ILLUSTRATIVE EXAMPLES

EXAMPLE-1 *Let $t : R^2 \to R^2$ be a linear transformation for which $t(1,2) = (2,3)$ and $t(0,1) = (1,4)$. Find a formula for t, find $t(x,y)$.*

SOLUTION We know that $B = \{(a_1,b_1),(a_2,b_2)\}$ forms a basis of R^2 iff $a_1b_2 \neq a_2b_1$. Clearly, $v_1 = (1,2), v_2 = (0,1)$ satisfy this relation. So, $\{v_1, v_2\}$ is a basis of R^2.

Let $v = (x,y)$ be an arbitrary vector in R^2. Then there exist unique scalars $a, b \in R$ such that

$$v = av_1 + bv_2$$

$$\Rightarrow \quad (x,y) = a(1,2) + b(0,1)$$

$$\Rightarrow \quad (x,y) = (a, 2a+b)$$

$$\Rightarrow \quad a = x,\ b = y - 2x$$

As $t : R^2 \to R^2$ is a linear transformation.

$$\therefore \quad v = av_1 + bv_2$$

$$\Rightarrow \quad t(v) = t(av_1 + bv_2)$$

$$\Rightarrow \quad t(v) = at(v_1) + bt(v_2)$$

$$\Rightarrow \quad t(x,y) = at(1,2) + bt(0,1)$$

$$\Rightarrow \quad t(x,y) = a(2,3) + b(1,4)$$

$$\Rightarrow \quad t(x,y) = (2a+b, 3a+4b)$$

$$\Rightarrow \quad t(x,y) = (y, -5x+4y)$$

Hence, $t : R^2 \to R^2$ is given by $t(x,y) = (y, -5x+4y)$ for all $(x,y) \in R^2$.

EXAMPLE-2 *Let $t : R^2 \to R^2$ be a linear transformation such that $t(1,1) = (1,3)$, $t(-1,1) = (3,1)$. Find a complete formula for t.*

SOLUTION Clearly, $B = \{(1,1),(-1,1)\}$ forms a basis for R^2. Let $(a,b) \in R^2$, then there exist scalars $\lambda, \mu \in R$ such that

$$(a,b) = \lambda(1,1) + \mu(-1,1)$$
$$\Rightarrow \quad (a,b) = (\lambda - \mu,\ \lambda + \mu)$$
$$\Rightarrow \quad \lambda - \mu = a \text{ and } \lambda + \mu = b \Rightarrow \lambda = \frac{a+b}{2} \text{ and } \mu = \frac{b-a}{2}$$

Since $t : R^2 \to R^2$ is a linear transformation.

$$\therefore \quad (a,b) = \lambda(1,1) + \mu(-1,1)$$
$$\Rightarrow \quad t(a,b) = \lambda\, t(1,1) + \mu\, t(-1,1)$$
$$\Rightarrow \quad t(a,b) = \lambda(1,3) + \mu\,(3,1)$$
$$\Rightarrow \quad t(a,b) = (\lambda + 3\mu,\ 3\lambda + \mu)$$
$$\Rightarrow \quad t(a,b) = (2b - a,\ a + 2b)$$

Hence, $t : R^2 \to R^2$ is given by $t(a,\ b) = (2b - a,\ a + 2b)$ for all $(a,\ b) \in R^2$.

EXAMPLE-3 *Let $B = \{(-1,0,1),\ (0,1,-1),\ (1,-1,1)\}$ be a basis of $R^3(R)$ and $t : R^3 \to R^3$ be a linear transformation such that $t(-1,0,1) = (1,0,0)$, $t(0,1,-1) = (0,1,0)$, $t(1,-1,1) = (0,0,1)$. Find formula for $t(x,y,z)$ and use it to compute $t(1,-2,3)$.*

SOLUTION Clearly, B forms a basis for R^3. Let $(x,y,z) \in R^3$. Then there exist scalars $a, b, c \in R$ such that

$$(x,y,z) = a(-1,0,1) + b(0,1,-1) + c(1,-1,1)$$
$$\Rightarrow \quad (x,y,z) = (-a + c,\ b - c,\ a - b + c)$$
$$\Rightarrow \quad -a + c = x,\ b - c = y,\ a - b + c = z$$
$$\Rightarrow \quad a = y + z,\ b = x + 2y + z,\ c = x + y + z$$

Since $t : R^3 \to R^3$ is a linear transformation. Therefore,

$$(x,y,z) = a(-1,0,1) + b(0,1,-1) + c(1,-1,1)$$
$$\Rightarrow \quad t(x,y,z) = a\, t(-1,0,1) + b\, t(0,1,-1) + c\, t(1,-1,1)$$
$$\Rightarrow \quad t(x,y,z) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$\Rightarrow \quad t(x,y,z) = (a,b,c)$$
$$\Rightarrow \quad t(x,y,z) = (y + z,\ x + 2y + z,\ x + y + z)$$
$$\therefore \quad t(1,-2,3) = (1,0,2)$$

EXAMPLE-4 *The range of a linear transformation $t : R^3 \to R^3$ has the subspace spanned by the vectors $v_1 = (1,0,-1)$ and $v_2 = (1,2,2)$. Find the transformation explicitly.*

