Binomial Distribution

Repeated trials

The probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$.

If a coin is tossed thrice, the sample space

$S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$

The probability of getting one head and two tails

$E = \{HTT, TTH, THT\}$

$= \frac{3}{8}$

The probability of each one (one head, one tail, one tail) ~~one~~ of these being $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ i.e, $\left(\frac{1}{2}\right)^3$, their total probability shall be $3 \times \left(\frac{1}{2}\right)^3$.

Similarly if a trial is repeated n times and if p is the probability of a success and q that of a failure, then the probability of r successes and $n-r$ failures is given by $p^r q^{n-r}$

But these r successes & $n-r$ failures can occur in any of ${}^nC_r$ ways in each of which the probability is same. Thus the probability of r successes in ${}^nC_r p^r q^{n-r}$

The probabilities of atleast r successes in n trials = Sum of probabilities of $r, r+1, \dots, n$ successes

$$= {}^nC_r\, p^r q^{n-r} + {}^nC_{r+1}\, p^{r+1} q^{n-r-1} + \dots + {}^nC_n\, p^n$$

<u>Binomial Distribution</u> is concerned with trials of a repetitive nature in which only the occurence or non-occurence, success or failure, acceptance or rejection, yes or no of a particular event is of interesst

If we perform a series of independent trials such that for each trial p is the probability of success and q that of a failure, then the probability of r successes in a series of n trials is given by ${}^nC_r\, p^r q^{n-r}$, where r takes any integral value from 0 to n. The probabilities of $0, 1, 2, \dots r \dots n$ successes are, therefore, given by

$$q^n,\ {}^nC_1\, p q^{n-1},\ {}^nC_2\, p^2 q^{n-2},\ \dots\ {}^nC_r\, p^r q^{n-r}, \dots, p^n.$$

The probability of the number of successes so obtained is called the binomial distribution.

$\therefore$ The sum of the probabilities

$$= q^n + {}^nC_1\, p q^{n-1} + {}^nC_2\, p^2 q^{n-2} + \dots + p^n = (q+p)^n = 1$$

Mean of a binomial distribution

The density function of a binomial distribution is given by $f(x) = {}^nC_x\, p^x q^{n-x}$,

where $0 \le p \le 1$, $0 \le q \le 1$, $p+q=1$.

We know that, the mean is given by,

$$\mu = \sum x f(x)$$

$$= \sum_{x=0}^{n} x \, {}^nC_x \, p^x \, q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\, x!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x q^{n-x}$$

Let $x-1=k$, so $x=k+1$.

$$= \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!\, k!} p^{k+1} q^{n-k-1}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!\, k!} p^k q^{n-k-1}$$

$$= np \underbrace{\sum_{k=0}^{n-1} {}^{n-1}C_k \, p^k q^{n-k-1}}_{=1}$$

$\left[\text{as } \sum_{k=0}^{n-1} {}^{n-1}C_k \, p^k q^{n-k-1} = (p+q)^{n-1} = 1^{n-1} = 1\right]$

$$\mu = np$$

Variance of a Binomial distribution

The variance is given by,

$$\sigma^2 = \sum x^2 f(x) - \mu^2$$

$$= \sum (x^2 - x + x) f(x) - \mu^2$$

$$= \sum (x^2 - x) f(x) + \sum x f(x) - \mu^2$$

$$= \sum x(x-1) f(x) + \mu - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^x q^{n-x} + \mu - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\, x!} p^x q^{n-x} + \mu - \mu^2$$

$$= \sum_{x=2}^{n} \frac{n!}{(n-x)!\,(x-2)!} p^x q^{n-x} + \mu - \mu^2$$

Let $x-2=k$, so $x=k+2$

$$= \sum_{k=0}^{n-2} \frac{n!}{(n-k-2)!\, k!} p^{k+2} q^{n-k-2} + \mu - \mu^2$$

$$= n(n-1)p^2 \underbrace{\sum_{k=0}^{n-2} \frac{(n-2)!}{(n-k-2)!\, k!} p^k q^{n-k-2}}_{=1} + \mu - \mu^2$$

$$= n(n-1)p^2 + \mu - \mu^2$$

$$= n^2p^2 - np^2 + \mu - \mu^2$$

$$\sigma^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\sigma^2 = np - np^2$$

$$= np(1-p)$$

$$\sigma^2 = npq$$

* The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. if 12 such pens are manufactured, find the probability that

(a) exactly two will be defective

(b) atleast two will be defective.

