Resource Management Techniques
CS635
Unit-2

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TEXT BOOKS

Frederick S. Hillier and Gerald J. Lieberman , “Introduction to Operations Research” Tata McGraw Hill, 9th Edition, 2010.
Hamdy A Taha ,”Operations Research: An Introduction”, 8th Edition, Prentice Hall India, 2012

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Wayne L. Winston , “Operations Research Applications and Algorithms”, Thomson Course Technology, 4th Edition, 2003.
Vohra, “Quantitative Techniques in Management”, Tata McGraw Hill, 2006.
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Winston ,“Operation Research”, Thomson Learning, 2003.
JK Sharma, “Operations Research theory and applications”, Macmillan India, 2009, Thomson Learning, 4th edition, 2009.

UNIT-2

- Simplex method (Big M)
- Duality Theory

Simplex Problems

- Minimization Problems : BIG M Method
- Surplus Variables
- Artificial Variables

BIG M Method

$$\text{Minimize } Z = 2X_1 + 3X_2$$

$$X_1 + X_2 \geq 6$$
$$7X_1 + X_2 \geq 14$$
$$X_1 \text{ and } X_2 \geq 0$$

Standard Form Method

$$\text{Minimize } Z = 2X_1 + 3X_2$$

$$X_1 + X_2 - S_1 = 6$$
$$7X_1 + X_2 - S_2 = 14$$
$$X_1, X_2, S_1 \text{ and } S_2 \geq 0$$

$$\text{Minimize } Z = 2X_1 + 3X_2 + 0S_1 + 0S_2 + MR_1 + MR_2$$

$$X_1 + X_2 - S_1 + R_1 = 6$$
$$7X_1 + X_2 - S_2 + R_2 = 14$$
$$X_1, X_2, S_1, S_2, R_1 \text{ and } R_2 \geq 0$$

$$\text{Minimize } Z = 2X_1 + 3X_2$$

$$X_1 + X_2 - S_1 = 6$$
$$7X_1 + X_2 - S_2 = 14$$
$$X_1, X_2, S_1 \text{ and } S_2 \geq 0$$

$$\text{Minimize } Z = 2X_1 + 3X_2 + 0S_1 + 0S_2 + MR_1 + MR$$

$$X_1 + X_2 - S_1 + R_1 = 6$$
$$7X_1 + X_2 - S_2 + R_2 = 14$$
$$X_1, X_2, S_1, S_2, R_1 \text{ and } R_2 \geq 0$$

Table 1

	C_j	2	3	0	0	M	M		
CB_i	Basic variable	X_1	X_2	S_1	S_2	R_1	R_2	Solution	Ratio
M	R_1								
M	R_2								
	Z_j								
	$C_j - Z_j$								

Table-2

	C_j	2	3	0	0	M	M		
CB_i	Basic variable	X_1	X_2	S_1	S_2	R_1	R_2	Solution	Ratio
M	R_1								
2	X_1								
	Z_j								
	$C_j - Z_j$								

Table -3

CB_i	C_j	2	3	0	0	M	M	Solution	Ratio
	Basic variable	X_1	X_2	S_1	S_2	R_1	R_2		
3	X_2								
2	X_1								
	Z_j								
	$C_j - Z_j$								

Table-4

CB_i	C_j	2	3	0	0	M	M	
	Basic variable	X_1	X_2	S_1	S_2	R_1	R_2	Solution
0	S_2							
2	X_1							
	Z_j							
	$C_j - Z_j$							

$X_1 = 6$, $X_2 = 0$, $S_1 = 0$, $S_2 = 28$, and $Z(\text{optimum}) = 12$

Problem-2 : Solve using Big M Method

$$\text{Minimize } Z = 10X_1 + 15X_2 + 20X_3$$

$$2X_1 + 4X_2 + 6X_3 \geq 24$$

$$3X_1 + 9X_2 + 6X_3 \geq 30$$

$$X_1, X_2, X_3 \geq 0$$

$$\text{Minimize } Z = 10X_1 + 15X_2 + 20X_3 + MR_1 + MR_2$$

$$2X_1 + 4X_2 + 6X_3 - S_1 + R_1 = 24$$

$$3X_1 + 9X_2 + 6X_3 - S_2 + R_2 = 30$$

$$X_1, X_2, X_3, S_1, S_2, R_1 \text{ and } R_2 \geq 0$$

Table 1: Obtained from initial basic feasible solution

	C_j	10	15	20	0	0	M	M		
CB_i	Basic variable	X_1	X_2	X_3	S_1	S_2	R_1	R_2	Solution	Ratio
M	R_1									
M	R_2									
	Z_j									
	$C_j - Z_j$									

