PROJECT

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Course : **Computational Methods [AE703]**

Question: Consider the three-element system to solve the differential equation with the source term $f(x)$,

$$\frac{d^2u}{dx^2} - 2u = f(x)\ 0 < x < 1$$
$$f(x) = 4x^2 - 2x - 4$$

subject to the following boundary conditions:

(i) Dirichlet boundary conditions:

$$u = 0 \qquad \text{at } x = 0$$
$$u = -1 \qquad \text{at } x = 1$$

(ii) Neumann boundary conditions:

$$u = 0 \qquad \text{at } x = 0$$
$$\frac{du}{dx} = -3 \qquad \text{at } x = 1$$

whose exact solution is given by $u = -2x^2 + x$.

(a) Solve the above-mentioned problem with FDM, FEM, FVM for both (i) & (ii)

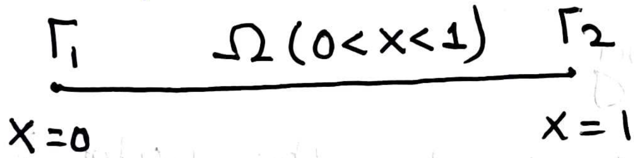
(b) Prepare a report with all mathematical formulations and results.

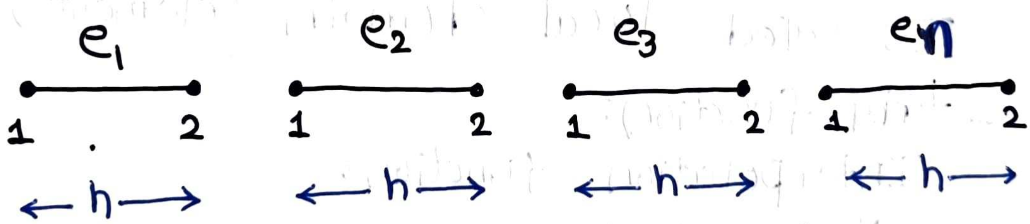
(c) Program the same and submit along with your report.

Finite Element Method

In FEM, the entire domain is divided into 3 element, for this problem, And the approximate function called shape function is used to Approximate the distribution of the given function over the entire element

Let the domain be divided into sub domains; say local elements $(e = 1, 2, 3)$ in the example





Assume that variable $u^e(x)$ is a linear function of x

$$u^e(x) = \alpha_1 + \alpha_2 x$$

For some element e, at $x=0$ (node 1)
$x=h$ (node 2)

the nodal values of variables are u_1^e and u_2^e

Solving for constant α_1 and α_2, we are getting

$$u^e(x) = \left(1-\frac{x}{h}\right)u_1^e + \left(\frac{x}{h}\right)u_2^e$$

$$= \phi_N^e(x)\, u_N^e \qquad (N=1,2)$$

Compact notation

- where, the index N Dummy variable, Implies Summing
- u_N^e Represents the nodal value of u at the local node N for the element e
- ϕ_N^e is a function of x for 1D problems

Is called local domain (element)

- trial functions
- Interpolation functions
- Shape functions
- basis functions

$$\phi_1^e(x) = 1-\frac{x}{h}$$

$$0 \le \phi_N^e(x) \le 1$$

$$\phi_2^e(x) = \frac{x}{h}$$

The shape functions values Assume the value of 1 under the consider
0 at the other node

→ Their are many different ways to formulate Finile element equations.

→ One of the simplest approaches is known as Galerkin Method.

→ The basic idea is to Construct an inner product of the Residual R^e of the local form of the governing equation with the test function chosen the same as the trail functions

$$(\phi_N^e(x), R^e)$$

$$\int_0^h \phi_N^e \left(a\frac{d^2u}{dx^2} + cu - f \right) dx = 0$$

where f, u and ϕ_N^e are functions of x

This Represents an Orthogonal projection of the Residual error into the subspace spanned by the test functions summed over the domain, which is then set equal to zero (implying that errors are minimized), leading to the best numerical approximation of the sol^n to the governing eq^n.

the source term 'f' may be linearly approximated in the form

$$f(x) = \phi_N^e f_N^e$$

For the element, we can write as

Dummy $M \rightarrow$ index 'm'

$$f^e = \phi_M^e f_M^e$$

$$u^e = \phi_M^e u_M^e$$

$$\int_0^h \phi_N^e \left[a\frac{d^2u^e}{dx^2} + cu^e - f^e \right] dx = 0$$

$$\boxed{\underbrace{a\int_0^h \phi_N^e \frac{d^2u^e}{dx^2}\,dx}_{\text{term1}} = -\underbrace{\int_0^h cu^e\phi_N^e\,dx}_{\text{term2}} + \underbrace{\int_0^h f^e\phi_N^e\,dx}_{\text{term2}}}$$

let us calculate this Integral term by term

term-1

$$= a\int_0^h \phi_N^e \frac{d^2u^e}{dx^2}\,dx$$

by Integration by parts

u	v
ϕ_N^e	$\frac{d^2u^e}{dx^2}$
$\frac{d\phi_N^e}{dx}$	$\frac{du^e}{dx}$

$$\Rightarrow a\phi_N^e \frac{du}{dx}\Big|_0^h - \int_0^h a\frac{d\phi_N^e}{dx}\frac{du^e}{dx}\,dx$$

This is known as weak form of the governing equation, because second derivative is transformed to first derivative.

The derivative $\frac{du}{dx}$ in the First term is no longer the variable with in the domain, but the Neumann boundary condition to be specified at $x=0$ or $x=h$.

Likewise, the test function ϕ_N is no longer $f(x)$.