(c) non will be defective.

Sol: The probability of defective pen is $\frac{1}{10} = 0.1 = p$

The probability of a non-defective pen is $1 - 0.1 = 0.9 = q$

The probability that exactly two will be defective

(a) probability that exactly two will be defective

$n = 12$

$r = 2$

$$= {}^{12}C_2\, p^2\, q^{12-2} = {}^{12}C_2\, (0.1)^2\, (0.9)^{10} = 0.2301.$$

(b) probability that atleast two will be defective

$= 1 -$ (probability that either none or one is defective)

$$= 1 - \left[{}^{12}C_0\, (0.1)^0\, (0.9)^{12} + {}^{12}C_1\, (0.1)^1\, (0.9)^{11}\right]$$

$= 0.3412$

(c) The probability that none will be defective

$$= {}^{12}C_0\, (0.1)^0\, (0.9)^{12}$$

$= 0.2833$

* In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads & 4 tails.

Sol^n

$P(\text{head}) = \frac{1}{2}$; $P(\text{tail}) = \frac{1}{2}$

probability of 8 heads & 4 tails in 12 trails

$= {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{495}{4096}$

Expected cases in 256 sets.

$= 256 \times \frac{495}{4096}$

$= 30.9$

$\approx \underline{31}$

The mean of a binomial distribution ~~with~~ {p being the probability of success and q being the probability of failure.} is given by $\mu = np$.

The standard deviation is given by $S.D = \sqrt{npq}$

* In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain atleast 3 defective parts.

Soln: Given $\mu = 2$, $n = 20$

$\mu = np \Rightarrow p = \frac{\mu}{n} = \frac{2}{20} = 0.1$

$\therefore$ probability of defective part $= p = 0.1$

probability of non-defective part $= q = 1 - p = 0.9$.

The probability of atleast 3 defective parts out of 20

$= 1 -$ [probability of 0, 1 or 2 defective parts]

$= 1 - \left[{}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_2 (0.1)^2 (0.9)^{18} \right]$.

$= 0.323$.

$\therefore$ The probability of atleast 3 defective parts out of 1000 $= 1000 \times 0.323$

$= 323$

* The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to this data.

x	0	1	2	3	4	5	6	7	8	9	10
f	6	20	28	12	8	6	0	0	0	0	0.

Sol^n: Here $n = 10$, $N = \Sigma f_i = 80$.

$$\text{mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0+20+56+36+32+30+0+0+0+0+0}{80}$$

$$\mu = 2.175 .$$

Also mean of B.D $\mu = np \Rightarrow p = \frac{\mu}{n} = \frac{2.175}{10} = 0.2175$

$\Rightarrow q = 1 - p = 0.7825$

The B.D to be fitted is

$$N(p+q)^n = 80\,[0.2175 + 0.7825]^{10}$$

$$= 80\left[{}^{10}C_0 (0.2175)^0 (0.7825)^{10} + {}^{10}C_1 (0.2175)^1 (0.7825)^9 + \cdots + {}^{10}C_{10} (0.2175)^{10} (0.7825)^0\right]$$

$$= 6.885 + 19.13 + 23.94 + \cdots + 0.0007 + 0.00002$$

$\therefore$

x	0	1	2	3	4	5	6	7	8	9	10
f	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0.

* Let x be a binomial variate with mean 6 and variance 4. Find the distribution of x.