Table-2

	C_j	10	15	20	0	0	M	M		
CB_i	Basic variable	X_1	X_2	X_3	S_1	S_2	R_1	R_2	Solution	Ratio
M	R_1									
15	X_2									
	Z_j								+50	
	$C_j - Z_j$									

Table-3

CB_i	C_j	10	15	20	0	0	M	M	
	Basic variable	X_1	X_2	X_3	S_1	S_2	R_1	R_2	Solution
20	X_3								
15	X_2								
	Z_j								82
	$C_j - Z_j$								

Problem 3: Solve the problem using Simplex.

$$\text{Maximize } Z = 20X_1 + 10X_2 + 15X_3$$

$$8X_1 + 6X_2 + 2X_3 \leq 60$$
$$5X_1 + X_2 + 6X_3 \geq 40$$
$$2X_1 + 6X_2 + 3X_3 \leq 30$$
$$X_1, X_2 \text{ and } X_3 \geq 0$$

Maximize $Z =$

Table: 1

Maximize $Z = 20X_1 + 10X_2 + 15X_3 + 0S_1 + 0S_2 + 0S_3 - MR_1$

$$8X_1 + 6X_2 + 2X_3 + S_1 = 60$$
$$5X_1 + X_2 + 6X_3 - S_2 + R_1 = 40$$
$$2X_1 + 6X_2 + 3X_3 + S_3 = 30$$
$$X_1, X_2, X_3, S_1, S_2, S_3 \text{ and } R_1 \geq 0$$

CB_i	C_j / Basic variable	20 / X_1	10 / X_2	15 / X_3	0 / S_1	0 / S_2	0 / S_3	$-M$ / R_1	Solution	Ratio
0	S_1	8	6	2	1	0	0	0	60	30
$-M$	R_1	5	1	$\boxed{6}$	0	-1	0	1	40	20/3
0	S_3	2	6	3	0	0	1	0	30	10
	Z_j	$-5M$	$-M$	$-6M$	0	M	0	$-M$	$-40M$	
	$C_j - Z_j$	$20 + 5M$	$10 + M$	$15 + 6M$	0	$-M$	0	0		

Table-2

CB_i	C_j	20	10	15	0	0	0	$-M$	Solution	Ratio
	Basic variable	X_1	X_2	X_3	S_1	S_2	S_3	R_1		
0	S_1	$\boxed{19/3}$	17/3	0	1	1/3	0	−1/3	140/3	140/19
15	X_3	5/6	1/6	1	0	−1/6	0	1/6	20/3	8
0	S_3	−1/2	11/2	0	0	1/2	1	−1/2	10	–
	Z_j	25/2	5/2	15	0	−5/2	0	5/2	100	
	$C_j - Z_j$	15/2	15/2	0	0	5/2	0	$-M-5/2$		

Table-3

C_j		20	10	15	0	0	0	$-M$		
	Basic variable	X_1	X_2	X_3	S_1	S_2	S_3	R_1	Solution	Ratio
20	X_1	1	17/19	0	3/19	1/19	0	–1/19	140/19	140
15	X_3	0	–11/19	1	–5/38	–4/19	0	4/19	10/19	–
0	S_3	0	113/19	0	3/38	$\boxed{10/19}$	1	–10/19	260/19	26
	Z_j	20	175/19	15	45/38	–40/19	0	40/19	2950/19	
	$C_j - Z_j$	0	15/19	0	–45/38	40/19	0	$-M$–40/19		

	C_j	20	10	15	0	0	0	$-M$	Solution	Ratio
	Basic variable	X_1	X_2	X_3	S_1	S_2	S_3	R_1		
20	X_1	1	3/10	0	3/20	0	−1/10	0	6	
15	X_3	0	9/5	1	−1/10	0	2/5	0	6	
0	S_2	0	113/10	0	3/20	1	19/10	−1	26	
	Z_j	20	33	15	3/2	0	4	0	210	
	$C_j - Z_j$	0	−23	0	−3/2	0	−4	$-M$		

Duality Theory

Rule 1: First we convert the problem into canonical form

- Min >=
- Max <=

Rule 2: Change the objective function of maximization in the primal into minimization in the dual and vice versa

Rule 3: The number of variable in the primal will be the number of constraints in dual and vice versa

Rule 3: Examples

Primal	Dual
Number of Variable : n	Number of Variable : m
Number of Constraints : m	Number of Contraints : n

Rule 4: Cost coefficient in objective of the primal will be RHS constant of the constraints in dual and vice versa

