Method of Variation of Parameters

Working rule to solve the Differential Equation of the form

$y'' + a_1 y' + a_0 y = f(x) \rightarrow ①$

Step:- write the complimentary function of ①

let it be $C.F = c_1 u + c_2 v$

Step 2:- Replace c_1 and c_2 by A and B respectively in C.F

to get the complete solution.

$y = Au + Bv \rightarrow ②$ where A and B are functions of x

Step 3:- Find the wronskian of the functions u and v by

using $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Step 4:- Find the functions A and B by using the

following:- $A = -\int \frac{v f(x)}{W} dx$

$B = \int \frac{u f(x)}{W} dx$

Step 5:- Substitute for A and B in ② to get the

general solution.

Note:-

Two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent

if $\frac{y_1}{y_2} \neq$ a constant i.e if neither is a constant times the

other.

Ex:- x and $2x$ are linearly dependent

If $y_2 = k y_1$ then $c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 k y_1 = (c_1 + c_2 k) y_1 = c y_1$.

In this case there is only one arbitrary constant c therefore

$c_1 y_1 + c_2 y_2$ is not the general solution.

x^2 and $2x^3$ are linearly independent

II Solve the following Differential Equations by the method of Variation of Parameters.

(1) $y'' + a^2 y = \sec ax$

Soln: The Auxiliary equation (A.Eqn) is

$m^2 + a^2 = 0$

$m = \pm ai$

$C.F = c_1 \cos ax + c_2 \sin ax$

Let $P.I = Au + Bv$

$= A\cos ax + B\sin ax \rightarrow (1)$

$u = \cos ax \qquad v = \sin ax \qquad f(x) = \sec ax$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax$$

$= a$

$$\therefore A = -\int \frac{v f(x)}{W} dx = -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a}\int \tan ax \, dx$$

$$= -\frac{1}{a} \frac{\log(\sec ax)}{a} = -\frac{1}{a^2}\log(\sec ax)$$

$$B = \int \frac{u f(x)}{W} dx = \int \frac{\cos ax \sec ax}{a} = \frac{1}{a}\int dx = \frac{x}{a}$$

(1) $\Rightarrow$

$$\therefore P.I = -\frac{1}{a^2}\log(\sec ax)\cos ax + \frac{x}{a}\sin ax$$

Hence $y = C.F + P.I$

$$= c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\log(\sec ax)\cos ax + \frac{x}{a}\sin ax$$

(2) $y'' + y = \operatorname{cosec} x$

Soln: A. Eqn is $m^2+1=0$

$$m = \pm i$$

$$C.F = c_1\cos x + c_2 \sin x$$

$$P.I = A\cos x + B\sin x$$

$u = \cos x \qquad v = \sin x \qquad f(x) = \operatorname{cosec} x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$A = -\int \frac{v f(x)}{W} dx = -\int \frac{\sin x \operatorname{cosec} x}{1} dx = -\int \sin x \times \frac{1}{\sin x} dx$$

$$= -x$$

$$B = \int \frac{u f(x)}{W} dx = \int \frac{\cos x \operatorname{cosec} x}{1} dx = \int \cos x \frac{1}{\sin x} dx$$

$$= \int \cot x \, dx$$

$$= \log(\sin x)$$

$\therefore$ $P.I = -x\cos x + \log(\sin x) \sin x$

$y = C.F + P.I$

(3) $\frac{d^2y}{dx^2} + y = \sec x \tan x$

Soln: A. Eqn is $m^2+1=0$

$$\Rightarrow m = \pm i$$

$$C.F = c_1\cos x + c_2 \sin x$$

$C.F = c_1\cos x + c_2\sin x$

$P.I = A\cos x + B\sin x$

$u = \cos x \qquad v = \sin x \qquad f(x) = \sec x \tan x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$A = -\int \frac{v f(x)}{W}\,dx = -\int \frac{\sin x \sec x \tan x}{1}\,dx$$

$$= -\int \tan^2 x\,dx$$

$$= -\int (\sec^2 x - 1)\,dx$$

$$= -[\tan x - x]$$

$$= x - \tan x$$

$$B = \int \frac{u f(x)}{W}\,dx = \int \frac{\cos x \sec x \tan x}{1}.$$

$$= \int \tan x\,dx$$

$$= \log(\sec x)$$

$\therefore P.I = (x - \tan x)\cos x + \log(\sec x)\sin x$

$y = C.F + P.I$

(4) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

Soln: A.Eqn is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$C.F = c_1 e^x + c_2 e^{-x}$

