ILLUSTRATIVE EXAMPLES

TYPE I ON CHECKING WHETHER A GIVEN SET OF VECTORS FORMS A BASIS OR NOT

EXAMPLE-1 *Determine whether or not each of the following sets form a basis of $R^3(R)$:*

(i) $B_1 = \{(1,1,1),(1,0,1)\}$

(ii) $B_2 = \{(1,1,1),(1,2,3),(2,-1,1)\}$

(iii) $B_3 = \{(1,2,3),(1,3,5),(1,0,1),(2,3,0)\}$

(iv) $B_4 = \{(1,1,2),(1,2,5),(5,3,4)\}$

SOLUTION (i) Since $\dim R^3 = 3$. So, a basis of $R^3(R)$ must contain ...

Hence, B_1 is not a basis of $R^3(R)$.

(ii) Since $\dim R^3 = 3$. So, B_2 will form a basis of $R^3(R)$ if and only if vectors in B_2 are linearly independent. To check this, let us form the matrix A whose rows are the given vectors as given below.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly independent. So, let us reduce A to echelon form.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 + 3R_2$$

Clearly, the echelon form of A has no zero rows. Hence, the three vectors are linearly independent and so they form a basis of R^3.

(iii) Since $(n+1)$ or more vectors in a vector space of dimension n are linearly dependent. So, B_3 is a linearly dependent set of vectors in $R^3(R)$. Hence, it cannot be a basis of R^3.

(iv) The matrix A whose rows are the vectors in B_4 is given by

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly independent. So, let us reduce A to echelon form.

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \quad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 5R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 + 2R_2$$

The echelon form of A has a zero row, hence the given vectors are linearly dependent and so B_4 does not form a basis of R^3.

EXAMPLE-2 *Let V be the vector space of all 2×2 matrices over a field F. Prove that V has dimension 4 by finding a basis for V.*

SOLUTION Let $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be four matrices in V, where 1 is the unity element in F.

Let $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$.

We shall now show that B forms a basis of V.

B is l.i.: Let x, y, z, t be scalars in F such that

$$xE_{11} + yE_{12} + zE_{21} + tE_{22} = 0$$

$$\Rightarrow \quad \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x = y = z = t = 0$$

So, B is l.i.

B spans v: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix in V. Then,

$$A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$$

$\Rightarrow$ A is expressible as a linear combination of matrices in B.

So, B spans V. Thus, B is a basis of V.

Hence, $\dim V = 4$.

EXAMPLE-3 *Let $v_1 = (1, i, 0), v_2 = (2i, 1, 1), v_3 = (0, 1 + i, 1 - i)$ be three vectors in $C^3(C)$. Show that the set $B = \{v_1, v_2, v_3\}$ is a basis of $C^3(C)$.*

SOLUTION We know that $\dim C^3(C) = 3$. Therefore, B will be a basis of $C^3(C)$ if B is a linearly independent set. Let $x, y, z \in C$ such that

$$xv_1 + yv_2 + zv_3 = 0$$

$$\Rightarrow \quad (x + 2iy, xi + y + z(1+i), y + z(1-i)) = (0, 0, 0)$$

$$\Rightarrow \quad x + 2iy = 0, xi + y + z(1+i) = 0, y + z(1-i) = 0$$

$$\Rightarrow \quad x = y = z = 0.$$

Hence, B is a basis of $C^3(C)$.

EXAMPLE-4 *Determine whether $(1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5)$ form a basis of $R^4(R)$. If not, find the dimension of the subspace they span.*

SOLUTION Given four vectors can form a basis of $R^4(R)$ iff they are linearly independent as the dimension of R^4 is 4.

The matrix A having given vectors as its rows is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$$

Since non-zero rows of a matrix in echelon form are linearly indep...
to echelon form

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \qquad \text{Applying } R_3 \to R_3 - 3R_2, R_4 \to R_4 - 4R_3$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Applying } R_4 \to R_4 - R_3$$

The echelon matrix has a zero row. So, given vectors are linearly ~~in~~dependent and do not form a basis of R^4. Since the echelon matrix has three non-zero rows, so the four vectors span a subspace of dimension 3.

TYPE II ON EXTENDING A GIVEN SET TO FORM A BASIS OF A GIVEN VECTOR SPACE

EXAMPLE-5 *Extend the set* $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ *to a basis of* R^4.

