Communication Systems

Xiong Shu Hua

Associate Professor

College of Electronics and information Engineering

Sichuan University

E-mail：xiongsh@scu.edu.cn

Chapter 1

Random Process

Contents

- **random process 随机过程**
- **Stationary process 平稳随机过程**
- **description of a random process: *mean* , *correlation, and covariance functions* 均值，相关函数，协方差函数**
- **ergodic stationary random process 各太历经平稳随机过程**
- **time averages and ensemble averages 时间平均与集平均.**

Contents…conted

- What happens to a stationary random process when it is transmitted through a linear time-invariant filter?
- The frequency-domain description of a random process in terms of power spectral density.
- The characteristics of Gaussian process.
- Sources of noise and their narrowband form.
- Rayleigh and Rician distributions, which represent two special probability distributions that arise in communication systems.

1.1 Introduction

Two classes of mathematical models:

Deterministic 确知的

Stochastic (random) 随机的

确知信号

随机信号

信号波形或参数具有随机性的信号称为随机信号。所谓随机性，是指信号的波形或信号在某时刻的值在观测之前不可能确切预知，重复观测时又不肯定重复已经观测到的结果。不可确切预知和不肯定重复是随机性的两个重要标志。

随机信号的描述方法

Although it is not possible to predict the exact value of the signal in advance, it is possible to describe the signal in terms of statistical parameters such as average power and power spectral density.

对于随机信号，由于不能给出确切的时间函数，它们的特性只能用统计数学的理论和方法加以描述和分析。

把随机信号归结为随机过程，随机信号就是用随机过程描述的信号。

Is the received signal in a communication system deterministic or stochastic?

The received signal in communication system includes:

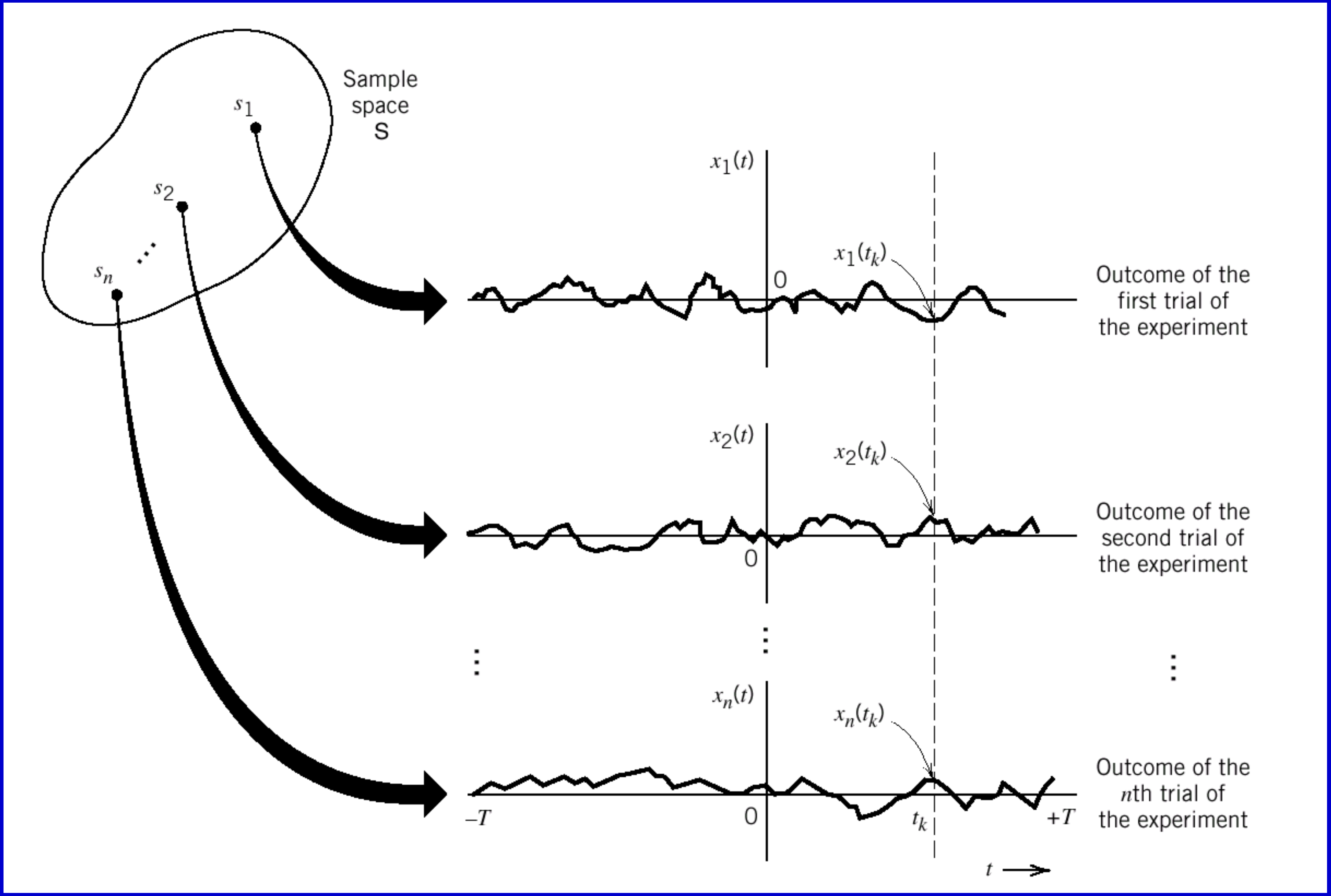
----Information-bearing signal 信息信号

----Random interference 随机干扰

----Channel noise 信道噪声

The received signal is random in nature. So we analyze it in terms of average power and power spectral density.

1.2 Definition of A Random Process



Sample point样本点 Sample space样本空间

sample function 样本函数 realization 实现

随机变量（Random variables）：

在统计数学中，随机变量定义为实验结果的函数。

随机函数、样本函数(sample function, realization)：

如果实验在时间上连续，实验结果将是一个随时间变化的随机函数。这样的随机函数在观测之前不可能预先知道，重复观测也不一定能再次重现，我们把它称为随机过程的样本函数或实现。

随机过程 random process

一次实验观测可以得到随机过程的一个样本函数 $x_n(t)$，所有观测可能获得的样本函数的总集(ensemble) $\{x_n(t)\}$，$n=1,2,\ldots,N$，定义为随机过程。

Random processes have two properties:

1. They are functions of time. 时间的函数
2. They are random. 随机性

- **Random variable随机变量: the outcome of a random experiment is mapped into a number.**
- **random process 随机过程: the outcome of a random experiment is mapped into a waveform that is a function of time.**

随机变量与随机过程的关系：

随机过程在任一时刻的观察值是一个随机变量。随机过程和随机变量两者的差异在于随机过程是一个观测波形的随机函数集，而随机变量反映了某一时刻的实验观测值，随机过程可以看作无数个随机变量的总体。

1.3 Stationary Process

Distribution function 分布函数

$$F_X(x) = P(X \le x)$$

X: random variable

P(X≤x): the probability of event X≤x

The distribution function $F_X(x)$ has properties:

1. $F_X(x)$ is bounded between 0 and 1.
2. $F_X(x)$ is a nondecreasing function of x; that is $F_X(x_1) \le F_X(x_2) \quad \text{if} \quad x_1 < x_2$

Probability density function 概率密度函数：

$$f_X(x) = \frac{d}{dx} F_X(x)$$

It has an important property: $\int_{-\infty}^{\infty} f_X(x)dx = 1$

K-dimensional Joint distribution function
联合分布函数

$$F_{X(t_1),\ldots X(t_k)}(x_1,\ldots x_k) = P[X(t_1) \le x_1,\ldots X(t_k) \le x_k]$$

K-dimensional Joint probability density function
联合概率密度函数

$$f_{X(t_1),\ldots X(t_k)}(x_1,\ldots x_k) = \frac{d}{dx} F_{X(t_1),\ldots X(t_k)}(x_1,\ldots x_k)$$

Stationary and Non-stationary
平稳与非平稳

- Strictly stationary 严平稳，狭义平稳
- Wide-sense stationary 广义平稳，宽平稳，弱平稳

The random process X(t) is said to be stationary in the strict sense or strictly stationary if the following condition holds: (严平稳条件)