Rule 5: For formulating constraints we considered the transpose of matrix

Summary of Primal – Dual Relationship

Primal Problem

- Minimization

Cannonical Form of Primal

- Minimization
- Constraints >=

- Dual

Maximization

Constraints <=

Primal Problem

- Maximization

Cannonical Form of Primal

- Maximization
- Constraints <=

- Dual

Minimization

Constraints >=

Summary of Primal – Dual Relationship

Primal Problem : Maximize $Z = x1 + 2x2 + x3$

Subject to constraints :

$2x1 + x2 - x3 \le 2$

$-2x1 + x2 - 5x3 \ge -6$

$4x1 + x2 + x3 \le 6$ where $x1, x2, x3 \ge 0$

Cannonical Form of Primal

- Maximization
- Constraints $\le$

Maximize $Z = x1 + 2x2 + x3$

Subject to constraints :

$2x1 + x2 - x3 \le 2$

$2x1 - x2 + 5x3 \le 6$

$4x1 + x2 + x3 \le 6$ where $x1, x2, x3 \ge 0$

Summary of Primal – Dual Relationship

Primal Problem : Maximize Z=x1 + 2x2 + x3

Subject to constraints :

2x1 + x2 –x3 <=2

-2x1 + x2 -5x3 >= -6

4x1 +x2 + x3 <= 6 where x1, x2 , x3 >=0

Cannonical Form :

Maximize Z = x1 + 2x2 + x3

Subject to constraints :

2x1 + x2 –x3 <=2 ------------y1

2x1 - x2 + 5x3 <= 6 ------------y2

4x1 +x2 + x3 <= 6 ----------y3

where x1, x2 , x3 >=0

Dual Problem from Primal Problem:

Dual

Minimize W = y1 + y2 + y3

Subject to constraints :

2	2	4	>=	1
1	**-1**	**1**	**>=**	**2**
-1	**5**	**1**	>=	1

where y1, y2 , y3 >=0

Duality Theory

Example: Suppose that the primal LP is

$$\text{Maximize} \qquad Z = 2x_1 + 3x_2$$

under constraints

$$\begin{aligned} 2x_1 &+ x_2 \leq 4 \\ -x_1 &+ x_2 \leq 1 \\ -3x_1 &+ x_2 \leq -1 \end{aligned}$$

Duality Theory

Primal

$$
\begin{array}{ll}
\text{Maximize} & Z = 2x_1 + 3x_2 \\
 & \\
 & \begin{array}{rcrcr}
2x_1 & + & x_2 & \leq & 4 \\
-x_1 & + & x_2 & \leq & 1 \\
-3x_1 & + & x_2 & \leq & -1
\end{array}
\end{array}
$$

Dual

In this case,

$$
A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -3 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}
$$

Therefore, the dual is:

$$
\text{Minmize} \qquad W = b^T y = 4y_1 + y_2 - y_3
$$

under constraints

$$
\begin{array}{rcrcrcr}
2y_1 & - & y_2 & - & 3y_3 & \geq & 2 \\
y_1 & + & y_2 & + & y_3 & \geq & 3
\end{array}
$$

and $y_1, y_2, y_3 \geq 0$.

Duality Theory

$$\text{Minimize } Z_1 = 24X_1 + 30X_2$$

$$2X_1 + 3X_2 \geq 10$$
$$4X_1 + 9X_2 \geq 15$$
$$6X_1 + 6X_2 \geq 20$$
$$X_1 \text{ and } X_2 \geq 0$$

$$\text{Maximize } Z = 10Y_1 + 15Y_2 + 20Y_3$$

$$2Y_1 + 4Y_2 + 6Y_3 \leq 24$$
$$3Y_1 + 9Y_2 + 6Y_3 \leq 30$$
$$Y_1,\ Y_2 \text{ and } Y_3 \geq 0$$

Example 2.34 Solve the following linear programming problem using the result of its dual problem.

$$\text{Minimize } Z_1 = 24X_1 + 30X_2$$

subject to

$$2X_1 + 3X_2 \geq 10$$

$$4X_1 + 9X_2 \geq 15$$

$$6X_1 + 6X_2 \geq 20$$

$$X_1 \text{ and } X_2 \geq 0$$

Solution Let, Y_1, Y_2 and Y_3 be the dual variables with respect to the constraints 1, 2 and 3 respectively. Then the corresponding dual problem is:

$$\text{Maximize } Z = 10Y_1 + 15Y_2 + 20Y_3$$

subject to

$$2Y_1 + 4Y_2 + 6Y_3 \leq 24$$

$$3Y_1 + 9Y_2 + 6Y_3 \leq 30$$

$$Y_1,\ Y_2 \text{ and } Y_3 \geq 0$$

Solve the dual of the problem using simplex

Maximize $Z = 10Y_1 + 15Y_2 + 20Y_3$

$$2Y_1 + 4Y_2 + 6Y_3 \le 24$$
$$3Y_1 + 9Y_2 + 6Y_3 \le 30$$
$$Y_1,\ Y_2 \text{ and } Y_3 \ge 0$$

