1D FEM H-P CODE TO SOLVE GOVERNING DIFFERENTIAL EQUATION OF A BAR SUBJECTED TO VARIOUS LOADS

A project report submitted in the fulfillment of the requirement for the course.

AE675 INTRODUCTION TO FINITE ELEMENT METHODS

Professor

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PROBLEM STATEMENT

Figure 1 shows an elastic bar under traction load and constrained at the ends A and B. Develop a generic finite element code to get the approximate solution to the resulting governing differential equation for the bar shown in Figure 1.

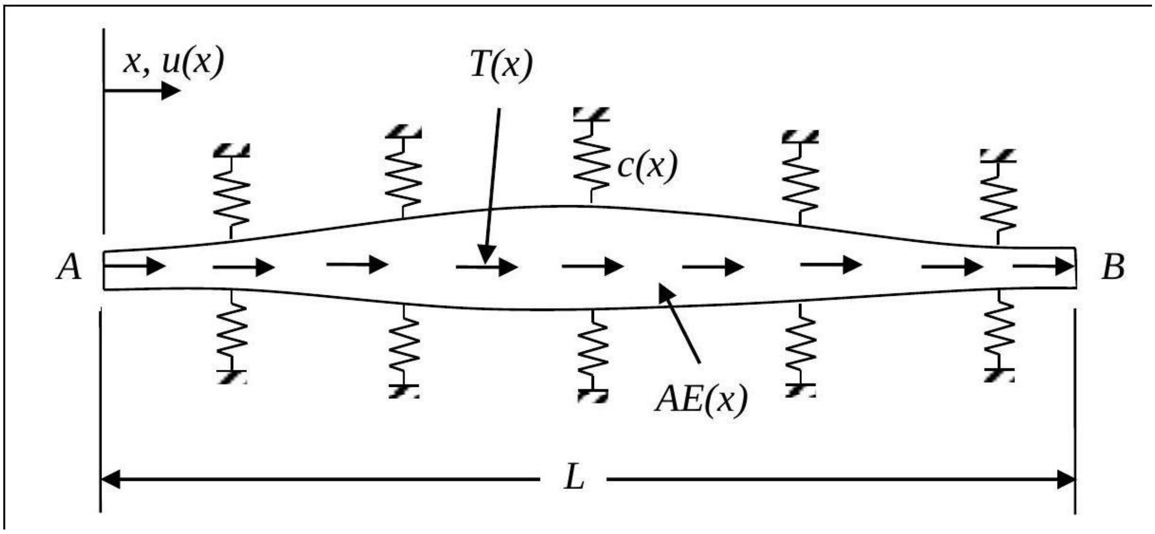


Figure 1: Elastic bar loaded in traction and constrained at ends A and B.

The code should have the following capabilities:

1. Boundary conditions/End Constraints: Both ends can be constrained by specifying (a) primary variable (Dirichlet/Displacement/Essential), (b) secondary variable or force (Force/Neumann/Natural) and (c) springs (Mixed/Robin)
2. The variables $T(x), c(x)$ and $AE(x)$ can vary from a constant to a quadratic function.
3. The length L and the number of elements will be input values. Discretize the domain into given number of elements with equal lengths.
4. There should be a provision to put at least one concentrated load at any given location (excluding the ends).
5. Use of either Lagrange interpolation or hierarchic shape functions up to quartic order should be possible.
6. Postprocessing must be able to represent the primary, secondary, and other variables over the domain either continuously or discretely as required.

Note:

1) In all above points take $L = 1$.

2. All the above should be done with both Lagrange and hierarchic shape functions.
3. Do not us inbuilt capabilities like integration and differentiation if you are going to use MATLAB for coding. You will be penalized for that.

Produce a detailed report to include the following:

1. Do the patch test for the following cases:

$AE(x) = 1$ and $c(x) = 0$ with $u(x)|_{x=0} = 0$ and $\left.\frac{du}{dx}\right|_{x=1} = 0$. When $T(x) = 1$, use 1,2,5,10 and 100 number of linear and quadratic elements and when $T(x) = x$ use 1,2,5,10 and 100 number of linear, quadratic, and cubic elements and superimpose your solutions with the respective exact solutions. Plot the error in the solution. Also plot the derivative of the exact solution and finite element solutions. Discuss the results.

2. For the problem in Point 1, plot the strain energy of the exact and finite element solution against the number of elements in the mesh for all the cases. Also plot the strain energy of the solution. Discuss the results.

3. Take $AE(x) = 1, c(x) = 1$ and $T(x) = 1$ with $u(x)|_{x=0} = 0$ and $\left.\frac{du}{dx}\right|_{x=1} = 0$. Obtain the finite element solution with linear, quadratic, cubic and quartic elements respectively for 1, 10, 20, 40, 80 and 100 number of elements. For these cases:

    a. Plot the exact and finite element solutions together for these cases.
    b. Plot the error in the solution for these cases.
    c. Plot the strain energy of the finite element and exact solution as a function of number of elements.
    d. Plot the strain energy of the error as a function of number of elements.
    e. Plot the log of the relative error in the energy norm versus the log of number of elements.
    f. Try to estimate the convergence rate. Discuss the results.

4. Take $AE(x) = 1, c(x) = 0$ and $T(x) = \sin\frac{\pi}{L}x$ with $AE\left.\frac{du}{dx}\right|_{x=0} = \frac{1}{\pi}$ and $AE\left.\frac{du}{dx}\right|_{x=1} = k_L(\delta_L - u(L))$ with $k_L = 10$ and $\delta_L = 0$. Then repeat the exercise given a) through f) in Point 3.

Produce a detailed report to include the following:

Question 1.

1.1 Part -1

Do the patch test for the following cases: $\boldsymbol{AE(x) = 1}$ ***and*** $\boldsymbol{c(x) = 0}$ ***with*** $\boldsymbol{u(x)|_{x=0} = 0}$ ***and*** $\boldsymbol{\left.\frac{du}{dx}\right|_{x=1} = 0}$***.***

When $\boldsymbol{T(x) = 1}$ ***and*** $\boldsymbol{T(x) = x}$

Solution:

1.1.1 Introduction

The *patch test* is a verification technique for Finite Element Method (FEM) codes, ensuring they can reproduce known polynomial solutions. The steps are as follows:

- **Problem Definition**: Choose a problem whose polynomial solution of degree p the FEM can represent.
- **Boundary Conditions**: Apply appropriate boundary conditions corresponding to the polynomial solution.
- **Mesh Creation**: Use a single element or a small element assembly as the mesh.
- **FEM Solution**: Solve the problem using the FEM code.
- **Solution Comparison**: Verify if the FEM solution matches the exact polynomial solution at all nodal points.
- **Result Assessment:** A match confirms the FEM code's accuracy; a mismatch suggests errors in the FEM formulation or implementation.

Passing the patch test is crucial before employing FEM codes for complex simulations.

1.1.2 Patch Test Analysis

The patch test is a crucial step in the verification of finite element codes, ensuring that the discretization can exactly reproduce field variables for simple polynomial cases. The following analysis details the application of the patch test for two distinct cases using a polynomial of degree p with the following conditions:

$$AE(x) = 1 \quad \text{and} \quad c(x) = 0$$

$$u(x)|_{x=0} = 0 \quad \text{and} \quad \left.\frac{du}{dx}\right|_{x=1} = 0$$

Case 1: Constant Distribution

For the case where $T(x) = 1$, we observe the following:

- **When $P = 1$:** The exact solution for the displacement field u is a constant function. However, due to the imposed boundary conditions, the resulting FE solution is a trivial case where u remains zero throughout the domain. This is represented by the interpolating function as $-\frac{x}{2}$ within the range $[0,1]$.
- When $P = 2$**:** With a second-order polynomial approximation, the exact equation for the displacement becomes more nuanced, allowing for a linear variation within the element. The interpolating function in this scenario is $\frac{x(x-2)}{2}$, also within the range $[0,1]$.

Case 2: Linear Distribution

Considering a case where $T(x) = x$, we have:

- When $P = 1$**:** A first-order polynomial approximation leads to a linearly varying displacement, where the FE solution and the exact solution should ideally match. The interpolating function is represented as $-\frac{x}{3}$, across the domain $[0,1]$.
- When $P = 2$**:** Increasing the order of the polynomial to second order, we find that the interpolating function becomes $\frac{x(3x-7)}{12}$, implying a quadratic displacement field that fits the boundary conditions.
- When $P = 3$**:** As we further increase the polynomial degree to three, the exact solution represented by the interpolating function is $\frac{x(x^2-3)}{6}$, providing an even more refined approximation of the displacement field.

1.1.3 Summary

The tables below present the exact solutions for different cases and compare them to the FEM solutions obtained for different orders of approximation p. The exactness of the FEM solution is demonstrated by its ability to match the exact solution for a given p.

Case 1: $T(x) = 1$

p	Exact Solution	FEM Interpolating Function	Range	Match
1	$\frac{x(x-2)}{2}$	$-\frac{x}{2}$	[0, 1]	No
2	$\frac{x(x-2)}{2}$	$\frac{x(x-2)}{2}$	[0, 1]	Yes

Case 2: $T(x) = x$

p	Exact Solution	FEM Interpolating Function	Range	Match
1	$\frac{x(x^2-3)}{6}$	$-\frac{x}{3}$	[0, 1]	No
2	$\frac{x(x^2-3)}{6}$	$\frac{x(3x-7)}{12}$	[0, 1]	No
3	$\frac{x(x^2-3)}{6}$	$\frac{x(x^2-3)}{6}$	[0, 1]	Yes

1.2 Part-2

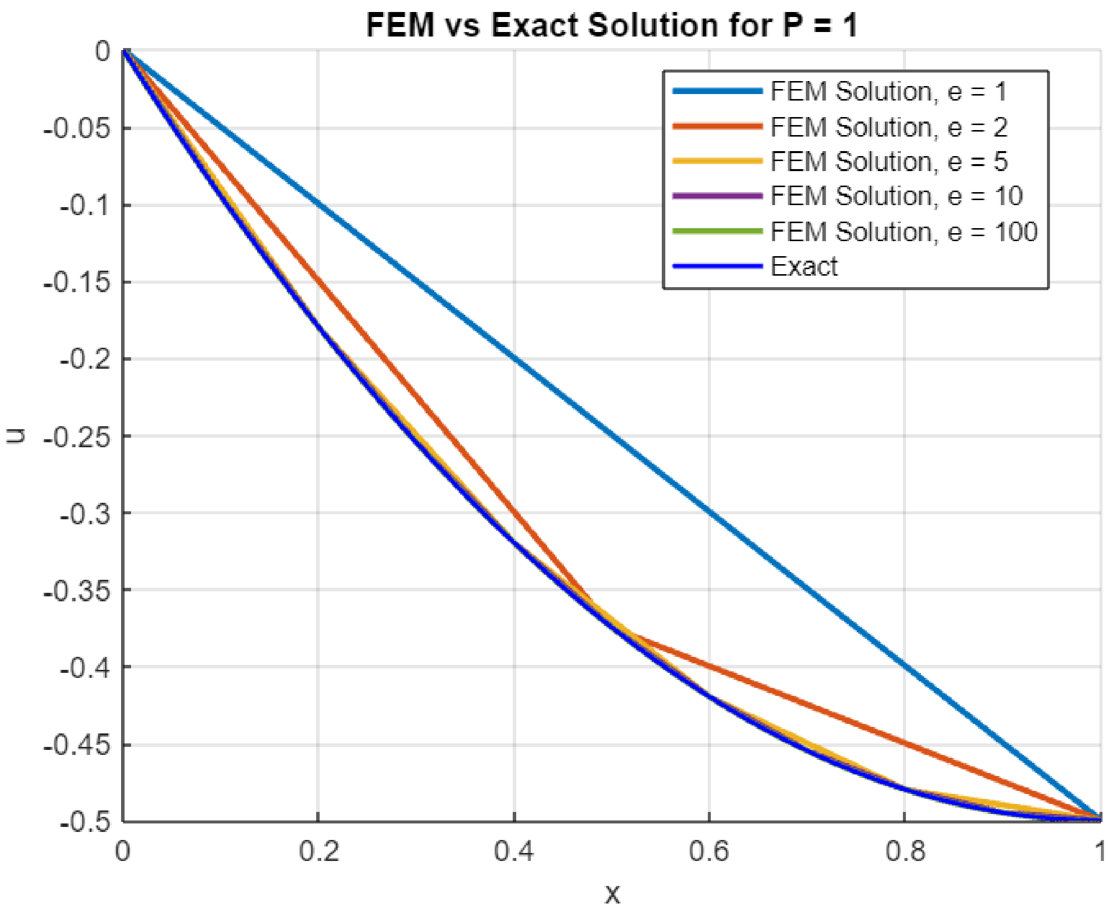
Do the patch test for the following cases: $\boldsymbol{AE(x) = 1}$ ***and*** $\boldsymbol{c(x) = 0}$ ***with*** $\boldsymbol{u(x)|_{x=0} = 0}$ ***and*** $\boldsymbol{\left.\frac{du}{dx}\right|_{x=1} = 0}$***.***

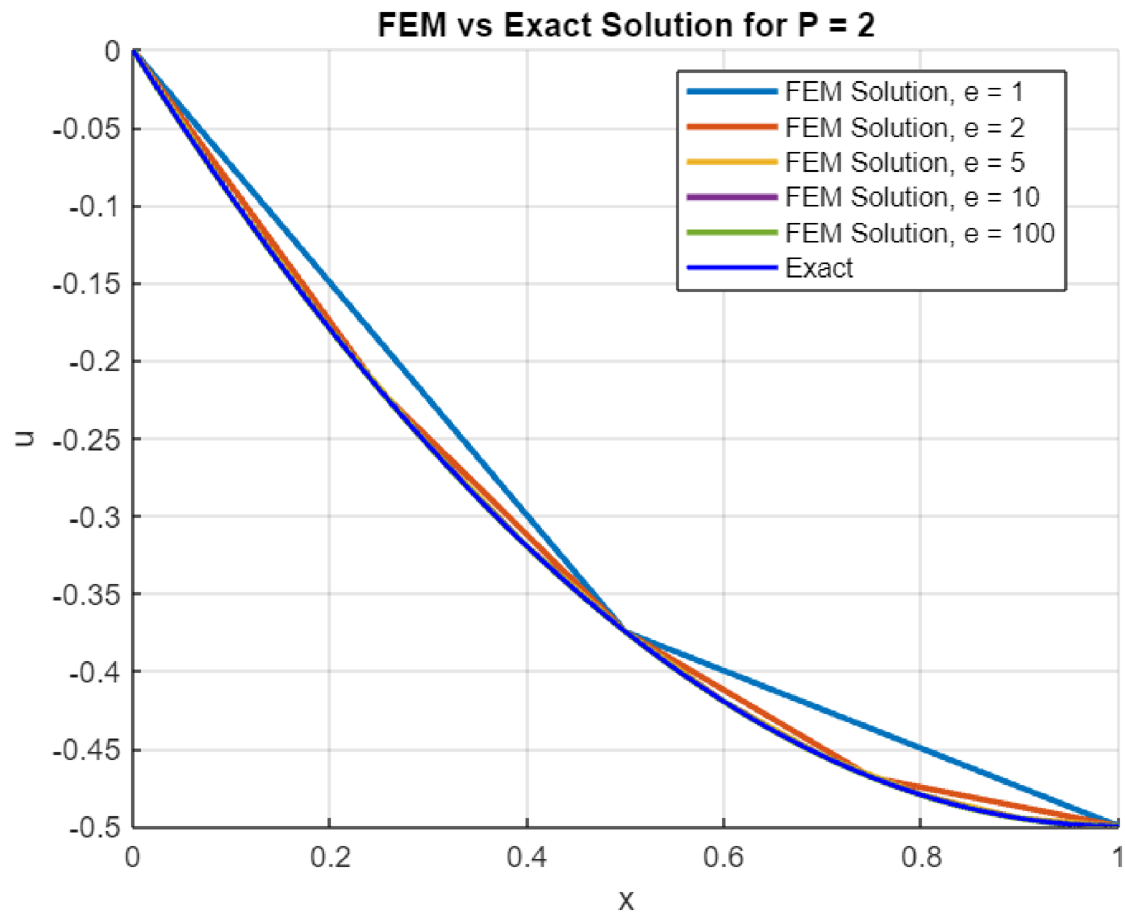
When $\boldsymbol{T(x) = 1}$***, use*** $\boldsymbol{1, 2, 5, 10}$ ***and 100 number of linear and quadratic elements and when*** $\boldsymbol{T(x) = x}$ ***use*** $\boldsymbol{1, 2, 5, 10}$ ***and 100 number of linear, quadratic, and cubic elements and superimpose your solutions with the respective exact solutions. Plot the error in the solution. Also plot the derivative of the exact solution and finite element solutions. Discuss the results.***

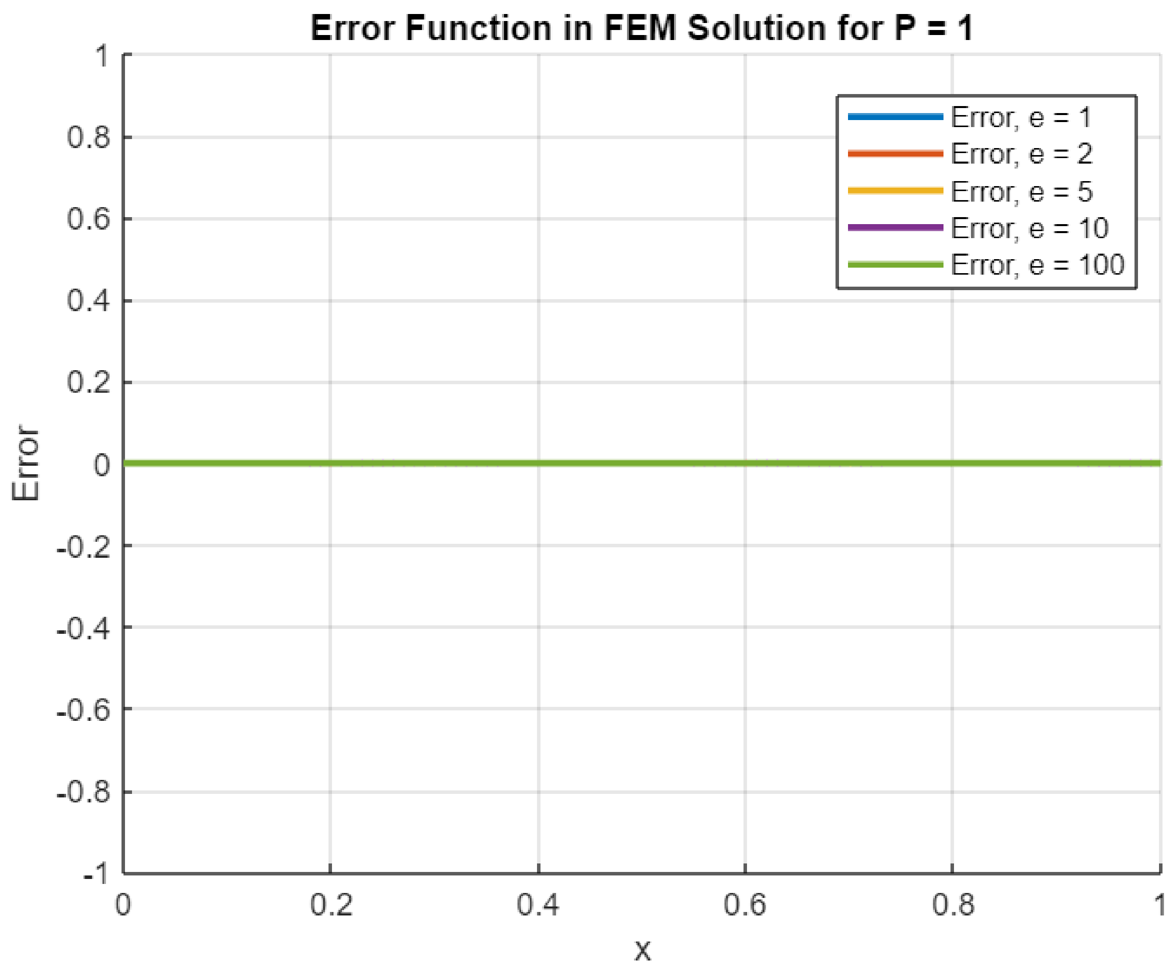
Solution:

1.2.1 Case-1

$AE(x) = 1$ *and* $c(x) = 0$ *with* $u(x)|_{x=0} = 0$ *and* $\left.\frac{du}{dx}\right|_{x=1} = 0$ $T(x) = 1$




FEM vs Exact Solution for P = 2
FEM Solution, e = 1
FEM Solution, e = 2
FEM Solution, e = 5
FEM Solution, e = 10
FEM Solution, e = 100
Exact
u
x
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
0
0.2
0.4
0.6
0.8
1


