3.1 PROCEDURE FOR SOLVING LPP BY GRAPHICAL METHOD

The steps involved in graphical method are as follows:

Step 1 Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph, as each equation will geometrically represent a straight line.

Step 3 Mark the region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the '*feasible region*'.

Step 4 Assign an arbitrary value, say zero, to the objective function.

Step 5 Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6 Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin, passing through at least one corner of the feasible region.

Step 7 Find the co-ordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution. That is, in the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optimal solution is the corner point which gives the minimum value for the objective function.

4.2 SIMPLEX ALGORITHM

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. Various steps for the computation of an optimum solution are as follows:

Step 1 Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by

$$\text{Min } Z = -\text{Max } (-Z)$$

Step 2 Check whether all b_i $(i = 1, 2 \cdots m)$ are positive. If any b_i is negative then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Step 3 Express the problem in standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

Step 4 Obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table as given below:

		C_j =	C_1	C_2	C_3	..	..	0	0	 0
C_B	S_B	x_B	x_1	x_2	x_3	x_4	..... x_n	S_1	S_2	..... S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	..... a_{1n}	1	0	 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	..... a_{2n}	1	0	 0

Step 5 Compute the net evaluations $Z_j - C_j$ by using the relation $Z_j - C_j = C_B\,(a_j - c_j)$
Examine the sign of $Z_j - C_j$

(*i*) If all $Z_j - C_j \geq 0$, then the initial basic feasible solution x_B is an optimum basic feasible solution.

(*ii*) If at least one $Z_j - C_j < 0$, then proceed to next step as the solution is not optimal.

Step 6 (To find the entering variable, i.e., key column)
If there are more than one negative $Z_j - C_j$, choose the most negative of them. Let it be $Z_r - C_r$ for some $j = r$. This gives the entering variable x_r and is indicated by an arrow at the bottom of the r^{th} column. If there are more than one variables having the same most negative $Z_j - C_j$ then, any one of them can be selected arbitrarily as the entering variable.

(*i*) If all $a_{ir} \leq 0$ $(i = 1, 2 \ldots m)$ then there is an unbounded solution to the given problem.

(*ii*) If at least one $a_{ir} > 0$ $(i = 1, 2 \ldots m)$ then the corresponding vector x_r enters the basis.

Step 7 (To find the leaving variable or key row)
Compute the ratio $(x_{Bi}/a_{ir},\ a_{ir} > 0)$
If the minimum of these ratios be x_{Bi}/a_{kr}, then choose the variable x_k to leave the basis called the *key row* and the element at the intersection of key row and key column is called the *key element*.

Step 8 Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using Gauss Elimination Method on the formula

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and column}}{\text{Key element}}\right]$$

Step 9 Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

BIG-M METHOD

The LPPs in which constraints may also have $\geq$ and = signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases basis matrix cannot be obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable called the *artificial variable*. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that the simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPPs there are two methods.

(*i*) The Charne's Big-M Method or Method of Penalties.

(*ii*) The Two-Phase Simplex Method.

5.2 THE CHARNE'S BIG-M METHOD

The following steps are involved in solving an LPP using the Big-M method.

Step 1 Express the problem in the standard form.

Step 2 Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type $\geq$ or =. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty ($-M$ for maximization and M for minimization) in the objective function.

Step 3 Solve the modified LPP by simplex method, until any one of the three cases may arise.

(i) If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

(ii) If at least one artificial variable is there in the basis at zero level and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution (though degenerated).

(iii) If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty M and is called *pseudo optimal solution*.

Note: While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

5.3 THE TWO-PHASE SIMPLEX METHOD

The two-phase simplex method is another method to solve a given LPP involving some artificial variables. The solution is obtained in two phases.

Phase I

In this phase, we construct an auxiliary LPP leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 Assign a cost -1 to each artificial variable and a cost 0 to all other variables and get a new objective function $Z^* = -A_1 - A_2 - A_3 \ldots$ where A_i are the artificial variables.

Step 2 Write down the auxiliary LPP in which the new objective function is to be maximized, subject to the given set of constraints.

Step 3 Solve the auxiliary LPP by simplex method until either of the following three cases arise:

(*i*) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level.
(*ii*) Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis at zero level.
(*iii*) Max $Z^* = 0$ and no artificial variable appears in the optimum basis.

In case (*i*), given LPP does not possess any feasible solution, whereas in cases (*ii*) and (*iii*) we go to phase II.

Phase II

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column that is eliminated from the basis in phase I from the table. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible solution is obtained or till there is an indication of unbounded solution.