Linear Independence:

The set of vectors $\{v_1, v_2, \dots v_n\}$ is linearly independent, if the $c_1=0, c_2=0 \dots c_n=0$ is the only solution for $c_1v_1 + c_2v_2 \dots + c_nv_n = 0$.

→ Method-1: find Det $[v_1\, v_2 \dots v_n]$, if Det value $\neq 0$. Then $[v_1\, v_2 \dots v_n]$ is L.I.

→ Method-2: write linear combination and solve the system of equation. If zero solution is the only solution for system, Then $[v_1, v_2 \dots v_n]$ is Linearly independent.

Example: Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ in R^2 linearly independent?

Sol:

method-1

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ forms a L.I set.

problem Determine whether the following set of Columns in R^3 is Linearly independent.

$$S = \left\{ \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix} \right\}$$

Method-1

Sol: Using Determinant method,

$$\begin{vmatrix} 2 & 3 & 8 \\ 6 & 1 & 16 \\ -2 & 2 & -3 \end{vmatrix}$$

$$= 2(-3-32) -3(-18+32)+8(12+2)$$

$$= -70 - 42 + 112$$

$$= 0$$

$\therefore$ S is Linearly Dependent set of vectors.

Method-2

writing Combinations with Vectors in S

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix} = 0$$

0 vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2c_1 \\ 6c_1 \\ -2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ c_2 \\ 2c_2 \end{bmatrix} + \begin{bmatrix} 8c_3 \\ 16c_3 \\ -3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2c_1 + 3c_2 + 8c_3 = 0$$
$$6c_1 + c_2 + 16c_3 = 0$$
$$-2c_1 + 2c_2 - 3c_3 = 0$$

Using Echelon method to solve

$$\left[\begin{array}{ccc:c} 2 & 3 & 8 & 0 \\ 6 & 1 & 16 & 0 \\ -2 & 2 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 3 & 8 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right] \quad \frac{R_2}{-8}, \frac{R_3}{5}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 3 & 8 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 3 & 8 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ie

$$2c_1 + 3c_2 + 8c_3 = 0$$
$$c_2 + c_3 = 0$$

Rank = 2, infinite solutions possible.

Taking $c_3 = k$

$c_2 + c_3 = 0 \Rightarrow c_2 = -k$

$2c_1 + 3c_2 + 8c_3 = 0 \Rightarrow 2c_1 - 3k + 8(k) = 0$

$\Rightarrow 2c_1 + 5k = 0$

$\boxed{c_1 = -\frac{5k}{2}}$

$\therefore \quad c_1 = -\frac{5k}{2}, \ c_2 = -k, \ c_3 = k$

which means $c_1 = 0, c_2 = 0, c_3 = 0$ not the only solution.

$\therefore$ S is Linearly Dependent.

<u>problem</u> Determine whether the set $\{t^2+2t-3, t^2+5t, 2t^2-4\}$ of vectors in $P_2(t)$ is L.I.

<u>Sol</u>: Consider

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1(t^2+2t-3) + c_2(t^2+5t) + c_3(2t^2-4) = 0$$

$$(c_1+c_2+2c_3)t^2 + (2c_1+5c_2)t + (-3c_1-4c_3) = 0$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 5c_2 &= 0 \\ -3c_1 - 4c_3 &= 0 \end{aligned}\right\} \quad \text{solving} \quad \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

$\therefore$ Linearly independent

problem Determine whether the set of matrices

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

in $M_{2\times 2}$ is linearly independent.

Sol: Consider

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 + c_5V_5 = 0$$

$$c_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_5\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1+c_4+c_5 & c_1+c_2+c_5 \\ c_3+c_4 & c_2+c_3+c_4+c_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\left.\begin{aligned} c_1+c_4+c_5 &= 0 \rightarrow (1) \\ c_1+c_2+c_5 &= 0 \rightarrow (2) \\ c_3+c_4 &= 0 \rightarrow (3) \\ c_2+c_3+c_4+c_5 &= 0 \rightarrow (4) \end{aligned}\right\}$$

~~solving (3) & (4)~~

~~$c_3+c_4 = 0$~~

~~and~~

~~$c_2+c_3+c_4+c_5 = 0$~~

~~we will get~~

~~$c_2+c_5 = 0$~~

4 - Equations

But 5 - variables.

non-Zero Solution Exists

it is system with 4 equations in 5 unknowns

so clearly will have infinitely many solutions

$\therefore$ Linearly Dependent set.

problem For what values of 'k' such that

$\left\{ \begin{bmatrix} 1 \\ k \end{bmatrix}, \begin{bmatrix} k \\ k+2 \end{bmatrix} \right\}$ is linearly independent.

sol: For Linearly independent, Det value $\neq 0$

$$\therefore \begin{vmatrix} 1 & k \\ k & k+2 \end{vmatrix} \neq 0$$

$$(k+2) - k^2 \neq 0$$

$$\Rightarrow k^2 - k - 2 \neq 0$$

$$\Rightarrow (k-2)(k+1) \neq 0$$

$$\therefore k \neq 2 \text{ and } k \neq -1$$

So for any value other than $k = 2$, $k = -1$

set is linearly independent.

But for $k = 2$, $k = -1$ it is Linearly Dependent.

problem For what values of h, set $\{v_1, v_2, v_3\}$ is Linearly Dependent.

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

Sol: If $\text{Det}[v_1 \ v_2 \ v_3] = 0$ Then Set is Linearly Dependent.

Consider

$$\begin{vmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{vmatrix} = 0$$

$$1(9h+42) + 3(-3h+14) + 5(18-18) = 0$$

$$\Rightarrow (9h+42) + (-9h+42) = 0$$

$$\Rightarrow 9h - 9h + 84 = 0$$

$$\Rightarrow 84 = 0$$

which is not possible

That means no 'h' value exist which makes set as Linearly Dependent.