Error Function in FEM Solution for P = 1
Error, e = 1
Error, e = 2
Error, e = 5
Error, e = 10
Error, e = 100
Error
x
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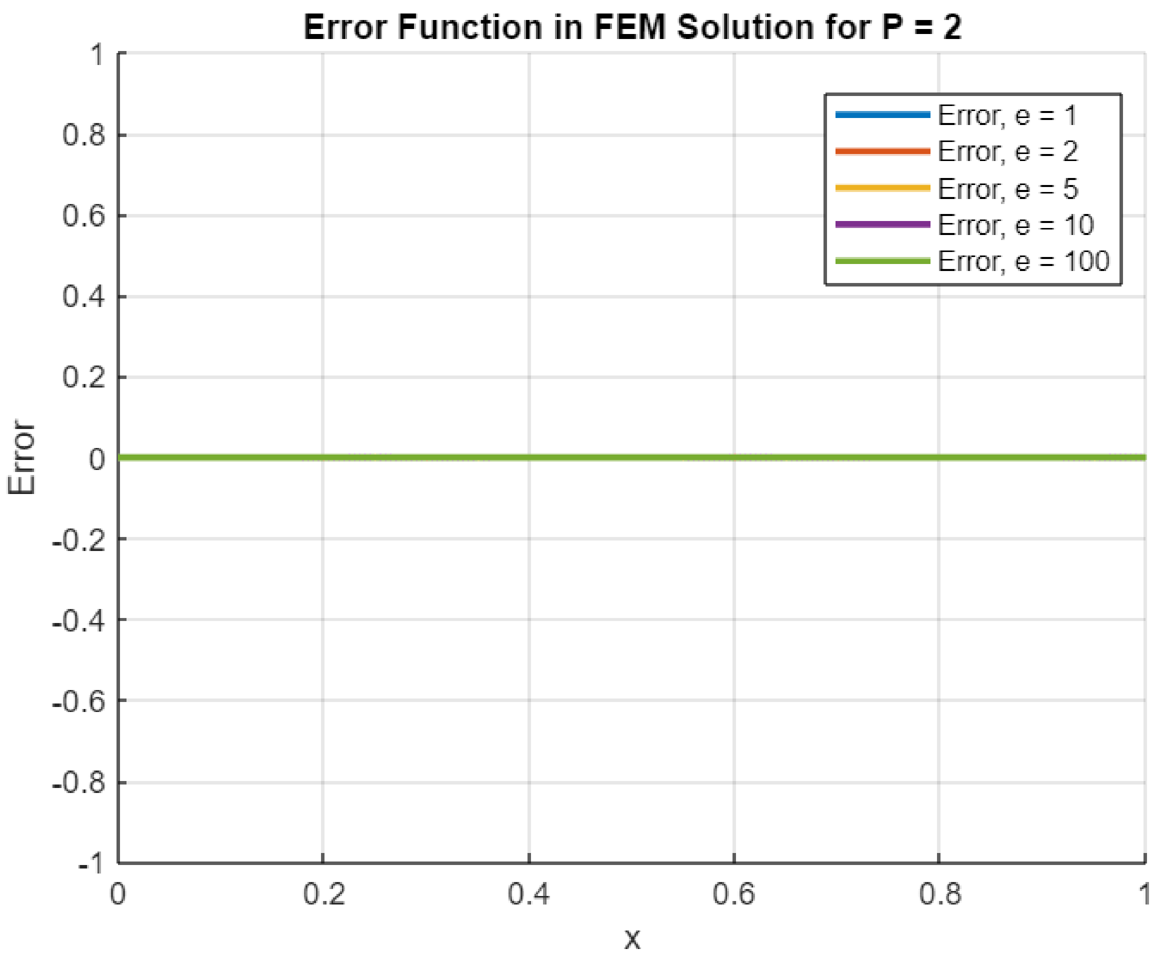
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1


Error Function in FEM Solution for P = 2
Error, e = 1
Error, e = 2
Error, e = 5
Error, e = 10
Error, e = 100
Error
x
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0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
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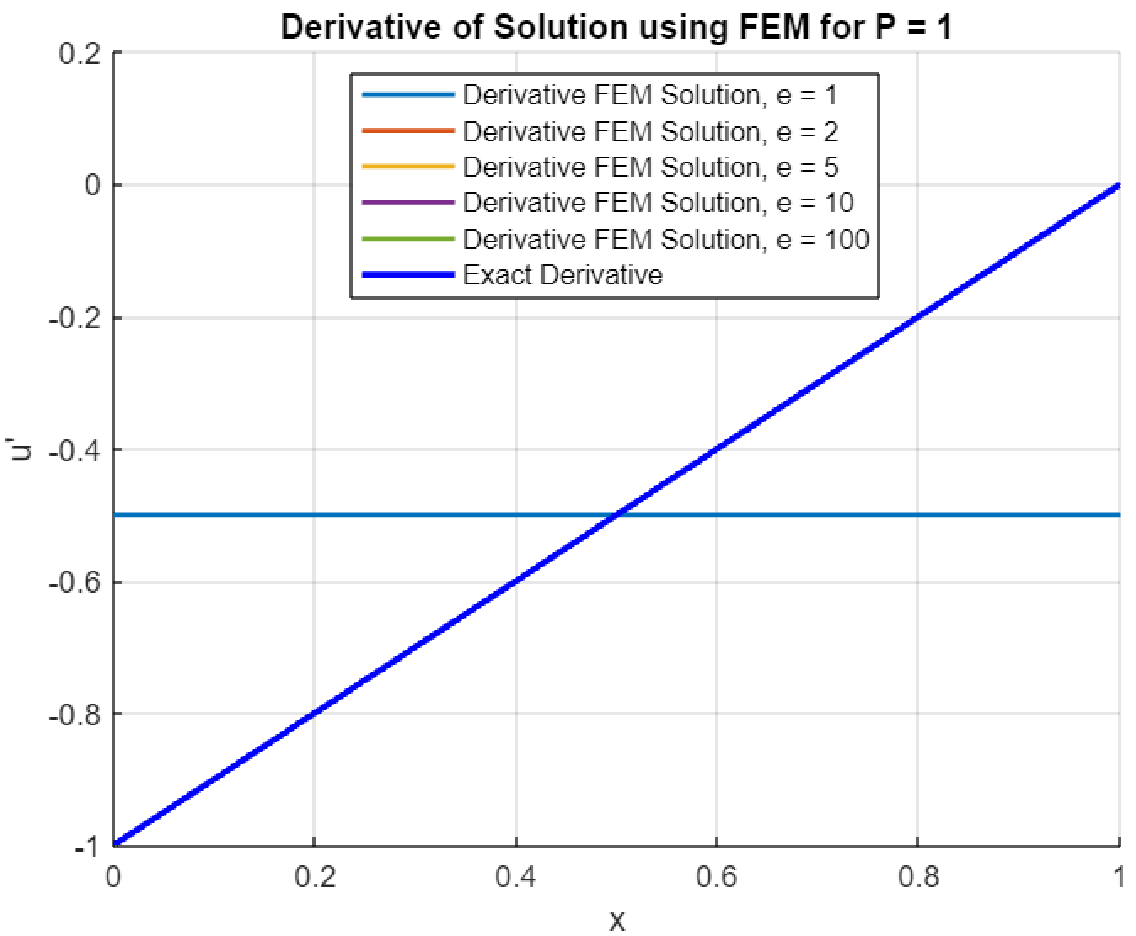
0.2
0.4
0.6
0.8
1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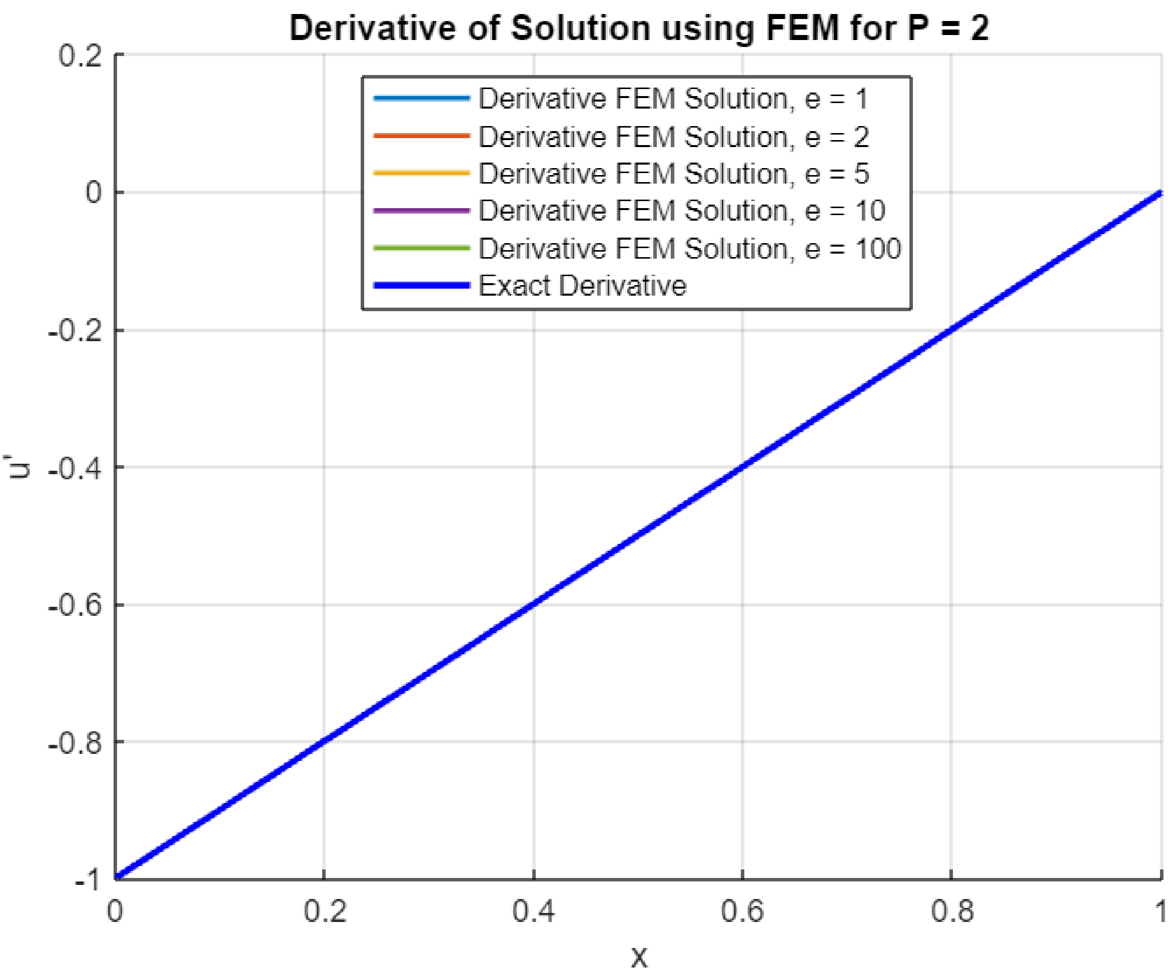
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 2
Derivative FEM Solution, e = 5
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 100
Exact Derivative
u'
x
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1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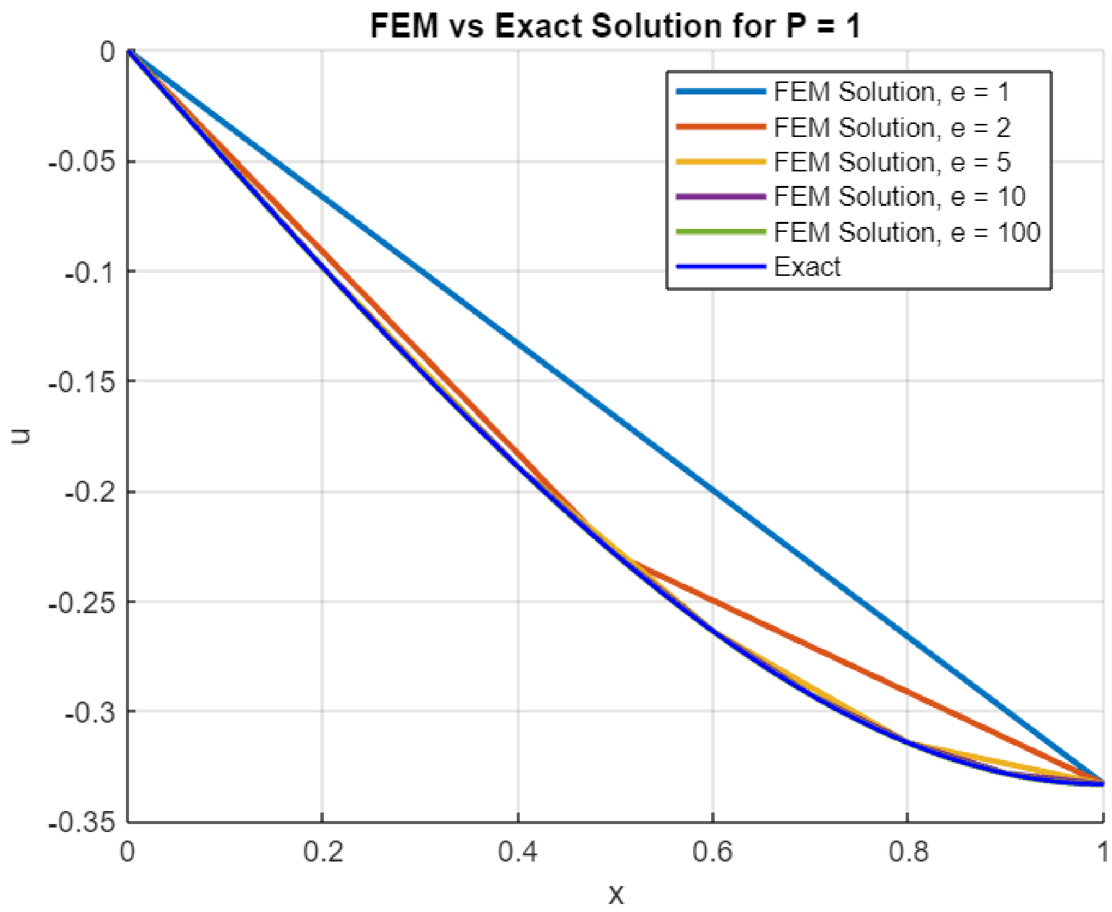
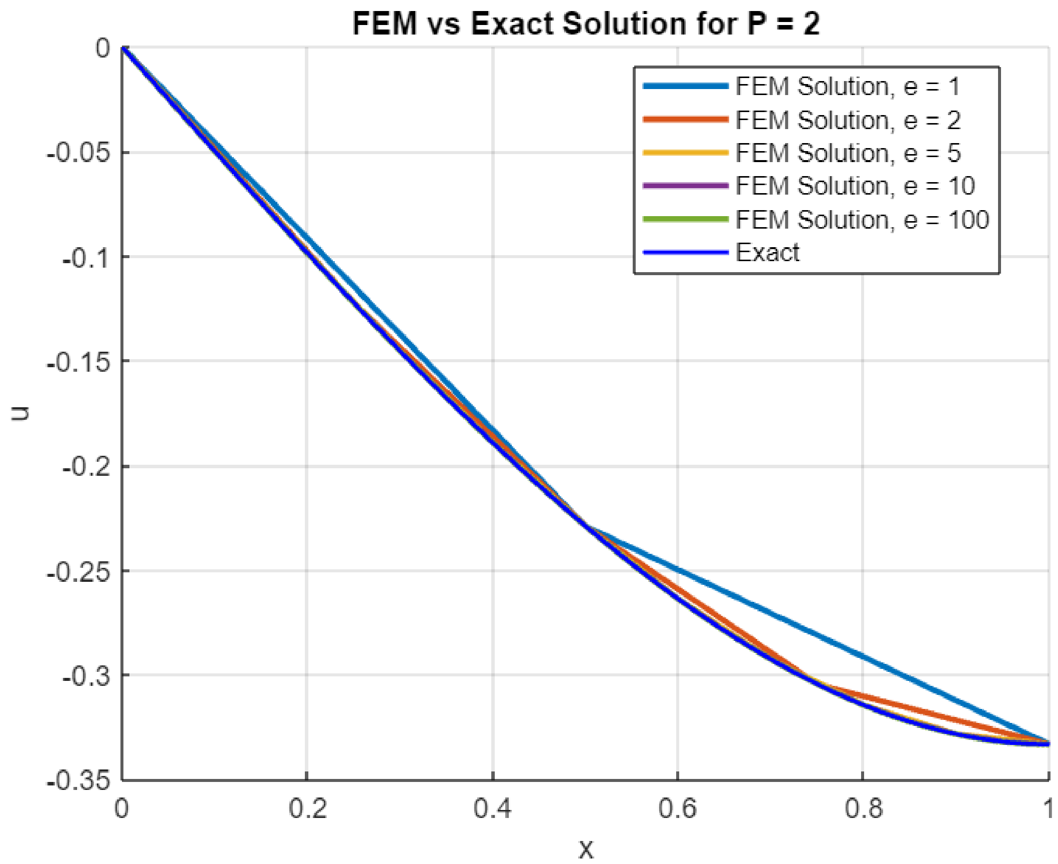
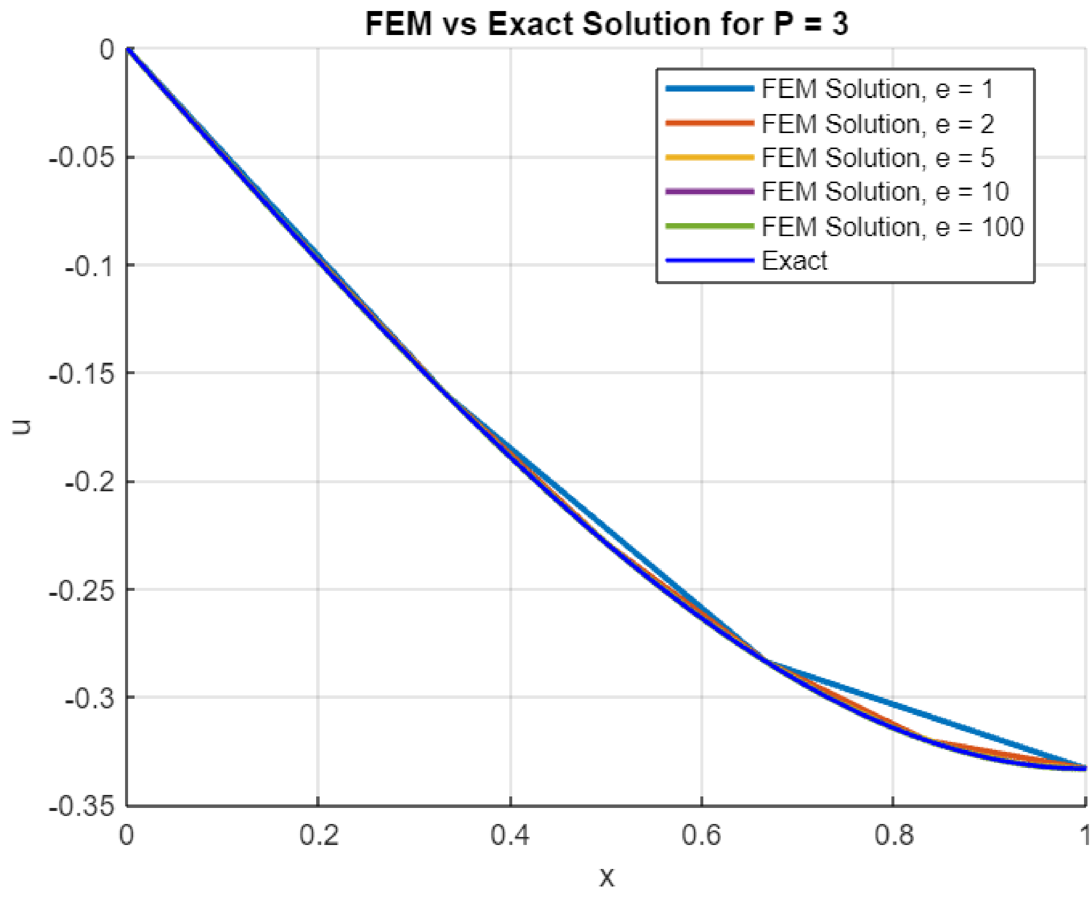
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 2
Derivative FEM Solution, e = 5
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 100
Exact Derivative
u'
x
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

1.2.2 Case-2

$AE(x) = 1$ *and* $c(x) = 0$ *with* $u(x)|_{x=0} = 0$ *and* $\frac{du}{dx}\Big|_{x=1} = 0$ $T(x) = x$.