$P.I = Ae^x + Be^{-x}$

$u = e^{x} \qquad v = e^{-x} \qquad f(x) = \frac{2}{1+e^{x}}$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}$$

$$= -e^{x}e^{-x} - e^{x}e^{-x}$$

$$= -2$$

$$A = -\int \frac{v f(x)}{w} dx = -\int \frac{e^{-x}\left(\frac{2}{1+e^{x}}\right)}{-2} dx$$

$$= \int \frac{e^{-x}}{1+e^{x}} dx$$

$$= \int \frac{e^{-x}}{e^{x}(1+e^{-x})} dx \longrightarrow \text{Taking } e^{x} \text{ common in denominator}$$

$$= \int \frac{e^{-x}\,e^{-x}}{(1+e^{-x})} dx \longrightarrow \text{Taking } e^{x} \text{ to the numertor}$$

put $e^{-x}+1 = t \Rightarrow -e^{-x}dx = dt$

$$\therefore A = -\int \frac{t-1}{t} dt = -\int \left(1 - \frac{1}{t}\right) dt$$

$$= \int \left(\frac{1}{t} - 1\right) dt$$

$$= \log t - t$$

$$= \log(e^{-x}+1) - (e^{-x}+1)$$

$$B = \int \frac{u f(x)}{w} dx = \int \frac{e^{x}\left(\frac{2}{e^{x}+1}\right)}{-2} dx$$

$$= -\int \frac{e^{x}}{e^{x}+1} dx$$

$$\int \frac{f'(x)}{f(x)} dx = \log f(x)$$

$$= -\log(e^{x}+1)$$

$\therefore$ P.I $= \left[\log(e^{-x}+1) - (e^{-x}+1)\right] e^{x} - \log(e^{x}+1)\, e^{-x}$

$\therefore\ y = C.F + P.I$

(5) $y'' - 2y' + 2y = e^{x}\tan x$

Soln: A. Eqn is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

$$C.F = e^{x}(c_1\cos x + c_2 \sin x)$$

$$P.I = Ae^{x}\cos x + Be^{x}\sin x$$

$u = e^{x}\cos x \qquad v = e^{x}\sin x \qquad f(x) = e^{x}\tan x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{x}\cos x & e^{x}\sin x \\ e^{x}(-\sin x + \cos x) & e^{x}(\sin x + \cos x) \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} \cos x & \sin x \\ \cos x - \sin x & \cos x + \sin x \end{vmatrix}$$

$$= e^{2x}\left[\cos^2 x + \cos x \sin x - \sin x \cos x + \sin^2 x\right]$$

$$= e^{2x}$$

$$A = -\int \frac{v f(x)}{W}\,dx = -\int \frac{e^{x}\sin x\, e^{x}\tan x}{e^{2x}}\,dx$$

$$= -\int \sin x \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\, dx$$

$$= -[\log(\sec x + \tan x) - \sin x]$$

$$A = \sin x - \log(\sec x + \tan x)$$

$$B = \int \frac{u f(x)}{W}\, dx = \int \frac{e^x \cos x\, e^x \tan x}{e^{2x}}\, dx$$

$$= \int \sin x\, dx$$

$$= -\cos x$$

$$P.I = [\sin x - \log(\sec x + \tan x)]\, e^x \cos x$$

$$- \cos x\, e^x \sin x$$

$$= e^x [\sin x \cos x - \cos x \log(\sec x + \tan x) - \sin x \cos x]$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

