\phi_0(t)|| = \sqrt{2}$$

$$||\phi_1(t)||^2 = < t, t > = \int_{-1}^{1} t^2 dt = \frac{2}{3} \quad \Rightarrow \quad ||\phi_1(t)|| = \sqrt{\frac{2}{3}}$$

$$||\phi_2(t)||^2 = \langle 3t^2 - 1, 3t^2 - 1 \rangle = \int_{-1}^{1} (3t^2 - 1)^2 dt = \frac{8}{5} \quad \Rightarrow \quad ||\phi_2(t)|| = 2\sqrt{\frac{2}{5}}$$

$$||\phi_3(t)||^2 = \langle 5t^3 - 3t,\ 5t^3 - 3t \rangle = \int_{-1}^{1} (5t^3 - 3t)^2 dt = \frac{8}{7} \quad \Rightarrow \quad ||\phi_3(t)|| = 2\sqrt{\frac{2}{7}}$$

Hence, an orthonormal basis of $P_3(t)$ is

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} t, \frac{1}{2}\sqrt{\frac{5}{2}}(3t^2 - 1), 2\sqrt{\frac{7}{2}}(5t^3 - 3t) \right\}.$$

EXAMPLE-3 *Let S be the subspace, of the inner product space R^4, spanned by the vectors $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 4, 5)$, $v_3 = (1, -3, -4, -2)$ in R^4. Apply the Gram-Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis of S.*

SOLUTION We observe that the vectors v_1, v_2, v_3 form a linearly independent set. So, $\{v_1, v_2, v_3\}$ is a basis for S. In order to orthogonalize this basis, let us define:

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \frac{12}{4} v_1 = v_2 - 3v_1 = (-2, -1, 1, 2)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = v_3 + \frac{8}{4} v_1 + \frac{7}{10} w_2 = \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5} \right)$$

Thus, $\{w_1, w_2, w_3\}$ forms an orthogonal basis of S.

Now,

$$||w_1||^2 = \langle w_1, w_1 \rangle = 4, ||w_2||^2 = \langle w_2, w_2 \rangle = 10,\ ||w_3||^2 = \langle w_3, w_3 \rangle = \frac{910}{100}$$

Let $u_i = \frac{w_i}{||w_i||}$, $i = 1, 2, 3$. Then,

$$\left\{ u_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), u_2 = \left(\frac{-2}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right), u_3 = \left(\frac{16}{\sqrt{910}}, \frac{-17}{\sqrt{910}}, \frac{-13}{\sqrt{910}}, \frac{14}{\sqrt{910}} \right) \right\}$$

is an orthonormal basis of S.

EXAMPLE-4 *Let S be the subspace of R^4 spanned by the vectors $v_1 = (1, 1, 1, 1)$, $v_2 = (1, -1, 2, 2)$, $v_3 = (1, 2, -3, -4)$. Apply the Gram-Schmidt orhtogonalization process to find an orthogonal basis of S and hence, find the projection of $v = (1, 2, -3, 4)$ onto S.*

SOLUTION Let $w_1 = v_1 = (1, 1, 1, 1)$.

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = v_2 - v_1 = (0, -2, 1, 1)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = v_3 + w_1 + \frac{11}{6} w_2 = \left(2, \frac{-2}{3}, \frac{-1}{6}, \frac{-7}{6}\right)$$

Let $w_3' = 6w_3 = (12, -4, -1, -7)$

Clearly, $\{w_1, w_2, w_3'\}$ forms an orthogonal basis of S.

$$\therefore \quad \text{proj } (v, S) = \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} w_2 + \frac{\langle v, w_3'\rangle}{\langle w_3', w_3'\rangle} w_3'$$

$$\Rightarrow \quad \text{proj } (v, S) = w_1 - \frac{1}{2} w_2 - \frac{1}{10} w_3' = \left(\frac{-1}{5}, \frac{12}{5}, \frac{3}{5}, \frac{6}{5}\right)$$

THEOREM-2 *Let S be a subspace of a finite dimensional inner product space V. If*