LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT AND VARIABLE COEFFICIENTS

Definition:-

A D.Eqn in which the Dependent Variable and its derivative appear only in first degree is called a linear Differential Equation.

A D.Eqn of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x) \rightarrow ①$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ and $f(x)$ are constants or functions of x only. If $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants, then the equation ① is called linear Differential equation with constant coeffients.

Eqn ① can also be written as

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_2 D^2 y + a_1 D y + a_0 y = f(x) \rightarrow ②$$

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots \; a_2 D^2 + a_1 D + a_0) y = f(x) \rightarrow ③$$

where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2} \cdots$, $D^n = \frac{d^n}{dx^n}$

using Polynomial operator

$$F(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_2 D^2 + a_1 D + a_0$$

③ can be written as

$$F(D) y = f(x)$$

If $f(x)=0$, then the differential Equation ① is called Homogeneous equation otherwise it is called non-homogeneous equation.

Since the general Solution of an O.D.E $F(D)y=0$ of order n contains n arbitrary constants. We conclude that

$C_1y_1+C_2y_2+\ldots+C_ny_n=u$ is the general solution of $F(D)y=0$

thus $F(D)u=0$

Again let v be any particular solution of $F(D)y=f(x)$

i.e $F(D)v=f(x)$

Hence $F(D)(u+v)=F(D)u+F(D)v$

$=0+f(x)$

$F(D)(u+v)=f(x)$

This shows that $y=u+v$ is the complete Soln of $F(D)y=f(x)$. Here the first part, u is called complimentary functions and the second part v is called particular integral.

Auxiliary Equation

Consider the Homogeneous L.D.E of order n with constant coefficients $F(D)y=0 \rightarrow$ ①

i.e $(a_nD^n+a_{n-1}D^{n-1}+\ldots+a_2D^2+a_1D+a_0)y=0$

Let $y=e^{mx}$ be the Solution

$\therefore (a_nD^n+a_{n-1}D^{n-1}+\ldots+a_2D^2+a_1D+a_0)e^{mx}=0$

$\Rightarrow a_nm^ne^{mx}+a_{n-1}m^{n-1}e^{mx}+\ldots+a_2m^2e^{mx}+a_1me^{mx}+a_0e^{mx}=0$

$$\Rightarrow (a_n m^n + a_{n-1} m^{n-1} + \dots + a_2 m^2 + a_1 m + a_0) e^{mx} = 0$$

$$\Rightarrow a_n m^n + a_{n-1} m^{n-1} + \dots + a_2 m^2 + a_1 m + a_0 = 0 \rightarrow (2)$$

This is called Auxiliary equation of the D.Eqn (1)

Rules for finiding Complimentary function (C.F)

We have equation (2) as the Auxiliary Eqn of D.Eqn (1)

and Let $m_1, m_2, \dots m_n$ be the roots of (2)

Case (i) :- If roots are real and distinct

i.e $m_1 \neq m_2 \neq m_3 \neq \dots \neq m_n$, then C.F of the D.Eqn is

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case (ii) If roots are real and repeated

i.e $m_1 = m_2 = \dots = m_n$

then $C.F = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}) e^{m_1 x}$

Case (iii) If the roots are complex

i.e $m_1, m_2 = \alpha \pm i\beta$ then

$$C.F = e^{\alpha x} [c_1 \cos\beta x + c_2 \sin\beta x]$$

Case (iv) If the roots are complex and repeated

i.e $m_1, m_2 = \alpha \pm i\beta = m_3, m_4$

then $C.F = e^{\alpha x} [(c_1 + c_2 x) \cos\beta x + (c_3 + c_4 x) \sin\beta x]$

1) Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$

Soln:- The given D.Eqn can be written as

$$(D^2 - 3D - 4)y = 0$$

The Auxiliary eqn is

$$m^2 - 3m - 4 = 0$$

$$m^2 - 4m + m - 4 = 0$$

$$m(m-4) + 1(m-4) = 0$$

$$(m-4)(m+1) = 0$$

$$\therefore m = 4, -1$$

Here the roots are real and Distinct

$$\therefore C.F = c_1 e^{4x} + c_2 e^{-x}$$

2) Solve $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$

Soln:- The given D.Eqn can be written as

$$(D^4 - 5D^2 + 4)y = 0$$

The A.Eqn is

$$m^4 - 5m^2 + 4 = 0$$

$$(m^2 - 1)(m^2 - 4) = 0$$

$$(m+1)(m-1)(m+2)(m-2) = 0$$

$$m = -1, 1, -2, 2$$

Hence $C.F = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x}$

3) Solve :- $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

Soln :- the Given D.Eqn can be written as

$$(D^3 - D^2 - D - 2)y = 0$$

The A.Eqn is $m^3 - m^2 - m - 2 = 0$

By inspection $m=2$ is a root of the equation

Now By Synthetic Division method

$$m=2 \left| \begin{array}{cccc} 1 & -1 & -1 & -2 \\ 0 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & \end{array} \right.$$

