Discrete Distributions

Uniform Distribution

Sample space $S = \{1,\ 2,\ 3, \ldots, k\}$.

Probability measure: equal assignment ($1/k$) to all outcomes, *ie* all outcomes are equally likely.

Random variable X defined by $X(i) = i$, $(i = 1,\ 2,\ 3, \ldots, k)$.

Distribution: $P(X = x) = \dfrac{1}{k}$ $\qquad (x = 1,\ 2,\ 3, \ldots, k)$

Moments:

$$\mu = E[X] = \frac{(1 + 2 + \cdots + k)}{k} = \frac{\frac{1}{2}k(k+1)}{k} = \frac{k+1}{2}$$

$$E[X^2] = \frac{(1^2 + 2^2 + \cdots + k^2)}{k} = \frac{\frac{1}{6}k(k+1)(2k+1)}{k} = \frac{(k+1)(2k+1)}{6}$$

$$\Rightarrow \sigma^2 = \frac{k^2 - 1}{12}$$

Uniform Distribution - Example

X = number that comes when we roll a fair dice; $x \in \{1,2,3,4,5,6\}$; $k = 6$

x	1	2	3	4	5	6
P(X=x)	1/6	1/6	1/6	1/6	1/6	1/6

Mean, $\mu = (k+1)/2 = (6+1)/2 = 3.5$

Variance $= (k^2 - 1)/12 = (6^2 - 1)/12 = 35/12$

Bernoulli Distribution

A Bernoulli trial is an experiment which has (or can be regarded as having) only two possible outcomes – *s* ("success") and *f* ("failure").

Sample space $S = \{s, f\}$. The words "success" and "failure" are merely labels – they do not necessarily carry with them the ordinary meanings of the words.

For example in life insurance, a success could mean a death!

Probability measure: $P(\{s\}) = \theta$, $P(\{f\}) = 1 - \theta$

Random variable X defined by $X(s) = 1$, $X(f) = 0$. X is the number of successes that occur (0 or 1).

Distribution: $P(X = x) = \theta^x (1-\theta)^{1-x}$, $x = 0,1;\ 0 < \theta < 1$

Moments: $\mu = \theta$

$$\sigma^2 = \theta - \theta^2 = \theta(1-\theta)$$

Bernoulli Distribution - Example

X = number of heads when we toss a coin; $x \in \{0, 1\}$; $\theta = 1/2$

x	0	1
P(X=x)	1/2	1/2

Mean, $\mu = \theta = 1/2$

Variance = $\theta(1-\theta) = 0.5 * (1 - 0.5) = 1/4$

Sequence of Bernoulli Trials

Consider a sequence of n Bernoulli trials as above such that:
i) the trials are independent of one another, ie the outcome of any trial does not depend on the outcomes of any other trials
ii) the trials are identical, i.e, for each trial, $P(\{s\}) = \theta$

Such a sequence is called a "**sequence of *n* independent, identical, Bernoulli (θ) trials**" or, for short, a "**sequence of n Bernoulli (θ) trials**".

A quick way of saying independent and identically distributed is IID.

Binomial Distribution

Sample space *S*: the joint set of outcomes of all *n* trials

Probability measure: as above for each trial

Random variable *X* is the number of successes that occur in the *n* trials.

Distribution: $P(X = x) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x}\,,\; x = 0,\, 1,\, 2, \ldots, n\,;\; 0 < \theta < 1$

Moments: $\mu = n\theta$

$$\sigma^2 = n\theta(1-\theta)$$

Binomial Distribution - Example

X = number of heads when we toss a coin 6 times;

$x \in \{0, 1, 2, 3, 4, 5, 6\}$; $\theta = \frac{1}{2}$; $n = 6$

x	0	1	2	3	4	5	6
P(X=x)	0.015625	0.09375	0.234375	0.3125	0.234375	0.09375	0.015625

Mean, $\mu = n\theta = 6 * \frac{1}{2} = 3$

Variance $= n\theta(1-\theta) = 6 * 0.5 * (1 - 0.5) = 1.5$

Geometric Distribution

Random variable X**: number of the trial on which the first success occurs**

Distribution: For $X = x$ there must be a run of $(x-1)$ failures followed by a success, so $P(X = x) = \theta(1-\theta)^{x-1}$, $x = 1, 2, 3, \ldots$ $(0 < \theta < 1)$

Moments:

$$\mu = \frac{1}{\theta}$$

$$\sigma^2 = \frac{(1-\theta)}{\theta^2}$$

Geometric Distribution - Example

X = number of coin tosses required for getting a head;

$x \in \{1, 2, 3, 4, 5, 6,\dots\}$; $\theta = ½$;

P(X=x) = $0.5 * (1 - 0.5)^{x-1}$

Mean, $\mu = 1/\theta = 1/(½) = 2$

Variance = $(1-\theta)/\theta^2 = (1 - ½)/(½)^2 = 2$

Poisson Distribution

This distribution models the number of events that occur in a specified interval of time. The parameter here is the rate of occurrence of the event, λ.

$$\textbf{Distribution:} \quad P(X = x) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3, \ldots; \ \lambda > 0$$

$$\mu = \sigma^2 = \lambda$$

Poisson Distribution - Example

If the rate at which goals are scored in a game of football is, on average, three every match, calculate the probability that 5 goals are scored in a match.

Here, $\lambda = 3$.

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$P(X = 5) = 3^5 e^{-3}/5! =$ **0.1**

Brain Teaser

If each of the billion people in India independently has a probability $5 * 10^{-9}$ of being killed by coronavirus in a given year, calculate the probability of exactly 4 such deaths occurring in a given year.