$\overset{*}{\phi}{}_N^e$ is a Special neumann boundary test Function as opposed to the domain test function $\phi_N^e(x)$.

$\overset{*}{\phi}{}_N^e = 1$, when Neumann b.c is applied at the node

$\overset{*}{\phi}{}_N^e = 0$, otherwise.

Depending on the Neumann boundary, condition being applied on either the left Hand side $(x=0)$ or Right Hand side $(x=h)$ we obtain

- $$\left.\frac{du}{dx}\right|_{x=0} = \left.\frac{du}{dx}\cos\theta\right|_{\theta=180} = -\frac{du}{dx}$$

- $$\left.\frac{du}{dx}\right|_{x=h} = \left.\frac{du}{dx}\cos\theta\right|_{\theta=0} = \frac{du}{dx}.$$

$$a\overset{*}{\phi}{}_N^e \frac{du}{dx}\Big|_0^h - a\int_0^h \frac{d\phi_N^e}{dx}\frac{du^e}{dx}\,dx$$

$$u^e = u_m^e \phi_m^e$$

$$\Rightarrow a\overset{*}{\phi}{}_N^e \frac{du}{dx}\Big|_0^h - a\int_0^h \frac{d\phi_N^e}{dx}\frac{d\phi_m^e}{dx}u_m^e\,dx$$

$$a\overset{*}{\phi}{}_N^e \frac{du}{dx}\Big]_0^h - a u_m^e \int_0^h \frac{d\phi_N^e}{dx}\frac{d\phi_M^e}{dx}\,dx$$

term2

$$= -\int_0^h c u^e \phi_N^e\,dx$$

$$= -c u_M^e \int_0^h \phi_N^e \phi_M^e\,dx$$

term3

$$= \int_0^h f^e \phi_N^e\,dx$$

$$= f_M^e \int_0^h \phi_M^e \phi_N^e\,dx$$

term1 = term2 + term 3

$$a\overset{*}{\phi}{}_N^e \frac{du}{dx}\Big]_0^h - a u_m^e \int_0^h \frac{d\phi_N^e}{dx}\frac{d\phi_M^e}{dx}\,dx$$

$$= -\int_0^h c u^e \phi_N^e\,dx + f_M^e \int_0^h \phi_N^e \phi_M^e\,dx$$

Rearraging the terms

$$u_m^e \Big[\int_0^h \Big(a \frac{d\phi_N^e}{dx}\frac{d\phi_M^e}{dx} - c\,\phi_M^e \phi_N^e\Big)\,dx$$

Neumann Boundary Vector

$$G_{NM} = a \overset{*}{\phi}{}^{e}_{N} \frac{du}{dx}$$

$$= a \begin{bmatrix} \overset{*}{\phi}{}^{e}_{1} \\ \overset{*}{\phi}{}^{e}_{2} \end{bmatrix} \frac{du}{dx}$$

If $\overset{*}{\phi}{}_1^e = \overset{*}{\phi}_2 = 0$ indicating Neumann boundary conditions are not to be applied to Any of the global nodes.

The Neumann boundary Vector is Zero even, if the gradient $\frac{du}{dx}$ is not Zero.

If Neumann boundary Conditions are to Applied, the Neumann test function $\overset{*}{\phi}{}_N^e$ Assume the Value 1, Given node under consideration or Neumann boundary is given at that Node.

$$K_{22}^e = \int_0^h a\frac{1}{h^2} - \frac{cx^2}{h^2}\, dx$$

$$= \frac{ax}{h^2} - \frac{cx^3}{3h^2}\Big]_0^h$$

$$= \frac{ah}{h^2} - \frac{ch^3}{3h^2}$$

$$\boxed{K_{22}^e = \frac{a}{h} - \frac{ch}{3}}$$

Force Vector Matrix $C_{NM}^e = \int_0^h \phi_M^e \phi_N^e \, dx$

$$C_{NM}^e = \int_0^h \phi_M^e \phi_N^e \, dx$$

$$C_{11}^e = \int_0^h 1 - \frac{2x}{h} + \frac{x^2}{h^2}\, dx$$

$$= x - \frac{2x^2}{2h} + \frac{x^3}{3h^2}\Big]_0^h$$

$$C_{11}^e = h - h + \frac{h}{3}$$

$$\boxed{C_{11}^e = \frac{h}{3}}$$

$$C_{22}^e = \int_0^h \frac{x^2}{h^2}\, dx$$

$$= \frac{x^3}{3h^2}\Big]_0^h = \frac{h}{3}$$

$$C_{12}^e = C_{21}^e = \int_0^h \frac{x}{h} - \frac{x^2}{h^2}\, dx$$

$$\boxed{C_{12}^e = C_{21}^e = \frac{h}{6}}$$

- $\dfrac{d\phi_1^e}{dx}\dfrac{d\phi_2^e}{dx} = \dfrac{-1}{h}\left(\dfrac{1}{h}\right) = \dfrac{-1}{h^2}$

- $\dfrac{d\phi_1^e}{dx}\dfrac{d\phi_1^e}{dx} = \dfrac{-1}{h}\left(\dfrac{-1}{h}\right) = \dfrac{1}{h^2}$

- $\dfrac{d\phi_2^e}{dx}\dfrac{d\phi_2^e}{dx} = \dfrac{1}{h} \times \dfrac{1}{h} = \dfrac{1}{h^2}$

- $\phi_1^e \phi_1^e = \left(1-\dfrac{x}{h}\right)\left(1-\dfrac{x}{h}\right) = 1-\dfrac{2x}{h}+\dfrac{x^2}{h^2}$