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$$

F( ) is joint distribution function. 联合分布函数

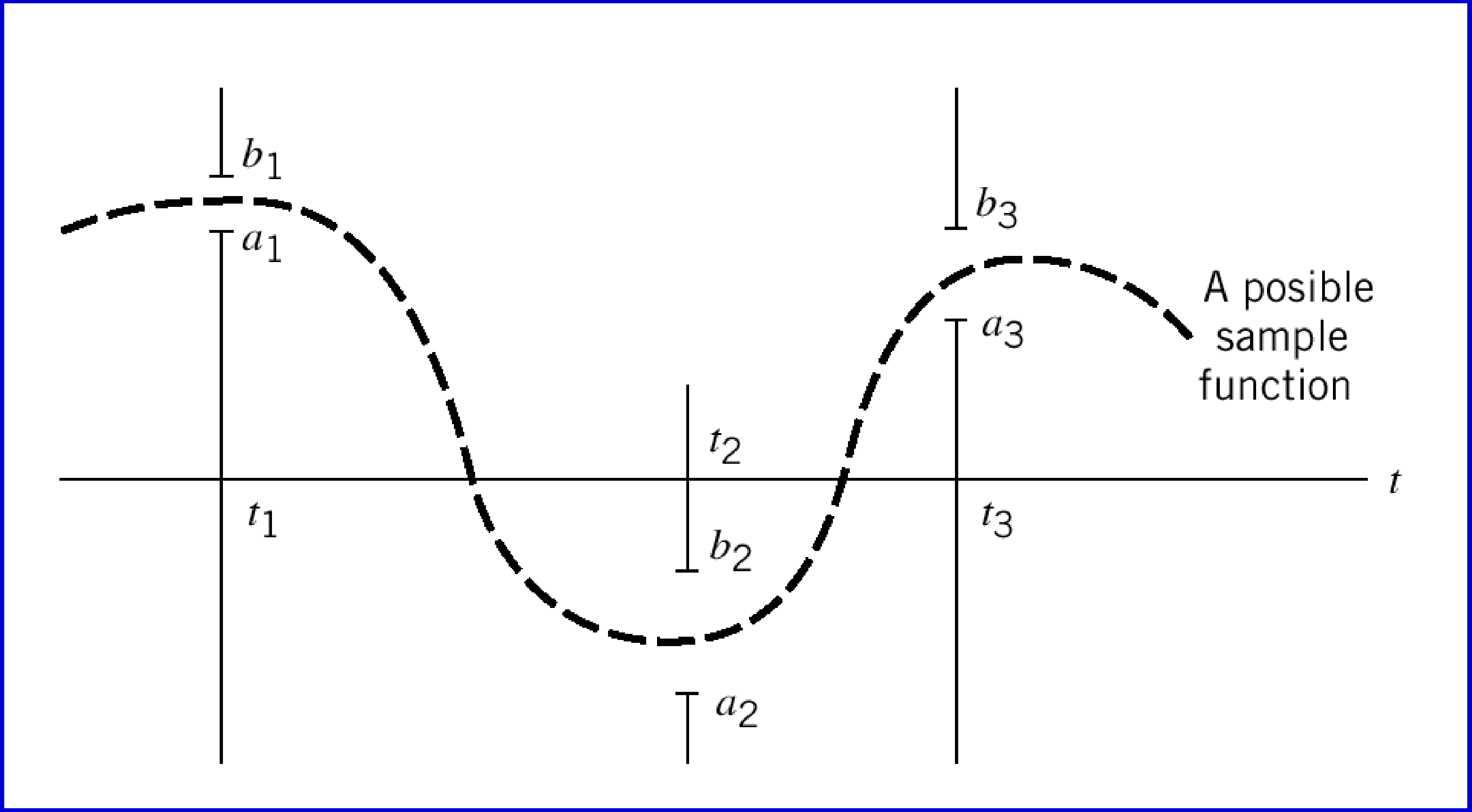
严平稳随机过程

如果一个随机信号经过时间平移后，其统计特性保持不变，或者说对于任意的k和时间平移τ，随机信号的所有多维联合概率密度函数满足：

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$$

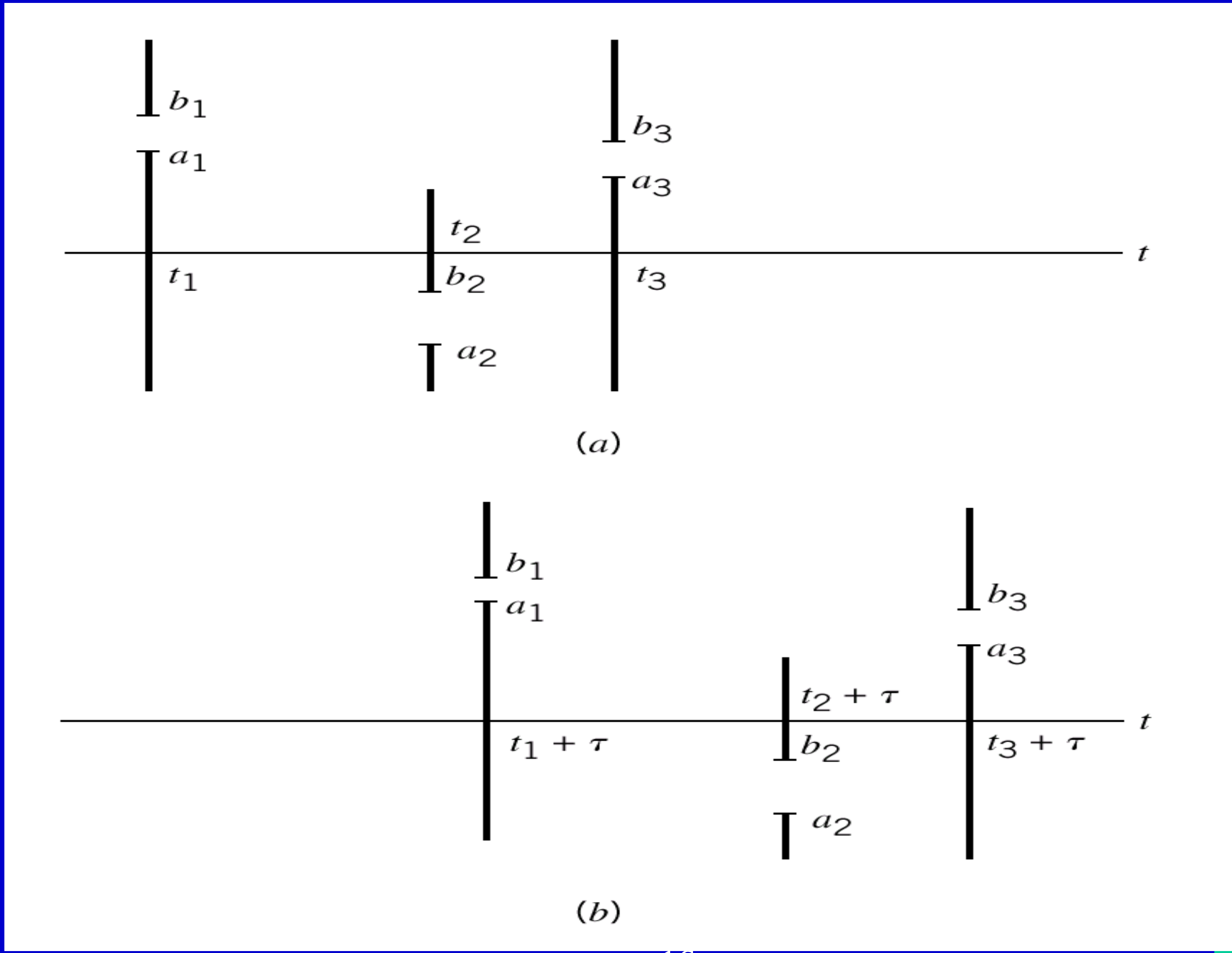
则称这种具有非时变概率密度函数的随机信号为严格平稳或狭义平稳随机信号，其它随机信号或者是广义平稳的，或者是非平稳的。

Figure 1.2 Illustrating the probability of a joint event



$$P(A) = F_{X(t_1),\,X(t_2),\,X(t_3)}(b_1, b_2, b_3) - F_{X(t_1),\,X(t_2),\,X(t_3)}(a_1, a_2, a_3)$$

Figure 1.3
Illustrating the concept of stationarity in Example 1.1.



Two Special situations for stationary process

1. For k=1, we have

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \quad \text{for all } t \text{ and } \tau$$

*The first-order distribution function of a stationary random process is **independent of time**. 一维分布函数独立于时间变量.*

2. For k=2 and τ=-t1, we have

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \textbf{ for all } t_1 \textbf{ and } \mathbf{t_2}$$

*The second-order distribution function of a stationary random process depends on **the time difference** between the observation times and not on the particular times at which the random process is observed. 二维分布函数只与时间间隔有关,与起始时间无关.*

1.4 Mean, Correlation, and Covariance Functions 均值，相关函数和协方差函数

Mean: (expectation 数学期望, or ensemble average 集平均)

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$

f(x) **is the first-order probability density function (一阶概率密度函数)**

The mean of a strictly stationary process is a constant.

$$\mu_X(t) = \mu_X$$

Autocorrelation function 自相关函数:

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)]$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X(t1)X(t2)}(x_1,x_2)dx_1dx_2 \quad (1.7)$$

f(x1,x2) is the second-order probability density function.

In a strictly stationary process:

$$R_X(t_1,t_2) = R_X(t_2 - t_1)$$

$$C_X(t_1,t_2) = E[(X(t_1)-\mu_X)(X(t_2)-\mu_X)]$$
$$= R_X(t_2 - t_1) - \mu^2{}_X \quad (1.9)$$

They depend only on time difference.

Wide-sense stationary 广义平稳

Definition: the random process, which mean is a constant, which autocorrelation function only depends on time difference, is called a wide-sense stationary process.

Strictly stationary $\xrightarrow{\surd}$ **Wide-sense stationary**

Strictly stationary $\xleftarrow{\times}$ **Wide-sense stationary**

Wide-sense stationary

second-order stationary

weakly stationary

We shall simply refer wide-sense stationary as stationary process.

Properties of the Autocorrelation Function

The autocorrelation function of a stationary process:

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

1.mean-square value:

$$R_X(0) = E[X^2(t)]$$

2. An even function of τ

$$R_X(\tau) = R_X(-\tau)$$

3. Maximum magnitude at τ=0

$$|R_X(\tau)| \le R_X(0)$$

1.5 Ergodic Process
遍历过程，各态历经过程

What is ergodic process?

----It is a special stationary process.

In order to know the exact definition, we should study **two concepts**:

Ensemble average 集平均

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$

Time average 时间平均

?

