Suppose that the joint density fn of two CRVs X & Y is

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

Find i) Marginal density fn of X & Y

ii) $P(X > 1/2)$ iii) $P(Y > 1)$ iii) $P(Y < X)$

iv) $P(Y < 1/2 \mid X < 1/2)$ v) $P(X+Y) \geq 1$ vi) Conditional density fn.

i) Marginal density fn of X & Y

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 x^2 + \frac{xy}{3}\,dy$$

$$= x^2 y + \frac{xy^2}{6}\Big|_0^2 = 2x^2 + \frac{2}{3}x \quad 0<x<1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{x=0}^{1} x^2 + \frac{xy}{3}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2 y}{6}\Big]_0^1 = \frac{1}{3} + \frac{y}{6} \quad 0<y<2$$

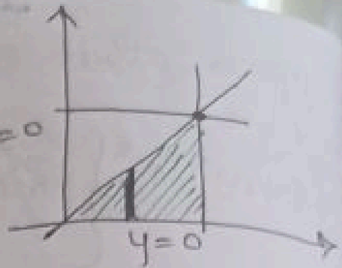
ii) $$P(X > 1/2) = \int_{y=0}^{2} \int_{x=1/2}^{1} f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_{1/2}^{1} \left[x^2 + \frac{xy}{3}\right] dx\,dy = \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_{1/2}^{1} dy$$

$$= \int_0^2 \frac{1}{3} + \frac{y}{6} - \frac{1}{24} - \frac{y}{24}\,dy = \int_0^2 \left(\frac{7}{24} + \frac{y}{8}\right) dy$$

$$= \frac{7}{24}[y]_0^2 + \frac{1}{8}\left[\frac{y^2}{2}\right]_0^2 = 5/6$$

$$P(Y<X) = \int_{x=0}^{1}\int_{y=0}^{x} f(x,y)\,dy\,dx$$



$$= \int_0^1\int_0^x \left(x^2 + \frac{xy}{3}\right) dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{6}\right]_0^x dx = \int_0^1 x^3 + \frac{x^3}{6}\,dx = \int_0^1 \frac{7x^3}{6}\,dx$$

$$= \frac{7x^4}{24}\Big|_0^1 = 7/24$$

iv) $$P(Y<1/2 \mid X<1/2) = \frac{P(X<1/2,\ Y<1/2)}{P(X<1/2)}$$

$$P(X<1/2) = \int_{x=0}^{1/2}\int_{y=0}^{2} f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^2 \left(x^2+\frac{xy}{3}\right) dy\,dx$$

$$= \int_0^{1/2}\left[x^2 y + \frac{xy^2}{6}\right]_0^2 dx = \int_0^{1/2} 2x^2 + \frac{4x}{6}\,dx$$

$$= \left[\frac{2x^3}{3} + \frac{4x^2}{12}\right]_0^{1/2} = \frac{2}{3}\left(\frac{1}{8}\right) + \frac{4}{12}\left(\frac{1}{4}\right) = \frac{2}{12}$$

$$= 1/6$$

$$P(X<1/2,\ Y<1/2) = \int_0^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy$$

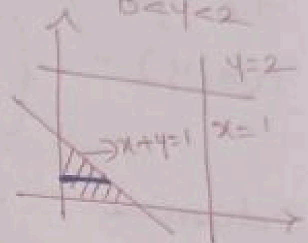
$$= \int_0^{1/2}\int_0^{1/2}\left(x^2+\frac{xy}{3}\right) dx\,dy = \int_0^{1/2}\left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^{1/2} dy$$

$$= \int_0^{1/2} \frac{1}{24} + \frac{1}{24}y\,dy = \frac{1}{24}y + \frac{1}{24}\frac{y^2}{2}\Big]_0^{1/2}$$

$$= \frac{1}{48} + \frac{1}{48}(1/4) = \frac{5}{192}$$

$$P(Y<1/2 \mid X<1/2) = \frac{5/192}{1/6} = 5/32$$

$0<x<1$
$0<y<2$



$x+y=1$

ii) $P[(X+Y) \geq 1] = 1 - P(X+Y<1)$

$$= 1 - \iint_R f(x,y)\,dx\,dy$$

$$= 1 - \int_{y=0}^{1}\int_{x=0}^{1-y}\left(x^2 + \frac{xy}{3}\right)dx\,dy$$

$$= 1 - \int_{y=0}^{1}\left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^{1-y} dy$$

$$= 1 - \int_0^1\left[\frac{(1-y)^3}{3} + \frac{(1-y)^2 y}{6}\right] dy$$

$$= 1 - \int_0^1 \frac{1}{3}\left[1 - y^3 - 3y + 3y^2\right] + \frac{1}{6}\left[y + y^3 - 2y^2\right] dy$$

$$= 1 - \left\{\frac{1}{3}\left[y - \frac{y^4}{4} - \frac{3y^2}{2} + \frac{3y^3}{3}\right]_0^1 + \frac{1}{6}\left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3}\right]_0^1\right.$$

$$= 1 - \frac{1}{3}\left[1 - \frac{1}{4} - \frac{3}{2} + 1 + \frac{1}{12} + \frac{1}{24} - \frac{1}{9}\right]$$

$$= 1 - 7/72 = 65/72$$

Conditional density fn

$$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{x^2 + xy/3}{1/3 + y/6} = \frac{2(3x^2 + xy)}{2+y} \quad 0<x<1,\ 0<y<2$$

$$f(y/x) = \frac{f(x,y)}{f_x(x)} = \frac{x^2 + xy/3}{2x^2 + 2/3\,x} = \frac{3x+y}{6x+2} \quad 0<x<1,\ 0<y<2$$