Unit 2 :- Differential Calculus-II

So far we have dealt with the calculus of functions of a Single Variable. But in the real world, physical Quantities often depend on two or more Variables, so in this chapter we turn our attention to functions of Several Variables and extend the basic ideas of differential calculus to such functions.

Functions of Two Variables.

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and lattitude y of the point. We can think of T as being a function of the two Variables x and y, or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The Volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Definition:-

A function f of two Variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

The set D is the domain of f and its range is the set of values that f takes on, that is $\{f(x, y) \mid (x, y) \in D\}$

Partial differentiation

Let $u = f(x, y)$ be a continuous function of x and y then the function "u" will change when either x or y or both x and y changes. If we keep y as a constant allow x to vary then it is called partial differentiation of "u" with respect to x and it is written as $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{\text{lt}} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

also written as u_x, f_x or $\frac{\partial f}{\partial x}$.

Similarly if we keep x as a constant and y to vary then it is called partial derivative of "u" with respect to y and it is written as $\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{\text{lt}} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

Here $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ are the first order partial derivatives of "u" with respect to x and y.

If "u" is a function of two or more independent variables, then the partial derivatives of u with respect to any one of the independent variables is the ordinary derivative of u with respect to that variable, treating all other variable as constant.

All the rules of differentiation applicable to functions of a single variable are applicable for partial differentiation also.

The second order partial derivatives are written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = u_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = u_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = u_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = u_{yy}$$

Note:- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, If $u = f(x, y)$ is continuous and possess continuous derivative at the point (x, y) then at this point $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Note:- In u_{xy} we differentiate first with respect to x then with respect to y, while $\frac{\partial^2 u}{\partial x \partial y}$ means, Differentiate with respect to y first and then with respect to x.

I. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following.

(1) $z = x^3 + y^3 + 3x^2y \rightarrow (1)$

Soln: Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy$$

Differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 3y^2 + 3x^2$$

(2) $z = e^{x/y}$

$$\frac{\partial z}{\partial x} = e^{x/y} \cdot \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = e^{x/y} - \frac{x}{y^2}$$

(3) $z = \log(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} ; \quad \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$

(4) $z = x\cos(xy) + y\sin(xy)$

$$\frac{\partial z}{\partial x} = -xy\sin(xy) + \cos(xy) + y^2\cos(xy)$$

$$\frac{\partial z}{\partial y} = -x^2\sin(xy) + xy\cos(xy) + \sin(xy)$$

(2) If $z = e^x(x\sin y + y\cos y)$ then Prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

Soln: $z = e^x(x\sin y + y\cos y) \rightarrow (1)$

Diff w.r.to "x" partially, we get

$$\frac{\partial z}{\partial x} = e^x(\sin y) + e^x(x\sin y + y\cos y)$$ (using product rule)

differentiating again w.r.to x, partially

$$\frac{\partial^2 z}{\partial x^2} = e^x\sin y + e^x\sin y + e^x(x\sin y + y\cos y)$$

$$\frac{\partial^2 z}{\partial x^2} = 2e^x\sin y + e^x x\sin y + e^x y\cos y \rightarrow (2)$$

Differentiating (1) partially w.r.to y, we get

$$\frac{\partial z}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$$

Differentiating, partially again w.r.to y, we get

$$\frac{\partial^2 z}{\partial y^2} = e^x[-x\sin y - \sin y - (y\cos y + \sin y)]$$

$$\frac{\partial^2 z}{\partial y^2} = e^x[-x\sin y - \underline{\underline{\sin y}} - y\cos y - \underline{\underline{\sin y}}]$$

$$\frac{\partial^2 z}{\partial y^2} = -2e^x\sin y - e^x x\sin y - e^x y\cos y \rightarrow (3)$$

$(2) + (3) \Rightarrow$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2e^x\sin y + e^x x\sin y + e^x y\cos y$$
$$- 2e^x\sin y - e^x x\sin y - e^x y\cos y$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

(3) If $u = x^y$ then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Soln:

$$u = x^y \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = y x^{y-1}$$

$$x^{y-1} = x^y x^{-1} = x^y \times \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{y x^y}{x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{x}\left[y\, x^y \log_e x + x^y\right]$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^y\left[y \log_e x + 1\right]}{x}$$