SOLUTION Let us first form a matrix A with rows u_1 and u_2, and reduce it to echelon form:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We observe that the vectors $v_1 = (1,1,1,1)$ and $v_2 = (0,0,1,2)$ span the same space as spanned by the given vectors u_1 and u_2. In order to extend the given set of vectors to a basis of $R^4(R)$, we need two more vectors u_3 and u_4 such that the set of four vectors v_1, v_2, u_3, u_4 is linearly independent. For this, we chose u_3 and u_4 in such a way that the matrix having v_1, v_2, u_3, u_4 as its rows is in echelon form. Thus, if we chose $u_3 = (0,a,0,0)$ and $u_4 = (0,0,0,b)$, where a, b are non-zero real numbers, then v_1, u_3, v_2, u_4 in the same order form a matrix in echelon form. Thus, they are linearly independent, and they form a basis of R^4. Hence, u_1, u_2, u_3, u_4 also form a basis of R^4.

EXAMPLE-6 *Let* $v_1 = (-1,1,0), v_2 = (0,1,0)$ *be two vectors in* $R^3(R)$ *and let* $S = \{v_1, v_2\}$. *Extend set S to a basis of* $R^3(R)$.

SOLUTION Let A be the matrix having v_1 and v_2 as its two rows. Then,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly, A is in echelon form. In order to form a basis of $R^3(R)$, we need one more vector such that the matrix having that vector as third row and v_1, v_2 as first and second rows is in echelon form. If we take $v_3 = (0, 0, a)$, where $a(\neq 0) \in R$, then matrix having its three rows as v_1, v_2, v_3 is in echelon form. Thus, v_1, v_2, v_3 are linearly independent and they form a basis of $R^3(R)$.

REMARK. *Sometimes, we are given a list of vectors in the vector space $R^n(R)$ and we want to find a basis for the subspace S of R^n spanned by the given vectors, that is, a basis of $[S]$. The following two algorithms help us for finding such a basis of $[S]$.*

ALGORITHM 1 *(Row space algorithm)*

Step I Form the matrix A whose rows are the given vectors

Step II Reduce A to echelon form by elementary row operations.

Step III Take the non-zero rows of the echelon form. These rows form a basis of the subspace spanned by the given set of vectors.

In order to find a basis consisting of vectors from the original list of vectors, we use the following algorithm.

ALGORITHM 2 *(Casting-out algorithm)*

Step I Form the matrix A whose columns are the given vectors.

Step II Reduce A to echelon form by elementary row operations.

Step III Delete (cast out) those vectors from the given list which correspond to columns without pivots and select the remaining vectors in S which correspond to columns with pivots. Vectors so selected form a basis of $[S]$.

TYPE III **ON FINDING THE DIMENSION OF SUBSPACE SPANNED BY A GIVEN SET OF VECTORS**

EXAMPLE-7 *Let S be the set consisting of the following vectors in R^5:*

$v_1 = (1,2,1,3,2), v_2 = (1,3,3,5,3), v_3 = (3,8,7,13,8), v_4 = (1,4,6,9,7), v_5 = (5,13,13,25,19)$

Find a basis of $[S]$ (i.e the subspace spanned by S) consisting of the original given vectors. Also, find the dimension of $[S]$.

SOLUTION Let A be the matrix whose columns are the given vectors. Then,

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 2 & 3 & 8 & 4 & 13 \\ 1 & 3 & 7 & 6 & 13 \\ 3 & 5 & 13 & 9 & 25 \\ 2 & 3 & 8 & 7 & 19 \end{bmatrix}$$

Let us now reduce A to echelon form by using elementary row operations.

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 5 & 8 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 1 & 2 & 5 & 9 \end{bmatrix} \quad \begin{matrix}\text{Applying } R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1, \\ R_4 \to R_4 - 3R_1 \text{ and } R_5 \to R_5 - R_1\end{matrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \quad \text{Applying } R_3 \to R_3 - 2R_2, R_4 \to R_4 - 2R_2, R_5 \to R_5 - R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Applying } R_4 \to R_4 - 2R_3, R_5 \to R_5 - 2R_3$$

We observe that pivots (encircled entries) in the echelon form of A appear in the columns C_1, C_2, C_4. So, we "cast out" the vectors u_3 and u_5 from set S and the remaining vectors u_1, u_2, u_4, which correspond to the columns in the echelon matrix with pivots, form a basis of $[S]$. Hence, $\dim[S] = 3$.

EXAMPLE-8 *Let S be the set consisting of following vectors in R^4:*

$$v_1 = (1, -2, 5, -3), v_2 = (2, 3, 1, -4), v_3 = (3, 8, -3, -5).$$

(i) Find a basis and dimension of the subspace spanned by S, i.e. $[S]$.

(ii) Extend the basis of $[S]$ to a basis of R^4.

(iii) Find a basis of $[S]$ consisting of the original given vectors.

SOLUTION (i) Let A be the matrix whose rows are the given vectors. Then,

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

Let us now reduce A to echelon form. The row reduced echelon form of A is as given below

$$A \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$