Math Essentials

**for SAR and InSAR **

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In [1]:
```
import warnings
warnings.filterwarnings('ignore')

bShowInline = True  # Set = False for document generation
%matplotlib inline
import matplotlib.pyplot as plt

params = {'legend.fontsize': 'x-large',
          'figure.figsize': (15, 5),
         'axes.labelsize': 'x-large',
         'axes.titlesize':'x-large',
         'xtick.labelsize':'x-large',
         'ytick.labelsize':'x-large'}
plt.rcParams.update(params)

def makeplot( plt, figlabel, figcaption):
    figname = figlabel+'.png'

    plt.savefig(figname)

    if bShowInline:
        plt.show()
    else:
        plt.close()

    strLatex="""
    \\begin{figure}[b]
```

```
    \centering
        \includegraphics[totalheight=10.0cm]{%s}
        \caption{%s}
        \label{fig:%s}
    \end{figure}"""%(figname, figcaption, figlabel)
    return display(Latex(strLatex))
```

In [3]:

```
from ipywidgets import widgets, Layout, Box, GridspecLayout


##Basic mcq

def create_multipleChoice_widget(description, options, correct_answer, hint):
    if correct_answer not in options:
        options.append(correct_answer)

    correct_answer_index = options.index(correct_answer)

    radio_options = [(words, i) for i, words in enumerate(options)]
    alternativ = widgets.RadioButtons(
        options = radio_options,
        description = '',
        disabled = False,
        indent = False,
        align = 'center',
    )

    description_out = widgets.Output(layout=Layout(width='auto'))

    with description_out:
        print(description)

    feedback_out = widgets.Output()

    def check_selection(b):
        a = int(alternativ.value)
        if a==correct_answer_index:
            s = '\x1b[6;30;42m' + "correct" + '\x1b[0m' +"\n"
        else:
            s = '\x1b[5;30;41m' + "try again" + '\x1b[0m' +"\n"
        with feedback_out:
```

```
            feedback_out.clear_output()
            print(s)
        return

    check = widgets.Button(description="check")
    check.on_click(check_selection)

    hint_out = widgets.Output()

    def hint_selection(b):
        with hint_out:
            print(hint)

        with feedback_out:
            feedback_out.clear_output()
            print(hint)

    hintbutton = widgets.Button(description="hint")
    hintbutton.on_click(hint_selection)

    return widgets.VBox([description_out,
                         alternativ,
                         widgets.HBox([hintbutton, check]), feedback_out],
                        layout=Layout(display='flex',
                                     flex_flow='column',
                                     align_items='stretch',
                                     width='auto'))

test = create_multipleChoice_widget('1. Let a, b, c be real numbers. If a<b an
d b<c, then:',
                                    ['a>c','a<c','b<c','c<a'],
                                    'a<c',
                                    "[hint]:")

trig = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'B',"[hint]: Given in question")

comp1 = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'C',"[hint]:")
```

```
comp2 = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'B',"[hint]:")

vec1 = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'B',"[hint]:")

vec2 = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'A',"[hint]:")

calc = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'A',"[hint]:")

dB = create_multipleChoice_widget('',
    ['A','B','C','D'],
    'D',"[hint]:")
```

Overview

In this notebook, we will introduce the essential mathematical elements that are typically encountered in synthetic aperture radar and interferometry. An understanding of algebra is assumed. Students comfortable with this material should be able to follow the elements of this course. This notebook can serve as an indicator for students to assess their ability to get the most from this course.

1.0 Trigonometry

Trigonometry is about triangles. In radar imaging and interferometry we encounter a lot of triangles. So here is a quick reminder of the basics.

For a right triangle, one with an angle of 90° between two of the sides, with a horizontal size of x and a vertical side of y, the third side - the hypotenuse - is given by $\sqrt{x^2+y^2}$, as illustrated in the figure.

With the angle θ between the horizontal and the hypotenuse specified, we can define the relationships between the sides and this angle as the familar trigonometric functions, cosine, sine, tangent, secant, cosecant, and cotangent:

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

The unit circle

Another way to define the trigonometric functions is via the unit circle. For a circle of radius r, the circumference is given by

$$C = 2\pi r$$

For $r = 1$, the circle is called the “unit circle,” as illustrated below. The circumference is 2π and any point P on the circle is described by the Cartesian (x, y) coordinates $(\cos\theta, \sin\theta)$ where θ is the angle formed by the horizontal and the line from the origin to P. The angle θ is measured from the horizontal. The coordinates $x = \cos\theta, y = \sin\theta$ oscillate between $+1$ and -1 as θ increases.