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1}\left[y \log_e x + 1\right] \rightarrow (2)$$

Consider $\frac{\partial u}{\partial y} = x^y \log_e x$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^y \frac{1}{x} + (\log_e x)\, y x^{y-1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + x^{y-1} y \log_e x$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1}\left[1 + y \log_e x\right] \rightarrow (3)$$

From (2) and (3) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

(4) If $z = f(ax+by)$ then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Soln:

$$\frac{\partial z}{\partial x} = f'(ax+by) \cdot a$$

$$\frac{\partial z}{\partial y} = f'(ax+by) \cdot b.$$

(5) If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then Show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}. \qquad (x \neq 0 \text{ and } y \neq 0)$$

Soln:

$$u = x^2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - y^2\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y}{\frac{x^2+y^2}{x^2}} - \frac{y}{\frac{x^2+y^2}{y^2}}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y\left[\frac{x^2+y^2}{x^2+y^2}\right]$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Now, Differentiating again w.r.to "y" we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = 2x \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} - 1$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2}{\frac{x^2+y^2}{x^2}} - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1$$

$$= \frac{2x^2 - x^2 - y^2}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2+y^2} = \frac{\partial^2 u}{\partial x \partial y}.$$

Symmetric function : A function $f(x,y)$ is said to be symmetric if $f(x,y) = f(y,x)$ and a function $f(x,y,z)$ is said to be symmetric if $f(x,y,z) = f(y,z,x) = f(z,x,y)$. In general we can say that a function of several variables is symmetric if the function remains unchanged when the variables are cyclically rotated.

ex: $u = x+y$, $u = x^2+y^2+z^2$, $u = xy+yz+xz$

$u = \log(x+y)$, $u = x^3+y^3+z^3-3xyz$.

Note: If we have symmetric function of three variables or two variables then just by computing $\frac{\partial u}{\partial x}$ or $\frac{\partial^2 u}{\partial x^2}$ we can write $\frac{\partial u}{\partial y}$ or $\frac{\partial^2 u}{\partial y^2}$.

(6) If $z = \frac{x^2+y^2}{x+y}$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$

Soln:

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Since the given function is symmetric, we get

$$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Consider $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2x^2-2y^2}{(x+y)^2} = \frac{2(x+y)(x-y)}{(x+y)^2}$

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2(x-y)}{x+y}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \rightarrow (1)$$

Consider $1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{4xy}{(x+y)^2}$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2}$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+y^2+2xy-4xy}{(x+y)^2}$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{(x-y)^2}{(x+y)^2}$$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{4(x-y)^2}{(x+y)^2} \rightarrow (2)$$

From (1) and (2) $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

(7) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan$

Soln:- $\sin u = \frac{x^2+y^2}{x+y}$

Applying $\log_e$ on both sides

$$\log_e(\sin u) = \log_e(x^2+y^2) - \log_e(x+y)$$

Diff partially w.r.to x, we get

$$\frac{1}{\sin u} \cos u \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} 2x - \frac{1}{x+y}$$

$$\cot u \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x+y}$$

$$\frac{\partial u}{\partial x} = \left[\frac{2x}{x^2+y^2} - \frac{1}{x+y}\right] \tan u$$

$$x\frac{\partial u}{\partial x} = \left[\frac{2x^2}{x^2+y^2} - \frac{x}{x+y}\right] \tan u$$

Since u is symmetric, we can write

$$\frac{\partial u}{\partial y} = \left[\frac{2y}{x^2+y^2} - \frac{1}{x+y}\right] \tan u$$

$$y\frac{\partial u}{\partial y} = \left[\frac{2y^2}{x^2+y^2} - \frac{y}{x+y}\right] \tan u$$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left[\frac{2x^2+2y^2}{x^2+y^2} - \frac{(x+y)}{x+y}\right] \tan u$$

$$= [2-1] \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

(8) If $u = \log_e(\tan x + \tan y + \tan z)$ Prove that

$$\sin 2x \, u_x + \sin 2y \, u_y + \sin 2z \, u_z = 2$$

Soln:-

$$u_x = \frac{1}{\tan x + \tan y + \tan z} \sec^2 x$$

$$\sin 2x \, u_x = \frac{1}{\tan x + \tan y + \tan z} \frac{2\sin x \cos x}{\cos^2 x}$$

