Subspace:

Let ω be a non-empty Subset of Vector space V. Then ω is said to be a subspace, if it satisfies the following conditions

(i) Zero vector exists in W

(ii) For vector $u, v \in W$, then $u+v$ must exists in W

(iii) For scalar c and vector v, Then cv must exist in W.

problem Let $\omega = \left\{ \begin{bmatrix} a \\ b \\ \frac{a}{2} - 2b \end{bmatrix}, a, b \in R \right\}$. verify ω is Subspace of R^3.

Method-1 Given $\omega = \{ (a, b, \frac{a}{2} - 2b) ; a, b \in R \}$

Condition-1: * Existence of Zero Vector

for $(a, b, c) = (0, 0, 0)$ we have $(0, 0, \frac{0}{2} - 0)$
$= (0, 0, 0)$

Condition-2 closure property

Let $u = (a_1, b_1, c_1) \in W$ and $v = (a_2, b_2, c_2)$

Then

$u+v$ becomes

$$= \left\{ \left(a_1, b_1, \frac{a_1}{2} - 2b_1\right) + \left(a_2, b_2, \frac{a_2}{2} - 2b_2\right) \right\}$$

$$= \left\{ \underbrace{a_1+a_2}_{A}, \underbrace{b_1+b_2}_{B}, \underbrace{\frac{a_1+a_2}{2}}_{\frac{A}{2}} \underbrace{- 2(b_1+b_2)}_{-2B} \right\}$$

which has required form as vectors W

$\therefore$ W has vectors of the form $\left(A, B, \frac{B}{2} - 2B\right)$

$\therefore$ $u+v \in W$.

Condition-3 Scalar multiplication

Consider $cu = c\left\{ \left(a_1, b_1, \frac{a_1}{2} - 2b_1\right) ; a_1, b_1 \in R \right\}$

$$= \left\{ \left(ca_1, cb_1, \frac{ca_1}{2} - 2(cb_1)\right) \right\}$$

$\therefore$ clearly ca_1, cb_1 are also in R

$\therefore$ Scalar multiplication satisfied

$\therefore$ W is subspace.

Method - 2

Given $W = \left\{ \begin{bmatrix} a \\ b \\ \frac{a}{2} - 2b \end{bmatrix}, \ a, b \in R \right\}$

which can be written as

$$a \underbrace{\begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}}_{u_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{u_2}$$

clearly

$w = \text{Span}\{u_1, u_2\}$.

$\therefore$ Spanning set is always subspace of V.

$\therefore$ w is subspace.

problem Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ Such that } b = c = 0 \right\}$

is a subspace of Vector space $M_{2 \times 2}$.

Sol: Given

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ s.t } b = c = 0 \right\}$$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \text{ where } a, d \in R \right\}$$

(i) Condition-1 Zero matrix

for $a, b, c, d = (0, 0, 0, 0)$

clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} \in H$

(ii) Condition-2 Closure property

Let $H_1 = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, a_1, d_1 \in R \right\}$

$$H_2 = \left\{ \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix}, a_2, d_2 \in R \right\}$$

clearly $H_1 + H_2 = \left\{ \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & d_1 + d_2 \end{bmatrix}, \begin{matrix} a_1 + a_2 \in R \\ d_1 + d_2 \in R \end{matrix} \right\}$

$\therefore$ closure property satisfied

(iii) Condition-3 Scalar multiplication.

For Scalar K, matrix $H_1 = \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}$

we get $KH_1 = \begin{bmatrix} Ka_1 & 0 \\ 0 & Ka_2 \end{bmatrix}$

Since $Ka_1 \in R$

$Ka_2 \in R$

$\therefore$ Scalar multiplication holds.

$\therefore$ H is Subspace of $M_{2\times2}$ vector space.

problem $W = \{ A_{2\times2}$ Such that $|A| = 0 \}$ verify

W forms Subspace of Vector Space with $M_{2\times2}$.

Sol: Condition-1 Zero matrix existence.

for $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ clearly $|A| = 0$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Condition-2   closure axiom

Let $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$ clearly $|A_1| = 0$

$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$ clearly $|A_2| = 0$

But $A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ whose $|A| = 1 \neq 0$

$\therefore \left.\begin{matrix} A_1 \in W \\ A_2 \in W \end{matrix}\right\}$ But $A_1 + A_2 \notin W$

$\therefore$ closure NOT holds.

$\therefore$ W is not a Subspace.