	C_j	10	15	20	0	0		
CB_i	Basic variable	Y_1	Y_2	Y_3	S_1	S_2	Solution	Ratio
0	S_1	2	4	$\boxed{6}$	1	0	24	4**
0	S_2	3	9	6	0	1	30	5
	Z_j	0	0	0	0	0	0	
	$C_j - Z_j$	10	15	20*	0	0		

	C_j	10	15	20	0	0		
CB_i	Basic variable	Y_1	Y_2	Y_3	S_1	S_2	Solution	Ratio
20	Y_3	1/3	2/3	1	1/6	0	4	12
0	S_2	$\boxed{1}$	5	0	−1	1	6	6**
	Z_j	20/3	40/3	20	10/3	0	80	
	$C_j - Z_j$	10/3*	5/3	0	−10/3	0		

CB_i	C_j / Basic variable	10 Y_1	15 Y_2	20 Y_3	0 S_1	0 S_2	Solution	Ratio
20	Y_3	1/3	2/3	1	1/6	0	4	12
0	S_2	$\boxed{1}$	5	0	−1	1	6	6**
	Z_j	20/3	40/3	20	10/3	0	80	
	$C_j - Z_j$	10/3*	5/3	0	−10/3	0		

CB_i	C_j / Basic variable	10 Y_1	15 Y_2	20 Y_3	0 S_1	0 S_2	Solution
20	Y_3	0	−1	1	1/2	−1/3	2
10	Y_1	1	5	0	−1	1	6
	Z_j	10	30	20	0	10/3	100
	$C_j - Z_j$	0	−15	0	0	−10/3	

Final Solution of Primal using Dual: solution of the primal is inferred as shown.

Basic variables in the initial table of dual problem	S_1	S_2
$-(C_j - Z_j)$ of the final table of dual problem	0	10/3
Corresponding primal variables	X_1	X_2

Exercise : Form the dual of the following primal problem.

Minimize $z= 5x1 + 8x2$

Subject to Constraints:

$4x1 + 9x2 >= 100$

$2x1 + x2 <= 20$

$2x1 + 5x2 >=120$

$x1,x2 >=0$

Cannocial Form :

Minimize $z= 5x1 + 8x2$

Subject to Constraints:

$4x1 + 9x2 >= 100$

$-2x1 -x2 >= -20$

$2x1 + 5x2 >=120$

$x1,x2 >=0$

Exercise : Form the dual of the following primal problem.

Cannocial Form :

Minimize $z= 5x1 + 8x2$

Subject to Constraints:

$4x1 + 9\,x2 >= 100$

$2x1 -x2 >= -20$

$2x1 + 5x2 >=120$

$x1,x2 >=0$

Dual Form

Maximize

Subject to constraints :

<=

<=

Q. 1 Solve the following:

Construct the dual of the problem

minimize $\quad Z = 3x_1 - 2x_2 + 4x_3,$

subject to the constraints
$$
\begin{aligned}
3x_1 + 5x_2 + 4x_3 &\geq 7, \\
6x_1 + x_2 + 3x_3 &\geq 4, \\
7x_1 - 2x_2 - x_3 &\leq 10, \\
x_1 - 2x_2 + 5x_3 &\geq 3, \\
4x_1 + 7x_2 - 2x_3 &\geq 2, \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

Solution:

The dual of the given problem will be

maximize $W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5,$

subject to

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3,$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \le -2,$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4,$$
$$y_1, y_2, y_3, y_4, y_5, \text{ all} \ge 0,$$

where y_1, y_2, y_3, y_4, and y_5 are the dual variables associated with the first, second, third, fourt[h] and fifth constraint respectively.

Q. 2 Solve the following:

big_M_dual_unrestrictedvat0306202109_24_03\CamScanner 03-06-2021 09.24.03.pdf

Construct the dual to the primal problem

maximize $Z = 3x_1 + 5x_2,$

subject to

$$2x_1 + 6x_2 \le 50,$$
$$3x_1 + 2x_2 \le 35,$$
$$5x_1 - 3x_2 \le 10,$$
$$x_2 \le 20,$$

where $x_1 \ge 0,\ x_2 \ge 0.$ [NIIFT Mohali, 2000]

Formulate the Primal Problem:

EXAMPLE 2.6-2 (Diet Problem)

A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in table 2.2.