为什么要讨论时间平均？

- 求集平均需要知道随机信号的概率密度函数，而概率密度函数难于用实验简单确定；
- 在实际工作中我们只有样本函数可供使用。

因此，讨论用“典型”样本函数或者随机信号的实现来求时间平均，并研究时间平均与集平均之间的关系是非常重要的。

Time average

The time average of the sample function x(t) is

$$\mu_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

The mean of time average is given by:

$$E[\mu_x(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]dt = \frac{1}{2T}\int_{-T}^{T} \mu_x dt = \mu_x$$

Process X(t) is ergodic in the mean if two conditions are satisfied:

$$\begin{cases} \lim\limits_{T\to\infty} \mu_x(T) = \mu_x \\ \lim\limits_{T\to\infty} \mathrm{var}[\mu_x(T)] = 0 \end{cases}$$

Time-averaged autocorrelation function 时间平均意义上的自相关

统计意义上的自相关函数:

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

Time-averaged autocorrelation function of the sample function x(t):

$$R_X(\tau, T) = \frac{1}{2T}\int_{-T}^{+T} x(t+\tau)x(t)dt$$

Process X(t) is ergodic in the autocorrelation function if two conditions are satisfied:

$$\begin{cases} \lim\limits_{T\to\infty} R_X(\tau, T) = R_X(\tau) \\ \lim\limits_{T\to\infty} \mathrm{var}[R_X(\tau, T)] = 0 \end{cases}$$

各态历经(ergodic)的物理含义

- 对于不同的样本函数，时间平均可能是不同的；而集平均如果存在的话，则只有一个。所以，时间平均一般不等于集平均。
- 如果一个随机信号的各种时间平均以概率1等于相应的集平均，则称该随机信号是遍历的，或各态历经的，或ergodic过程。
- 简言之，如果随机信号是遍历的，则每一个样本函数在统计意义上看起来非常象其它的样本函数。一般来说，遍历性难以证明，所以常常假定它是正确的。

conclusion

Ensemble average 集平均 = **Time average 时间平均**

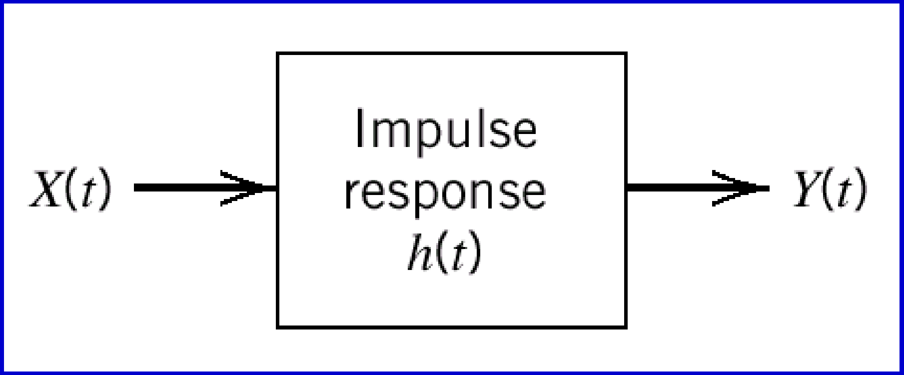
If the random process is ergodic.

In ergodic processes, we can substitute time averages for ensemble average.

$$\text{Ergodic} \underset{X}{\rightleftarrows} \text{stationary}$$

For a random process to be ergodic, it has to be stationary; however, the converse is not necessarily true.

1.6 Transmission of a random Process Through a Linear Time-Invariant Filter



Convolution integral relation between input and output

$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t-\tau_1) d\tau_1$$

Conclusion: if the input to a stable linear time-invariant filter is a stationary process, then the output of the filter is also a stationary process.

Relations between input and output

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau_1) d\tau_1$$

$$= \mu_X H(0)$$

It means that the mean of Y(t) is equal to the mean of X(t) multiplied by the DC response H(0) of the system.

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2$$

$R_Y(\tau)$ is only related to time difference τ.

Since $\boldsymbol{R_Y(0) = E[Y^2(t)]}$

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2$$

The mean-square value of the Y(t) is a constant.

Brief review for last class

The received signal in communication system includes information-bearing signal, random interference and channel noise, so it is random in nature.

strictly stationary random process X(t) (严平稳)， its distribution function satisfies:

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k)$$

wide-sense stationary process (宽平稳）: mean is a constant, autocorrelation function only depends on time difference.

Strictly stationary $\xrightarrow{\surd}$ **Wide-sense stationary / second-order stationary / weakly stationary**

Strictly stationary $\xleftarrow{X}$ **Wide-sense stationary / second-order stationary / weakly stationary**

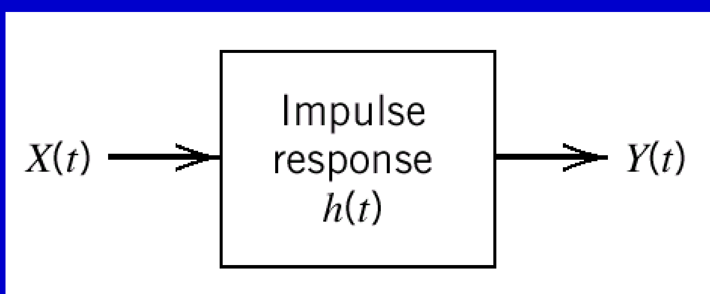
1.7 Power Spectral Density (PSD)

Definition:

Physical significance is the power per unit frequency;

Mathematical description is defined as

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$



H(f): frequency response of the system

Relation between output PSD and input PSD

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

Viener-Khintchine relations
维纳-辛钦关系

$$S_X(f)=\int_{-\infty}^{\infty}R_X(\tau)e^{-j2\pi f\tau}d\tau$$

FT **IFT**

$$R_X(\tau)=\int_{-\infty}^{\infty}S_X(f)e^{j2\pi f\tau}df$$

Method to calculate the power of a random process:

$$R_X(\tau)\Rightarrow S_X(f)\Rightarrow power\quad P=\int S_X(f)df$$

Example 1.2 and 1.5 sinusoidal wave with random phase

Consider a sinusoidal wave with random phase, defined by

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

Θ is uniformly distributed over the interval [-π, π], that is

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$= E[A^2\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c \tau + \Theta)]$$

$$= \frac{A^2}{2}E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] + \frac{A^2}{2}E[\cos(2\pi f_c \tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)d\theta + \frac{A^2}{2}\cos(2\pi f_c \tau)$$

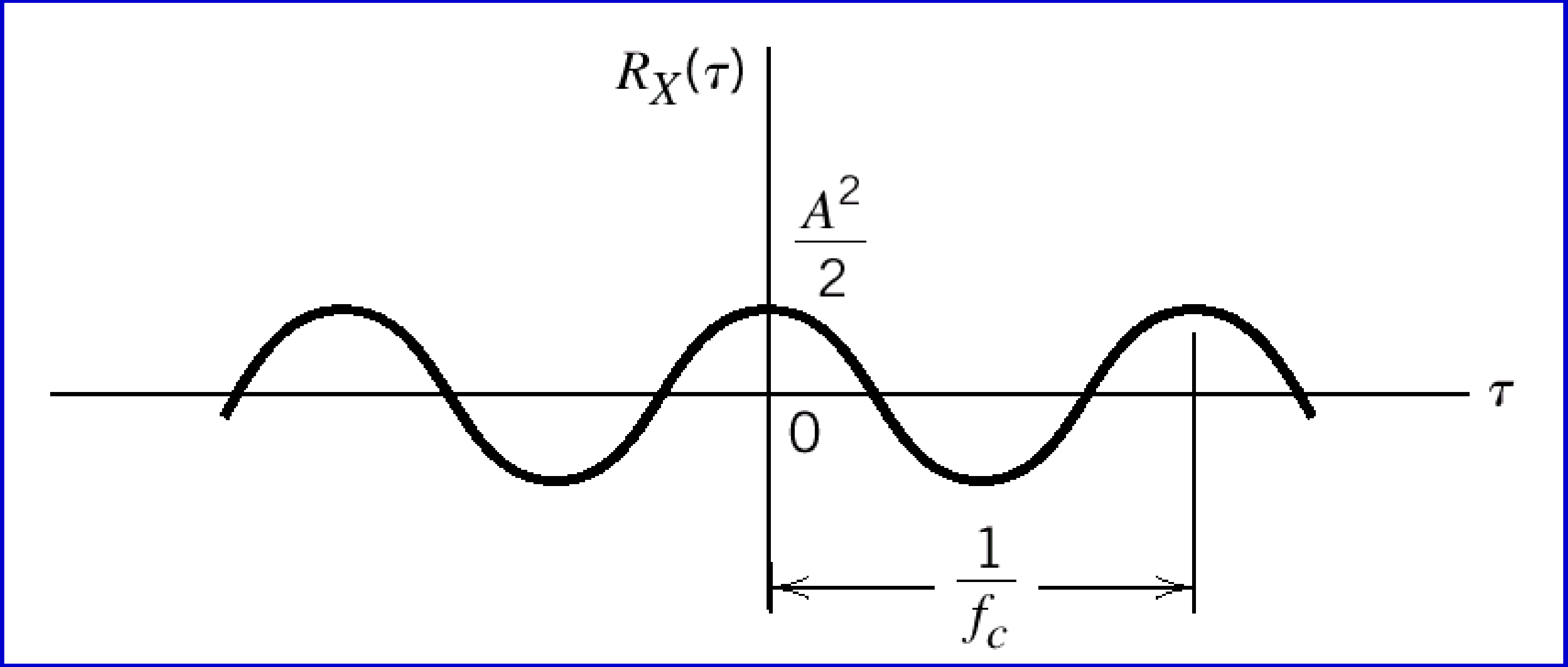
The first term integrates to zero, and so we get

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

$$\int_{-\pi}^{\pi}\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)d\theta = 0$$

Why?