FEM vs Exact Solution for P = 3
FEM Solution, e = 1
FEM Solution, e = 2
FEM Solution, e = 5
FEM Solution, e = 10
FEM Solution, e = 100
Exact
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x
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-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
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0.2
0.4
0.6
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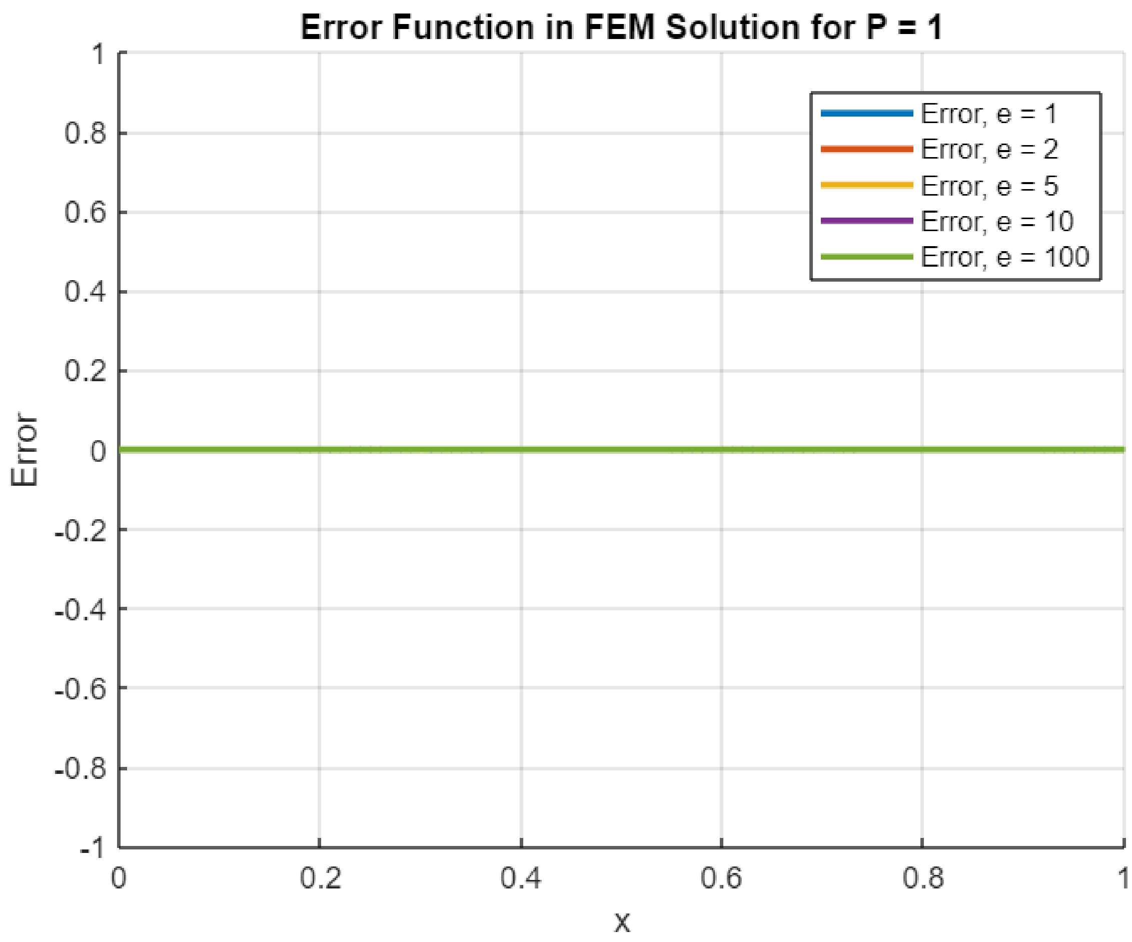
1


Error Function in FEM Solution for P = 1
Error, e = 1
Error, e = 2
Error, e = 5
Error, e = 10
Error, e = 100
Error
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
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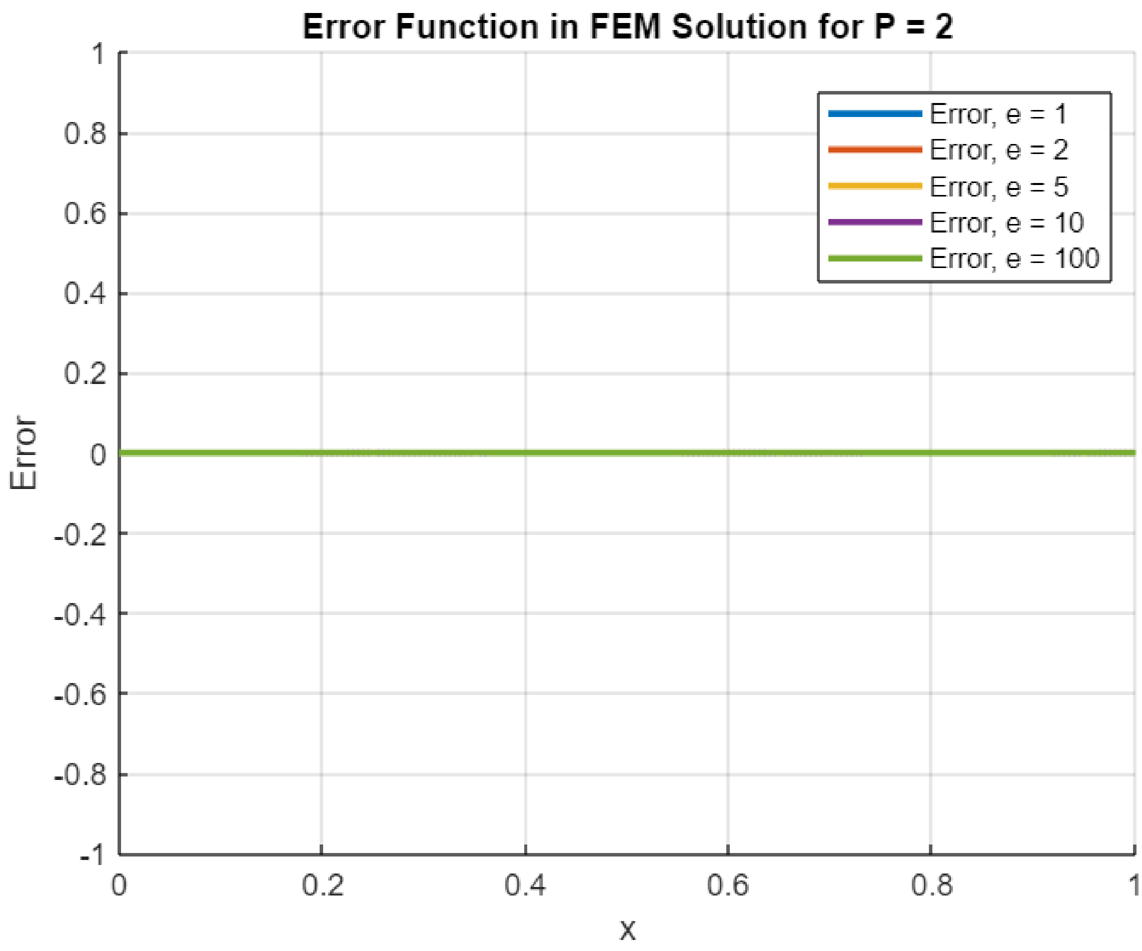
0.2
0.4
0.6
0.8
1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Error, e = 1
Error, e = 2
Error, e = 5
Error, e = 10
Error, e = 100
Error
x
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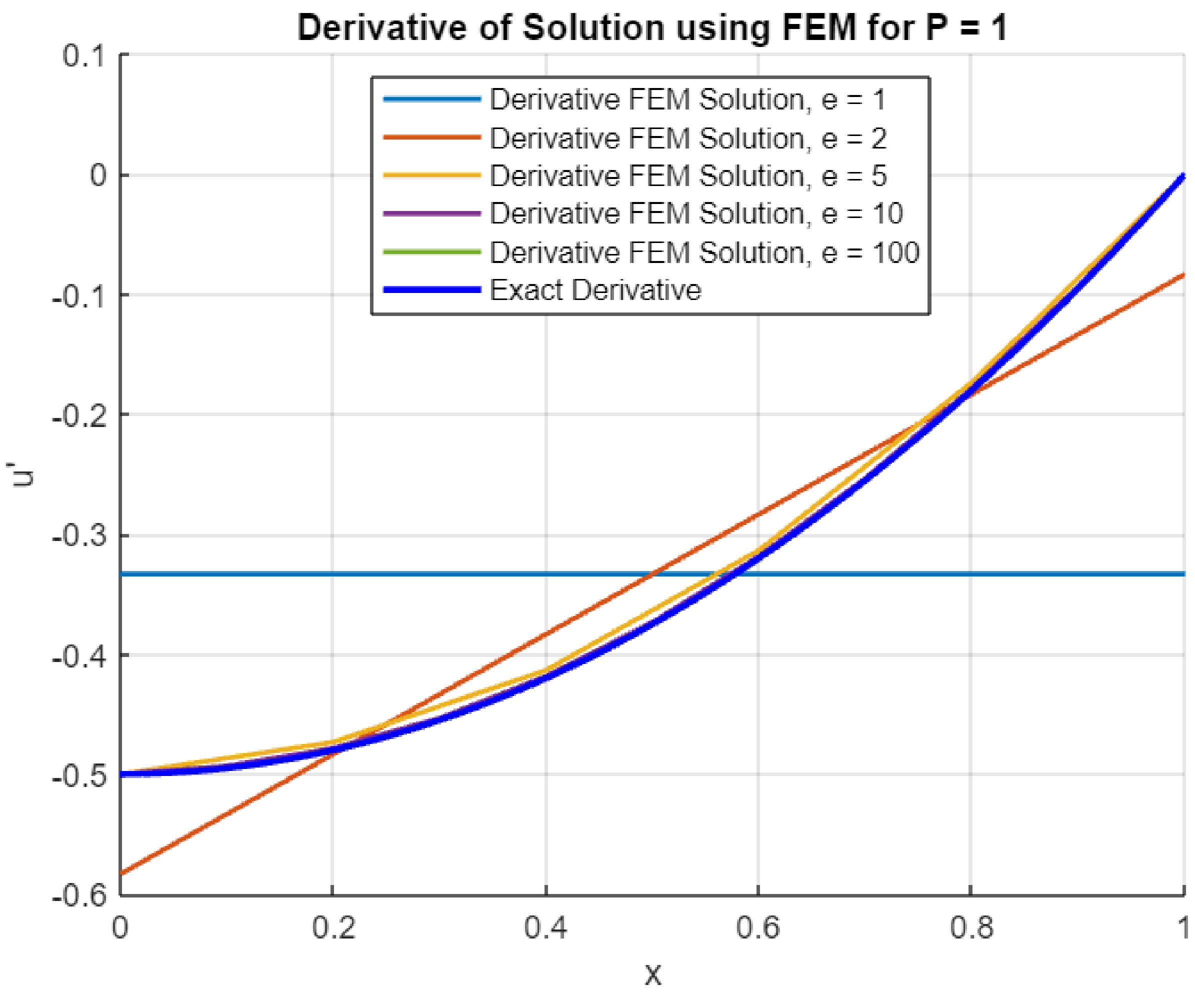
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Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 2
Derivative FEM Solution, e = 5
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 100
Exact Derivative
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-0.5
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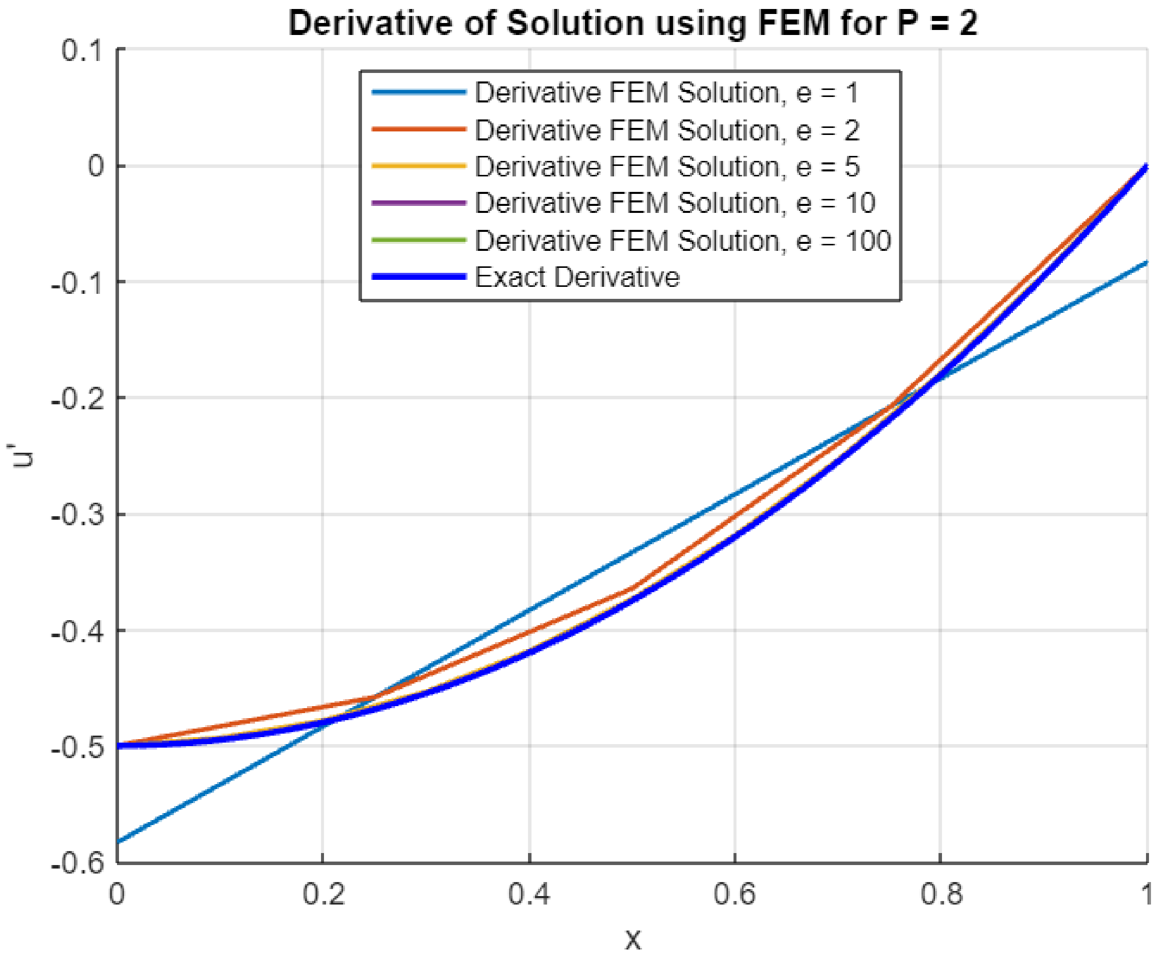
0.6
0.8
1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Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 2
Derivative FEM Solution, e = 5
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 100
Exact Derivative
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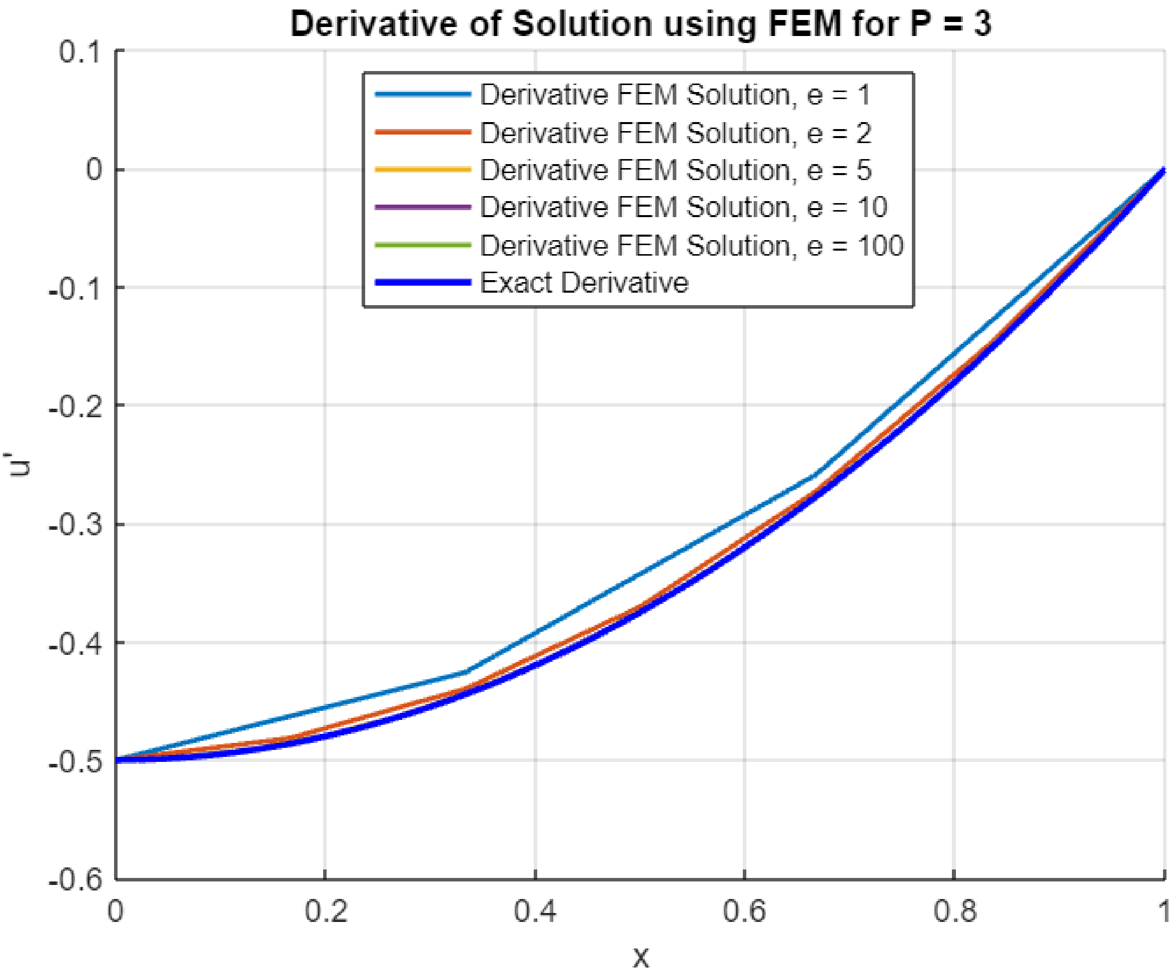
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Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 2
Derivative FEM Solution, e = 5
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 100
Exact Derivative
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-0.5
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0.6
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1

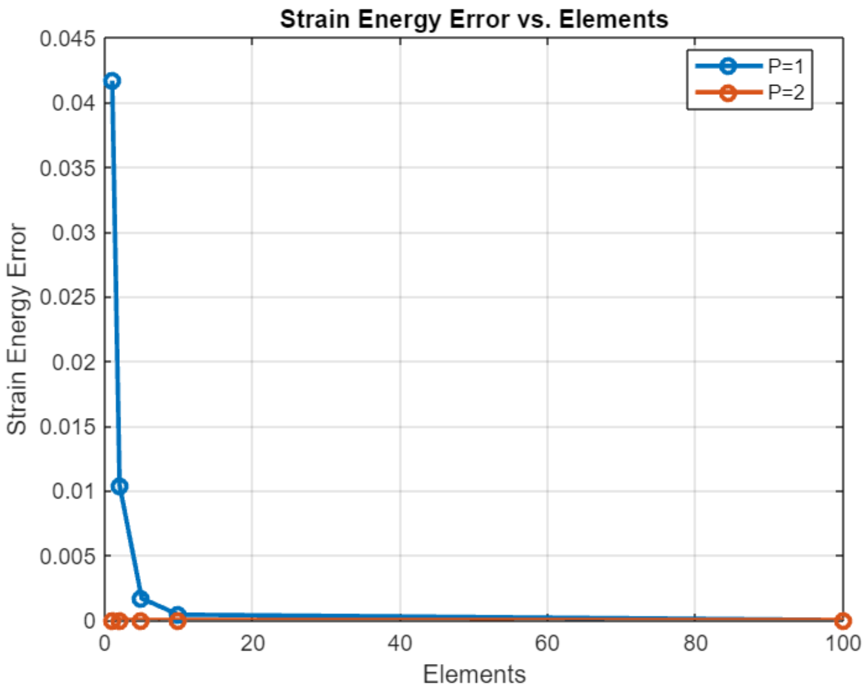
Question 2.

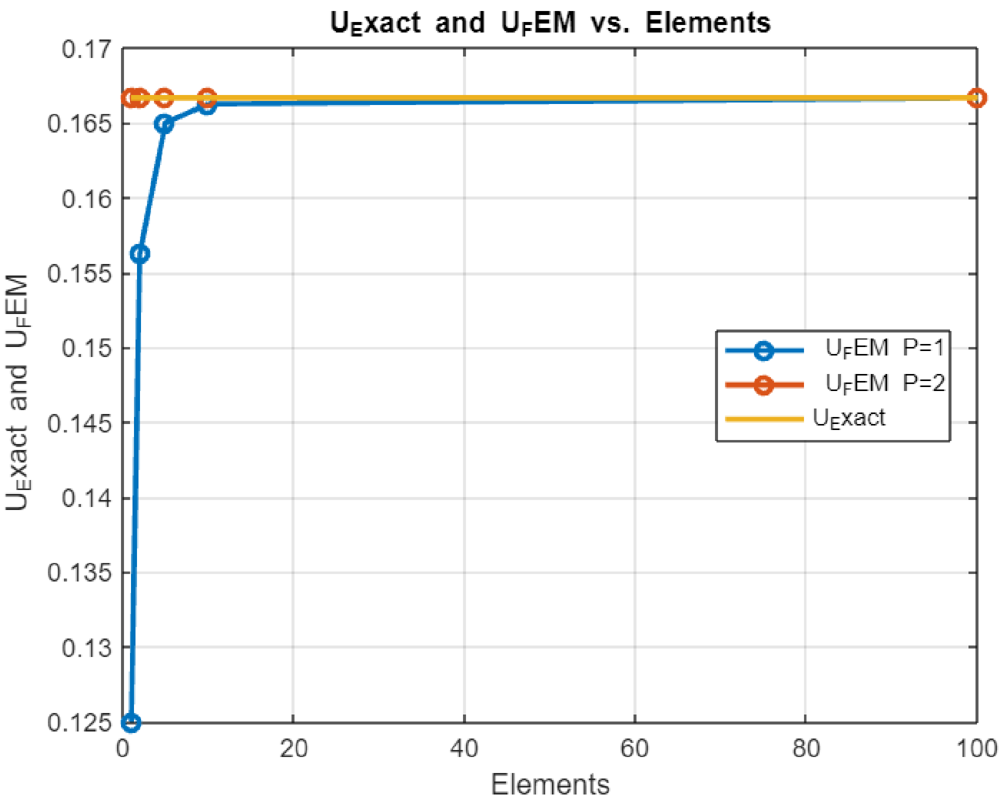
For the problem in Point 1, plot the strain energy of the exact and finite element solution against the number of elements in the mesh for all the cases. Also plot the strain energy of the solution. Discuss the results.