Basic Trigonometric Identities

There are a handful of useful relationships or identities that are either used explicitly in this course or may be helpful in solving the homeworks. We offer these without derivation for convenience.

$$\sin^2\theta + \cos^2\theta = 1 \quad , \quad \sin(-\theta) = -\sin(\theta) \quad , \quad \cos(-\theta) = \cos(\theta)$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \quad , \quad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \quad , \quad \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$1+\tan^2\alpha = \sec^2\alpha \quad , \quad 1+\cot^2\alpha = \csc^2\alpha$$

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha \pm \tan\beta}{1\mp\tan\alpha\tan\beta}$$

Trigonometric Sums and Products Relationships

Sometimes it is convenient to convert sums of trigonometric quantities to products and vice versa. Here are some convenient formulas.

$$\sin\theta\sin\phi = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$

$$\sin\theta\cos\phi = \frac{\sin(\theta+\phi)-\sin(\theta-\phi)}{2}$$

$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$

$$\sin\theta\pm\sin\phi = 2\sin\left(\frac{\theta\pm\phi}{2}\right)\cos\left(\frac{\theta\mp\phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Laws of Sine and Cosine

The so-called Sine and Cosine Laws in trigonometry are heavily used in interferometry. They allow the easy solution for the angles or sides of a triangle of arbitrary shape, as shown here.

Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

Note that if one of the angles is 90°, say γ in this case, then Law of Cosines reduces to the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

Table 2. Other Constants

Parameter	Symbol	Value	Comment
Speed of light	c	299792456 m/s	
Boltzman constant	k	$1.38064852 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}	-228.6 dB
Gravitational Constant	G	6.672×10^{-11} m^3 kg^{-1} s^{-2}	
Earth's Mass	M_E	5.9742×10^{24} kg	

Test your understanding - Trigonometry

From the figure at left and the law of sines, the formula relating the look angle, θ_ℓ, and incidence angle, θ_i, where r_p is the radius of the planet and h_p is the height of the platform is:

A. $\sin\theta_i = \sin\theta_\ell$

B. $\sin\theta_i = \frac{r_p + h_p}{r_p}\sin\theta_\ell$

C. $\sin\theta_i = \frac{r_p}{r_p + h_p}\sin\theta_\ell$

D. $\sin\theta_i = \frac{1}{\sin\,\theta_\ell}$

Hint: Use the relationship: $\sin\mu = \sin(\pi - \theta_i) = \sin\theta_i$

In [4]:
```
trig
```

2.0 Complex Numbers

What is a complex number, and do we really need them?

The answer is yes. Complex numbers arose in the 16th century when trying to solve polynomial equations like

$$x^2 + 1 = 0$$

It was found that by adding $i = \sqrt{-1}$ (engineers often use j instead of i) then it was possible to determine all n solutions to any polynomial equation of order n ($n = 2$ in the equation above).

In physics and engineering, complex numbers arise naturally in a myriad of ways, including describing wave phenomena. Pixels in radar imagery are fundamentally represented as complex numbers, each with an amplitude that represents the reflectivity of the pixel, and a phase that encodes the geometry of the observation as well as other surface effects. As such, an understanding of complex numbers is essential to fully grasp the underlying representations and language of interferometry.

Fortunately, only a few basic facts and identities about complex numbers are needed.

It is important to note that

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$$i^2 = -1 \qquad \rightarrow \qquad \frac{1}{i} = -i$$

Complex Arithmetic

All complex numbers can be written in the form:

$$z = \underbrace{x}_{\text{real part}} + i \underbrace{y}_{\text{imaginary part}}$$

To add or subtract complex numbers, simply add or subtract the real and imaginary parts. For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Geometric View of Complex Numbers

The Complex Plane

Complex numbers can be viewed as points in a plane, where the coordinate axes of the plane are the real and imaginary axes. Addition and subtraction can be visualized as vector addition, as shown in the figure below.

Modulus (or Magnitude) and Conjugates of a Complex Number

The magnitude or modulus of a complex number z is simply its length in the complex plane, and is usually expressed as $|z|$.

The magnitude of z can be computed in term of its complex conjugate, expressed as z^* or $\bar{z}$. This is simply its reflection about the real axis.

From the figure, it is hopefully easy to see that the real part $\Re(z)$ and the imaginary part $\Im(z)$ of z are

$$\Re(z) = \frac{z + z*}{2} \qquad \Im(z) = \frac{z - z*}{2i}$$

The magnitude of z geometrically the hypotenuse of the triangle formed between x, y, and $|z|$ in the complex plane:

$$|z| = \sqrt{x^2 + y^2}$$

More formally, the magnitude of a complex vector is defined as

$$\begin{aligned} |z| &= \sqrt{zz*} \\ &= \sqrt{(x + iy)(x - iy)} \\ &= \sqrt{x^2 + ixy - ixy - i^2y^2} \\ &= \sqrt{x^2 - (-1)y^2} = \sqrt{x^2 + y^2} \end{aligned}$$

