LINEAR PROGRAMMING PROBLEM

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as *objective functions*. It is a subject consisting of a set of linear equalities and/or inequalities known as *constraints*. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.

In this chapter, properties of Linear Programming Problems (LPP) have been discussed. The graphical method of solving an LPP is applicable where two variables are involved. The most widely used method for solving LPP problems consisting of any number of variables is called *simplex method*, developed by G. Dantzig in 1947 and made generally available in 1951.

Formulation of LPP

The procedure for mathematical formulation of a LPP consists of the following steps:

Step 1 To write down the decision variables of the problem.

Step 2 To formulate the objective function to be optimized (maximized or minimized) as a linear function of the decision variables.

Step 3 To formulate the other conditions of the problems such as resource limitation, market constraints, interrelations between variables, etc. as linear inequations or equations in terms of the decision variables.

Step 4 To add the non-negativity constraints from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative restrictions together form a Linear Programming Problem (LPP).

GENERAL FORMULATION OF LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $x_1\ x_2 \dots x_n$ to maximize or minimize the objective function.

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \qquad (1)$$

and also satisfy m-constraints

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \dots b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \dots b_2 \\ &\vdots \\ &a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \dots b_i \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \dots b_m \end{aligned}\right\} \qquad (2)$$

where constraints may be in the form of inequality ≤ or ≥ or even in the form of an equation (=) and finally satisfy the non-negative restrictions

$$x_1 \geq 0,\ x_2 \geq 0 \dots x_n \geq 0 \qquad (3)$$

MATRIX FORM OF LPP

The LPP can be expressed in the matrix form as follows:

$$\text{Maximize or}$$
$$\text{Minimize } Z = cx \longrightarrow \text{Objective function}$$
$$\text{Subject to, } Ax\ (\leq = \geq)\ b \quad \text{Constraint equation}$$
$$b > 0,\ x \geq 0 \quad \text{Non-negativity restrictions}$$

where, $x = (x_1\ x_2\ \cdots\ x_n)$

$c = (c_1\ c_2\ \ldots\ c_n)$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m x 1} \qquad A = \begin{pmatrix} a_{11}\ a_{12} & \cdots & a_{1n} \\ a_{21}\ a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1}\ a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m x n}$$

NOTE

1. A set of values $x_1, x_2 \ldots x_n$ that satisfies the constraints (2) of the LPP is called its *solution.*
2. Any solution to a LPP, which satisfies the non-negativity restrictions (3) of the LPP is called its *feasible solution*.
3. Any feasible solution, which optimizes (minimizes or maximizes) the objective function (1) of the LPP is called its *optimum solution*.
4. Given a system of m linear equations with n variables ($m < n$), any solution that is obtained by solving for m variables keeping the remaining n–m variables zero is called a *basic solution.* Such m variables are called *basic variables* and the remaining are called *non-basic variables.*

   The number of basic solutions $= \leq \dfrac{n!}{m!(n-m)!}$

5. A *basic feasible* solution is a basic solution which also satisfies (3), that is all basic variables are non-negative. Basic feasible solutions are of two types:
   (*a*) *Non-degenerate*: A non-degenerate basic feasible solution is the basic feasible solution that has exactly m positive x_i ($i = 1, 2 \ldots m$) i.e., None of the basic variables are zero.
   (*b*) *Degenerate*: A basic feasible solution is said to degenerate if one or more *basic variables* are zero.
6. If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called *unbounded solutions.*

STANDARD FORM

In *standard form*, irrespective of the objective function, namely, maximize or minimize, all the constraints are expressed as equations. Moreover RHS of each constraint and all variables are non-negative.

Following are the characteristics of Standard form of LPP.

(*i*) The objective function is of maximization type.

(*ii*) All constraints are expressed as equations.

(*iii*) Right hand side of each constraint is non-negative.

(*iv*) All variables are non-negative.

CANONICAL FORM

Following are the characteristics of Canonical form of LPP.

(*i*) The objective function is of maximization type.

(*ii*) All constraints are of ($\leq$) type.

(*iii*) All variables x_i are non-negative.

Notes:

(*i*) Minimization of a function Z is equivalent to maximization of the negative expression of this function, i.e., Min Z = – Max (– Z)

(*ii*) An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).

NOTE:

(*i*) **Slack Variables:** If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad (i = 1, 2 \ldots m).$$

Then the non-negative variables S_i, which are introduced to convert the inequalities (≤) to the equalities,

$\sum_{j=1}^{n} a_{ij}x_j + S_i = b_i \ (i = 1, 2 \ldots m)$ are called 'slack variables'.

Slack variables are also defined as the non-negative variables that are added in the LHS of the constraint to convert the inequality '≤' into an equation.

(*ii*) **Surplus Variables:** If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i \quad (i = 1, 2 \ldots m).$$

Then the non-negative variables S_i, which are introduced to convert the inequalities ≥ to the equalities

$\sum_{j=1}^{n} a_{ij}x_j - S_i = b_i \ (i = 1, 2 \ldots m)$ are called surplus variables.

Surplus variables are defined as the non-negative variables that are removed from the LHS of the constraint to convert the inequality (≥) into an equation.