Solution:

Case-1

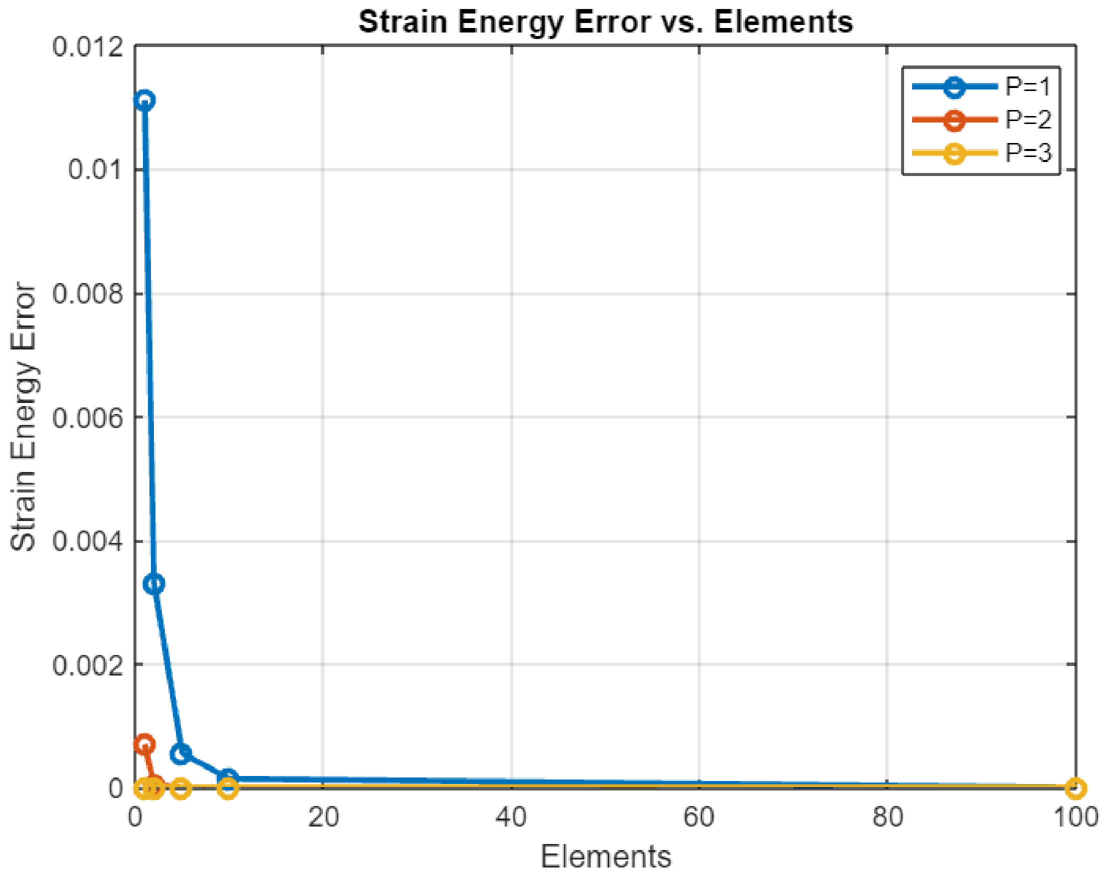
$AE(x) = 1 \; and \; c(x) = 0 \; with \; u(x)|_{x=0} = 0 \; and \; \left.\frac{du}{dx}\right|_{x=1} = 0 \;\; T(x) = 1$

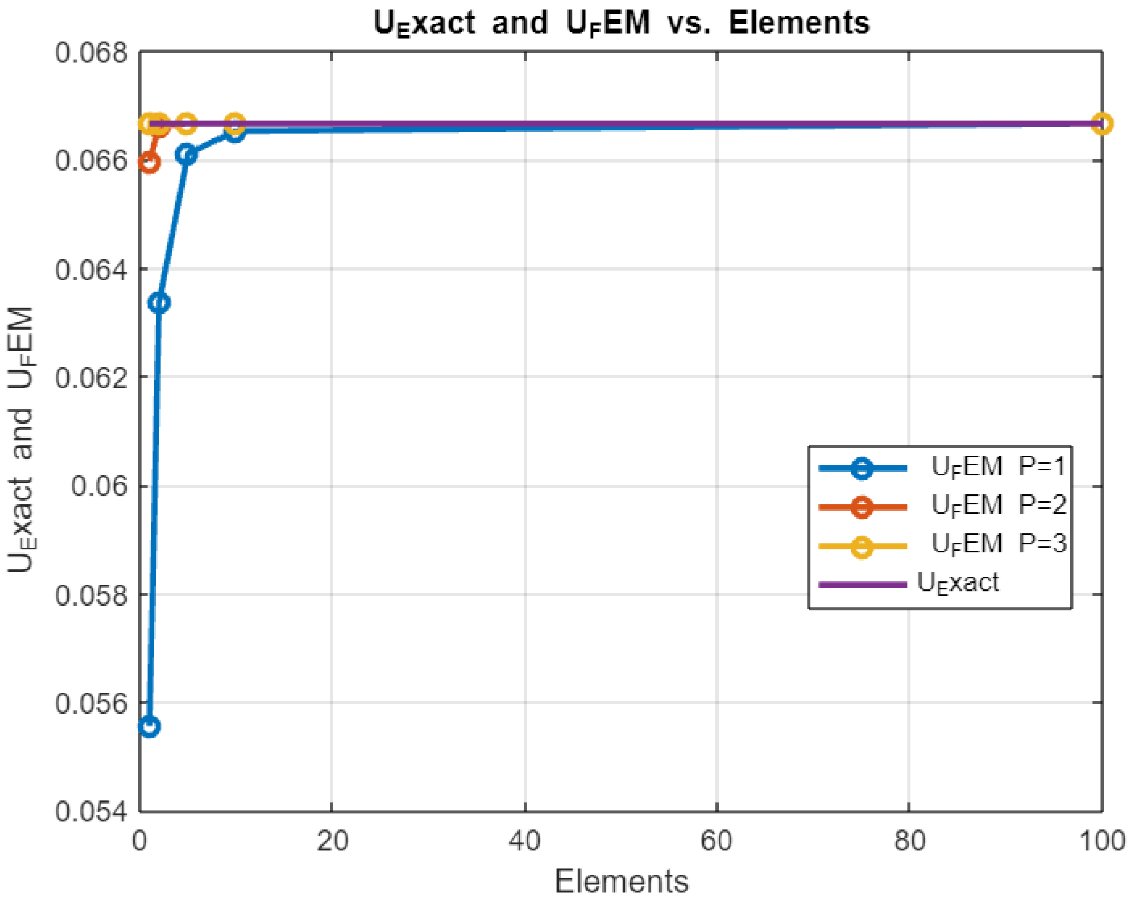




Case-2

$AE(x) = 1 \text{ and } c(x) = 0 \text{ with } u(x)|_{x=0} = 0 \text{ and } \left.\frac{du}{dx}\right|_{x=1} = 0 \;\; T(x) = \mathrm{x}$





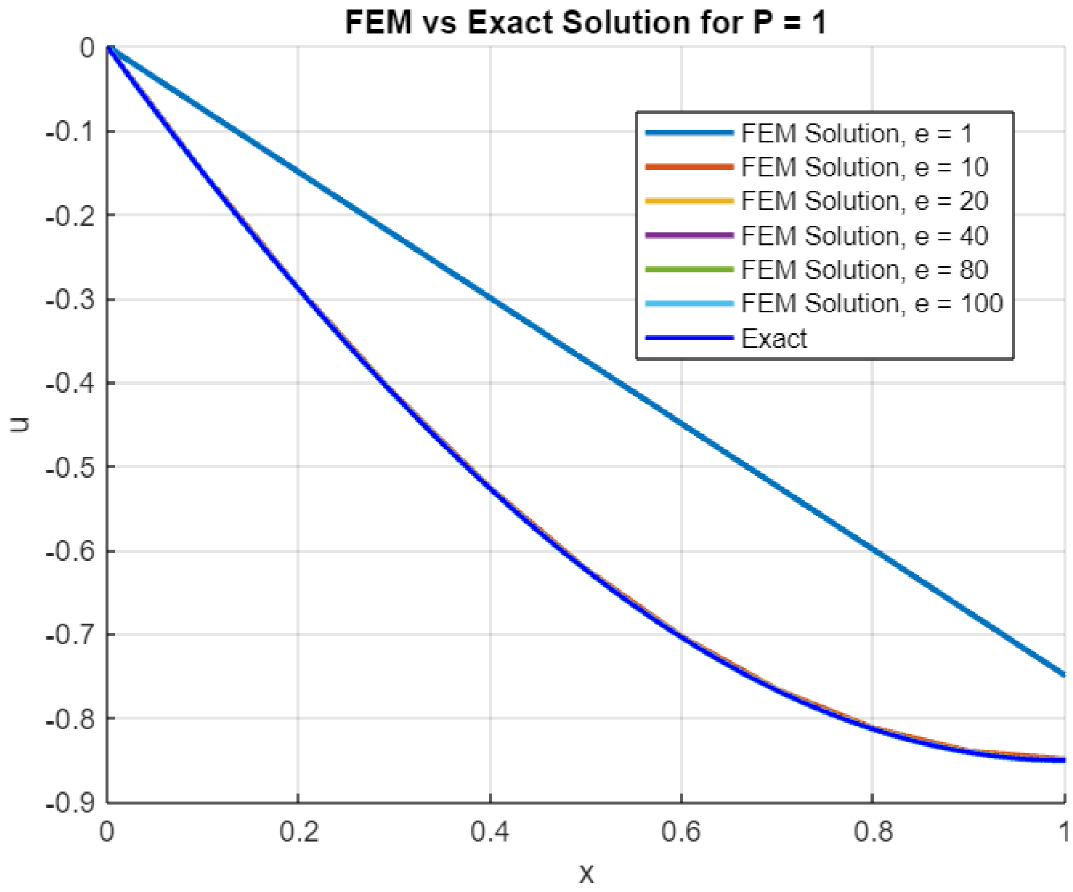
Question 3

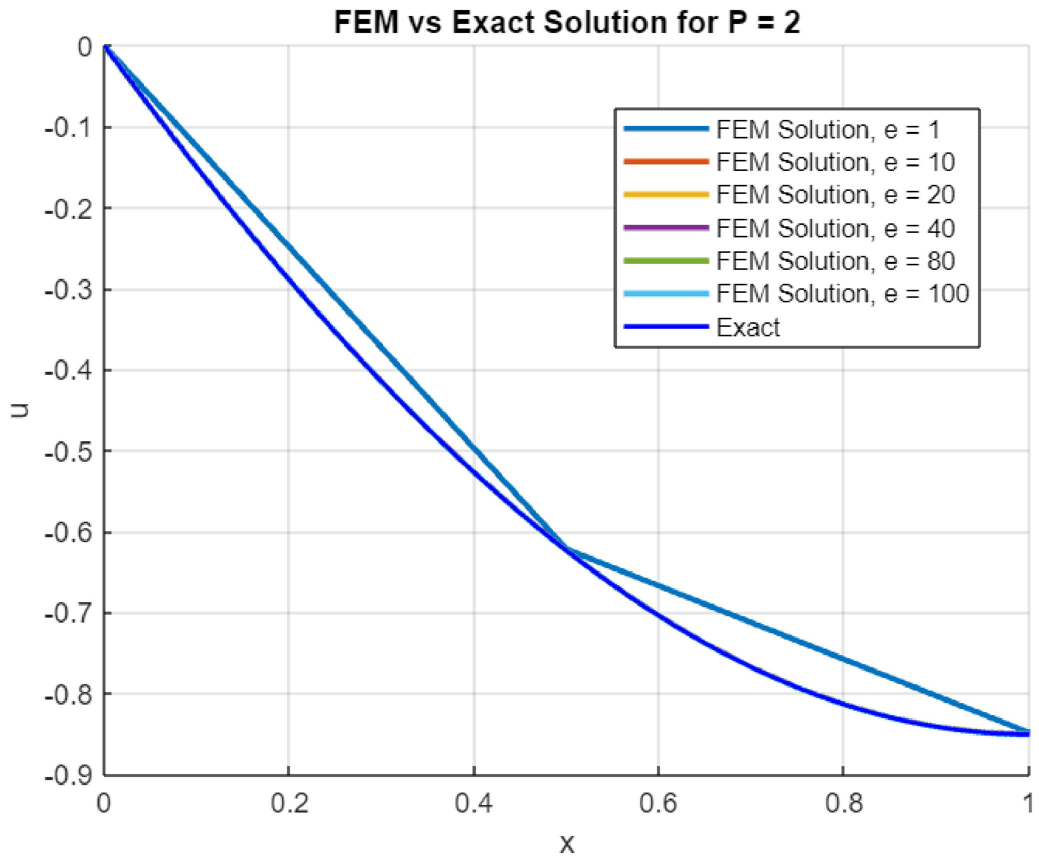
Take $AE(x) = 1, c(x) = 1$ and $T(x) = 1$ with $u(x)|_{x=0} = 0$ and $\left.\frac{du}{dx}\right|_{x=1} = 0$. Obtain the finite element solution with linear, quadratic, cubic and quartic elements respectively for 1, 10, 20, 40, 80 and 100 number of elements. For these cases:

a. Plot the exact and finite element solutions together for these cases.
b. Plot the error in the solution for these cases.
c. Plot the strain energy of the finite element and exact solution as a function of number of elements.
d. Plot the strain energy of the error as a function of number of elements.
e. Plot the log of the relative error in the energy norm versus the log of number of elements.
f. Try to estimate the convergence rate. Discuss the results.

Solution:

a. Plot the exact and finite element solutions together for these cases.




FEM vs Exact Solution for P = 2
FEM Solution, e = 1
FEM Solution, e = 10
FEM Solution, e = 20
FEM Solution, e = 40
FEM Solution, e = 80
FEM Solution, e = 100
Exact
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x
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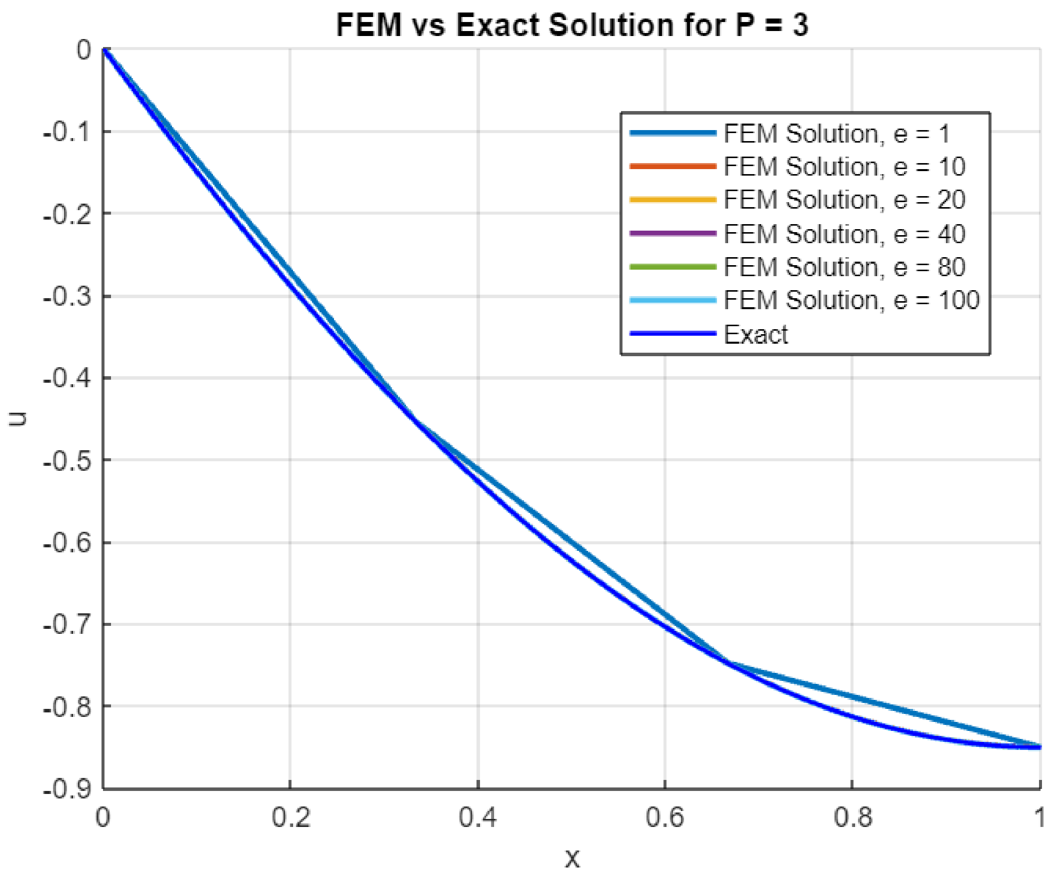
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
0.2
0.4
0.6
0.8
1