$\therefore\ y = C.F + P.I$

(6) $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

Soln: A. Eqn $m^2 - 6m + 9 = 0$

$m = 3, 3$

$C.F = (c_1 + c_2 x)e^{3x}$

$P.I = Ae^{3x} + Bxe^{3x}$

$u = e^{3x}$ $\quad v = xe^{3x}$ $\quad f(x) = \frac{e^{3x}}{x^2}$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(3x+1) \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & x \\ 3 & 3x+1 \end{vmatrix}$$

$$= \cancel{e^{6x}}\ e^{6x}$$

$$A = -\int \frac{v f(x)}{W} dx = -\int \frac{x e^{3x} \, (e^{3x}/x^2)}{e^{6x}} dx$$

$$= -\int \frac{1}{x} dx = -\log x$$

$$B = \int \frac{u f(x)}{W} dx = \int \frac{e^{3x} \, (e^{3x})}{e^{6x} \, x^2} dx$$

$$= -\frac{1}{x}$$

$$P.I = -\log x \; e^{3x} - \frac{1}{x} x e^{3x}$$

$$= e^{3x}(-\log x) - e^{3x}$$

$\therefore \quad y = C.F + P.I$

$$= (c_1 + c_2 x) e^{3x} - e^{3x} (\log x + 1)$$

(7) $\quad y'' + y = \dfrac{1}{1+\sin x}$

Soln:. $\quad m^2 + 1 = 0$

$m = \pm i$

$C.F = c_1 \cos x + c_2 \sin x$

$P.I = A \cos x + B \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$A = -\int \frac{v f(x)}{W} dx = -\int \frac{\sin x}{1} \left(\frac{1}{1+\sin x} \right) dx$$

$$= -\int \frac{\sin x \, (1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= -\int \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= -\int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= -\int (\tan x \sec x - \tan^2 x)\, dx$$

$$= -\int (\tan x \sec x - (\sec^2 x - 1))\, dx$$

$$= -[\sec x - (\tan x - x)]$$

$$= \tan x - \sec x - x$$

$$B = \int \frac{u f(x)}{w} dx = \int \frac{\cos x \frac{1}{1+\sin x}}{1} dx$$

$$= \int \frac{\cos x}{1+\sin x} dx$$

put $1+\sin x = t$

$\cos x\, dx = dt$

$$B = \int \frac{dt}{t} = \log t$$

$$= \log(1+\sin x)$$

$$P.I = [\tan x - \sec x - x] \cos x + \log(1+\sin x) \sin x$$

$$\therefore y = C.F + P.I$$

8) $y'' - y = e^{-2x} \sin(e^{-x})$

soln: $m^2 - 1 = 0$

$m = 1, -1$

$C.F = c_1 e^{x} + c_2 e^{-x}$

$P.I = Ae^{x} + Be^{-x}$

$$w = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$A = -\int \frac{v f(x)}{w} dx$$

$$= -\int \frac{e^{-x} e^{-2x} \sin(e^{-x})}{-2} dx$$

Put $e^{-x} = t$

$$-e^{-x} dx = dt$$

$$A = -\frac{1}{2} \int t^2 \sin t \, dt$$

Integrating by parts we get

$$= -\frac{1}{2}\left[t^2(-\cos t) - 2t(-\sin t) + 2\cos t\right]$$

$$= -\frac{1}{2}\left[-t^2\cos t + 2t\sin t + 2\cos t\right]$$

$$= \frac{e^{-2x}\cos(e^{-x})}{2} - e^{-x}\sin(e^{-x}) - \cos(e^{-x})$$

$$B = \int \frac{u f(x)}{w} dx = \int \frac{e^{x} e^{-2x} \sin(e^{-x})}{-2} dx$$

$$= -\frac{1}{2}\int e^{-x} \sin(e^{-x}) dx$$

Put $e^{-x} = t$

$$-e^{-x} dx = dt$$

$$\therefore B = \frac{1}{2}\int \sin t \, dt$$

$$= -\frac{\cos t}{2}$$

$$= -\frac{\cos(e^{-x})}{2}$$

$$P.I = \left[\frac{1}{2} e^{-2x}\cos(e^{-x}) - e^{-x}\sin(e^{-x}) - \cos(e^{-x})\right]e^{x}$$
$$- \frac{1}{2}\cos(e^{-x})\, e^{-x}$$
$$= -\sin(e^{-x}) - e^{x}\cos(e^{-x})$$

$\therefore y = c_1 e^{x} + c_2 e^{-x} - \sin(e^{-x}) - e^{x}\cos(e^{-x})$

(9) Solve $x^2y'' - 4xy' + 6y = x^4 \sin x$ by the method of Variation of Parameters.