TABLE 2.2

Food type	Yield per unit			Cost per unit (Rs.)
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming model for the problem.

Solution:

Formulation of L.P. Model

Let x_1, x_2, x_3 and x_4 denote the number of units of food of type 1, 2, 3 and 4 respectively. *Objective* is to minimize the cost *i.e.*,

$$\text{Minimize } Z = \text{Rs. } (45x_1 + 40x_2 + 85x_3 + 65x_4).$$

Constraints are on the fulfilment of the daily requirements of the various constituents.

i.e., for proteins, $3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$,

for fats, $2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$,

and for carbohydrates, $6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$,

where x_1, x_2, x_3, x_4, each ≥ 0.

Apply Big M method

The final product of a firm must weigh exactly 150 kg. It uses two raw materials A and B with costs Rs. 2 and Rs. 8 per unit. At least 14 units of B and no more than 20 units of A must be used. Each unit of A weighs 5 kg and each unit of B weigh 10 kg. How much of each raw material be used per unit of the final product if its cost is to be minimized?

Apply Big M

$$
\begin{aligned}
\text{Minimize:} \quad & 2X_1 + 8X_2 \\
\text{Subject to:} \quad & 5X_1 + 10X_2 = 150 \\
& X_1 \le 20 \\
& X_2 \ge 14 \\
& X_1 \; ; \; X_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Minimize:} \quad & 2X_1 + 8X_2 + 0S_1 + 0S_2 + 1A_1 + 1A_2 \\
\text{Subject to:} \quad & 5X_1 + 10X_2 + A_1 = 150 \\
& X_1 + S_1 = 20 \\
& X_2 - S_2 + A_2 = 14 \\
& X_1 \; ; \; X_2 \; ; \; S_1 \; ; \; S_2 \; ; \; A_1 \; ; \; A_2 \ge 0
\end{aligned}
$$

Modified objective function

$$\text{Minimize: } 2X_1 + 8X_2 + 0S_1 + 0S_2 + 1A_1 + 1A_2$$

$$\begin{aligned}
\text{Subject to: } 5X_1 + 10X_2 + A_1 &= 150 \\
X_1 + S_1 &= 20 \\
X_2 - S_2 + A_2 &= 14 \\
X_1 \;;\; X_2 \;;\; S_1 \;;\; S_2 \;;\; A_1 \;;\; A_2 &\geq 0
\end{aligned}$$

Big M Method

$$\text{Minimize: } 2X_1 + 8X_2 + 0S_1 + 0S_2 + MA_1 + MA_2$$

$$\text{Subject to: } 5X_1 + 10X_2 + A_1 = 150$$

$$X_1 + S_1 = 20$$

$$X_2 - S_2 + A_2 = 14$$

$$X_1 \; ; \; X_2 \; ; \; S_1 \; ; \; S_2 \; ; \; A_1 \; ; \; A_2 \geq 0$$

Table 3.45

Tableau 1									
C_i	Basis	X_1	X_2	S_1	S_2	A_1	A_2	X_b	Min
M	A_1	5	10	0	0	1	0	150	15
0	S_1	1	0	1	0	0	0	20	–
M	A_2	0	1	0	–1	0	1	14	14
	C_j	2	8	0	0	M	M		
	Z_j	5M	11M	0	–M	M	M	164M	
	Z_j-C_j	5M–2	11M–8	0	–M	0	0		

Tableau 2

C_i	Basis	X_1	X_2	S_1	S_2	A_1	X_b	Min
M	A_1	5	0	0	10	1	10	1
0	S_1	1	0	1	0	0	20	–
8	X_2	0	1	0	−1	0	14	–
	C_j	2	8	0	0	M		
	Z_j	5M	8	0	10M−8	M	10M+112	
	Z_j-C_j	5M−2	0	0	10M−8	0		

Table 3.47

Tableau 3

C_j	Basis	X_1	X_2	S_1	S_2	X_b	Min
0	S_2	0.5	0	0	1	1	2
0	S_1	1	0	1	0	20	20
8	X_2	0.5	1	0	0	15	30
	C_j	2	8	0	0		
	Z_j	4	8	0	0	90	
	Z_j-C_j	2	0	0	0		

Tableau 4

C_j	Basis	X_1	X_2	S_1	S_2	X_b
2	X_1	1	0	0	2	2
0	S_1	0	0	1	−2	18
8	X_2	0	1	0	−1	14
	C_j	2	8	0	0	
	Z_j	2	8	0	−4	116
	Z_j-C_j	0	0	0	0	