FEM vs Exact Solution for P = 3
FEM Solution, e = 1
FEM Solution, e = 10
FEM Solution, e = 20
FEM Solution, e = 40
FEM Solution, e = 80
FEM Solution, e = 100
Exact
u
x
0
-0.1
-0.2
-0.3
-0.4
-0.5
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-0.7
-0.8
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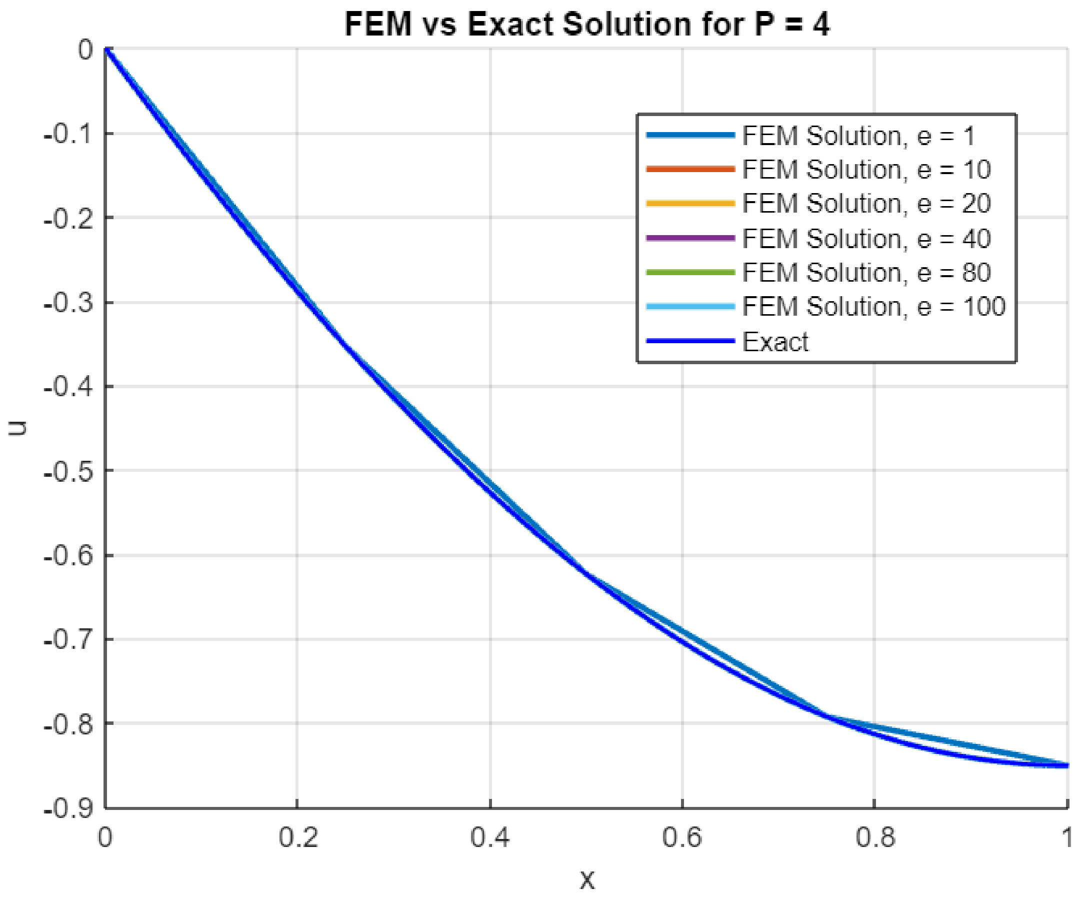
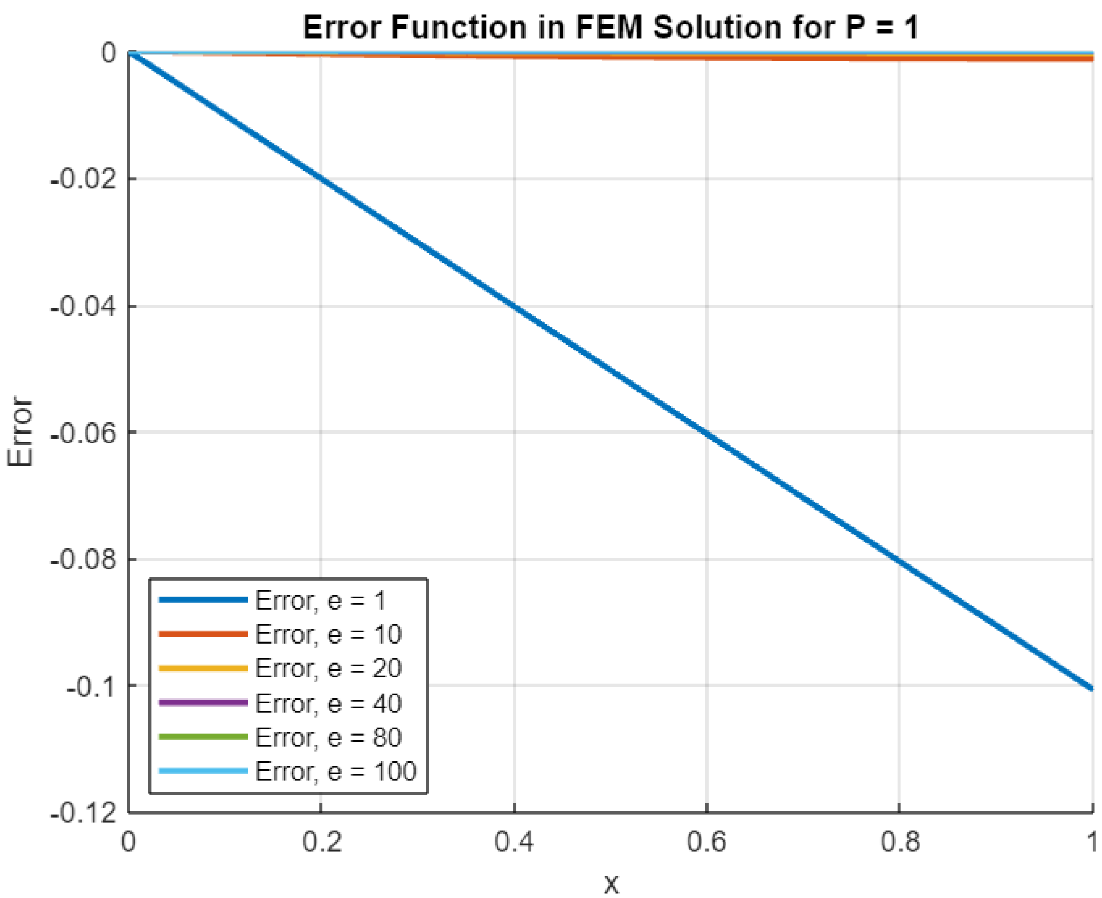
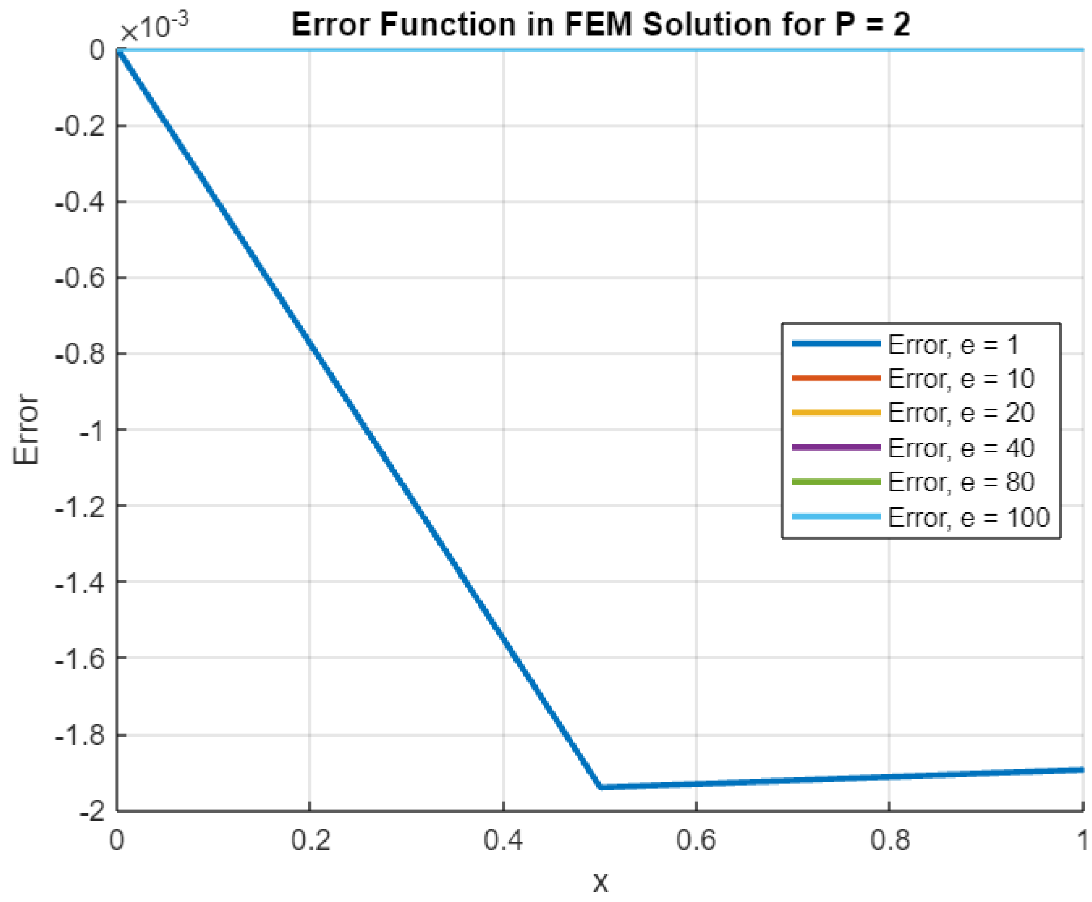
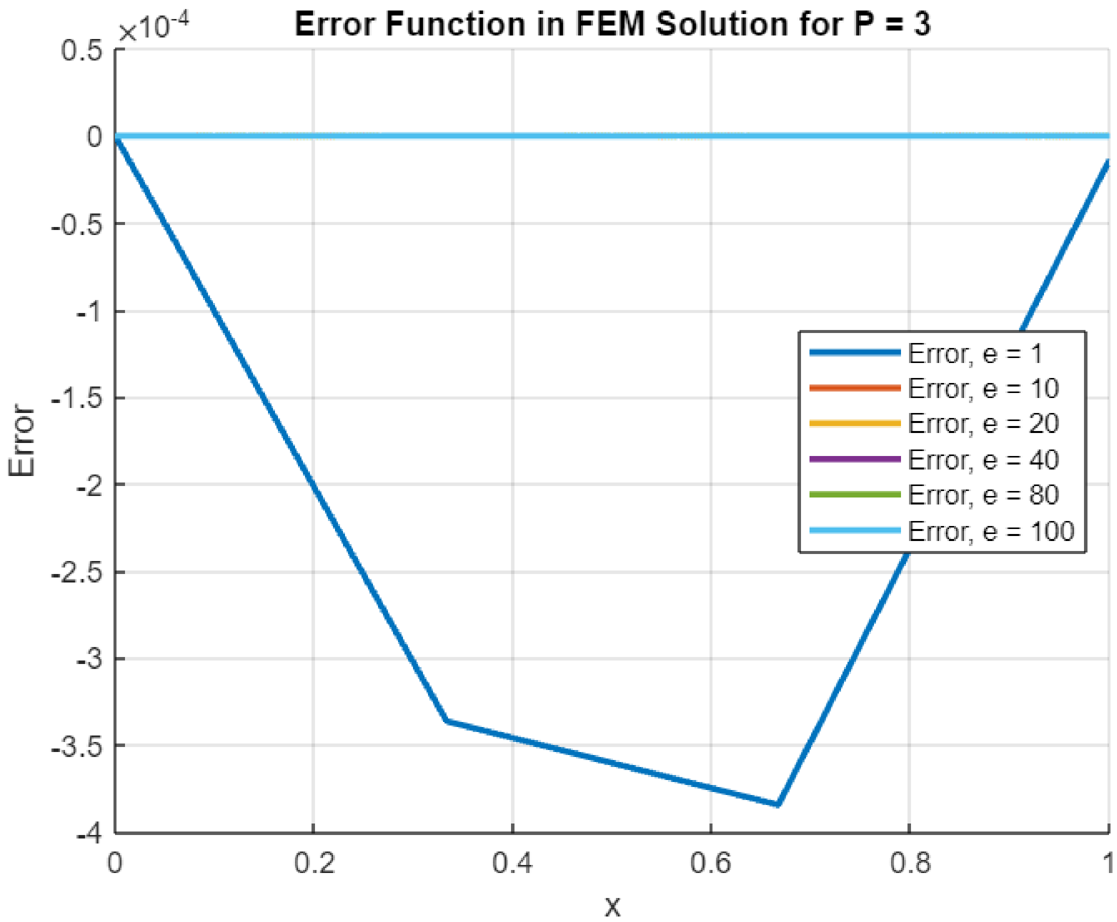
0.6
0.8
1



b. Plot the error in the solution for these cases.




Error Function in FEM Solution for P = 2
×10-3
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
Error
0
0.2
0.4
0.6
0.8
1
x
Error, e = 1
Error, e = 10
Error, e = 20
Error, e = 40
Error, e = 80
Error, e = 100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×10-4
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0
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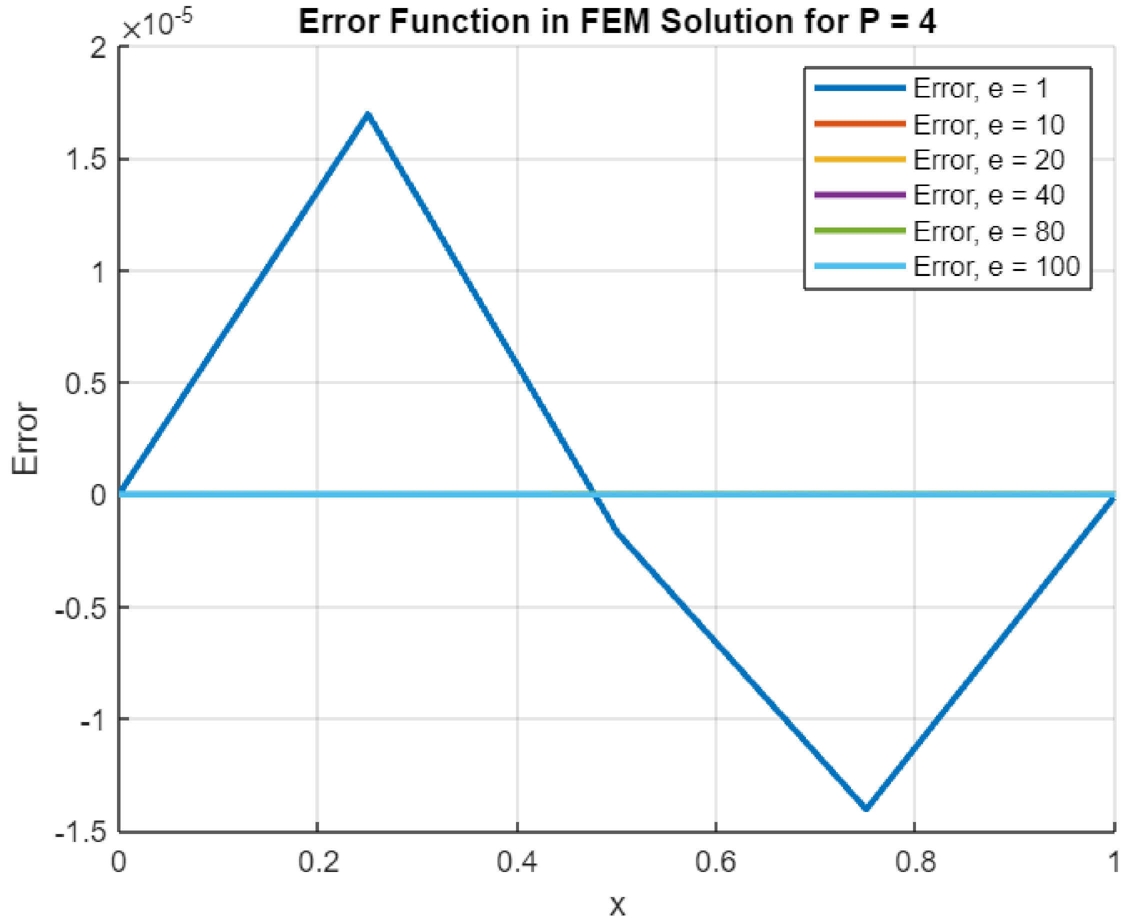
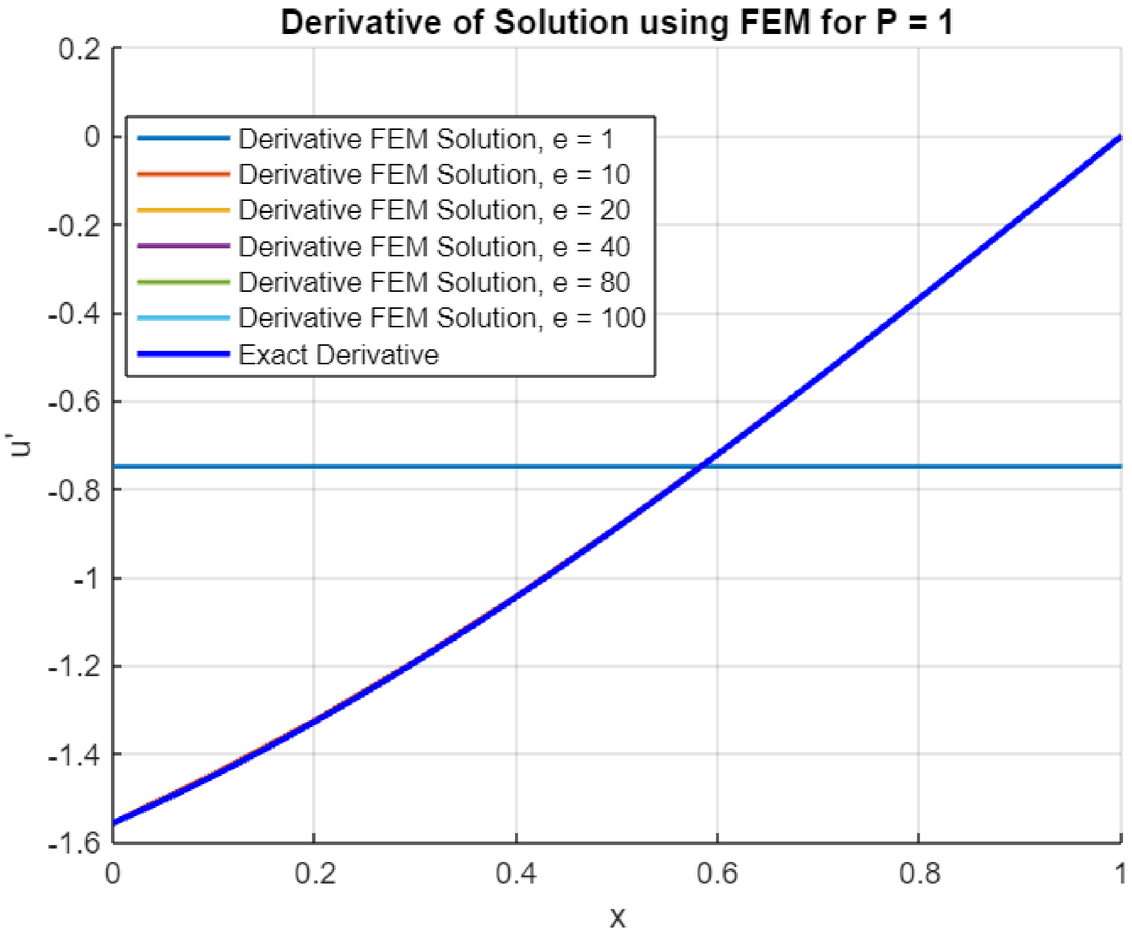
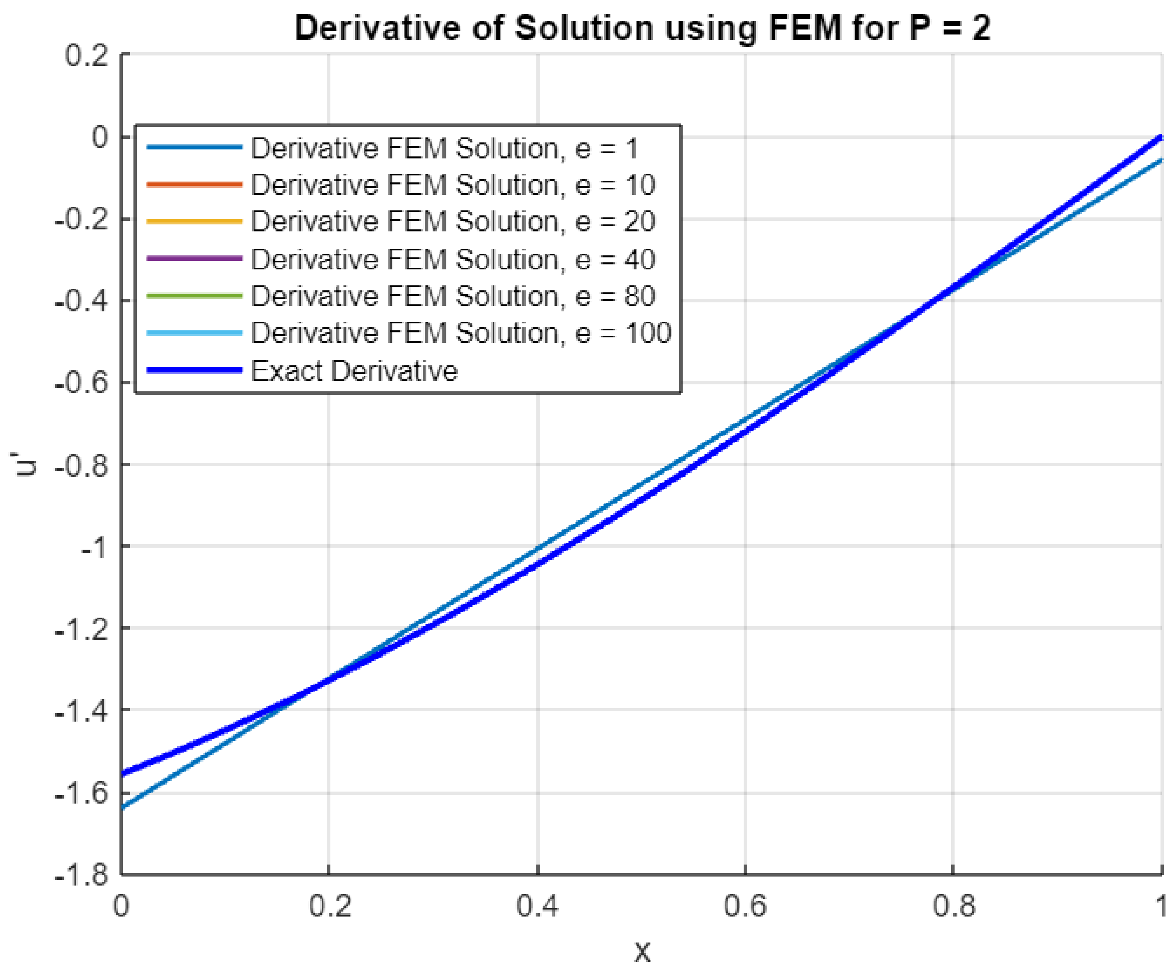
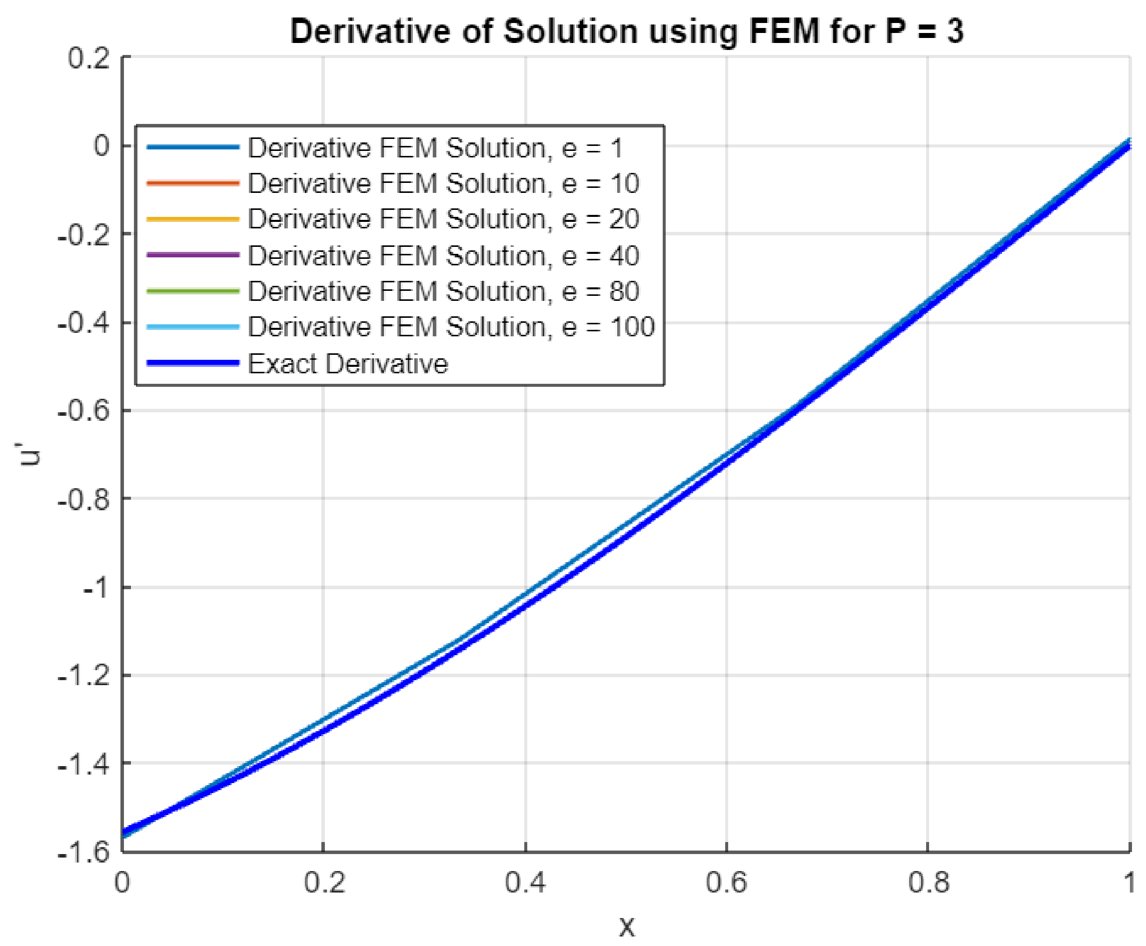
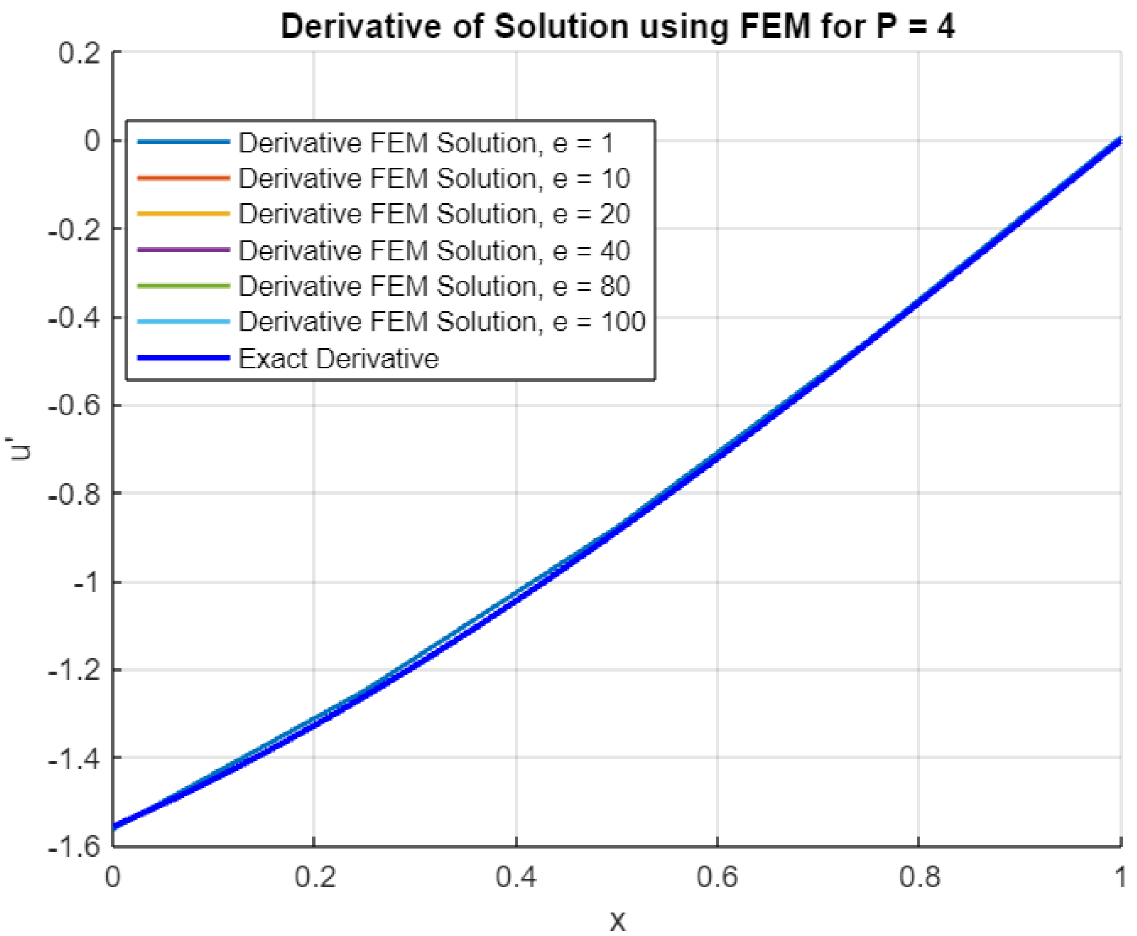
-1
-1.5
-2
-2.5
-3
-3.5
-4
Error
0
0.2
0.4
0.6
0.8
1
x
Error, e = 1
Error, e = 10
Error, e = 20
Error, e = 40
Error, e = 80
Error, e = 100