Figure 1.5 Autocorrelation function of a sine wave with random phase.



The power spectral density of X(t) is

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) \xrightarrow{FT} S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)]$$

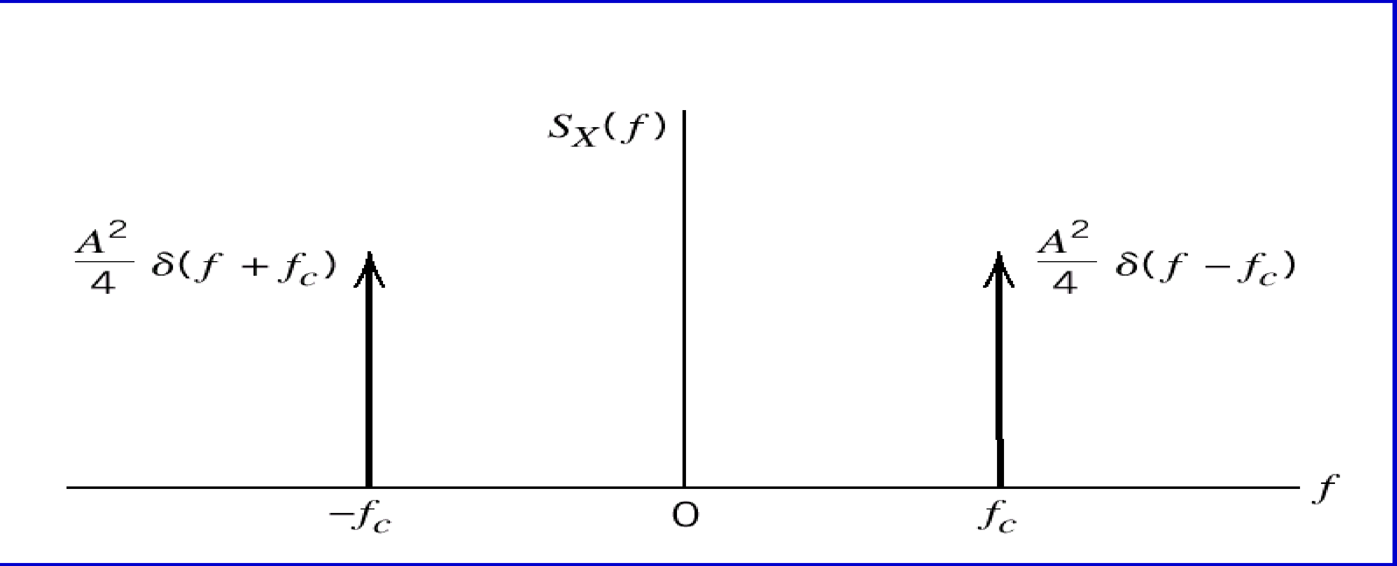


Figure 1.10 Power spectral density of sine wave with random phase; δ(*f*) denotes the delta function at *f* = 0.

Power?

$$P = \int_{-\infty}^{+\infty} S_X(f)df = \frac{A^2}{2}$$

1.8 Gaussian Process

The process is a Gaussian process if its every linear functional 泛函 is a Gaussian random variable. The random variable Y has a Gaussian distribution if its probability density function has the form:

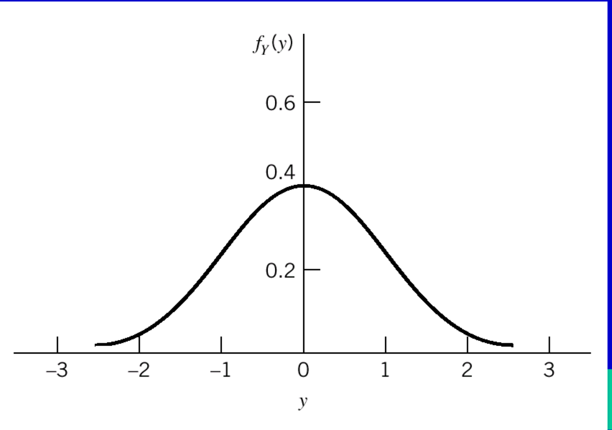
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2} \right]$$

Key parameters: Mean and variance 均值和方差

Normalized Gaussian Distribution: *N(0,1)*

归一化高斯分布，标准正太分布

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right)$$



A Gaussian process has two main virtues.

1. It has many properties that make analytic results possible.
2. The random processes produced by physical phenomena are often such that a Gaussian model is appropriate because of **Central Limit Theorem**(中心极限定理）.

Gaussian process is very important in the study of communication systems.

Properties of a Gaussian Process

① 如果高斯过程X(t)通过一个稳定线性滤波器，那么滤波器的输出Y(t)也是高斯的。

② 设在时刻$t_1, t_2, \ldots t_n$ 观测随机过程X(t)所得的样本或随机变量组为$X(t_1), X(t_2), \ldots X(t_n)$。如果X(t)是高斯型的，则这组随机变量是联合高斯分布的，它们的联合概率密度函数可由它们的均值组和协方差组完全决定。

③ 如果高斯过程是广义平稳的，那么也一定是严格平稳的。

④ 如果在时刻t1,t2,…tn对高斯过程抽样的随机变量X(t1),X(t2),…X(tn)是不相关的，那么这些随机变量也是统计独立的。

Central Limit Theorem
中心极限定理

在特定时刻观测到的随机变量实际上是大量相互独立的随机事件的总和，可以选择高斯过程来作为这些互不相同的自然现象模型，中心极限定理为此提供了数学依据。

在一定条件下，当个数无限增加时，大量独立的随机变量的和的极限分布是正态分布。这类定理就是中心极限定理。

中心极限定理的数学描述

Suppose that:

X_i $(i=1,2, \ldots,n,)$ **are a set of independently and identically distributed (IID) random variables :** 独立同分布的随机变量

1. **The X_i are statistically independent.**
2. **The X_i have the same probability distribution with mean μ and variance σ^2.**

Normalization 归一化处理:

$$Y_i = \frac{1}{\sigma_X}(X_i - \mu_X), \qquad i=1,2,.....,N$$

So that we have $E[Y_i] = 0 \qquad \mathrm{var}[Y_i] = 1$

Define the random variable $V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$

THEN the probability distribution of V_N approaches a normalized Gaussian distribution N(0,1) in the limit as the number of random variables N approaches infinity.

编程验证中心极限定理

Step1: 利用计算机编程产生一组（N个）均匀分布的随机数X；归一化得到Y; 得到一个输出V;

Step2: 重复循环step1 M次，共得到M个V;

Step3: 统计M个v的概率分布，画出概率密度曲线；

Step4: 改变N与M，重复上述过程，观察v的概率密度曲线是否与高斯分布吻合。

1.9 Noise

Noise: unwanted signals that tend to disturb the transmission and processing of signals.

- **External noise(外部噪声): atmospheric noise, galactic noise, man-made noise…**
- **Internal noise(内部噪声): shot noise, thermal noise**

Additive noise	加性噪声	$s(t)+n(t)$
Multiplying noise	乘性噪声	$s(t)\cdot n(t)$
Convolution noise	卷积噪声	$s(t)*n(t)$

Shot Noise 散弹噪声

- Shot noise arises in electronic devices such as diodes (二极管) and transistors (晶体管) because of the discrete nature of current flow(电流） in these devices.
- It is difficult to describe statistical characterization of the shot-noise process.
- However, we know that the number of electrons is very big.

Thermal Noise 热噪声

It is the name given to the electrical noise arising from the random motion of electrons(电子) in a conductor (导体).

Mean-square value of the thermal noise voltage:

$$E[V_{TN}^2] = 4kTR\Delta f \quad volts^2$$

k =1.38x10^{-23} joules 焦耳, Boltzmann's constant 波尔兹曼常数

T: absolute temperature in degrees Kelvin 绝对温度T=273+C^o

It is of interest to note that the number of electrons in a resistor is very large and their random motions inside the resistor are statistically independent of each other, the central limit theorem indicates that thermal noise is Gaussian distributed with zero mean.

White Noise

Its power spectral density is independent of the operating frequency.

$$S_W(f) = \frac{N_0}{2}$$

The dimensions of *No* are in watts per Hertz.

Why is it called white noise (白噪声)?
How about colored noise (有色噪声)?