- $\phi_1^e \phi_2^e = \left(1-\dfrac{x}{h}\right)\left(\dfrac{x}{h}\right) = \dfrac{x}{h}-\dfrac{x^2}{h^2}$

- $\phi_2^e \phi_2^e = \dfrac{x}{h}\cdot\dfrac{x}{h} = \dfrac{x^2}{h^2}$

$$K_{11}^e = \int_0^h a\left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right) - c\left(1-\frac{x}{h}\right)\left(1-\frac{x}{h}\right) dx$$

$$= \frac{a}{h} - \frac{ch}{3}$$

$$K_{21}^e = K_{12}^e = \int_0^h a\left(\frac{-1}{h^2}\right) - \frac{xc}{h} + \frac{x^2c}{h^2}\, dx$$

$$= \frac{-ax}{h^2} - \frac{x^2c}{2h} + \frac{x^3c}{3h^2}\Bigg|_0^h$$

$$= -\frac{a}{h} - \frac{hc}{2} + \frac{hc}{3}$$

$$K_{12}^e = K_{21}^e = -\frac{a}{h} - \frac{ch}{6}$$

$$= -f_M^e \int_0^h \phi_M^e \phi_N^e \, dx + a \phi_N^{*e} \frac{du}{dx} \Big|_0^h$$

considering

$$K_{NM}^e = \int_0^h \left(a \frac{d\phi_M^e}{dx} \frac{d\phi_N^e}{dx} - c \phi_N^e \phi_M^e \right) dx$$

$$C_{NM}^e = - \int_0^h \phi_M^e \phi_N^e \, dx$$

$$F_N^e = C_{NM}^e f_M^e$$

$$G_N^e = a \phi_N^{*e} \frac{du}{dx} \Big|_0^h$$

Using Compact Notation

$$\boxed{K_{NM}^e u_m^e = F_N^e + G_N^e}$$

let us calculate the Stiffness (Diffussion or Viscosity Matrix)

$$K_{NM}^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

useful Calculation function

$$\phi_N^e \quad N=1,2$$

$$\phi_1^e = \left(1 - \frac{x}{h}\right) \qquad \frac{d\phi_1^e}{dx} = -\frac{1}{h}$$

$$\phi_2^e = \frac{x}{h} \qquad \frac{d\phi_2^e}{dx} = \frac{1}{h}$$

Global Stiffness Matrix

- local stiffness matrix for the element

$$K_{NM}^{e} = \begin{bmatrix} K_{11}^{e} & K_{12}^{e} \\ K_{21}^{e} & K_{22}^{e} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{h} - \frac{ch}{3} & -\frac{ch}{6} - \frac{a}{h} \\ -\frac{ch}{6} - \frac{a}{h} & \frac{a}{h} - \frac{ch}{3} \end{bmatrix}$$

Here $\left.\begin{array}{l} a = 1 \\ c = -2 \end{array}\right]$ Coefficients of DE

$h = 1/3$ — element length

$$K_{NM} = \begin{bmatrix} 3.2222 & -2.8889 \\ -2.8889 & 6.4444 \end{bmatrix}$$

- Global stiffness matrix, for 3 element

$$K_{NM} = \begin{bmatrix} K_{11}^{1} & K_{12}^{1} & 0 & 0 \\ K_{21}^{1} & K_{22}^{1} + K_{11}^{2} & K_{12}^{2} & 0 \\ 0 & K_{21}^{2} & K_{22}^{2} + K_{11}^{3} & K_{12}^{3} \\ 0 & 0 & K_{21}^{3} & K_{11}^{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3.2222 & -2.8889 & 0 & 0 \\ -2.8889 & 6.4444 & -2.8889 & 0 \\ 0 & -2.8889 & 6.4444 & -2.8889 \\ 0 & 0 & -2.8889 & 3.2222 \end{bmatrix}$$

Force coefficient for 3 element, similar K_{NM} Matrix

$$C_{NM} = \begin{bmatrix} C_{11}^{1} & C_{12}^{1} & 0 & 0 \\ C_{21}^{1} & C_{22}^{1} + C_{11}^{2} & C_{12}^{2} & 0 \\ 0 & C_{21}^{2} & C_{22}^{2} + C_{11}^{3} & C_{12}^{3} \\ 0 & 0 & C_{21}^{3} & C_{11}^{3} \end{bmatrix}$$

$$C_{NM}^{e} = \begin{bmatrix} C_{11}^{e} & C_{12}^{e} \\ C_{21}^{e} & C_{22}^{e} \end{bmatrix} = \begin{bmatrix} -\frac{h}{3} & -\frac{h}{6} \\ -\frac{h}{6} & -\frac{h}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{9} \end{bmatrix}$$

Therefore

$$C_{NM} = \begin{bmatrix} -0.1111 & -0.0556 & 0 & 0 \\ -0.0556 & -0.2222 & -0.0556 & 0 \\ 0 & -0.0556 & -0.222 & -0.0556 \\ 0 & 0 & -0.0556 & -0.1111 \end{bmatrix}$$

$$f_{NM} = C_{NM} f_M$$

$$f_M = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -4.2222 \\ -3.556 \\ -2 \end{bmatrix}$$

$$K_{NM} U_M = F_N + G_{NM}^e = RHS$$

$$F_N = C_{NM} f_M$$

$$= \begin{bmatrix} 0.6790 \\ 1.3580 \\ 1.1358 \\ 0.4198 \end{bmatrix}$$

$$\rightarrow G_{NM}^e = \begin{bmatrix} \phi_1^e \\ \phi_2^e \end{bmatrix} a \cdot \frac{du}{dx}$$