Error Function in FEM Solution for P = 4
×10^-5
Error
x
Error, e = 1
Error, e = 10
Error, e = 20
Error, e = 40
Error, e = 80
Error, e = 100
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1

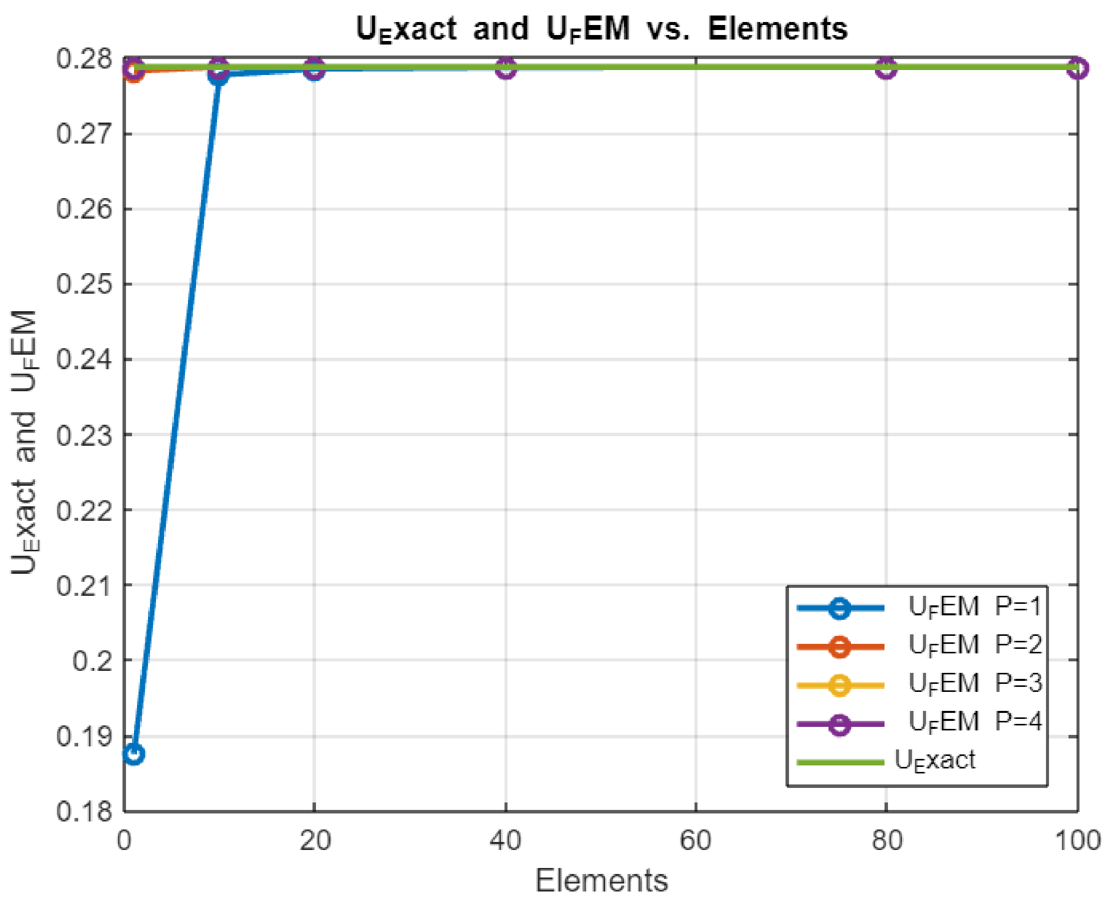

Derivative of Solution using FEM for P = 1
u'
x
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Derivative FEM Solution, e = 100
Exact Derivative
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0
0.2
0.4
0.6
0.8
1


Derivative of Solution using FEM for P = 2
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Derivative FEM Solution, e = 100
Exact Derivative
u'
x
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
0
0.2
0.4
0.6
0.8
1

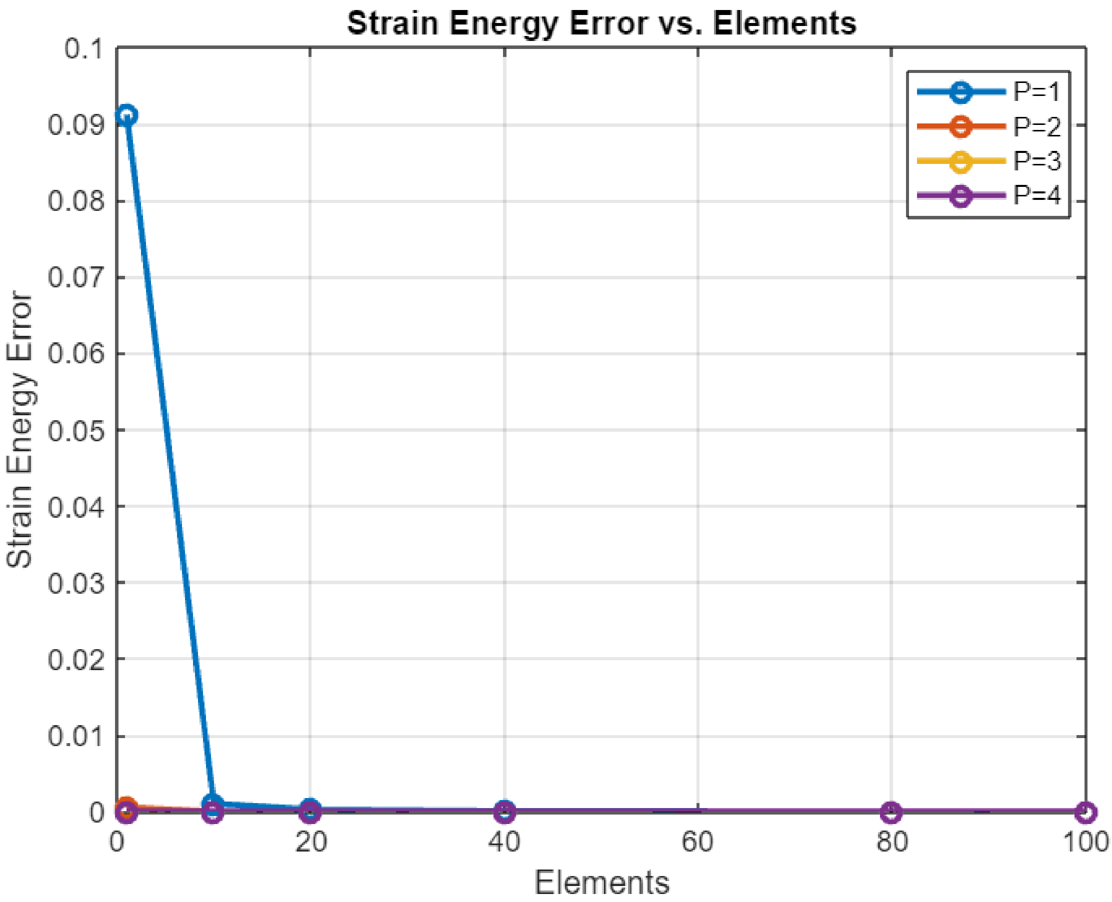

Derivative of Solution using FEM for P = 3
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Derivative FEM Solution, e = 100
Exact Derivative
u'
x
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
0
0.2
0.4
0.6
0.8
1



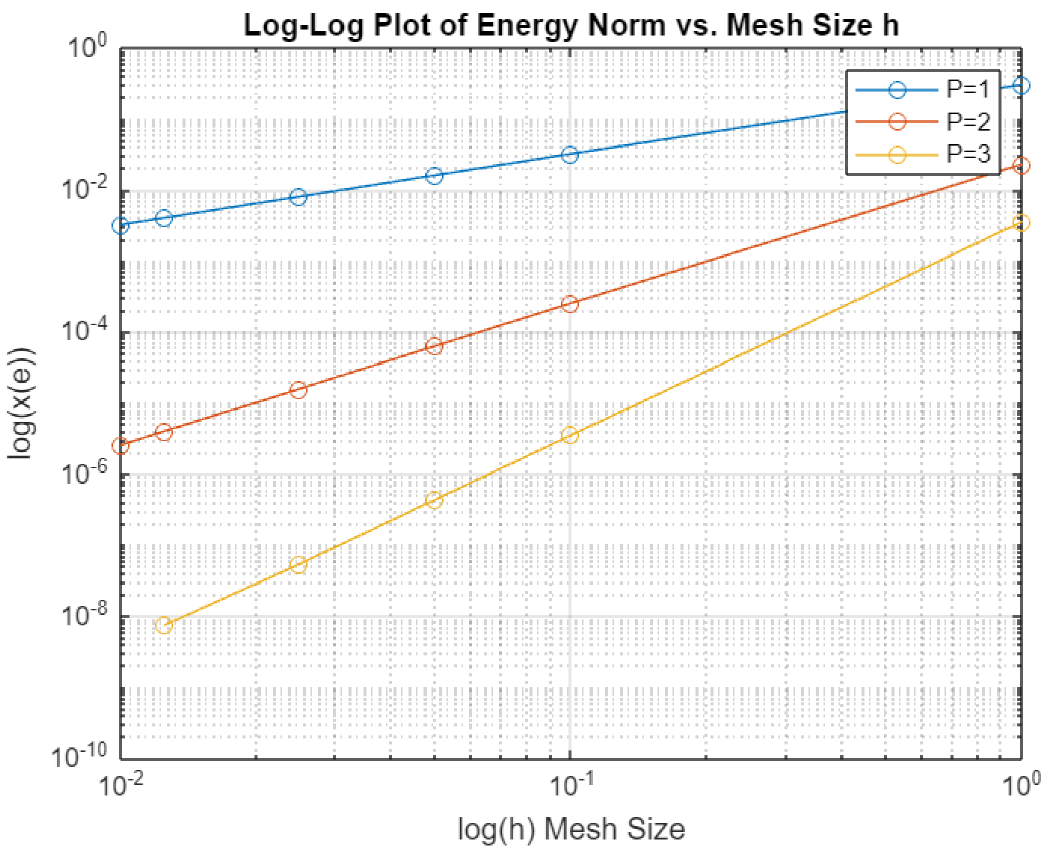
c. Plot the strain energy of the finite element and exact solution as a function of number of elements.

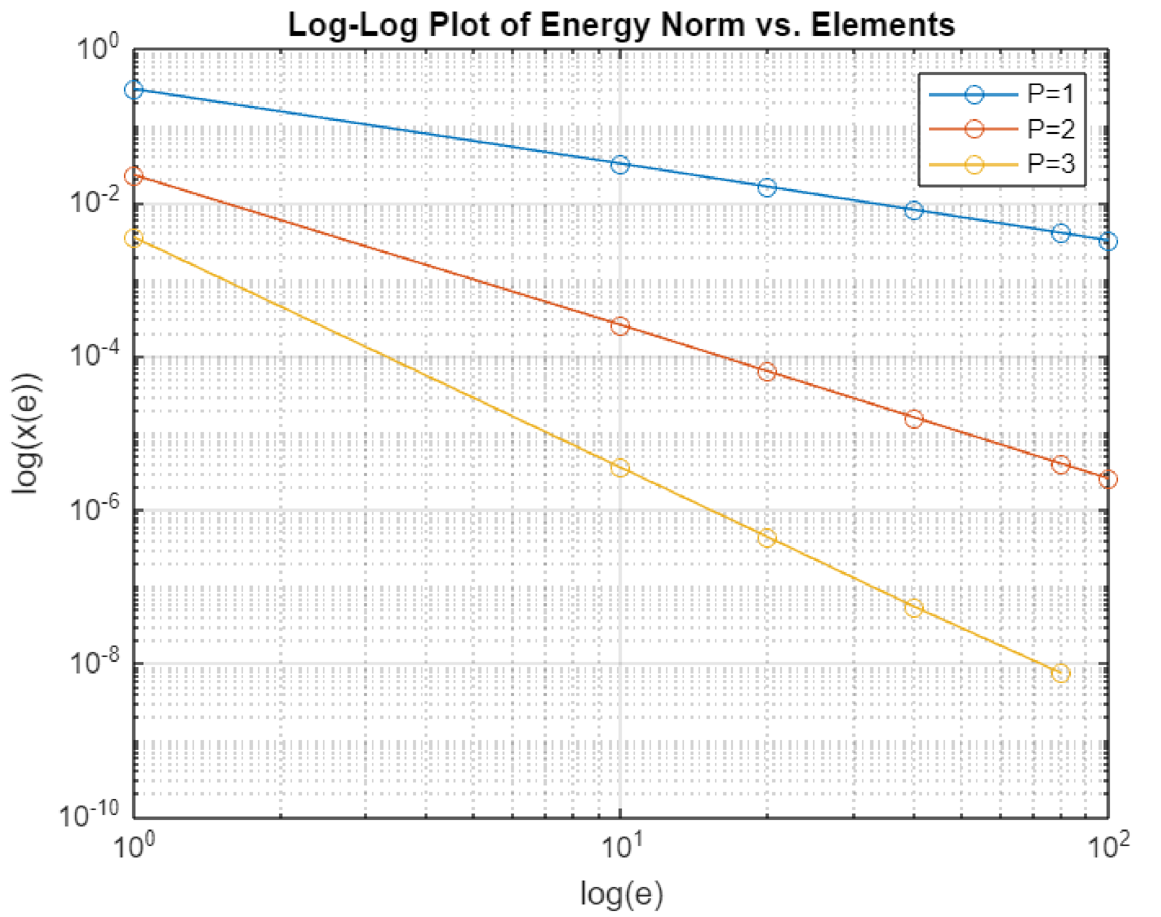


d. Plot the strain energy of the error as a function of number of elements .

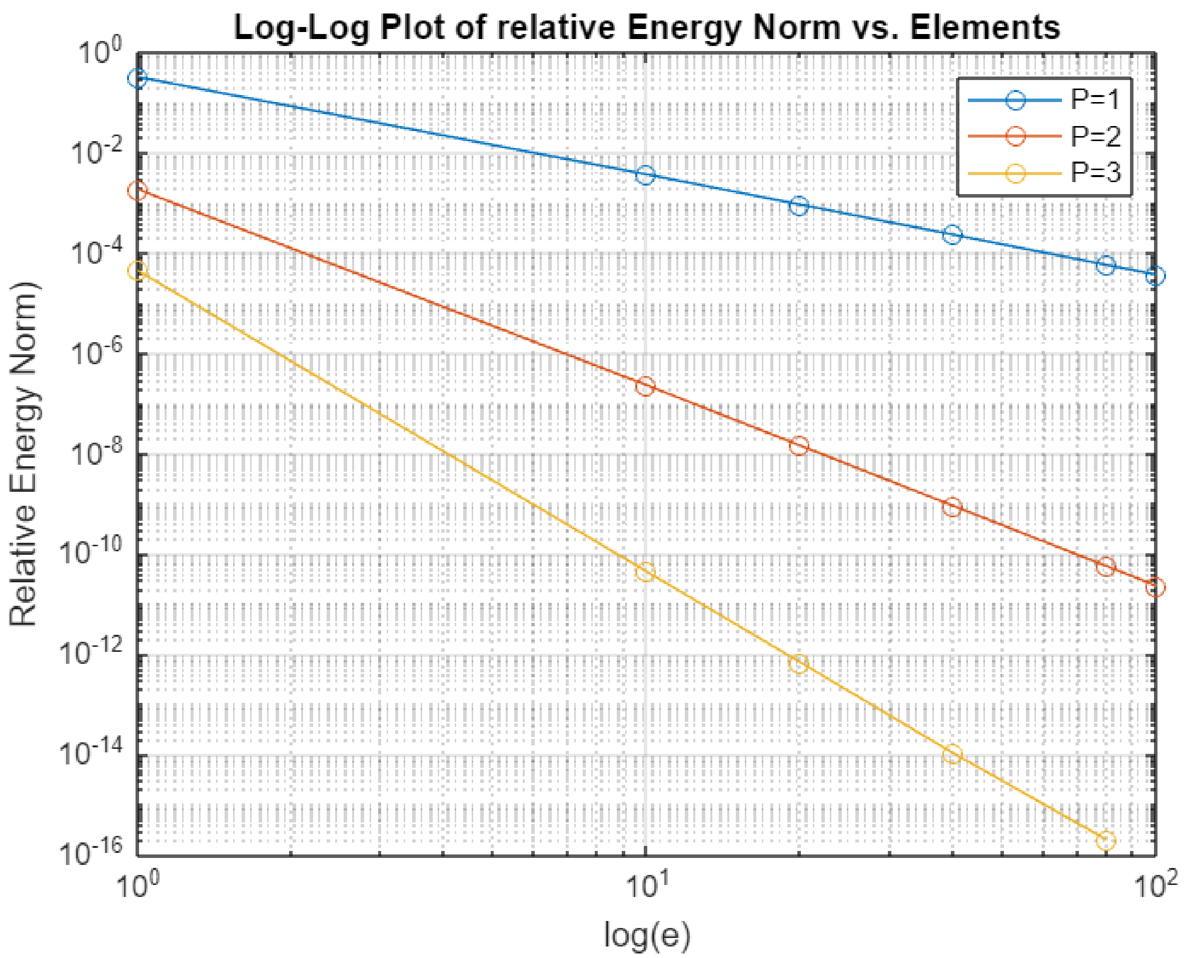


***e. Plot the log of the relative error in the energy norm versus the log of number of elements*.**





Log-Log Plot of Energy Norm vs. Elements
P=1
P=2
P=3
log(x(e))
log(e)



Log-Log Plot of relative Energy Norm vs. Elements
P=1
P=2
P=3
Relative Energy Norm)
log(e)

f. Try to estimate the convergence rate. Discuss the results.