Soln:- $(x^2D^2 - 4xD + 6)y = x^4 \sin x \rightarrow ①$

Put $\log x = z$ or $e^z = x$

$xD = D_1$

$x^2D^2 = D_1(D_1 - 1)$ where $D_1 = \frac{d}{dz}$

$\therefore ① \Rightarrow (D_1^2 - D_1 - 4D_1 + 6)y = e^{4z}\sin(e^z)$

A.Eqn is $m^2 - 5m + 6 = 0$

$m = 2, 3$

$C.F = c_1 e^{2z} + c_2 e^{3z} = c_1 x^2 + c_2 x^3$

$P.I = Ae^{2z} + Be^{3z}$

$\cancel{A=e^{2z}},\ \cancel{B=e^{3z}}\ \cancel{A=e^{2z}},\ \cancel{B=e^{3z}},\ \cancel{f(z)=e^{4z}\sin(e^z)}$

$u = e^{2z}, \quad v = e^{3z}, \quad f(z) = e^{4z}\sin(e^z)$

$$W = \begin{vmatrix} e^{2z} & e^{3z} \\ 2e^{2z} & 3e^{3z} \end{vmatrix} = e^{5z}$$

$$\therefore A = -\int \cancel{e^{3z}}\ \frac{v f(z)}{W}\, dz$$

$$= -\int \frac{e^{3z} e^{4z} \sin(e^z)}{e^{5z}}\, dz$$

$$= -\int e^{2z} \sin(e^z)\, dz$$

put $e^z = t$

$e^z dz = dt$

$\therefore A = -\int t \sin t \, dt$

Integrating by parts

$= -[t(-\cos t) - (-\sin t)]$

$= t\cos t - \sin t$

$= e^z \cos(e^z) - \sin(e^z)$

$B = \int \frac{e^{2z} e^{4z} \sin(e^z)}{e^{5z}} dz$

$= \int e^z \sin(e^z) dz$ Put $e^z = t$

$e^z dz = dt$

$= \int \sin t \, dt$

$= -\cos t$

$= -\cos(e^z)$

$\therefore P.I = [e^z \cos(e^z) - \sin(e^z)] e^{2z} - e^{3z} \cos(e^z)$

$= e^{3z}\cos(e^z) - e^{2z}\sin(e^z) - e^{3z}\cos(e^z)$

$= -e^{2z}\sin(e^z)$

$= -x^2 \sin x$

$\therefore y = c_1 x^2 + c_2 x^3 - x^2 \sin x$

(10) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ by the method of variation of parameters.

Soln:- $(x^2D^2 + 3xD + 1)y = \frac{1}{(1-x)^2}$

Put $\log x = z$ or $e^z = x$

$xD = D_1$

$x^2D^2 = D_1(D_1 - 1)$ where $D_1 = \frac{d}{dz}$

$\therefore \quad (D_1^2 - D_1 + 3D_1 + 1)y = \frac{1}{(1-e^z)^2}$

$\therefore \quad (D_1^2 + 2D_1 + 1)y = \frac{1}{(1-e^z)^2}$

A. Eqn $\quad m^2 + 2m + 1 = 0$

$m = -1, -1$

$C.F = (C_1 + C_2 z)e^{-z} \quad = \quad (C_1 + C_2 \log x)x^{-1}$

$P.I = Ae^{-z} + Bze^{-z}$

$u = e^{-z}, \quad v = ze^{-z}, \quad f(z) = \frac{1}{(1-e^z)^2}$

$$w = \begin{vmatrix} e^{-z} & ze^{-z} \\ -e^{-z} & e^{-z}(z-1) \end{vmatrix} = e^{-2z}$$