$$\therefore \sin 2x \, u_x = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

|||ly $\sin 2y \, u_y = \dfrac{2\tan y}{\tan x + \tan y + \tan z}$ ($\because$ Since u is symmetric)

$$\sin 2z \, u_y = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

$$\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$$

(9) If $u = \log(x^3+y^3+z^3-3xyz)$ then Show that

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$. Hence deduce

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

Soln:- $$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$$

III'ly $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$ ($\because$ since u is symmetric)

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2 - xy - yz - xz)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3[(x^2+y^2+z^2) - xy - yz - xz]}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \rightarrow (1)$$

consider $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad (\because \text{by } (1))$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

(10) If $x^x y^y z^z = c$, show that at $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -(x\log_e x)^{-1}.$$

Soln:

$$x^x y^y z^z = c$$

Taking logarithm on B.S

$$\log_e(x^x y^y z^z) = \log_e c$$

$$x\log_e x + y\log_e y + z\log_e z = \log_e c$$

Taking ($z = f(x,y)$) z in a function of x and y

Diff w.r.to x partially, we get

$$x \times \frac{1}{x} + \log_e x + z \times \frac{1}{z}\frac{\partial z}{\partial x} + \log_e z \frac{\partial z}{\partial x} = 0$$

(y in a constant when Diff w.r.to x)

$$1+\log_e x + \frac{\partial z}{\partial x} + \log_e z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log_e x)}{1+\log_e z}$$

III'ly

$$\frac{\partial z}{\partial y} = -\frac{(1+\log_e y)}{1+\log_e z}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -\frac{(1+\log_e x)}{(1+\log_e z)^2} \times \frac{1}{z}\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(1+\log_e x)}{(1+\log_e z)^2} \times \frac{1}{z}\left(-\frac{(1+\log_e y)}{(1+\log_e z)}\right)$$

$$= -\frac{(1+\log_e x)^2}{(1+\log_e x)^3} \times \frac{1}{x} \qquad (\because x=y=z)$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{x} \times \frac{1}{(\log_e x+1)}$$

$$= -\frac{1}{x} \times \frac{1}{(\log_e x + \log_e e)}$$

$$= -\frac{1}{x} \times \frac{1}{\log_e ex}$$

$$\frac{\partial z}{\partial y \partial x} = -(x \log_e ex)^{-1}$$

(11) If $\theta = t^n e^{-r^2/4t}$, what value of n will make

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right)\right) = \frac{\partial \theta}{\partial t} ?$$

Soln: $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial \theta}{\partial t} = t^n e^{-r^2/4t}\left[-\frac{r^2}{4}\right]\left[-\frac{1}{t^2}\right] + n t^{n-1} e^{-r^2/4t}$$

$$= e^{-r^2/4t}\left[n t^{n-1} + \frac{r^2}{4} t^{n-2}\right] \rightarrow (1)$$

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left[\frac{-2r}{4t}\right]$$

$$= -\frac{r}{2} t^{n-1} e^{-r^2/4t}$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{r^3}{2} t^{n-1} e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\left[\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} + \frac{r^3}{2} t^{n-1} e^{-r^2/4t}\left(\frac{-2r}{4t}\right)\right]$$

$$= -\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} + \frac{r^4}{4} t^{n-2} e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = e^{-r^2/4t}\left[-\frac{3r^2}{2} t^{n-1} + \frac{r^4}{4} t^{n-2}\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = e^{-r^2/4t}\left[-\frac{3}{2} t^{n-1} + \frac{r^2}{4} t^{n-2}\right] \quad \to (2)$$

Since $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

$$e^{-r^2/4t}\left[-\frac{3}{2} t^{n-1} + \frac{r^2}{4} t^{n-2}\right] = e^{-r^2/4t}\left[n t^{n-1} + \frac{r^2}{4} t^{n-2}\right]$$

$$\Rightarrow -\frac{3}{2} t^{n-1} + \frac{r^2}{4} t^{n-2} = n t^{n-1} + \frac{r^2}{4} t^{n-2}$$

$$\Rightarrow -\frac{3}{2} t^{n-1} = n t^{n-1}$$

$$\Rightarrow \boxed{n = -\frac{3}{2}}$$

(12) Show that $V(x,y,z) = \cos(3x)\cos(4y)\sinh(5z)$ satisfies the Laplace equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$