for 3 element system

$$G_{NM} = \begin{bmatrix} \phi_1^1 \\ \phi_2^1 + \phi_1^2 \\ \phi_2^2 + \phi_1^3 \\ \phi_2^3 \end{bmatrix} a \cdot \frac{du}{dx}$$

let us Apply Boundary conditions for this problem

1. Dirichlet Boundary condition

$u = 0 \quad x = 0$

$u = -1 \quad x = 1$

$$K_{NM} U_M = F_M + G_M$$

$$\boxed{K_{NM} U_M = RHS_M}$$

$$\begin{bmatrix} 3.222 & -2.889 & 0 & 0 \\ -2.8889 & 6.444 & -2.889 & 0 \\ 0 & -2.8889 & 6.444 & -2.8889 \\ 0 & 0 & -2.8889 & 3.2222 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \begin{bmatrix} -0.111 & -0.055 & 0 & 0 \\ -0.055 & -0.222 & -0.0556 & 0 \\ 0 & -0.055 & -0.2222 & -0.0556 \\ 0 & 0 & -0.0556 & -0.1111 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 1 \cdot \frac{du}{dx} \qquad \text{— eq } Ⓐ$$

Applying boundary condition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.889 & 6.444 & -2.889 & 0 \\ 0 & -2.889 & 6.44 & -2.889 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0. \\ 1.3580 \\ 1.1358 \\ -1 \end{bmatrix}$$

$$Ax = b$$

$$x = A^{-1} b$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1111 \\ -0.2222 \\ -1 \end{bmatrix}$$

② Neumann Boundary Condition

$$u = 0 \quad \text{at} \quad x = 0$$

$$\frac{du}{dx} = -3 \quad \text{at} \quad x = L$$

till eq Ⓐ same, but G_m is changed

$$G_{NM} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} 1 \cdot \frac{du}{dx}$$

Applying boundary condition

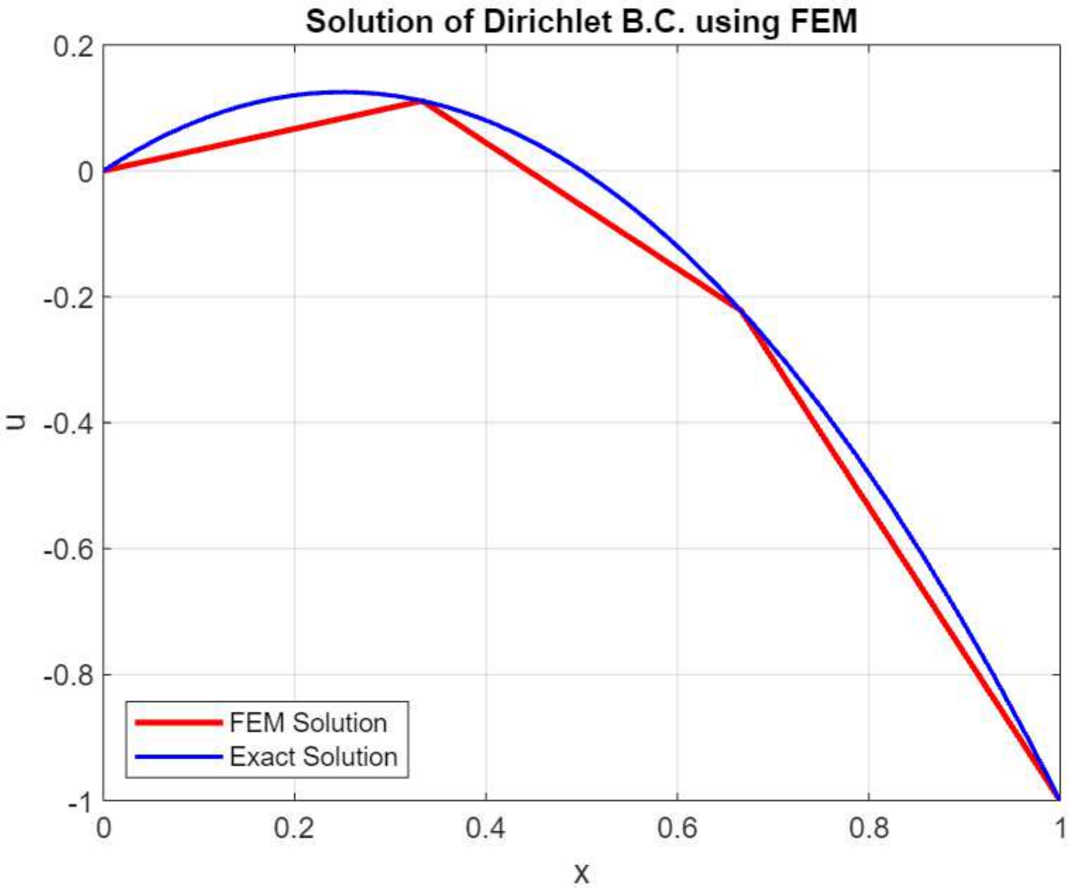
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2.889 & 6.444 & -2.889 & 0 \\ 0 & -2.889 & 6.44 & -2.889 \\ 0 & 0 & -2.889 & 3.222 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.3580 \\ 1.1358 \\ -2.5802 \end{bmatrix}$$

