Continuous Distributions

Notation

For **continuous distributions**, point probabilities are always zero, i.e, $P(X=x) = 0$.

Hence, we will use $P(X<=x)$ and $P(X<x)$ interchangeably, **$P(X<=x) = P(X<x)$**.

Also, $P(X>=x) = P(X>x)$

Also, since sum of probabilities over the entire range of x is 1, hence,

$P(X<x) = 1 - P(X>x)$

or

$P(X>x) = 1 - P(X<x)$

Uniform Distribution

X takes values between two specified numbers α and β say,

Probability density function: $$f_X(x) = \frac{1}{\beta - \alpha} \quad \alpha < x < \beta$$

$X \sim U(\alpha, \beta)$ is often written as shorthand for "the random variable X has a continuous uniform distribution between α and β".

Moments: $\mu = \dfrac{\alpha + \beta}{2}$, **by symmetry, the mid-point of the range of possible values**

$$\sigma^2 = \frac{(\beta - \alpha)^2}{12}$$

Normal Distribution

This distribution is of fundamental importance in both statistical theory and practice.

It has a symmetrical "bell-shaped" density curve.

The distribution has 2 parameters, which can conveniently be expressed directly as the mean, **μ** and the standard deviation, **σ** of the distribution.
The distribution is symmetrical about **μ**.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty$$

Standard Normal Distribution

It is not possible to find an explicit expression for $F_X(x) = P(X <= x)$, so tables have to be used.
These are provided for the distribution of **$Z = (X-\mu)/\sigma$**, which is the **standard normal variable** – it has mean 0 and standard deviation 1.

The x-values **μ, $\mu+\sigma$, $\mu+2\sigma$, $\mu+3\sigma$** correspond to the z-values 0, 1, 2, 3 respectively, and so on. The z-value measures how many standard deviations the corresponding x value is above or below the mean.

$P(X < x) = P(Z < (x-\mu)/\sigma)$

$P(Z<z)$ can be looked up from tables and is also denoted by $\Phi(z)$.

Since Z is symmetric about zero, it follows that: $P(Z < -z) = P(Z>z)=1-P(Z<z)$
$P(Z>-z) = P(Z<z)$

Exponential Distribution

PDF: $f_X(x) = \lambda e^{-\lambda x}, \; x > 0$

$X \sim Exp(\lambda)$ is often written as shorthand for "the random variable X has an exponential distribution with parameter λ".

Moments: $\mu = \dfrac{1}{\lambda}, \; \sigma^2 = \dfrac{1}{\lambda^2}$

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$$

Exponential <> Poisson

The exponential distribution is used as a simple model for the lifetimes of certain types of equipment. Very importantly, it also gives the distribution of the waiting-time, T, from one event to the next in a Poisson process with rate λ.

$$P(T > t) = P(0 \text{ events in time t})$$

$$= P(X = 0) \qquad \text{where } X \sim \text{Poisson}(\lambda t) \text{, so}$$

$$= e^{-\lambda t}$$

$$P(T < t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \lambda e^{-\lambda t}$$

Gamma Function

First note that the gamma function $\Gamma(\alpha)$ is defined for $\alpha > 0$ as follows:

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

Note in particular that $\Gamma(1) = 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$ (*ie* if α is an integer $\Gamma(\alpha) = (\alpha - 1)!$), and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Chi-square Distribution

So the PDF of a χ^2 distribution is:

$$f_X(x) = \frac{(1/2)^{\frac{1}{2}\upsilon}}{\Gamma(\frac{1}{2}\upsilon)} x^{\frac{1}{2}\upsilon - 1} e^{-\frac{1}{2}x} \quad \text{for } x > 0.$$

Moments: $\mu = \upsilon,\ \sigma^2 = 2\upsilon$

t-Distribution

If the variable X has a χ_{ν}^{2} distribution and another independent variable Z has the standard normal distribution of the form $N(0,1)$ then the function:

$$\frac{Z}{\sqrt{X/\nu}}$$

is said to have a t-distribution with parameter "degrees of freedom" ν .

The t-distribution, like the normal, is symmetrical about 0.

The PDF of the *t*-distribution is defined by:

$$f(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\cdot\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \qquad \text{for } -\infty<x<\infty$$

F-Distribution

If two independent random variables, X and Y have χ^2 distributions with parameter n_1 and n_2 respectively, then the function:

$$\frac{X / n_1}{Y / n_2}$$

is said to have an F distribution with parameters "degrees of freedom" n_1 and n_2.

The PDF of the F distribution is defined by:

$$f(x) = \frac{\Gamma\left(\frac{n_1 + n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} . x^{\frac{n_1}{2}-1}\left(1 + \frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}$$

for $x > 0$ and $f(x) = 0$ elsewhere.

Notations

Uniform : U(40, 50)

Normal : N(5, 10^2)

Standard Normal : Z

Exponential : Exp(5)

Chi-square : χ^2_4

t : t_4

F : $F_{(7, 4)}$