<u>Sol</u>

Sol^n

Given $\mu = 6$, $\sigma^2 = 4$.

we have, $\mu = np$ $\quad$ $\sigma^2 = npq$

$\Rightarrow 6 = np$ $\quad$ $4 = 6 \times q$

$$\Rightarrow q = \frac{2}{3}$$

Also $p + q = 1 \Rightarrow p = 1 - \frac{2}{3}$

$$p = \frac{1}{3}$$

$6 = np \Rightarrow 6 = n \times \frac{1}{3}$

$$\Rightarrow n = 18$$

$$\therefore f(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$f(x) = {}^{18}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{18-x}$$

* In a binomial distribution consisting of 5 independent trials, probability of 1 & 2 successes are 0.4096 & 0.2048 respectively. Find the parameter p of the distributive function.

Sol^n: Given $n = 5$.

Let x	1	2	3	4	5
$f(x)$	0.4096	0.2048	-	-	-

We have, $f(x) = {}^nC_x \, p^x q^{n-x}$

with $n=5$, $x=1$

$$\Rightarrow f(1) = {}^5C_1 \, p^1 q^{5-1}$$

$$\Rightarrow 0.4096 = 5 \times p^1 \times q^4 \Rightarrow pq^4 = \frac{0.4096}{5} \quad -(1)$$

with $n=5$, $x=2$

$$\Rightarrow f(2) = {}^5C_2 \, p^2 q^{5-2}$$

$$\Rightarrow 0.2048 = 10p^2q^3 \Rightarrow p^2q^3 = \frac{0.2048}{10} \quad -(2)$$

$(2) \div (1) \Rightarrow$

$$\frac{p^2q^3}{pq^4} = \frac{0.2048}{\cancel{10}\,2} \times \frac{\cancel{5}\,1}{0.4096} \Rightarrow \frac{p}{q} = \frac{0.2048}{2 \times 0.4096}$$

$$\Rightarrow 2 \times 0.4096 \, p = 0.2048 q$$

$$\Rightarrow 0.8192p = 0.2048(1-p)$$

$$\Rightarrow p = 0.2 \quad \& \quad q = 0.8$$

Poisson Distribution

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence.

example:

number of plane crashes.

failure of a machine in one month.

number of deaths by dog bite.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed.

The Binomial distribution function is given by

$$f(x) = {}^nC_x \, p^x \, q^{n-x}$$

let $np = \mu$ is a positive real number, then

$$p = \frac{\mu}{n} \quad \& \quad q = 1 - p = 1 - \frac{\mu}{n}$$

$$\therefore f(x) = {}^nC_x \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

As $n \to \infty$, the Binomial distribution reduces to Poisson distribution.

i.e, $$f(x) = \lim_{n \to \infty} {}^nC_x \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{(n-x)!\, x!} \cdot \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-(x-1)) \times (n-x)!}{(n-x)!\;\; x!} \times \frac{\mu^x}{n^x}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{\mu^x}{x!} \frac{n(n-1)(n-2)\cdots(n-(x-1))}{n \cdot n \cdot n \cdots n}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{\mu^x}{x!} \times \frac{n \times 1 \times n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right) \cdots n\left(1-\frac{x-1}{n}\right)}{n \times n \times n \cdots \times n}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$= \lim_{n\to\infty} \frac{\mu^x}{x!} \underbrace{\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]}_{=1} \underbrace{\left(1-\frac{\mu}{n}\right)^n}_{e^{-\mu}} \underbrace{\left(1-\frac{\mu}{n}\right)^{-x}}_{=1}$$

$$f(x) = \frac{\mu^x}{x!} e^{-\mu} \quad \text{for } x = 0, 1, 2, \ldots, \infty$$

$$f(x) = \frac{\mu^x}{x!} e^{-\mu} \quad \text{for } x = 0, 1, 2 \cdots \infty$$

$$\Rightarrow f(x) = e^{-\mu}\left[\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots\right]$$

$$= e^{-\mu} \times \left(1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots\right)$$