$\therefore$ we get $m^2+m+1 = 0$

$$\Rightarrow m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ the roots are $m = 2, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$C.F = c_1e^{2x} + e^{-\frac{1}{2}x}\left[c_2\cos\left(\sqrt{3}/2\right)x + c_3\sin\left(\sqrt{3}/2\right)x\right]$$

4) Solve $(D^3 - 2D^2 + 4D - 8)y = 0$

Soln :- the A.Eqn is given by

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$m^2(m-2) + 4(m-2) = 0$$

$$(m^2+4)(m-2) = 0$$

$$\therefore m = 2, \pm 2i$$

$$C.F = c_1e^{2x} + e^{0x}(c_2\cos 2x + c_3\sin 2x)$$

(5) Solve $y'' + y' - 2y = 0$ given that $y(0) = 0$ and $y'(0) = 3$

Soln:- given $(D^2 + D - 2)y = 0$

A.Eqn $m^2 + m - 2 = 0$

$m = -2, 1$

$\therefore$ C.F $= c_1 e^{-2x} + c_2 e^{x}$

i.e $y = c_1 e^{-2x} + c_2 e^{x} \to (1)$ $\Rightarrow$ $y' = -2c_1 e^{-2x} + c_2 e^{x} \to (2)$

Given $y(0) = 0$ → Meaning when $x = 0$, $y = 0$

x

(1) $\Rightarrow$ $0 = c_1 + c_2$ $\Rightarrow$ $c_1 = -c_2$

$y'(0) = 3$

(2) $\Rightarrow$ $3 = -2c_1 + c_2$ $\Rightarrow$

$3 = -2c_1 - c_1$

$\Rightarrow 3 = -3c_1 \Rightarrow \boxed{c_1 = -1}$

$\therefore \boxed{c_2 = 1}$

Hence $y = -e^{-2x} + e^{x}$.

(6) Solve the initial value problem $y'' - 4y' + 5y$ given that $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$.

Soln:- Given $(D^2 - 4D + 5)y = 0$

A.Eqn $m^2 - 4m + 5 = 0$

$m = 2 \pm i$

$\therefore$ C.F $= e^{2x}(c_1 \cos x + c_2 \sin x)$

Hence $y = e^{2x}(c_1 \cos x + c_2 \sin x) \to (1)$

given that $y=1$ at $x=0$ and $\frac{dy}{dx}=2$ at $x=0$

$\therefore$ (1) $\Rightarrow$ $\boxed{1=c_1}$

Differentiating eqn (1), we get

$$\frac{dy}{dx}= e^{2x}[-c_1\sin x + c_2\cos x] + 2e^{2x}[c_1\cos x + c_2\sin x]$$

we have $\frac{dy}{dx}=2$ at $x=0$.

$$2 = c_2 + 2c_1$$

$$\Rightarrow 2 = c_2+2$$

$$\Rightarrow c_2 = 0$$

$$\therefore y = e^{2x}\cos x.$$

(7) Solve $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$

Soln:. $(D^4-5D^2+4)y=0$

A. Eqn $m^4-5m^2+4=0$

$1-5+4=0$

By Trial and Error $m=1$, is a root

$\therefore$ By Synthetic Division method

$$\begin{array}{c|cccccc} m=1 & 1 & 0 & & -5 & 0 & 4 \\ & 0 & 1 & & 1 & -4 & -4 \\ \hline & 1 & 1 & & -4 & -4 & \underline{0} \end{array}$$

$\therefore$ we get $m^3+m^2-4m-4=0$

$$m^2(m+1)-4(m+1)=0$$

$$(m^2-4)(m+1)=0$$

$\Rightarrow \quad m^2 = 4 \quad$ or $\quad m = -1$

$m = \pm 2$

Hence the roots are $1, -1, 2, -2$

$\therefore \quad C.F = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

(8) Solve $(D^3 - 8)y = 0$

Soln:- A. Eqn $\quad m^3 - 8 = 0$

$m^3 - 2^3 = 0$

$(m-2)(m^2+2m+4) = 0$

$\therefore m = 2 \quad$ or $\quad m^2 + 2m + 4 = 0$

$m = -1 \pm i\sqrt{3}$

$\therefore \quad C.F = c_1 e^{2x} + e^{-x}[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x]$