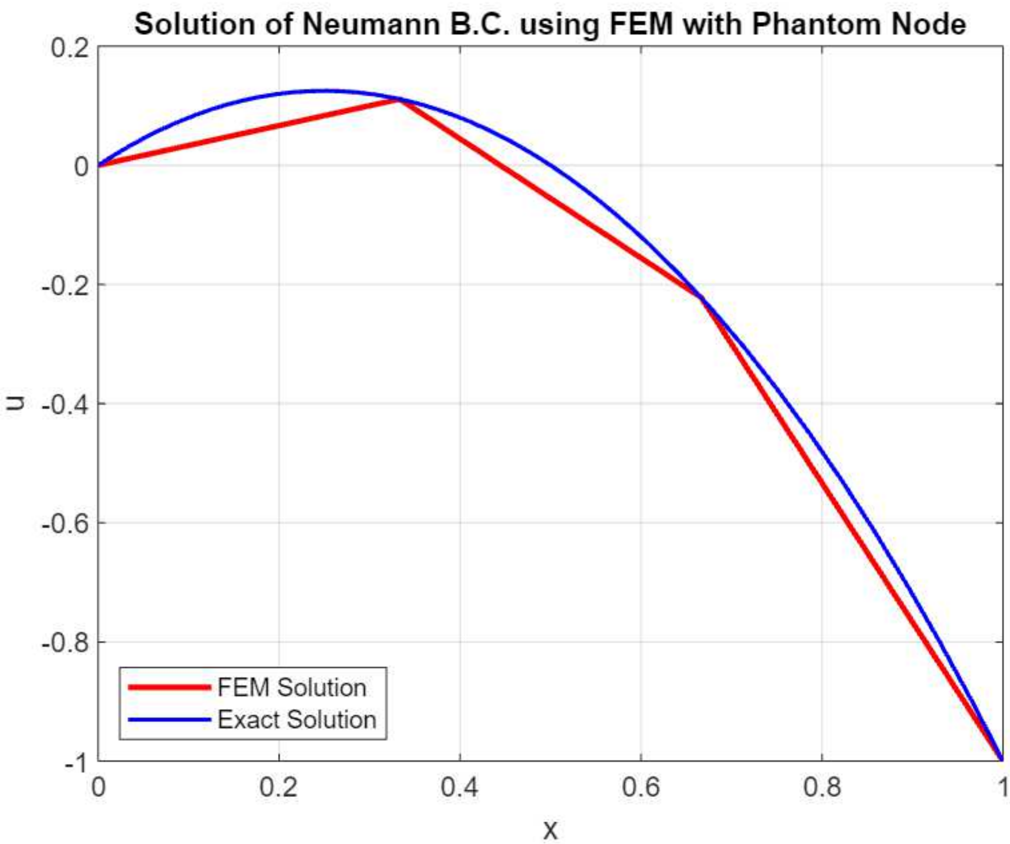
$$Ax = b$$

$$x = A^{-1} b$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1111 \\ -0.2222 \\ -1 \end{bmatrix}$$

Finite Element Method





Finite Difference Method

→ 3-element System

Given equation $\frac{d^2u}{dx^2} - 2u = f$

where $f = 4x^2 - 2x - 4$

Domain $0 < x < 1$

Instead of writing for this System, let us write for General D.E with constant Coefficients.

$$a\frac{d^2u}{dx^2} + \cancelto{0}{b\frac{du}{dx}} + cu = f(x)$$

$$a\frac{d^2u}{dx^2} + cu = f(x)$$

where $a = 1$ For this problem
$c = -2$

by changing this Coefficients, we can solve, Many differential equation problems.

→ Considering Uniform grid of 3 elements with Spacing of $h = \frac{1}{3}$

→ Total Nodal points $n = 4$
elements $e = 3$

each nodal points are located at
X-distance $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$

Useful Finite differences for this System

	Derivative	Forward Difference	central Difference	backward Difference
1	$\frac{du}{dx}$	$\frac{u_{i+1} - u_i}{h}$	$\frac{u_{i+1} - u_{i-1}}{2h}$	$\frac{u_i - u_{i-1}}{h}$
2.	$\frac{d^2u}{dx^2}$	$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$	not needed	not needed

$$a\frac{d^2u}{dx^2} + cu - f = 0$$

$$\left.\frac{d^2u}{dx^2}\right|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$a\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right] + cu_i - f_i = 0$$

$$\frac{a}{h^2}u_{i-1} - \frac{2a}{h^2}u_i + \frac{a}{h^2}u_{i+1} + cu_i = f_i$$

$$\boxed{\frac{a}{h^2}u_{i-1} + \left(c - \frac{2a}{h^2}\right)u_i + \frac{a}{h^2}u_{i+1} = f_i}$$

This formula is useful for Any ith node

Boundary Conditions

1. Dirichlet bc

node 1 $u = 0$ at $x = 0$ $\rightarrow u_1 = 0$ —— (1)

node 4 $u = -1$ at $x = 0$ $\rightarrow u_4 = -1$ —— (2)

node 2

$$\frac{a}{h^2} u_1 + \left(c - \frac{2a}{h^2}\right) u_2 + \frac{a}{h^2} u_3 = f_2 \quad \text{—(3)}$$

node 3

$$\frac{a}{h^2} u_2 + \left(c - \frac{2a}{h^2}\right) u_3 + \frac{a}{h^2} u_4 = f_3 \quad \text{—(4)}$$

writing all equation in Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{h^2} & c - \frac{2a}{h^2} & \frac{a}{h^2} & 0 \\ 0 & \frac{a}{h^2} & c - \frac{2a}{h^2} & \frac{a}{h^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \\ 0 \end{bmatrix}$$

Point	x	$f(x)$	$4x^2-2x-4$
1	0	f_1	-4
2	$1/3$	f_2	-4.2222
3	$2/3$	f_3	-3.5556
4	1	f_4	-2