$$= e^{-\mu} \times e^{\mu}$$

$$f(x) = 1$$

The mean of a Poisson distribution

$$\text{Mean} = \sum x f(x)$$

$$= \sum_{x=0}^{\infty} x \times \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} x \times \frac{\mu^x}{x(x-1)!} e^{-\mu}$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!}$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \mu \cdot \frac{\mu^{x-1}}{(x-1)!}$$

$$= \mu e^{-\mu} \times \left[\frac{\mu^0}{0!} + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots \right]$$

$$= \mu e^{-\mu} \times \left(1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots \right)$$

$$= \mu e^{-\mu} \times e^{\mu}$$

$$\text{Mean} = \mu.$$

Variance of a Poisson Distribution

$$\text{Variance} = \sum x^2 f(x) - \mu^2$$

$$= \sum (x^2 - x) f(x) + \sum x f(x) - \mu^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \times \frac{\mu^x}{x!} e^{-\mu} + \mu - \mu^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \times \frac{\mu^{x-2+2}}{x(x-1)(x-2)!} e^{-\mu} + \mu - \mu^2$$

$$= \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} \mu^2 e^{-\mu} + \mu - \mu^2$$

$$= \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} + \mu - \mu^2$$

$$= \mu^2 e^{-\mu} \left(\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} + \cdots \right) + \mu - \mu^2$$

$$= \mu^2 e^{-\mu} \times e^{\mu} + \mu - \mu^2$$

$$= \mu^2 + \mu - \mu^2$$

$$\underline{\text{Variance} = \mu}$$

$$\text{S.D} = \sqrt{\text{Variance}} = \sqrt{\mu}$$

* In a Poisson distribution if $P(2) = \frac{2}{3} P(1)$, find $P(0)$

<u>Soln</u> $P(\mu, x) = \frac{e^{-\mu} \mu^x}{x!}$

$$P(\mu, 2) = \frac{2}{3} P(1)$$

$$\Rightarrow \frac{\cancel{e^{-\mu}} \mu^{\cancel{2}}}{2!} = \frac{2}{3} \frac{\cancel{e^{-\mu}} \cancel{\mu^1}}{1!}$$

$$\Rightarrow \frac{\mu}{2} = \frac{2}{3} \Rightarrow \mu = \frac{4}{3}$$

$$P(\mu, 0) = \frac{e^{-\mu} \mu^0}{0!} = e^{-\mu} = e^{-\frac{4}{3}} = \underline{3.79}$$

* For a poisson variable $3\,P(2) = P(4)$, find S.D

<u>Soln</u> $P(\mu, x) = \frac{e^{-\mu} \mu^x}{x!}$

$$3\,P(2) = P(4)$$

$$\Rightarrow 3 \times \frac{\cancel{e^{-\mu}} \cancel{\mu^2}}{2!} = \frac{\cancel{e^{-\mu}} \mu^{\cancel{4}\,2}}{4!}$$

$$\Rightarrow \frac{3}{2} = \frac{\mu^2}{24}$$

$$\Rightarrow \mu^2 = 36$$

$$\Rightarrow \mu = 6$$

$$S.D = \sqrt{\text{variance}}$$

$$S.D = \sqrt{6} = 2.45$$

*. Assume that the probability of an individual Coalminer being killed in a mine accident during a year is $\frac{1}{2400}$. Calculate the probability that in a mine employing 200 miners there will be atleast one fatal accident in a year.

Sol^n $p = \frac{1}{2400}$, $n = 200$,

$$\mu = np \Rightarrow \mu = 200 \times \frac{1}{2400} = 0.83$$

$$\left.\begin{array}{l}\text{probability of atleast} \\ \text{one fatal accident}\end{array}\right\} = 1 - \left(\begin{array}{l}\text{probability of zero} \\ \text{fatal accidents}\end{array}\right)$$

$$= 1 - \frac{(0.83)^0}{0!} e^{-0.83}$$

$$= 1 - e^{-0.83}$$

$$= 0.5640$$

* A car hire firm has 2 cars, which it hires out day by day. The number of demands for a car on each day is distributed as poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused.