The adjective white is used in the sense that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation.

$$S_W(f) = \frac{N_0}{2} \quad \Longleftrightarrow \quad R_W(\tau) = \frac{N_0}{2}\delta(\tau)$$

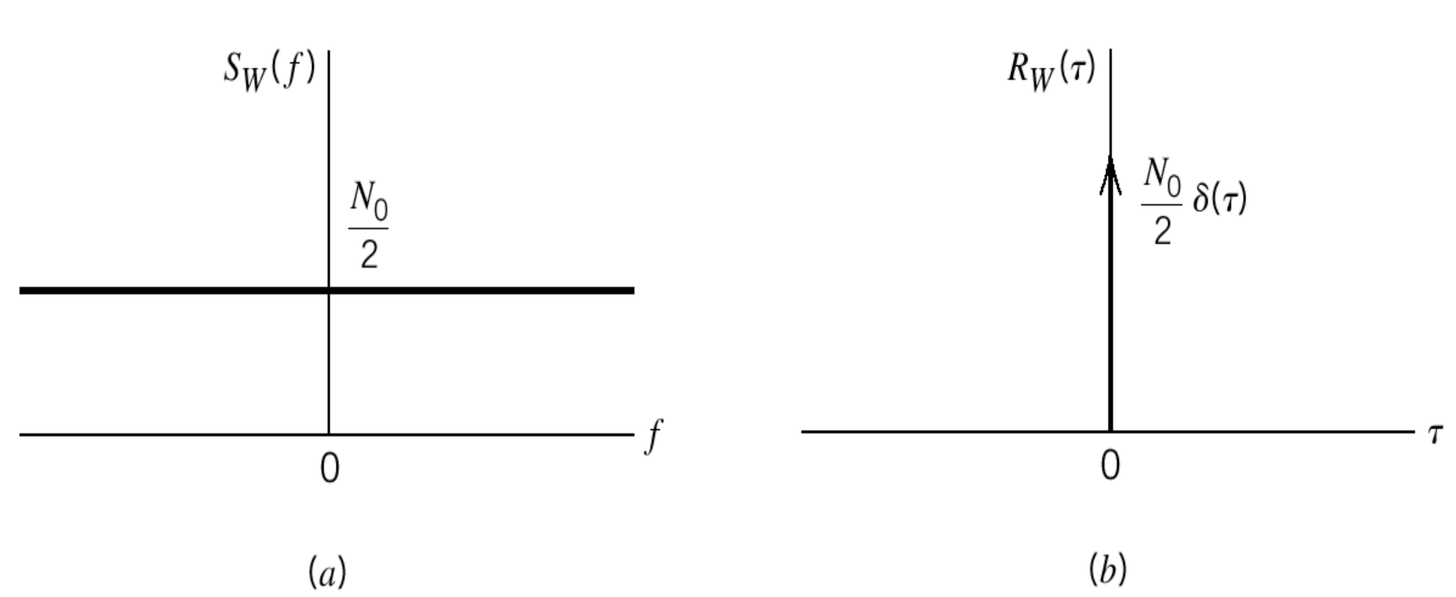


Figure 1.16 Characteristics of white noise. (*a*) Power spectral density. (*b*) Autocorrelation function.

单边带功率谱密度　双边带功率谱密度

Discussion—white noise

Strictly speaking, white noise has infinite average power and it is not physically realizable.

Is it useful in practical system analysis?

As long as the bandwidth of a noise process at the input of a system is appreciable larger than that of the system itself, then we may model the noise process as white noise.

Additive white Gaussian noise AWGN 加性高斯白噪声 —补充内容

Additive noise:

$$s(t)+n(t)$$

Probability density function:

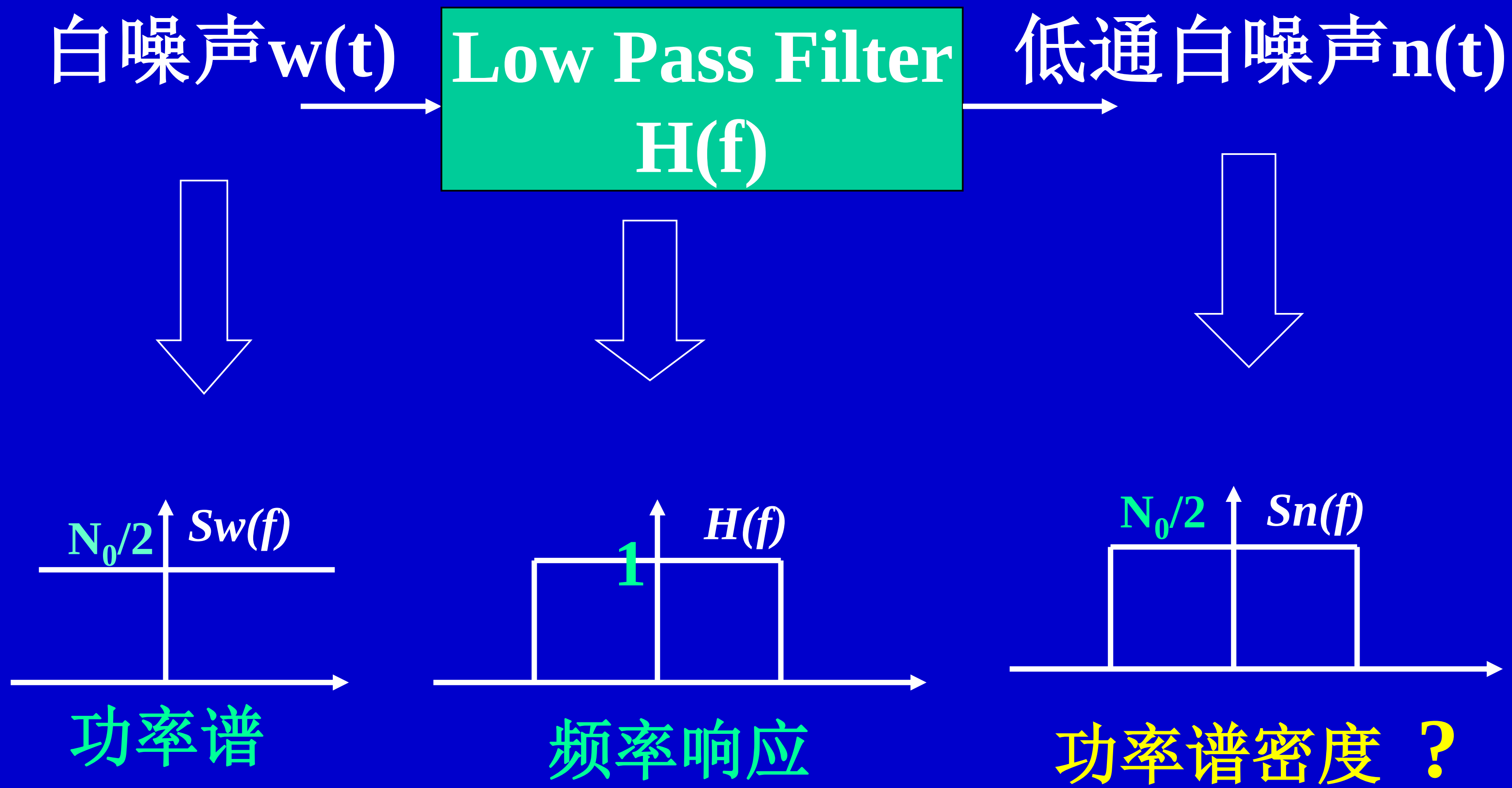
$$f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$$

Power spectral density:

$$S_W(f)=\frac{N_0}{2}$$

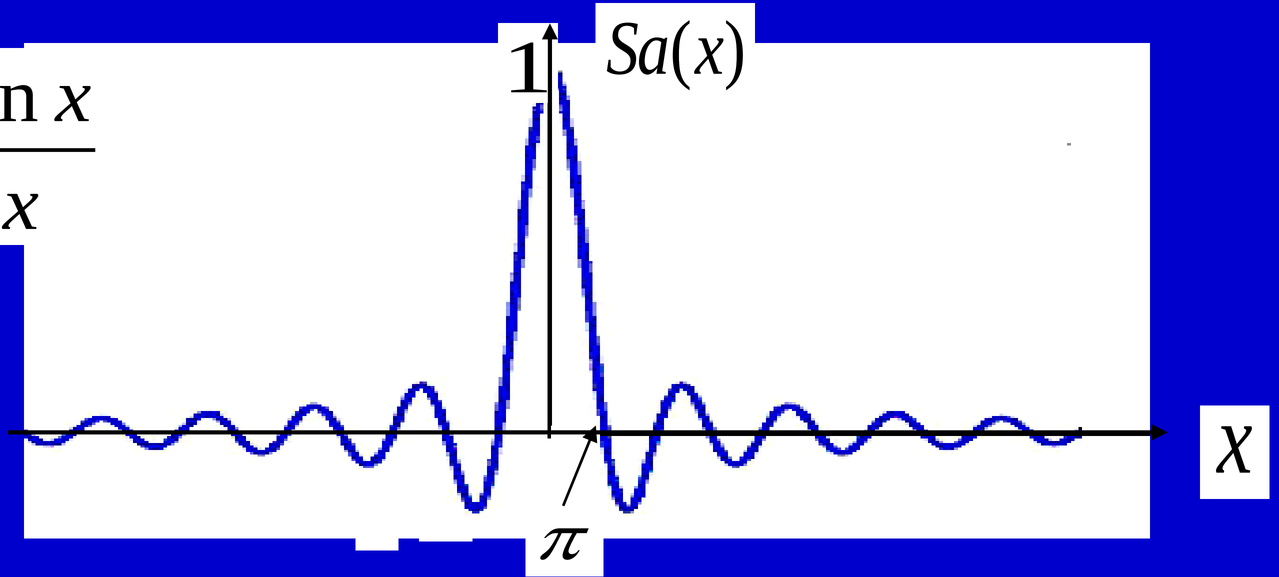
Example 1.10

Ideal Low-Pass Filtered White Noise 低通白噪声

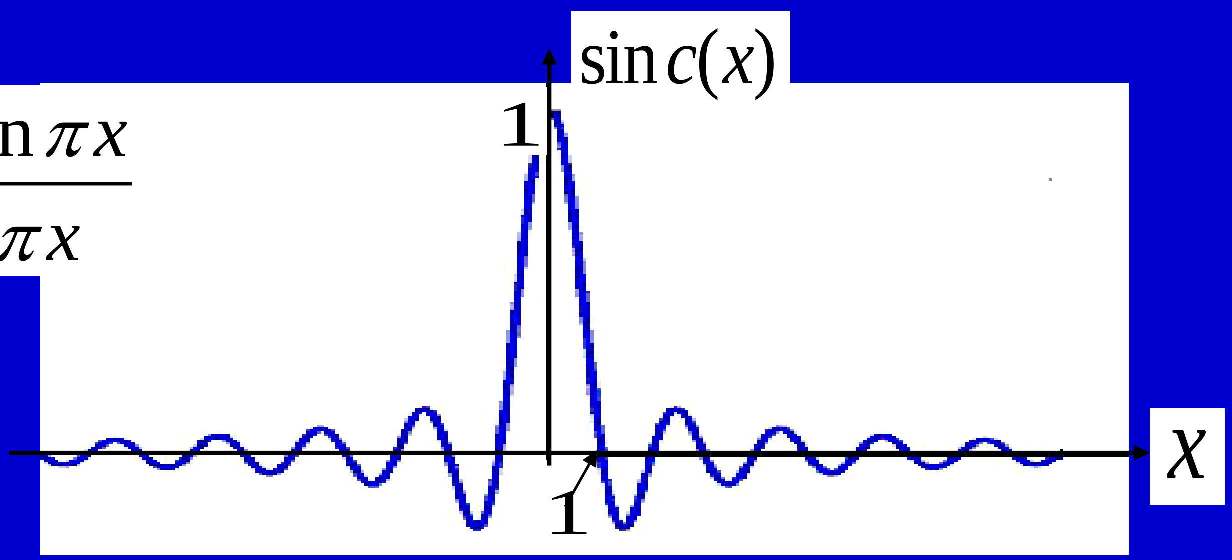


辛格函数

$$\mathrm{Sa}(x)=\frac{\sin x}{x}$$



$$\mathrm{sinc}(x)=\frac{\sin \pi x}{\pi x}$$



$$S_N(f) = \begin{cases} \dfrac{N_0}{2}, & -B < f < B \\ 0, & |f| > B \end{cases}$$

$$R_N(\tau) = \int_{-B}^{+B} \frac{N_0}{2} \exp(j2\pi f\tau)df = N_0 B \operatorname{sinc}(2B\tau)$$

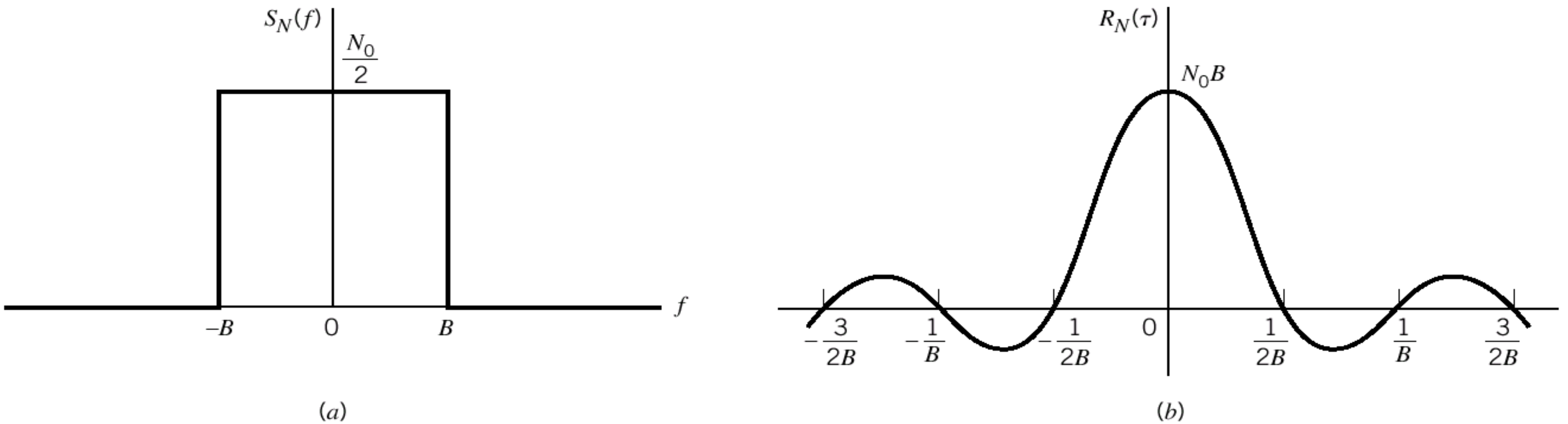
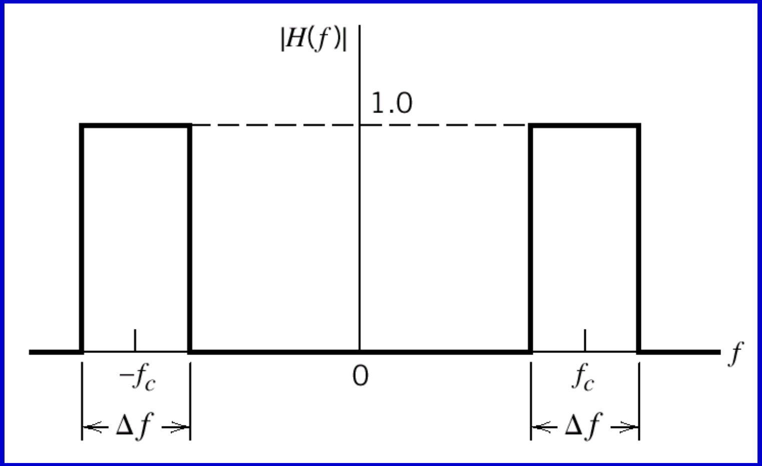


Figure 1.17 Characteristics of low-pass filtered white noise. (*a*) Power spectral density. (*b*) Autocorrelation function.

Ideal narrowband filter



$$|H(f)| = \begin{cases} 1, & \left| f \pm fc \right| < \frac{1}{2}\Delta f \\ 0, & \left| f \pm fc \right| > \frac{1}{2}\Delta f \end{cases}$$

Figure 1.9 Magnitude response of ideal narrowband filter.

Narrowband:

$$\Delta(f) << f_c$$

1.10 Narrowband Noise

The noise process appearing at the output of a narrow band filter is called narrowband noise.

白噪声w(t) → **Narrow band-pass Filter** → **窄带带通噪声n(t)**

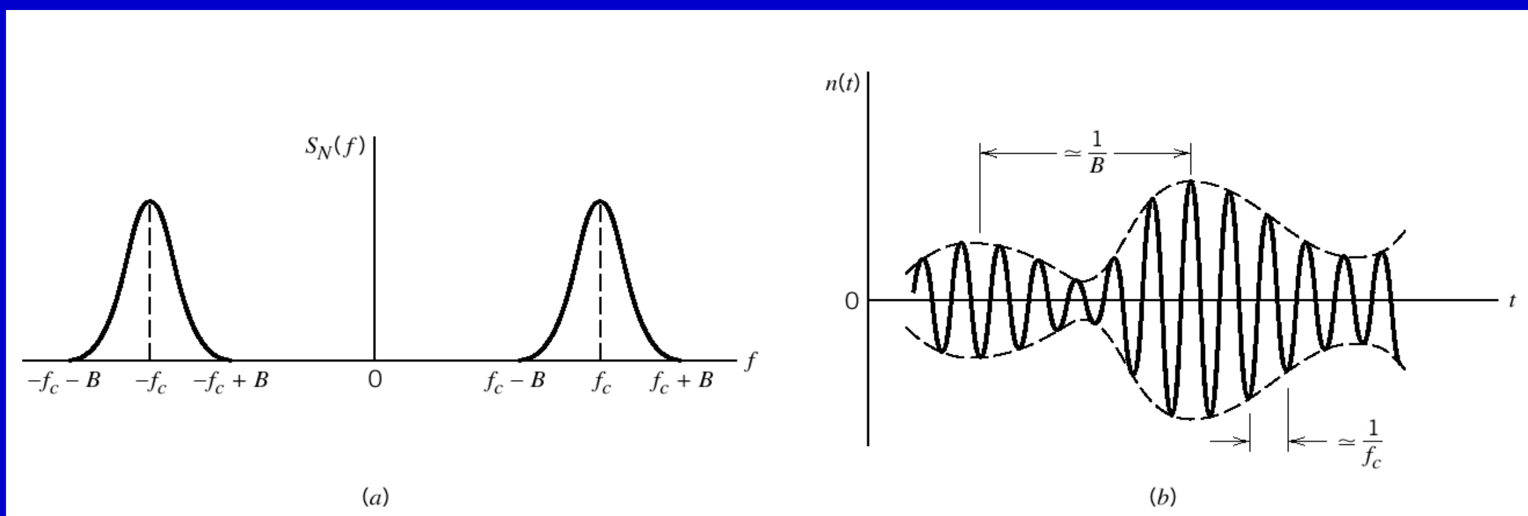


Figure 1.18 (*a*) Power spectral density of narrowband noise. (*b*) Sample function of narrowband noise.