(9) Solve $(D^4 - m^4)y = 0$

Soln:- A. Eqn $\quad k^4 - m^4 = 0$ $\qquad a^2 - b^2 = (a+b)(a-b)$

$(k^2+m^2)(k^2-m^2) = 0$

$\therefore \quad k^2 = -m^2 \quad$ or $\quad k^2 = m^2$

$k = \pm im \quad$ or $\quad k = \pm m$

$\therefore \quad$ Roots are $m, -m, \pm im$

$C.F = c_1 e^{mx} + c_2 e^{-mx} + e^{0x}(c_3 \cos mx + c_4 \sin mx)$

(10) $(D^2 - 2D + 1)y = 0$

Soln: A. Eqn $\quad m^2 - 2m + 1 = 0$

$m = 1, 1$ $\qquad$ Roots are repeated.

$C.F = (c_1 + c_2 x)e^{x}$

Rules for finding the particular integral.

Case (i) If $f(x) = Ke^{ax}$ then the particular integral is

$$P.I = \frac{1}{F(a)} Ke^{ax} \quad \text{provided} \quad F(a) \neq 0$$

i.e Replace D by a

If $F(a) = 0$, then $P.I = x \cdot \frac{1}{F'(a)} Ke^{ax}$ provided $F'(a) \neq 0$

If $F'(a) = 0$, then $P.I = x^2 \frac{1}{F''(a)} Ke^{ax}$ provided $F''(a) \neq 0$

Note:- If $f(x) = K$ (constant), then express the constant as Ke^{0x}, then

$$P.I = \frac{K}{F(D)} = \frac{1}{F(D)} Ke^{0x} = \frac{1}{F(0)} Ke^{0x} = \frac{1}{F(0)} K$$

Provided $F(0) \neq 0$

Note:- If the RHS function is $\sinh ax$ or $\cosh ax$

then $\sinh ax = \dfrac{e^{ax} - e^{-ax}}{2}$ $\qquad$ $\cosh ax = \dfrac{e^{ax} + e^{-ax}}{2}$

Note:- If we have a^x, then

$a^x = e^{(\log a)x}$ $\qquad$ $[\because a^x = e^{\log a^x} = e^{x \log a}]$

Case (ii) When $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

then in finding the particular integral replace

D^2 by $-a^2$, D^4 by a^4, D^6 by $-a^6$, D^8 by a^8

Note :-

① If denominator becomes a constant, it will be final step in finding particular integral.

② If denominator reduces to D only, we are then only to integrate the given function $f(x)$ once.

③ If denominator reduces to a factor of the form $\alpha D+\beta$ then we operate by its conjugate $\alpha D-\beta$ on both numerator and denominator such as

$$\frac{\alpha D-\beta}{\alpha D-\beta}\left[\frac{1}{\alpha D+\beta}\sin(ax+b)\right]$$

By doing so, denominator will become $\alpha^2 D^2-\beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.

Case of failure :- If $F(-a^2)=0$, the above method fails.

Then $P.I = x \cdot \frac{1}{F'(D^2)}\sin(ax+b)$ Provided $F'(-a^2) \neq 0$.

If the RHS is of the form $\sin A\cos B$, $\cos A\sin B$ then use transformation formulae

① $\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

② $\cos A\sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)]$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = -\frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

If RHS is $\sin^2 x$, $\cos^2 x$, $\sin^2(x/2)$, $\sin^3 x$ then

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \sin^2(x/2) = \frac{1-\cos x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \cos^2(x/2) = \frac{1+\cos x}{2}$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Case (iii) If $f(x)$ is a polynomial then

$$P.I = \frac{1}{F(D)} f(x)$$

$$= \frac{1}{1 \pm \phi(D)} f(x)$$

$$= \left[1 \pm \phi(D)\right]^{-1} f(x)$$

Expand $[1 \pm \phi(D)]^{-1}$ using binomial series expansion and apply on $f(x)$.

Note:- $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Case (iv)

If $f(x) = e^{ax} v$, where v is any function of x

then Particular integral is given by

$$P.I = \frac{1}{F(D)} e^{ax} v$$

$$= e^{ax} \cdot \frac{1}{F(D+a)} v$$

i.e First replace D by D+a then operate it on v.