<u>Sol^n</u> x - demand of the car, $\mu = 1.5$.

Proportion of days on which neither car is used } = probability of there being no demand of the car.

$$= P(x=0)$$

$$= \frac{\mu^0 e^{-\mu}}{0!} = e^{-1.5} = 0.2231$$

proportion of days on which demand is refused } = probability of there being demand, more than 2.

$$= 1 - \text{probability of demand being} \leq 2$$

$$= 1 - P(x \leq 2)$$

$$= 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[\frac{\mu^0 e^{-\mu}}{0!} + \frac{\mu^1 e^{-\mu}}{1!} + \frac{\mu^2 e^{-\mu}}{2!}\right]$$

$$= 0.1913$$

* The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is the probability that in a group of seven ~~4~~, 5 or more will suffer from it.

Sol^n:- $p = 10\% = 0.1$, $n = 7$.

$\mu = np \Rightarrow \mu = 0.1 \times 7 = 0.7$

$$P(x \geq 5) = P(5) + P(6) + P(7)$$

$$= \frac{e^{-\mu}\mu^5}{5!} + \frac{e^{-\mu}\mu^6}{6!} + \frac{e^{-\mu}\mu^7}{7!}$$

$$= \frac{e^{-0.7}(0.7)^5}{5!} + \frac{e^{-0.7}(0.7)^6}{6!} + \frac{e^{-0.7}(0.7)^6}{7!}$$

$$= 0.0008$$

* If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction.

Sol^n

Probability that more than 2 will get a bad reaction

$$= 1 - [\text{prob that no one gets a bad reaction} + \text{prob} \ldots \text{one} \ldots + \text{prob} \ldots \text{two} \ldots]$$

$$= 1 - \left[\frac{\mu^0 e^{-\mu}}{0!} + \frac{\mu^1 e^{-\mu}}{1!} + \frac{\mu^2 e^{-\mu}}{2!}\right]$$

But $\mu = np = 2000 \times 0.001 = 2$

$$= 1 - \left[e^{-2} + 2e^{-2} + \frac{2^2 e^{-2}}{2}\right]$$

$$= 1 - \left(\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right)$$

$$= 1 - \frac{5}{e^2}$$

$$= 0.32$$

* In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.

Sol^n. ~~Given~~ Here $p = 0.002$, $n = 10$ $\therefore \mu = 10 \times 0.002 = 0.02$

probability of no defective blades $= \frac{\mu^0}{0!} e^{-\mu} = e^{-0.02} = 0.9802$

$\therefore$ no of packets containing no defective blades out of 10,000 packets $\Big\} = 10,000 \times 0.9802$

$= 9802$.

III^ly no of packets containing one defective blade out of 10,000 packets $\Big\} = 10,000 \times \frac{\mu^1}{1!} e^{-\mu} = 10,000 \times 0.02 \times e^{-0.02}$

$= 196$

no of packets containing two defective blades out of 10,000 packets $\Big\} = 10,000 \times \frac{\mu^2}{2!} e^{-\mu} = 10,000 \times \frac{(0.02)^2}{2} \times e^{-0.02}$

$= \cancel{0}1.96 \simeq 2$

* Fit a Poisson distribution to the set of observations:

x	0	1	2	3	4
$f(x)$	122	60	15	2	1

Sol^n

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{122\times 0 + 60\times 1 + 15\times 2 + 2\times 3 + 1\times 4}{122+60+15+2+1}$$

$$\mu = \frac{100}{200} = 0.5$$

$$\therefore f(x) = \frac{\mu^x}{x!} e^{-\mu} \quad \text{for } x = 0, 1, 2, 3, 4$$

$$f(x) = \frac{(0.5)^x}{x!} e^{-0.5}$$

$\therefore$ The P.D to be fitted is

$$N \times f(x) = 200 \times \frac{(0.5)^x}{x!} e^{-0.5} \quad \text{for } x = 0, 1, 2, 3, 4$$

i.e.,

x	0	1	2	3	4
$f(x)$	121	61	15	2	0