$a_1 = 1$

$c = -2$

$h = \frac{1}{3}$

Substituting a, c and h values in the equation above

$$\begin{bmatrix} -20 & 9 & 0 & 0 \\ 9 & -20 & 9 & 0 \\ 0 & 9 & -20 & 9 \\ 0 & 0 & 9 & -20 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.2222 \\ -3.5556 \\ 0 \end{bmatrix}$$

$$A \qquad x \qquad b$$

$$x = A^{-1} b$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1111 \\ -0.2222 \\ -1 \end{bmatrix}$$

ii) Neumann boundary condition (with phantom node)

$u = 0$ at $x = 0$

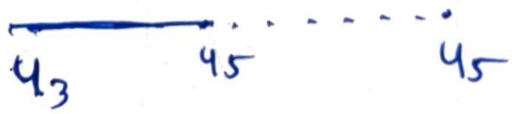
$\frac{du}{dx} = -3$ at $x = 1$

Given $u_1 = 0$

$\left(\frac{du}{dx}\right)_4 = -3$

$u_4' = -3$

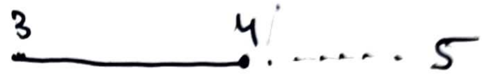
Consider phantom node u_5



We have from the General equation

$$U_{i-1}\left(\frac{a}{h^2}\right) + \left(C - \frac{2a}{h^2}\right)U_i + U_{i+1}\left(\frac{a}{h^2}\right) = f_i$$

Phantom node case



let us write for point 4

$$U_3\frac{a}{h^2} + \left(C - \frac{2a}{h^2}\right)U_4 + \frac{a}{h^2}U_5 = f_4$$

$$\boxed{U_5 = U_3 + 2h\,U_4'}$$

$$\frac{a}{h^2}U_3 + \left(C - \frac{2a}{h^2}\right)U_4 + \frac{a}{h^2}\left[U_3 + 2h\,U_4'\right] = f_4$$

$$\boxed{\frac{2a}{h^2}U_3 + \left(C - \frac{2a}{h^2}\right)U_4 = f_4 - \frac{2a}{h}U_4'}$$

for Nodal point 4.

let us Represent In Matrix form

node-1	$U_1 = 0$
node-2	$U_1\frac{a}{h^2} + \left(C - \frac{2a}{h^2}\right)U_2 + U_3\frac{a}{h^2} = f_2$
node-3	$U_2\frac{a}{h^2} + \left(C - \frac{2a}{h^2}\right)U_3 + U_4\frac{a}{h^2} = f_3$
node-4	$\frac{2a}{h^2}U_3 + \left(C - \frac{2a}{h^2}\right)U_4 = f_4 - \frac{2a}{h}U_4'$

Substituting the values of a, h, c, f_i and u_4'

In Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{h^2} & c-\frac{2a}{h^2} & \frac{a}{h^2} & 0 \\ 0 & \frac{a}{h^2} & c-\frac{2a}{h^2} & \frac{a}{h^2} \\ 0 & 0 & \frac{2a}{h^2} & c-\frac{2a}{h^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \\ f_4 - \frac{2a u_4'}{h} \end{bmatrix}$$