$$A = -\int \frac{ze^{-z}}{e^{-2z}} \frac{1}{(1-e^z)^2} dz$$

$$= -\int \frac{ze^z}{(1-e^z)^2} dz$$

Put $1 - e^z = t$

$\Rightarrow e^z = 1 - t$

$e^z dz = -dt$

$\Rightarrow z = \log(1-t)$

$$= \int \frac{\log(1-t)}{t^2} dt$$

Integrating by parts, we get

$$A = \log(1-t)\left(-\frac{1}{t}\right) - \int \left(-\frac{1}{t}\right)\left(-\frac{1}{1-t}\right) dt$$

$$= -\frac{\log(1-t)}{t} - \int \frac{1}{t(1-t)} dt$$

By Partial fractions

$$= -\frac{\log(1-t)}{t} - \int \left[\frac{1}{t} + \frac{1}{1-t}\right] dt$$

$$= \frac{-\log(1-t)}{t} - \log t + \log(1-t)$$

$$\therefore A = \frac{-\log(e^z)}{1-e^z} - \log(1-e^z) + \log(e^z)$$

$$= -\frac{\log x}{1-x} - \log(1-x) + \log x$$

$$B = \int \frac{e^{-z}}{e^{-2z}} \frac{1}{(1-e^z)^2} dz$$

$$= \int \frac{e^z}{(1-e^z)^2} dz$$

Put $e^z = t$

$e^z dz = dt$

$$= \int \frac{dt}{(1-t)^2} = \int (1-t)^{-2} dt$$

$$= \frac{1}{1-t}$$

$$= \frac{1}{1-e^z} \qquad [\because e^z = x]$$

$$= \frac{1}{1-x}$$

$$P.I = \left[-\frac{\log x}{1-x} - \underbrace{\log(1-x) + \log x}_{\log a - \log b = \log(a/b)}\right] x^{-1} + \left[\frac{1}{1-x} \log x\right] x^{-1}$$

$$= x^{-1} \log\left(\frac{x}{1-x}\right)$$

$$y = \frac{c_1 + c_2 \log x}{x} + \left[\log\left(\frac{x}{1-x}\right)\right] \frac{1}{x}$$

(11) Solve $(x+1)^2 y'' + (x+1) y' + y = 4\cos\log(1+x)$ by the method of variation of parameters.

Soln: $[(x+1)^2 D^2 + (x+1) D + 1] y = 4\cos\log(1+x)$

Put $\log(x+1) = z$ or $e^z = x+1$

$(x+1) D = D_1$

$(x+1)^2 D^2 = D_1(D_1 - 1)$ where $D_1 = \frac{d}{dz}$

$\therefore \quad (D_1^2 - D_1 + D_1 + 1)y = 4\cos z$

$(D_1^2 + 1)y = 4\cos z$

A.Eqn is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ C.F $= c_1 \cos z + c_2 \sin z$

P.I $= A\cos z + B\sin z$

$u = \cos z \qquad v = \sin z \qquad f(z) = 4\cos z$

$$w = \begin{vmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{vmatrix} = 1$$

$$A = -\int \frac{v f(z)}{w}\, dz = -\int \frac{\sin z \, 4\cos z}{1}\, dz$$

$$= -\int 2\sin 2z \, dz = -2\left(\frac{-\cos 2z}{2}\right)$$

$$= \cos 2z$$

$$B = \int \frac{u f(z)}{w}\, dz = \int \cos z \, 4\cos z \, dz$$

$$= 4\int \cos^2 z \, dz$$

$$= 4\int \left(\frac{1+\cos 2z}{2}\right) dz$$

$$= 2z + \sin 2z$$

$\therefore$ P.I $= \cos 2z \cos z + 2z\sin z + \sin 2z \sin z$

$$= \frac{1}{2}\left[\cancel{\cos 3z} + \cos z\right] + 2z\sin z + \frac{1}{2}\left[\cos z - \cancel{\cos 3z}\right]$$

$$= \cos z + 2z\sin z$$

$$= \cos\log(x+1) + 2\log(x+1)\sin\log(x+1)$$

$y =$ C.F + P.I //