Rate of Convergence for Different P Values

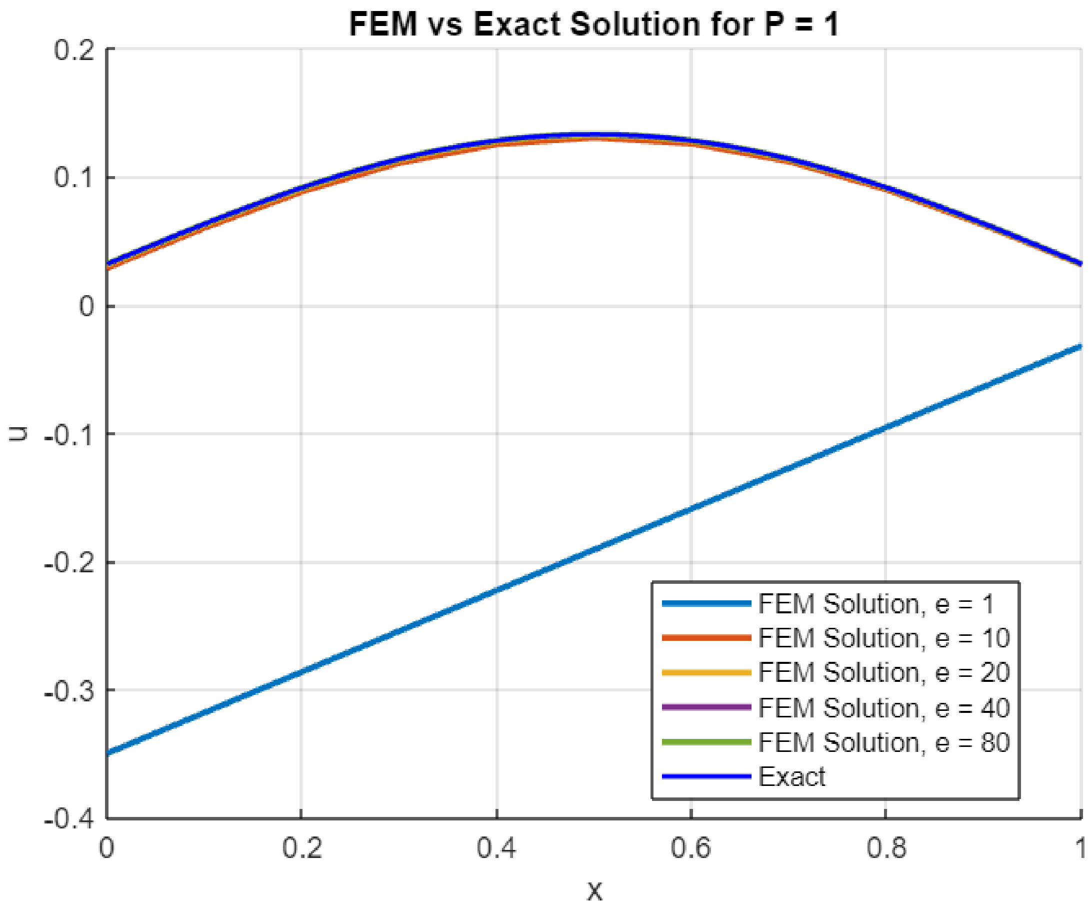
P	Rate of Convergence
1	0.98655
2	1.9783
3	Inf (exact solution)
4	Inf (exact solution)

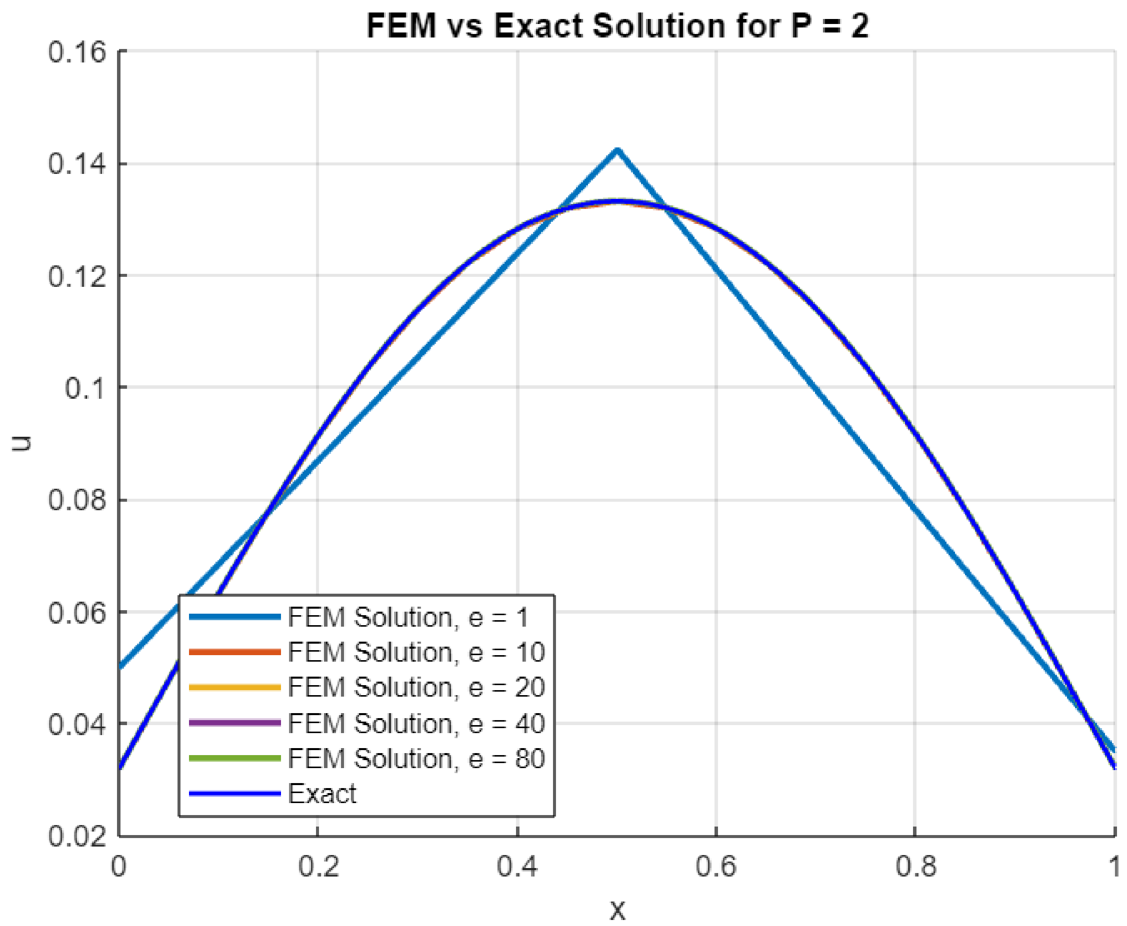
Question 4

Take $AE(x) = 1, c(x) = 0$ and $T(x) = sin\ \frac{\pi}{L}x$ with $AE\left.\frac{du}{dx}\right|_{x=0} = \frac{1}{\pi}$ and $AE\left.\frac{du}{dx}\right|_{x=1} = k_L(\delta_L - u(L))$ with $k_L = 10$ and $\delta_L = 0$. Then repeat the exercise given a) through f) in Point 3.

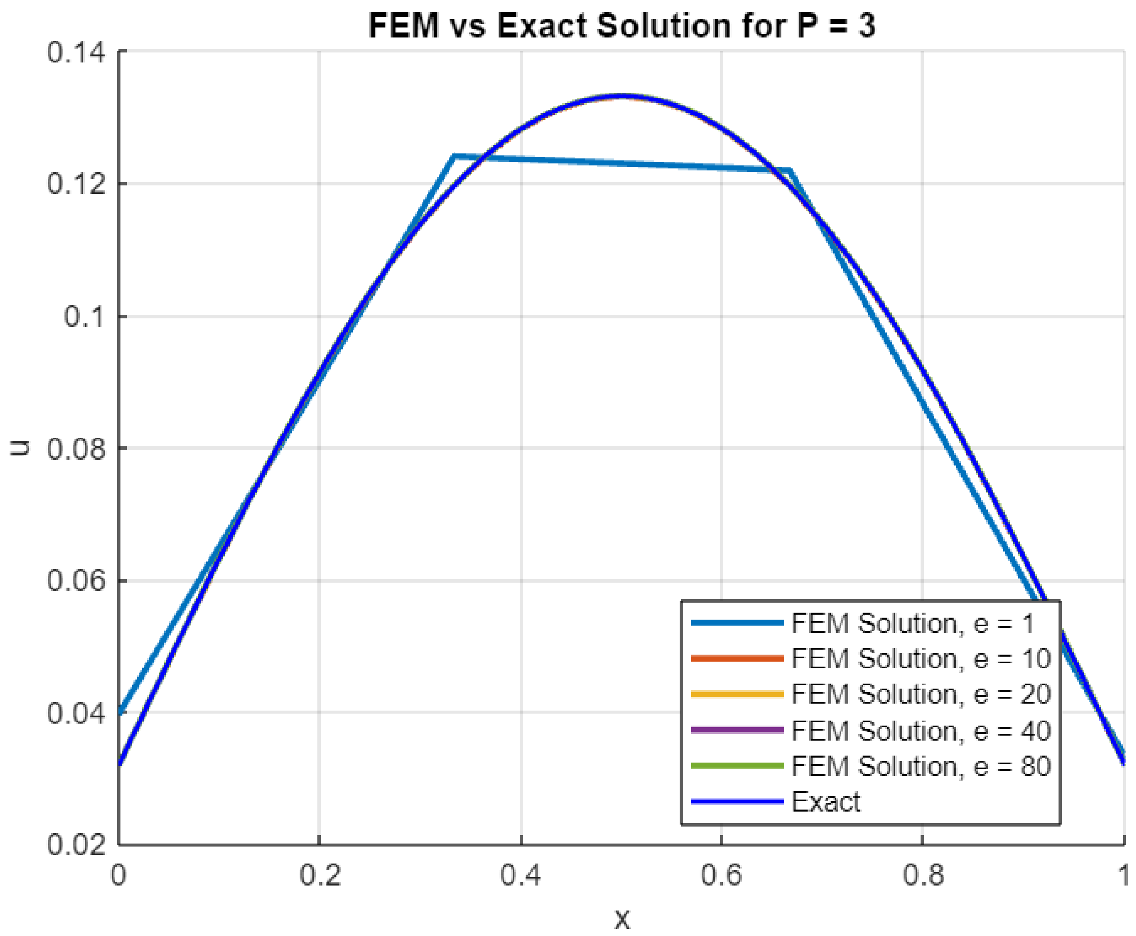
a. Plot the exact and finite element solutions together for these cases.

Solution:



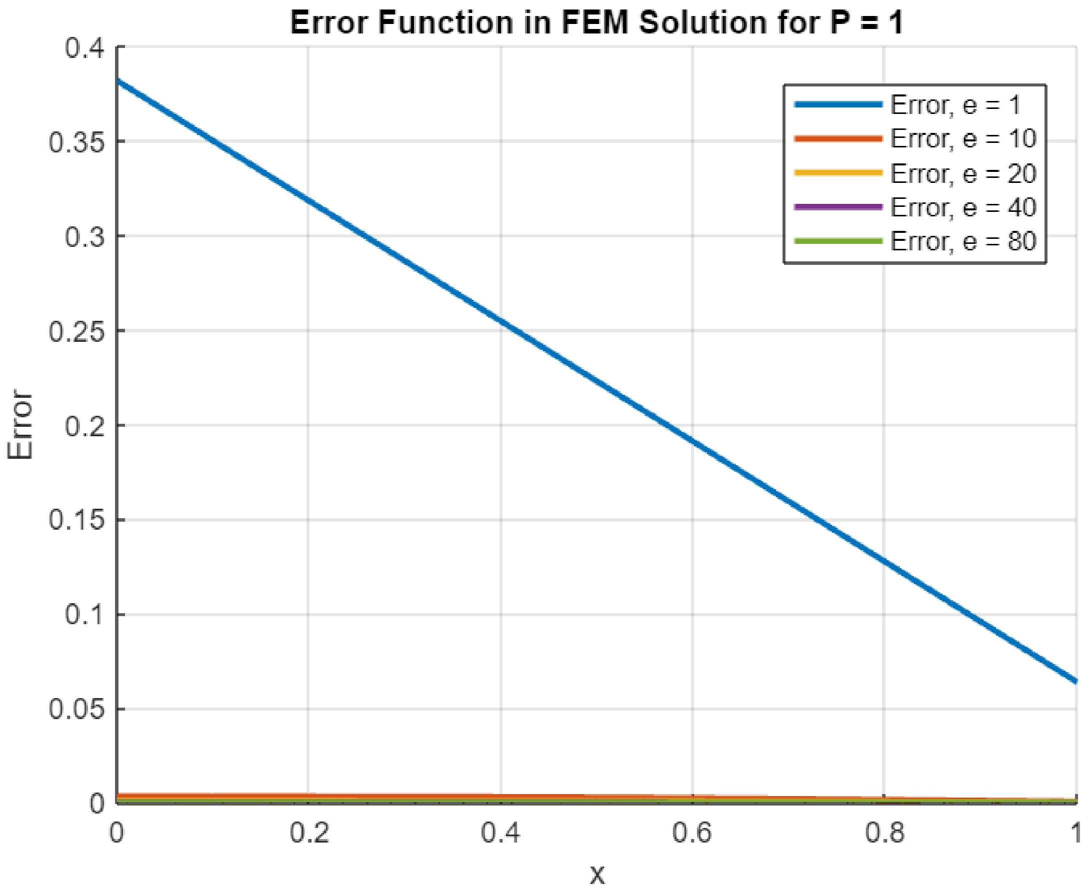


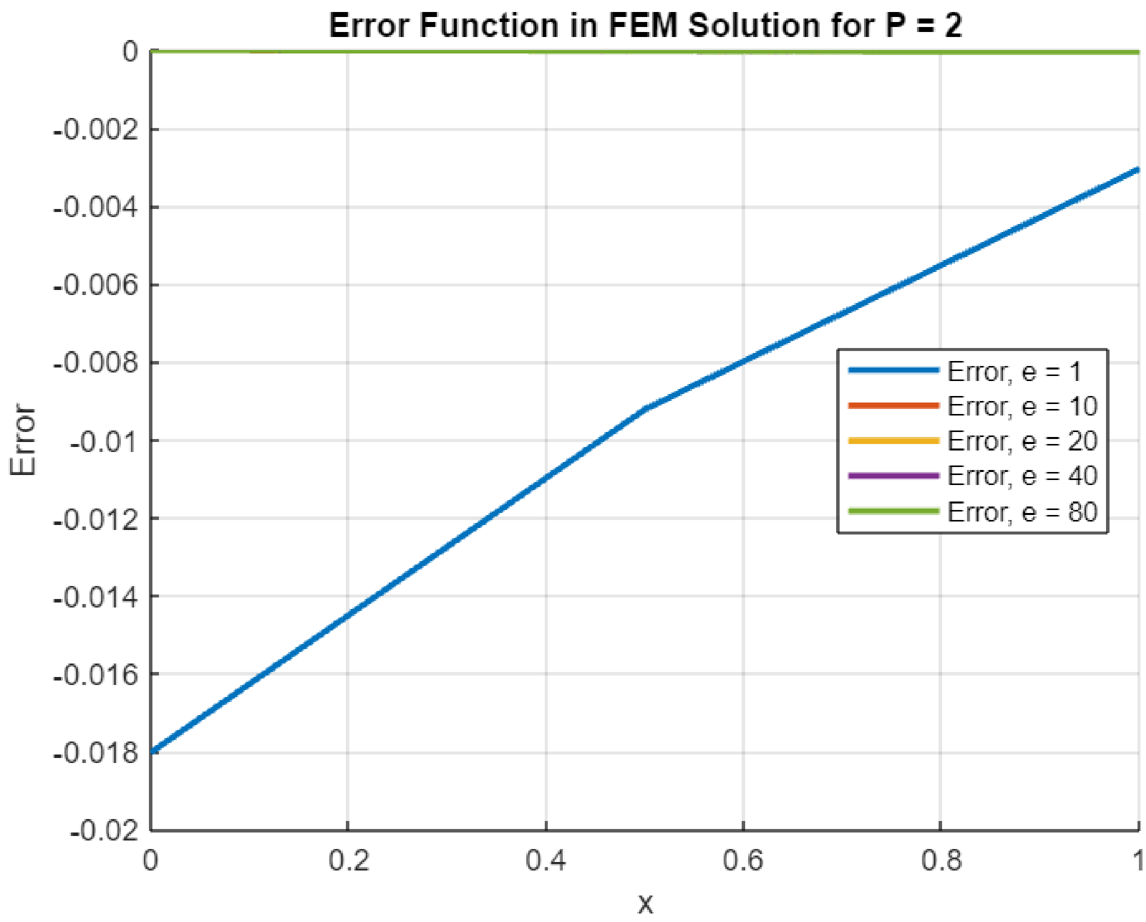
FEM vs Exact Solution for P = 2
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
u
0
0.2
0.4
0.6
0.8
1
x
FEM Solution, e = 1
FEM Solution, e = 10
FEM Solution, e = 20
FEM Solution, e = 40
FEM Solution, e = 80
Exact