Soln:

$$\frac{\partial V}{\partial x} = -3\sin 3x \cos 4y \sinh(5z)$$

$$\frac{\partial^2 V}{\partial x^2} = -9\cos 3x \cos 4y \sinh 5z$$

$$\frac{\partial^2 V}{\partial x^2} = -9V$$

|||ly $\frac{\partial^2 v}{\partial y^2} = -16v$, $\frac{\partial^2 v}{\partial z^2} = 25v$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -9v - 16v + 25v$$

$$= 0.$$

(13) If $u = f(x+at) + g(x-at)$, then prove that

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Soln: $u = f(x+at) + g(x-at)$

$$\frac{\partial u}{\partial x} = f'(x+at) + g'(x-at)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+at) + g''(x-at) \rightarrow (1)$$

$$\frac{\partial u}{\partial t} = f'(x+at)(a) - ag'(x-at)$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x+at)a^2 + a^2 g''(x-at)$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 [f''(x+at) + g''(x-at)]$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \text{(from (1))}$$

$\hookrightarrow$ 1-D- wave equation.

(14) Find the value of n so that the equation $V = r^n (3\cos^2\theta - 1)$, satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0.$$

Soln:

$$V = r^n (3\cos^2\theta - 1)$$

$$\frac{\partial V}{\partial \theta} = r^n(-6\cos\theta\sin\theta)$$

$$\sin\theta\frac{\partial V}{\partial \theta} = -6r^n\sin^2\theta\cos\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = -6r^n\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = 6r^n\left[\sin^2\theta - 2\cos^2\theta\right]$$

$$= 6r^n\left[1-\cos^2\theta - 2\cos^2\theta\right]$$

$$= 6r^n\left[1-3\cos^2\theta\right]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = -6V$$

$$\frac{\partial V}{\partial r} = n r^{n-1}(3\cos^2\theta - 1)$$

$$r^2\frac{\partial V}{\partial r} = n r^{n+1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = n(n+1)r^n(3\cos^2\theta - 1)$$

$$= n(n+1)V$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)\right) = 0$$

$$n(n+1)V - 6V = 0$$

$$(n^2+n-6)V = 0$$

$$\Rightarrow \quad n^2+n-6 = 0$$

$$n = -3, 2$$

(15) If $v = x^y y^x$ Prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x+y+\log_e v)$

Soln:

$$V = x^y y^x$$

$$\log_e v = y\log_e x + x\log_e y$$

Differentiating partially w.r.t x, we get

$$\frac{1}{v}\frac{\partial v}{\partial x} = \frac{y}{x} + \log_e y$$

$$\frac{\partial v}{\partial x} = v\left[\frac{y}{x} + \log_e y\right]$$

$$x\frac{\partial v}{\partial x} = v\left[y + x\log_e y\right]$$

III'y $$y\frac{\partial v}{\partial y} = v\left[x + y\log_e x\right]$$

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v\left[x + y + x\log_e y + y\log_e x\right]$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v\left[x + y + \log_e v\right]$$

(16) If $u = \frac{y}{z} + \frac{z}{x}$ then Show that $xu_x + yu_y + zu_z = 0$.

Soln: $$\frac{\partial u}{\partial x} = -\frac{z}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{z}, \quad \frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$

$$xu_x + yu_y + zu_z = 0$$

(17) If $z = e^{ax+by} f(ax-by)$ then prove that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Soln:.

$$z = e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)(a) + f(ax-by)e^{ax+by}(a)$$

$$b\frac{\partial z}{\partial x} = ab\, e^{ax+by}[f'(ax-by) + f(ax-by)] \rightarrow ①$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)e^{ax+by}(b)$$

$$a\frac{\partial z}{\partial y} = ab\, e^{ax+by}[-f'(ax-by) + f(ax-by)] \rightarrow ②$$

① + ② ⇒

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab\, e^{ax+by}[2f(ax-by)]$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$