representations of narrowband noise 窄带带通噪声的表示

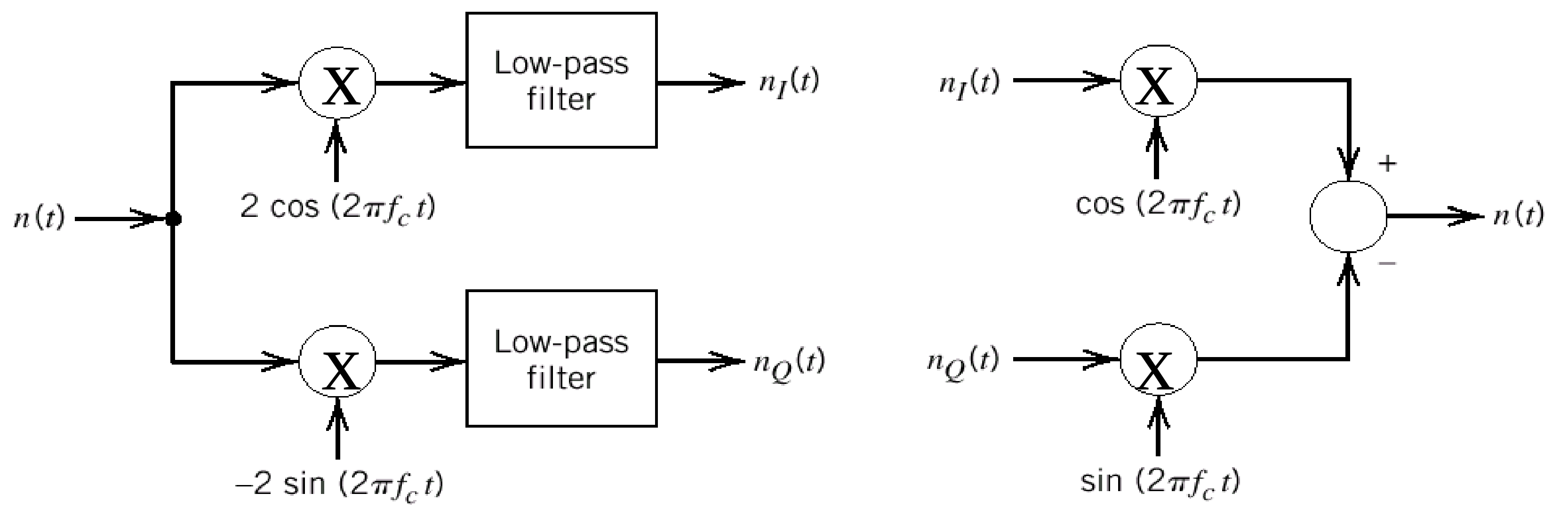
$$n(t) = r(t)\cos[2\pi ft_c + \psi(t)]$$
$$= r(t)\cos[\psi(t)]\cos(2\pi f_c t) - r(t)\sin[\psi(t)]\sin(2\pi f_c t)$$
$$= n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

1. **Represented in terms of Envelope and phase** 包络与相位表示
2. **Represented in terms of In-phase and quadrature components** 同相分量与正交分量表示
3. 复数表示

1.11 Representation of Narrowband Noise in terms of In-Phase and Quadrature Components

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$n_I(t)$: in-phase component $n_Q(t)$: quadrature component both of them are low-pass signals.



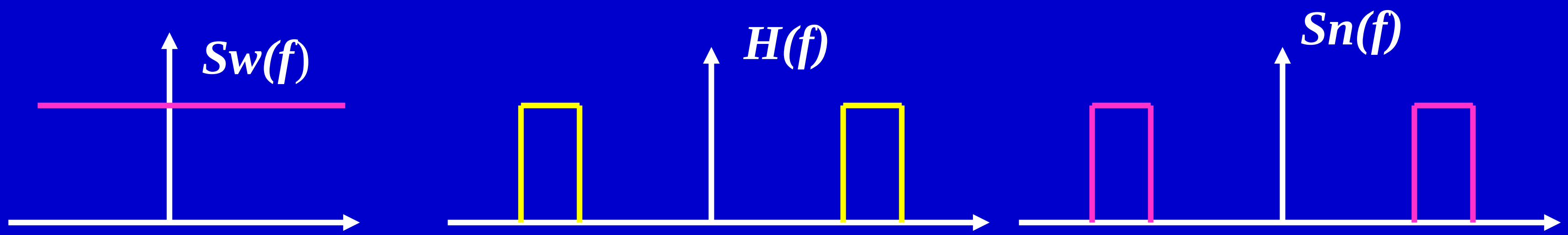
Narrowband noise analyzer and synthesizer

Properties of Narrowband Noise

1. $n_I(t)$ and $n_Q(t)$ have zero mean.
2. If $n(t)$ is Gaussian, then $n_I(t)$ and $n_Q(t)$ are Gaussian.
3. If $n(t)$ is stationary, then $n_I(t)$ and $n_Q(t)$ are jointly stationary.
4. $$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f-f_c)+S_N(f+f_c) & -B \le f \le B \\ 0, & otherwise \end{cases}$$
5. $n_I(t)$ and $n_Q(t)$ have the same variance as $n(t)$.

Example 1.12 Ideal Band-Pass Filtered White Noise 理想带通白噪声

w(t) → **Band Pass Filter H(f)** → n(t)



$$R_N(\tau)=\int_{-f_c-B}^{-f_c+B}\frac{N_0}{2}\exp(j2\pi f\tau)df+\int_{f_c-B}^{f_c+B}\frac{N_0}{2}\exp(j2\pi f\tau)df$$

$$=N_0B\sin c(2B\tau)[\exp(-j2\pi f_c\tau)+\exp(j2\pi f_c\tau)]$$

$$=2N_0B\sin c(2B\tau)\cos(2\pi f_c\tau)$$

$$R_{N_I}(\tau)=R_{N_Q}(\tau)=2N_0B\sin c(2B\tau)$$

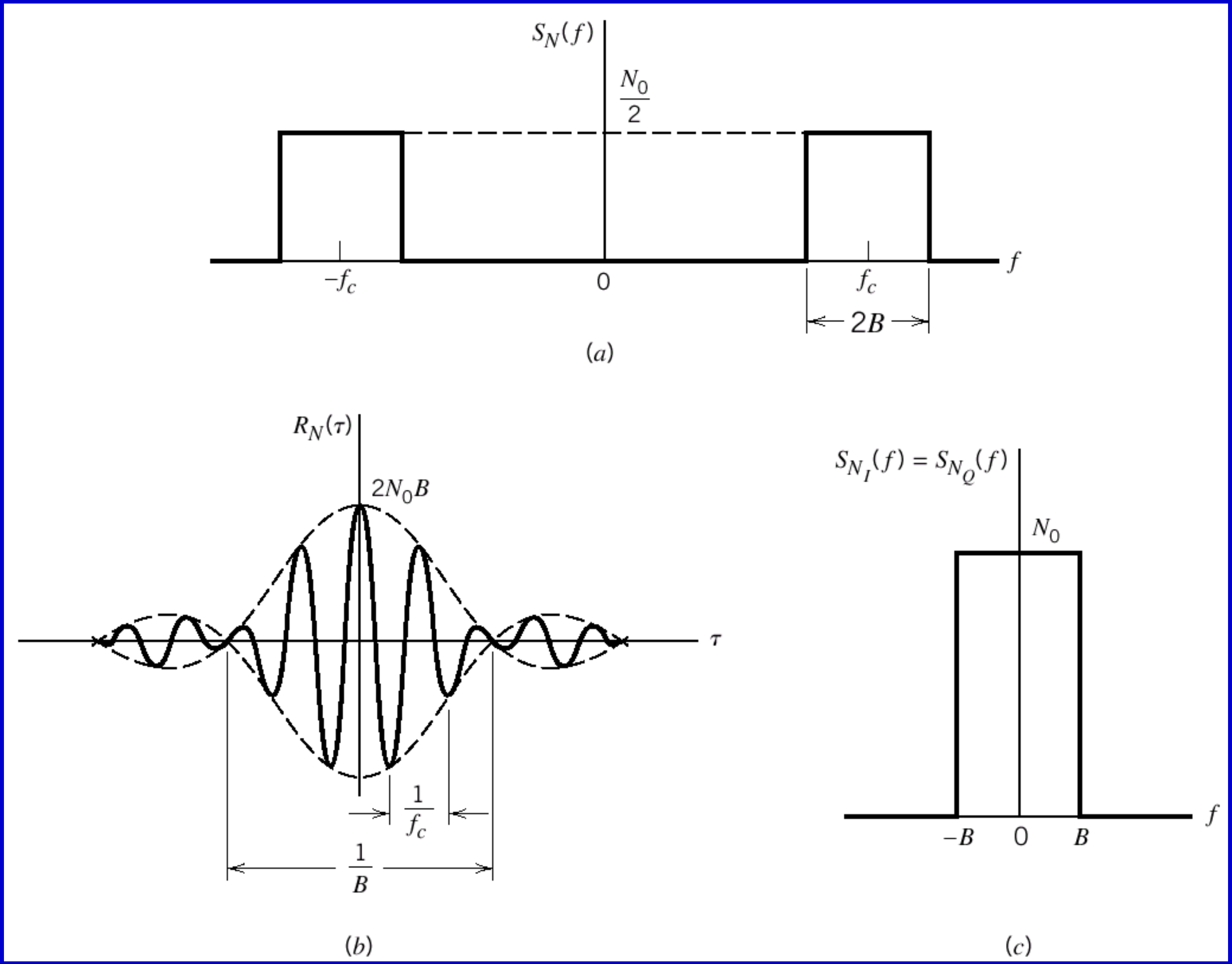


Figure 1.20 Characteristics of ideal band-pass filtered white noise. (*a*) Power spectral density. (*b*) Autocorrelation function. (*c*) Power spectral density of in-phase and quadrature components.