$$f_4 - \frac{2a u_4'}{h}$$

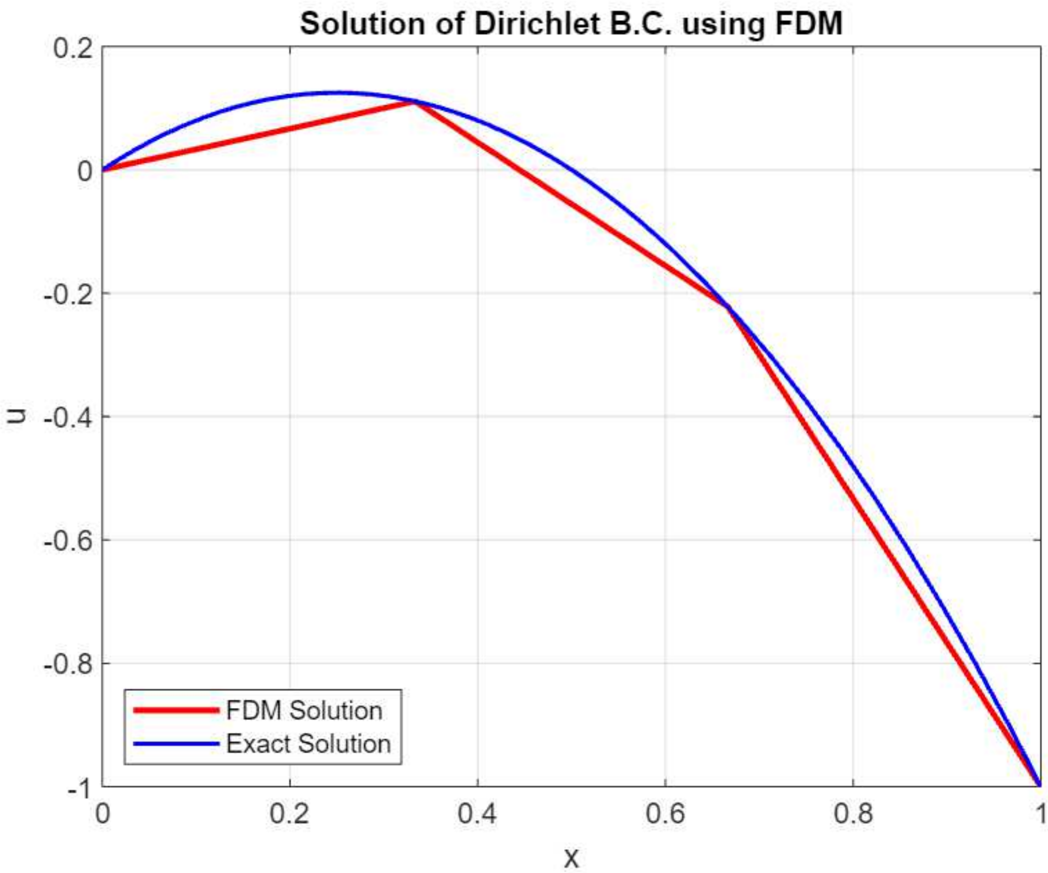
$a = 1$
$h = 1/3$
$c = -2$
$u_4' = -3$

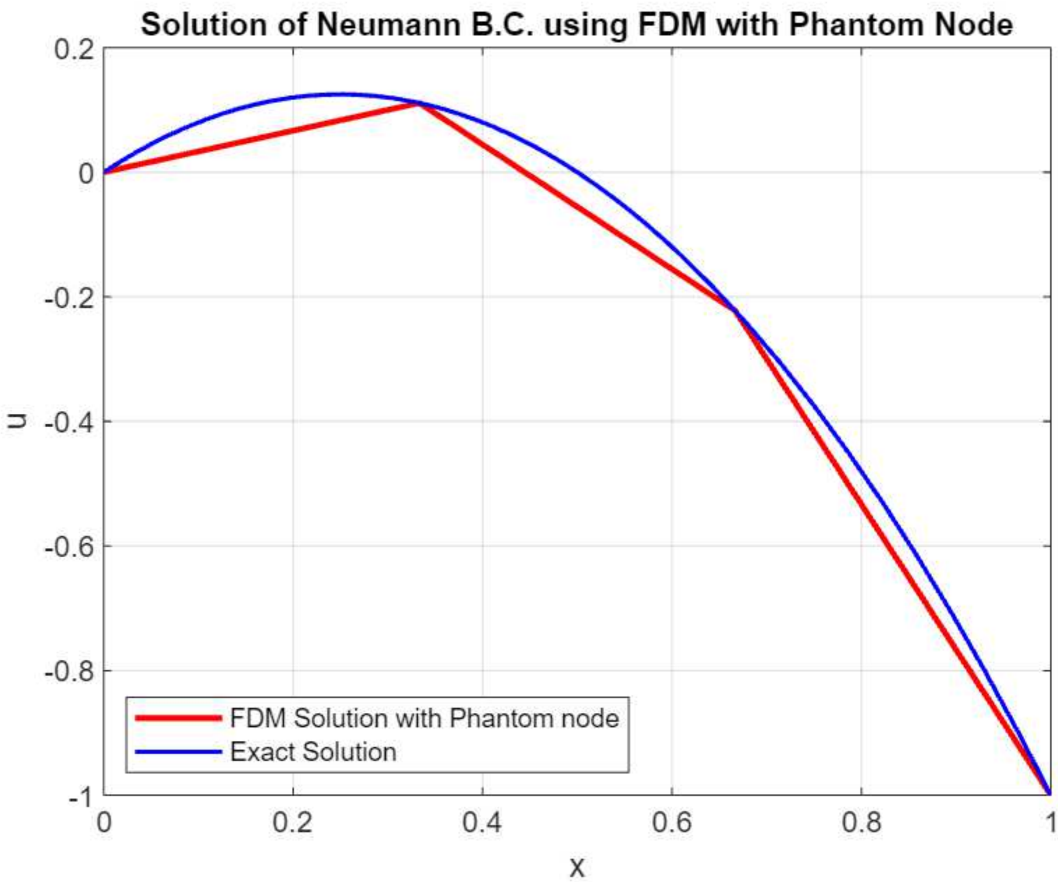
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & -20 & 9 & 0 \\ 0 & 9 & -20 & 9 \\ 0 & 0 & 18 & -20 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.2222 \\ -3.5556 \\ 16 \end{bmatrix}$$

$$A \qquad x \quad = b$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1111 \\ -0.2222 \\ -1 \end{bmatrix}$$

Finite Difference Method

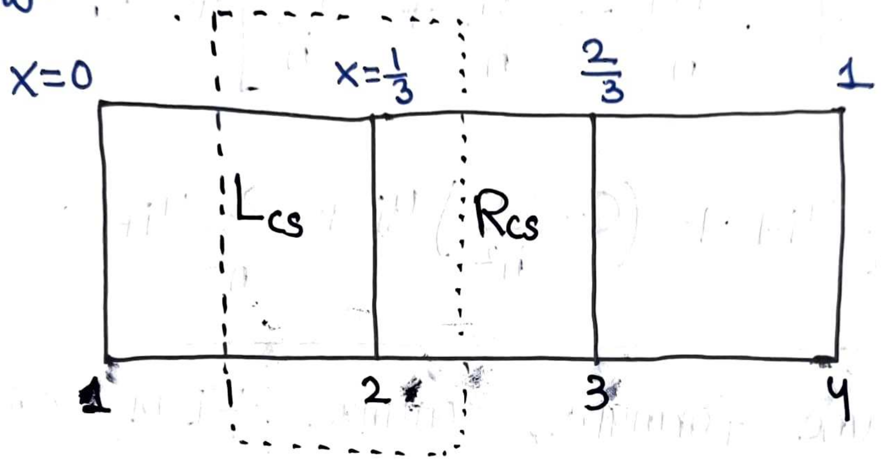




Finite Volume Method.

FVM Require the Use of Control volumes and Surfaces Centered around a node

The governing differential equation is Derived below.