SOLUTION We know that $B = \left\{e_1^{(3)} = (1,0,0),\ e_2^{(3)} = (0,1,0),\ e_3^{(3)} = (0,0,1)\right\}$ is a basis of $R^3(R)$. Also, $v_1 = (1,0,-1)$ and $v_2 = (1,2,2)$ are two vectors spanning the image of t. Therefore, $v_1 = (1,0,-1), v_2 = (1,2,2)$ and $v_3 = (0,0,0)$ also span $I_m(t)$. By Theorem 4, there exists a unique linear transformation $t : R^3 \to R^3$ such that

$t(e_1{}^{(3)}) = v_1,\ t(e_2{}^{(3)}) = v_2$ and $t(e_3{}^{(3)}) = v_3$

Let $v = (a,b,c)$ be an arbitrary vector in R^3 whose basis is B.

Clearly,

$$v = a\, e_1^{(3)} + b\, e_2^{(3)} + c\, e_3^{(3)}$$

$$\Rightarrow \quad t(v) = a\, t\left(e_1^{(3)}\right) + b\, t\left(e_2^{(3)}\right) + c\, t\left(e_3^{(3)}\right)$$

$$\Rightarrow \quad t(v) = av_1 + bv_2 + cv_3$$

$$\Rightarrow \quad t(a,b,c) = a(1,0,-1) + b(1,2,2) + c(0,0,0)$$

$$\Rightarrow \quad t(a,b,c) = (a+b, 2b, -a+2b)$$

REMARK. *In the above example, we may choose v_3 as a linear combination of v_1 and v_2, accordingly t will also change. So, t is not unique.*

EXAMPLE-5 *Find a linear transformation $t : R^3 \to R^4$ whose image is spanned by the vectors $v_1 = (1,2,0,-4)$ and $v_2 = (2,0,-1,-3)$.*

SOLUTION We know that $B = \left\{e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right\}$ is the standard basis of $R^3(R)$. It is given that $v_1 = (1,2,0,-4), v_2 = (2,0,-1,-3)$ span $I_m(t)$. Therefore, $v_1 = (1,2,0,-4), v_2 = (2,0,-1,-3)$ and $v_3 = (0,0,0,0)$ also span $I_m(t)$. By Theorem 4, there exists a unique linear transformation $t : R^3 \to R^4$ such that

$$t\left(e_1^{(3)}\right) = v_1,\ t\left(e_2^{(3)}\right) = v_2 \text{ and } t\left(e_3^{(3)}\right) = v_3$$

Let $v = (a,b,c)$ be an arbitrary vector in R^3 whose basis is B.

Clearly,

$$v = a\, e_1^{(3)} + b\, e_2^{(3)} + c\, e_3^{(3)}$$

$$\Rightarrow \quad t(v) = a\, t\left(e_1^{(3)}\right) + b\, t\left(e_2^{(3)}\right) + c\, t\left(e_3^{(3)}\right)$$

$$\Rightarrow \quad t(v) = av_1 + bv_2 + cv_3$$

$$\Rightarrow \quad t(a,b,c) = a(1,2,0,-4) + b(2,0,-1,-3) + c(0,0,0,0)$$

$$\Rightarrow \quad t(a,b,c) = (a+2b,\ 2a-b,\ -4a-3b)$$

REMARK. *In the above example t is not unique. Instead of taking v_3 as the null vector in R^4, we may take $v_3 = v_1$ or, $v_3 = v_2$ or, v_3 as any linear combination of v_1 and v_2. Accordingly linear transformation t changes.*

EXERCISE 3.2

1. Let $t : R^3 \to R^2$ be a linear transformation given by
$$T(x, y, z) = (x+y, y+z).$$
Find a basis and the dimension of (i) the image of t (ii) the kernel of t.
2. Let $t : R^3 \to R^2$ be the linear transformation such that $t(1,2,3) = (1,0,0)$, $t(1,2,0) = (0,1,0)$, $t(1,-1,0) = (0,1,0)$. Find $t(a,b,c)$ for any $(a,b,c) \in R^3$.
3. Let F be a field and $t : F^2 \to F^2$ be a linear transformation such that $t(1,0) = (a,b)$ and $t(0,1) = (c,d)$. Find $t(x,y)$ for any $(x,y) \in F^2$.
4. Describe explicitly the linear transformation $t : R^2 \to R^2$ such that $t(2,3) = (4,5)$ and $t(1,0) = (0,0)$.
5. Find the linear transformation $t : R^2 \to R^2$ such that $t(1,\ 0) = (1,\ 1)$, $t(0,\ 1) = (-1,\ 2)$. Prove that t maps the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ into a parallelogram.

ANSWERS

1. (i) $\{(1,0),\ (0,1)\}$, $\dim I_m(t) = 2$ (ii) $\{(-1,\ 1,\ -1)\}$, $\dim \mathrm{Ker}(t) = 1$
2. $t(a,b,c) = \left(\frac{c}{3}, \frac{3a-c}{3}, 0\right)$
3. $t(x,y) = (xy+yc,\ xb+yd)$
4. $t(x,\ y) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$