1.12 Representation of Narrowband Noise in terms of Envelope and Phase

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$\Downarrow$$

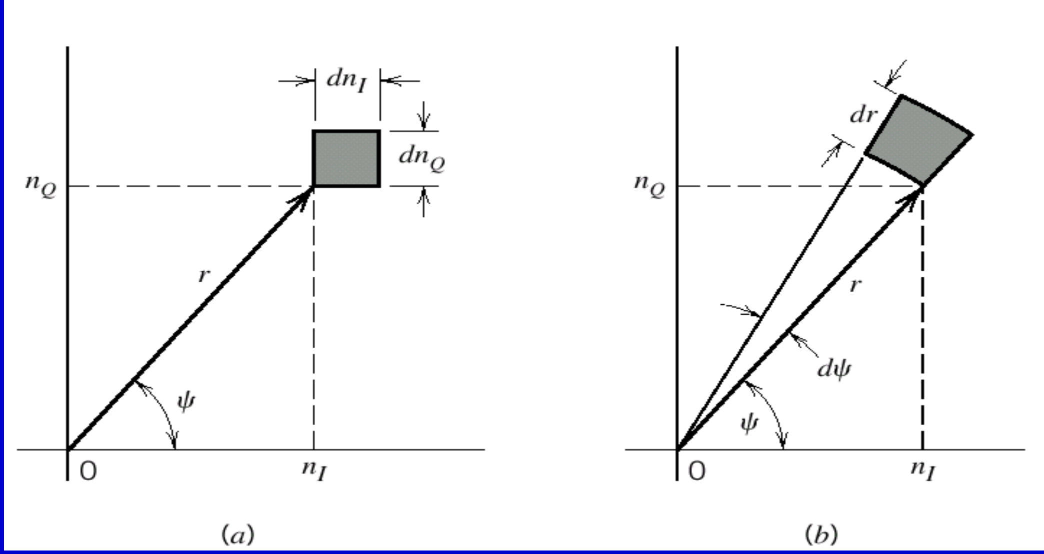
$$n(t) = r(t)\cos[2\pi fct + \psi(t)]$$

Envelope:

$$r(t) = [n_I^2(t) + n_Q^2(t)]^{\frac{1}{2}}$$

Phase:

$$\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$$



r(t) and ψ(t) belong to low-pass random processes.

Joint probability density function of r and ψ:

$$f_{R,\psi}(r,\psi)=\frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)$$

Phase ψ is uniformly distributed 均匀分布

$$f_{\psi}(\psi)=\begin{cases}\dfrac{1}{2\pi}, & 0\le\psi\le 2\pi\\ 0, & elsewhere\end{cases}$$

Envelope r is Rayleigh distributed 瑞利分布

$$f_R(r)=\begin{cases}\dfrac{r}{\sigma^2}\exp\left(-\dfrac{r^2}{2\sigma^2}\right), & r\ge 0\\ 0, & elsewhere\end{cases}$$

For convenience of graphical presentation, Let

$$v = \frac{r}{\sigma} \qquad f_V(v) = \sigma f_R(r)$$

Then we may get the normalized form of Rayleigh distribution 归一化瑞利分布

$$f_V(v) = \begin{cases} v\exp(-\dfrac{v^2}{2}), & v \geq 0 \\ 0, & elsewhere \end{cases}$$

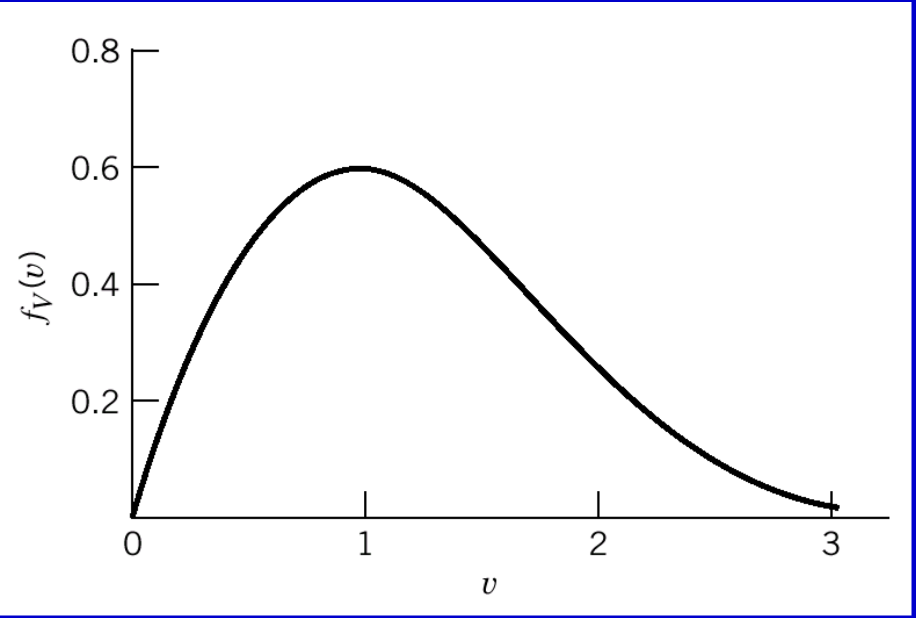


Figure 1.22 Normalized Rayleigh distribution.

1.13 Sine Wave Plus Narrowband Noise 正弦波加窄带噪声

A sample function of the sinusoidal wave plus noise is expressed by

$$x(t) = A\cos(2\pi f_c t) + n(t)$$

Narrowband noise:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

So:

$$x(t) = n_I'(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$n_I'(t) = A + n_I(t)$$

If n(t) is Gaussian with zero mean and variance σ^2, then we have the following properties

1. Both $n'_I(t)$ and $n_Q(t)$ are Gaussian and statistically independent.
2. The mean of $n'_I(t)$ is A and that of $n_Q(t)$ is zero.
3. The variance of both $n'_I(t)$ and $n_Q(t)$ is σ^2

Let r(t) denote the envelope of x(t) and Ψ(t) denote its phase:

$$
\begin{cases}
r(t) = \{[n_I^{'}(t)]^2 + n_Q^{'}(t)\}^{1/2} \\
\psi(t) = \tan^{-1}[\dfrac{n_Q(t)}{n_I^{'}(t)}]
\end{cases}
$$

Rician Distribution 莱斯分布

The probability density function of R is Rician distributed:

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2} \right) I_0\left(\frac{Ar}{\sigma^2} \right)$$

Where, $I_0(x)$ is the modified Bessel function of the first kind of zero order第一类零阶贝塞尔函数; that is

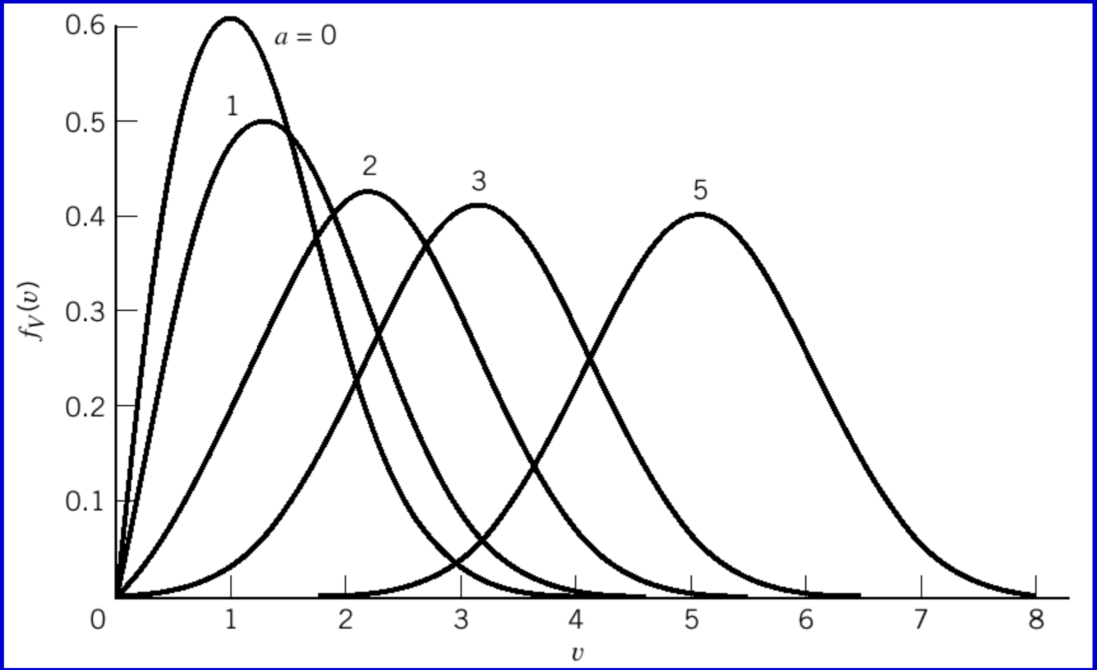
$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos\psi) d\psi$$

Normalized Rician distribution

归一化莱斯分布:

$$v = \frac{r}{\sigma} \qquad a = \frac{A}{\sigma} \qquad f_V(v) = \sigma f_R(r)$$

$$f_V(v) = v \exp\left(-\frac{v^2 + a^2}{2}\right) I_0(av)$$



1. 当a=0时，

   莱斯分布退化为瑞利分布；

2. 当a足够大时，

   莱斯分布近似于高斯分布。

Summary for this chapter

- Ergodic stationary random process
- Power spectral density
- Additive white Gaussian noise AWGN
- Narrowband noise

Homework

1. Problem 1.26 .
2. Problem 1.27.
3. Translate Example 1.10 into Chinese.
4. Translate Example 1.12 into Chinese.