For Any node, not at the ends (all except 1^{st} and n^{th} node) let us write the General equation

$$\int_{L_{cs}}^{R_{cs}} \left[a \frac{d^2u}{dx^2} + cu - f\right] dx = 0$$

$$\int_{L}^{R} \left[a u_i'' + c u_i - f_i\right] dx = 0$$

$$a u_i' \Big|_{L_{cs}}^{R_{cs}} + c u_i h - f_i h = 0$$

$$(u_i')_{R \text{ control surface}} = \frac{u_{i+1} - u_i}{h}$$

$$(u_i')_{Lcs} = \frac{u_i - u_{i-1}}{h}$$

$$a\left[\frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h}\right] + c u_i h - f_i h = 0$$

$$a\left[\frac{u_{i+1}}{h} - \frac{2u_i}{h} + \frac{u_{i-1}}{h}\right] + c u_i h - f_i h = 0$$

$$\boxed{\frac{a}{h^2} u_{i-1} + \left(c - \frac{2a}{h^2}\right) u_i + \frac{a}{h^2} u_{i+1} = f_i}$$

Same formula, continue FDM calculation for FVM

Boundary Conditions

1. Dirichlet bc

node 1 $\quad u = 0$ at $x = 0 \quad \rightarrow u_1 = 0 \quad — (1)$

node 4 $\quad u = -1$ at $x = 0 \quad \rightarrow u_4 = -1 \quad — (2)$

node 2

$$\frac{a}{h^2} u_1 + \left(c - \frac{2a}{h^2}\right) u_2 + \frac{a}{h^2} u_3 = f_2 \quad — (3)$$

node 3

$$\frac{a}{h^2} u_2 + \left(c - \frac{2a}{h^2}\right) u_3 + \frac{a}{h^2} u_4 = f_3 \quad — (4)$$

writing all equation in Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{h^2} & c - \frac{2a}{h^2} & \frac{a}{h^2} & 0 \\ 0 & \frac{a}{h^2} & c - \frac{2a}{h^2} & \frac{a}{h^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \\ 0 \end{bmatrix}$$

Point	x	f(x)	$4x^2-2x-4$
1	0	f_1	-4
2	$1/3$	f_2	-4.2222
3	$2/3$	f_3	-3.5556
4	1	f_4	-2

$a_1 = 1$
$c = -2$
$h = \frac{1}{3}$

Substituting a, c and h values in the equation abov

$$\begin{bmatrix} -20 & 9 & 0 & 0 \\ 9 & -20 & 9 & 0 \\ 0 & 9 & -20 & 9 \\ 0 & 0 & 9 & -20 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.2222 \\ -3.5556 \\ 0 \end{bmatrix}$$

$$A \qquad x \qquad b$$

$$x = A^{-1} b$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1111 \\ -0.2222 \\ -1 \end{bmatrix}$$

ii) Neumann boundary condition (with phantom node)

$$u = 0 \quad \text{at } x = 0$$

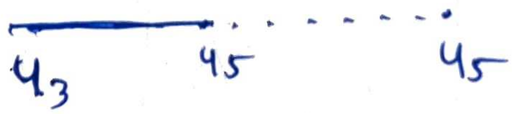
$$\frac{du}{dx} = -3 \quad \text{at } x = 1$$

Given $u_1 = 0$

$$\left(\frac{du}{dx}\right)_4 = -3$$

$$u_4' = -3$$

consider phantom node u_5



We have from the General equation

$$u_{i-1}\left(\frac{a}{h^2}\right) + \left(c-\frac{2a}{h^2}\right)u_i + u_{i+1}\left(\frac{a}{h^2}\right) = f_i$$

Phantom node case

let us write for point 4

$$u_3\frac{a}{h^2} + \left(c-\frac{2a}{h^2}\right)u_4 + \frac{a}{h^2}u_5 = f_4$$

$$\boxed{u_5 = u_3 + 2h\,u_4'}$$

$$\frac{a}{h^2}u_3 + \left(c-\frac{2a}{h^2}\right)u_4 + \frac{a}{h^2}\left[u_3 + 2h\,u_4'\right] = f_4$$

$$\boxed{\frac{2a}{h^2}u_3 + \left(c-\frac{2a}{h^2}\right)u_4 = f_4 - \frac{2a}{h}u_4'}$$

for Nodal point 4.

let us Represent In Matrix form

node-1	$u_1 = 0$
node-2	$u_1\frac{a}{h^2} + \left(c-\frac{2a}{h^2}\right)u_2 + u_3\frac{a}{h^2} = f_2$
node-3	$u_2\frac{a}{h^2} + \left(c-\frac{2a}{h^2}\right)u_3 + u_4\frac{a}{h^2} = f_3$
node-4	$\frac{2a}{h^2}u_3 + \left(c-\frac{2a}{h^2}\right)u_4 = f_4 - \frac{2a}{h}u_4'$

Substituting the values of a, h, c, f_i and u_4'

In Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{h^2} & c-\frac{2a}{h^2} & \frac{a}{h^2} & 0 \\ 0 & \frac{a}{h^2} & c-\frac{2a}{h^2} & \frac{a}{h^2} \\ 0 & 0 & \frac{2a}{h^2} & c-\frac{2a}{h^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \\ f_4 - \frac{2au_4'}{h} \end{bmatrix}$$

$f_4 - \frac{2au_4'}{h}$

$a = 1$
$h = 1/3$
$c = -2$
$u_4' = -3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 9 & -20 & 9 & 0 \\ 0 & 9 & -20 & 9 \\ 0 & 0 & 18 & -20 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -4.2222 \\ -3.5556 \\ 16 \end{bmatrix}$$

$$A \qquad x \quad = b$$

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Finite Volume Method