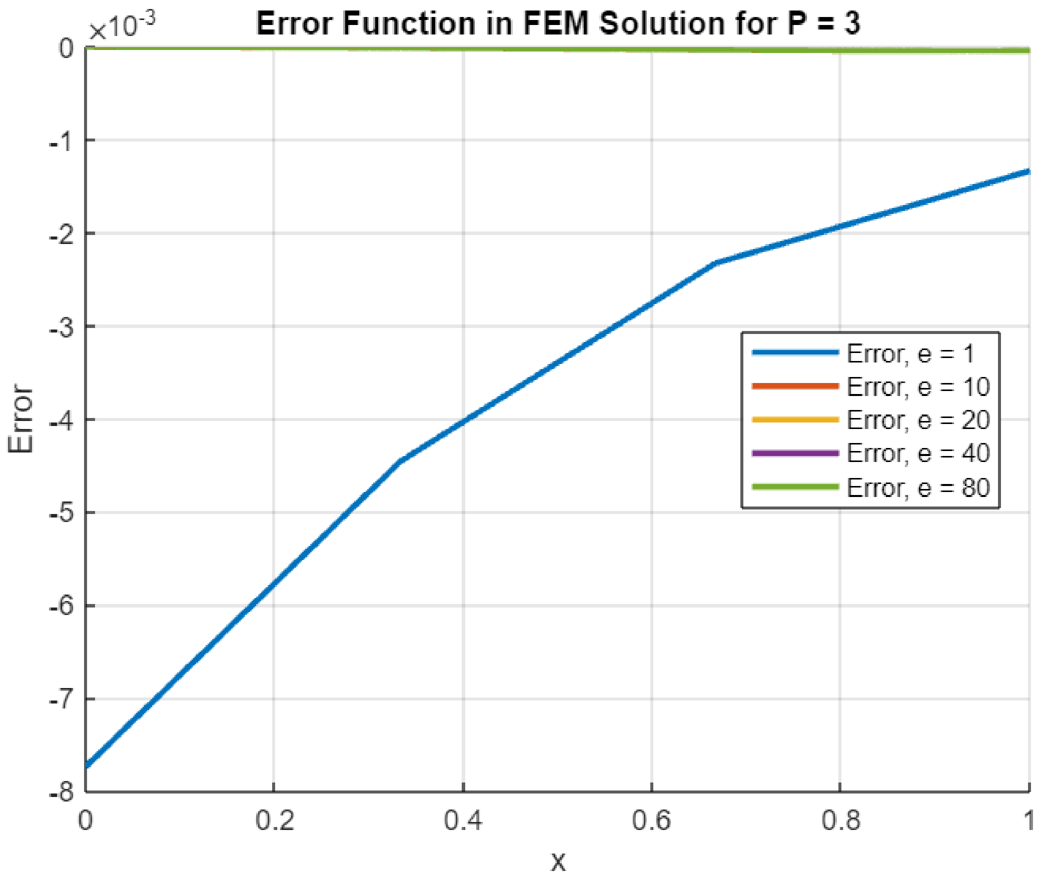


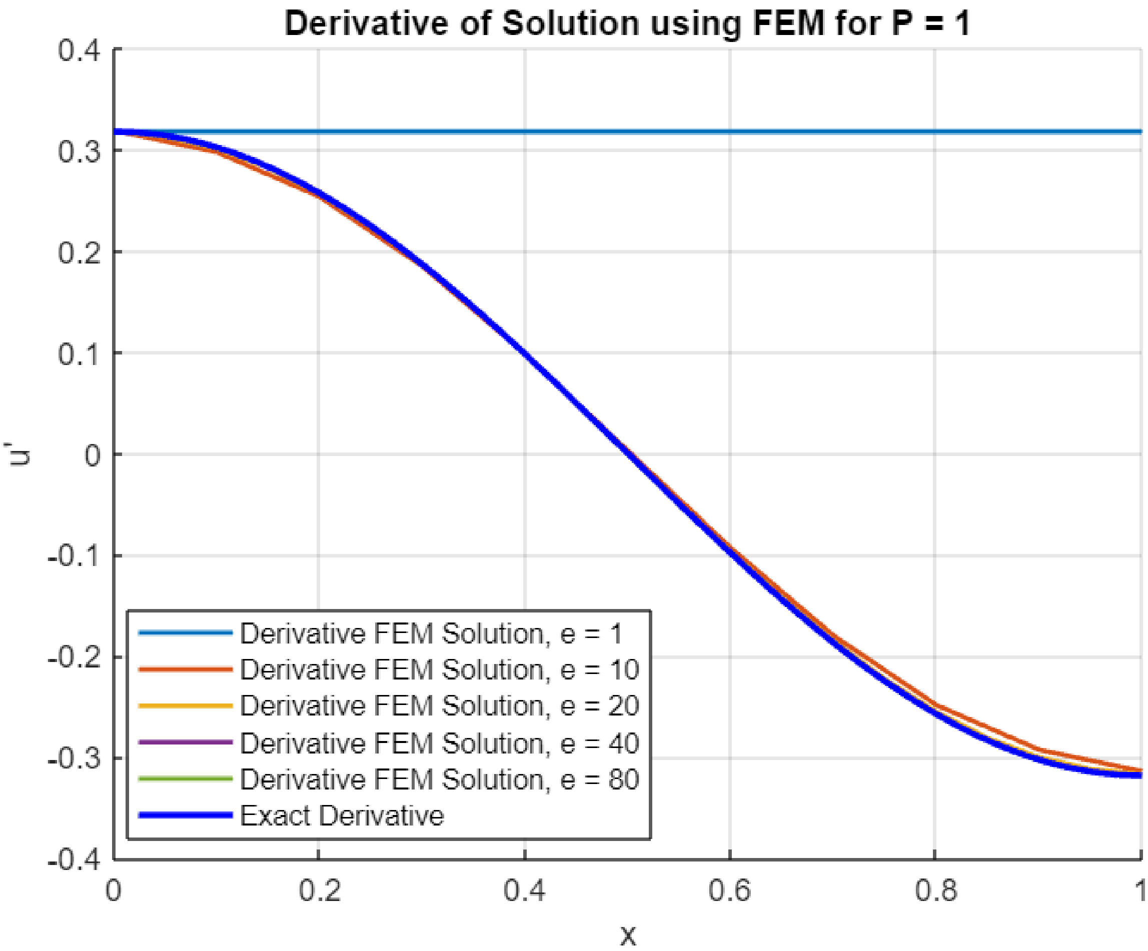
FEM vs Exact Solution for P = 3
0.14
0.12
0.1
0.08
0.06
0.04
0.02
u
0
0.2
0.4
0.6
0.8
1
x
FEM Solution, e = 1
FEM Solution, e = 10
FEM Solution, e = 20
FEM Solution, e = 40
FEM Solution, e = 80
Exact

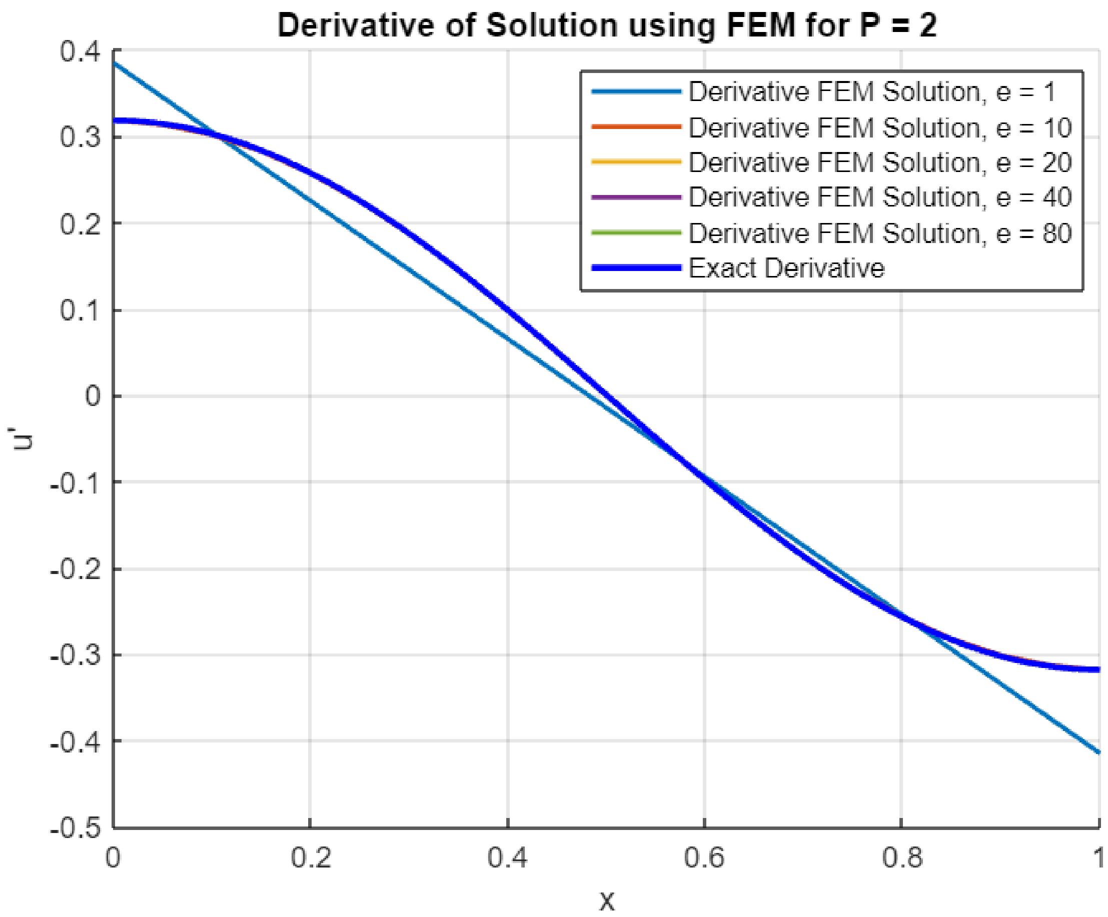
b. Plot the error in the solution for these cases.

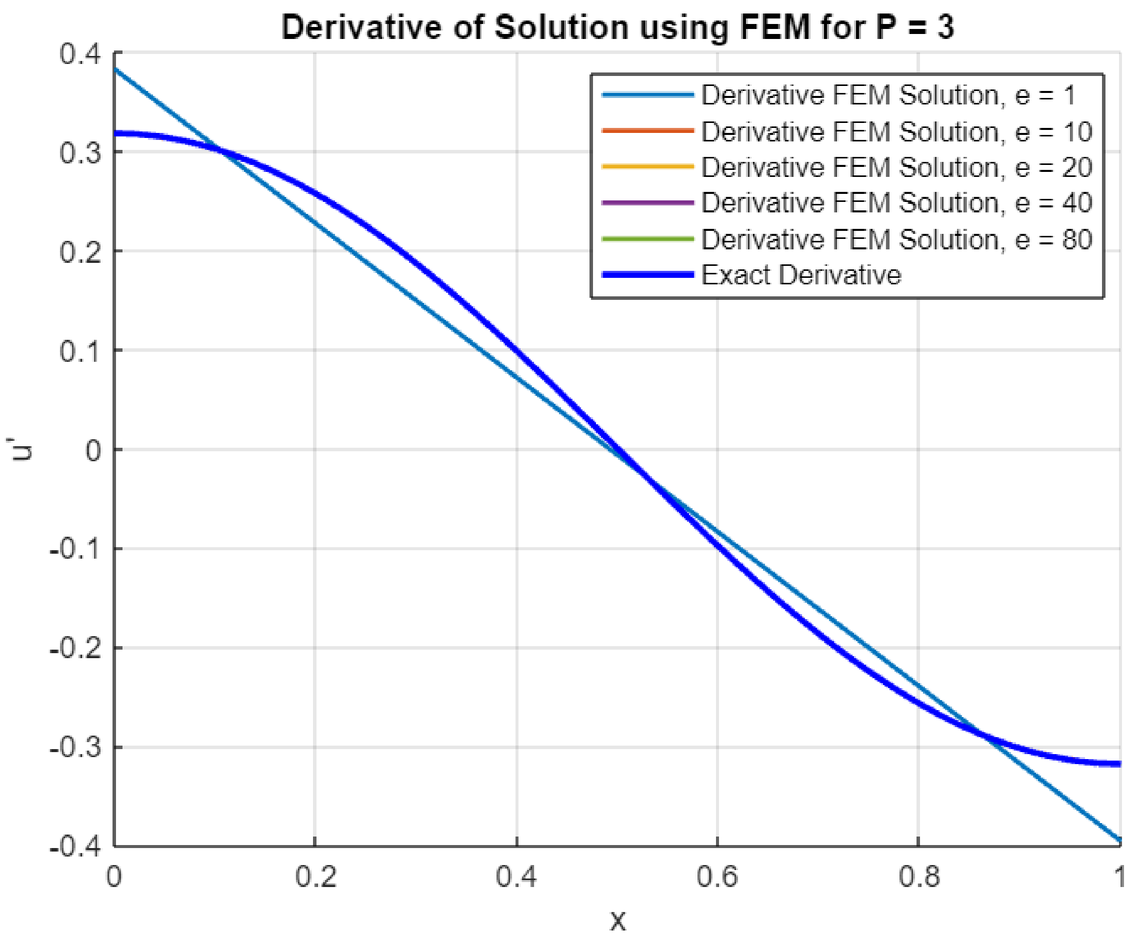




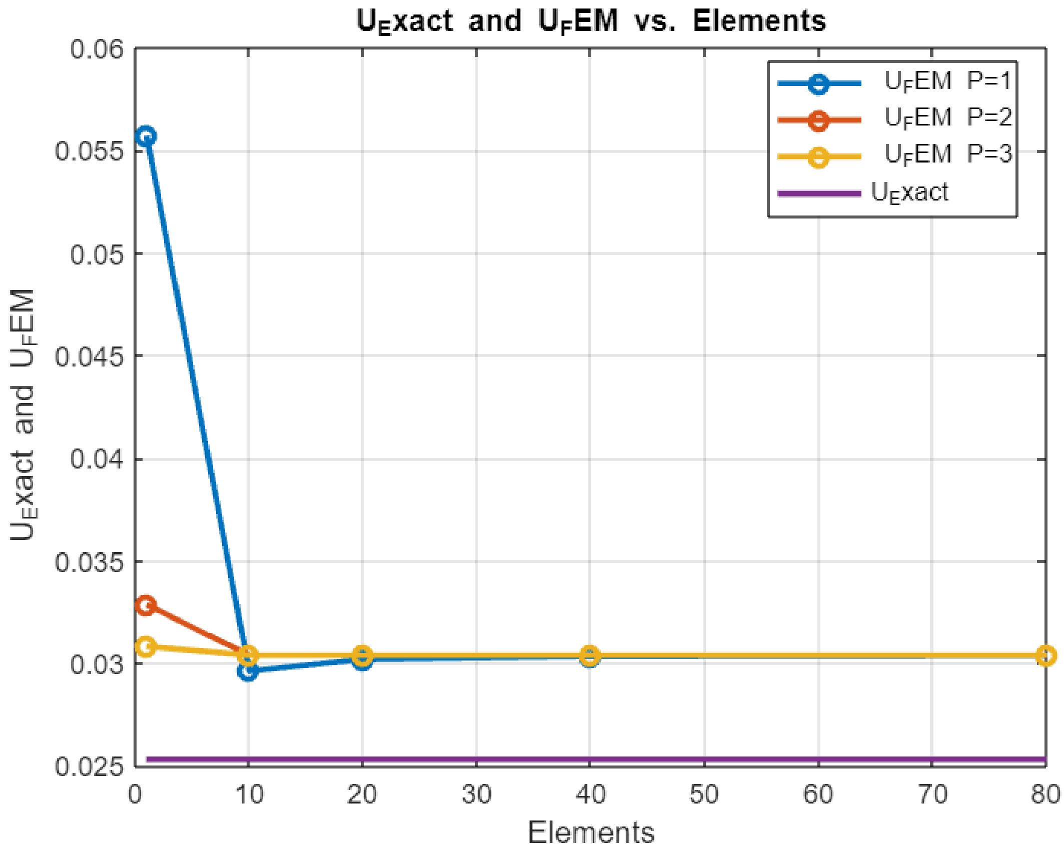

Error Function in FEM Solution for P = 3
×10^-3
0
-1
-2
-3
-4
-5
-6
-7
-8
Error
0
0.2
0.4
0.6
0.8
1
x
Error, e = 1
Error, e = 10
Error, e = 20
Error, e = 40
Error, e = 80


Derivative of Solution using FEM for P = 1
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
u'
0
0.2
0.4
0.6
0.8
1
x
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Exact Derivative

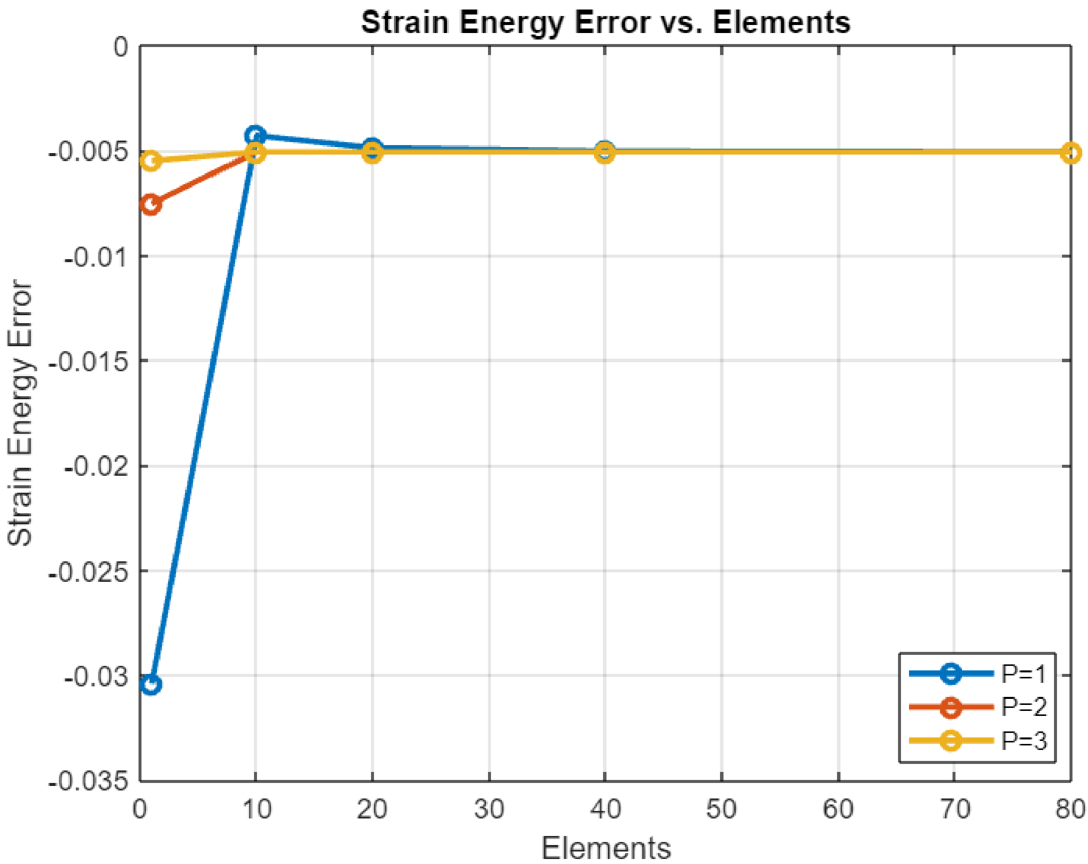

Derivative of Solution using FEM for P = 2
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Exact Derivative
u'
x


Derivative of Solution using FEM for P = 3
Derivative FEM Solution, e = 1
Derivative FEM Solution, e = 10
Derivative FEM Solution, e = 20
Derivative FEM Solution, e = 40
Derivative FEM Solution, e = 80
Exact Derivative
u'
x

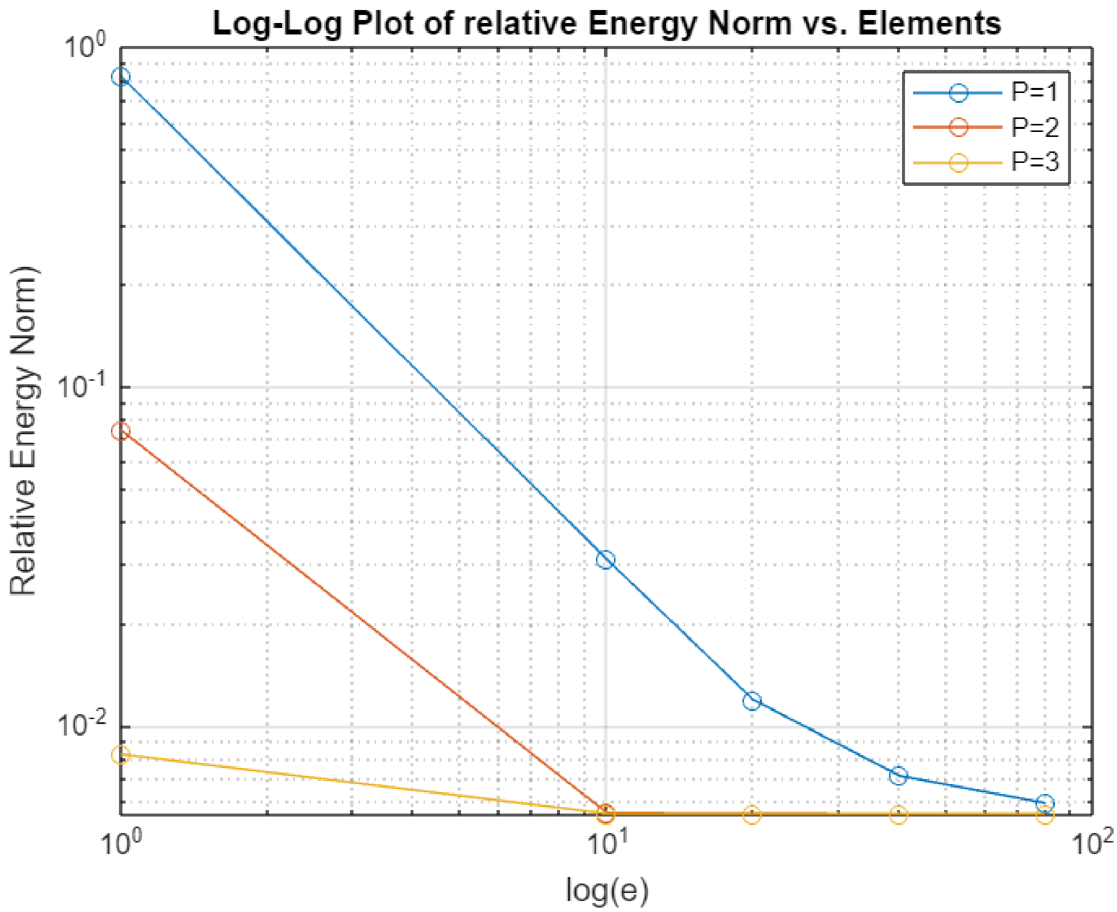
c. Plot the strain energy of the finite element and exact solution as a function of number of elements.



d. Plot the strain energy of the error as a function of number of elements .



e. Plot the log of the relative error in the energy norm versus the log of number of elements.



f. Try to estimate the convergence rate. Discuss the results.

Rate of Convergence for Different P Values

P	Rate of Convergence
1	0.98655
2	1.9783
3	2.8723
4	3.7345