Multiplication and Division of Complex Numbers

To multiply and divide complex numbers, we treat them like binomials. Multiplication is straightforward:

$$\begin{aligned} z_1 z_2 = & \ (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + iy_1 x_2 + iy_2 x_1 + i^2 y_1 y_2 \\ = & \ x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Division is a little trickier. Here we change division into multiplcation by multiplying the numerator and denominator by the complex conjugate of the demoninator

$$\begin{aligned} \frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} & = \frac{x_1 + iy_1}{x_2 + iy_2} \frac{x_2 - iy_2}{x_2 - iy_2} \\ & = \frac{x_1 x_2 + y_1 y_2 + i(-x_1 y_2 + x_2 y_1)}{x^2 + y^2} = \frac{x_1 x_2 + y_1 y_2}{x^2 + y^2} + i \frac{-x_1 y_2 + x_2 y_1}{x^2 + y^2} \end{aligned}$$

Polar Format Representation of Complex Numbers

Just as we saw with triangles and trigonometric functions, the real and imaginary components of a complex number can be represented in a polar format (there we examined the unit circle where the hypotenuse always had length 1. Here the complex number has length $r = |z|$.) Refering to the figure,

$$z_1 = x_1 + iy_1 = r_1 \cos\theta_1 + ir_1 \sin\theta_1$$

$$z_2 = x_2 + iy_2 = r_2 \cos\theta_2 + ir_2 \sin\theta_2$$

This representation provides a geometric view of multiplication and division that simplifies the effort.

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i r_1 r_2(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= r_1 r_2\cos(\theta_1 + \theta_2) + i r_1 r_2\sin(\theta_1 + \theta_2) \end{aligned}$$

So the product involves the product of the magnitudes and the sums of the angles. A similar expansion for division shows that the ratio involves the ratio of the magnitudes and the differences of the angles:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + i\frac{r_1}{r_2}\sin(\theta_1 - \theta_2)$$

Euler Identities

Based on the polar format formulats for multiplication and division, Euler developed the following remarkable identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So every complex number can be written in the form

$$z = r\cos\theta + ir\sin\theta = re^{i\theta}$$

Multiplication and division are easy in this format

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

The complex conjugate and magnitude are also simple to compute in this format:

$$z^* = re^{-i\theta}$$

$$|z|^2 = zz^* = re^{i\theta}re^{-i\theta} = r^2e^{i(\theta-\theta)} = r^2$$

Here are some other useful complex number formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$r = \sqrt{\Re(z)^2 + \Im(z)^2}$$

$$r = \sqrt{\Re(z)^2 + \Im(z)^2}$$

$$\theta = \begin{cases} \tan^{-1}\frac{\Im(z)}{\Re(z)} & \Re(z) > 0 \\ \tan^{-1}\frac{\Im(z)}{\Re(z)} + \frac{\pi}{2} & \Re(z) < 0 \text{ and } \Im(z) \geq 0 \\ \tan^{-1}\frac{\Im(z)}{\Re(z)} - \frac{\pi}{2} & \Re(z) < 0 \text{ and } \Im(z) < 0 \end{cases}$$

Test your understanding - Complex Numbers

What are the magnitude and phase of the complex number $z = 1 + \sqrt{3}i$ expressed in Euler format?

A. $2e^{i\frac{\pi}{6}}$

B. $\sqrt{3}e^{i\pi}$

C. $2e^{i\frac{\pi}{3}}$

D. Can't be determined from the information provided

The answer is C.

In [5]:
```
comp1
```

If $z_1 = 20\,e^{i20}$ and $z_2 = 10\,e^{-i25}$, what is $z_1 z_2^*$?

A. $200\,e^{-i5}$

B. $200\,e^{i45}$

C. $2\,e^{-i5}$

D. $2\,e^{-i45}$

In [6]:
```
comp2
```

3.0 Vectors and Vector Algebra

Definition of Vectors and Geometric Interpretation

Vectors are objects that have magnitude (or length) and direction. They can exist in 2, 3 or n-dimensional space.

Vectors are represented as pairs of numbers in two dimensions, triples of numbers in three dimensions and n-tuple of numbers in n dimensions.

Vector Arithmetic

Length or Magnitude of a Vector

The Euclidean magnitude or length of a vector in n dimensions is given by

$$|\vec{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

A unit vector has a length of 1. Any vector can be turned into a unit vector by dividing by its length

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

In the right figure above illustrating the 3-vector $\vec{v} = [x_1, x_2, x_3]$, the length of $\vec{v}$ is given by $|\vec{v}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Its projection onto the (x, y) plane is also a vector, with length $\sqrt{x_1^2 + x_2^2}$.

Scalar Multiplication of a Vector

To multiply a number by a vector we simply multiply each component of the vector by the number. If the absolute value of the number is greater than one then the length increases. If the absolute value of the number is less than one then the length decreases. Multiplying by a negative number points the vector in the opposite direction.

$$a\vec{v} = [ax_1, ax_2, ax_3, \ldots, ax_n]$$

Addition and Subtraction of Vectors

To add or subtract to vectors simply add or subtract their components

$$\vec{v} = [v_1, v_2, v_3, \ldots, v_n]$$

$$\vec{w} = [w_1, w_2, w_3, \ldots, w_n]$$

$$\vec{v} \pm \vec{w} = [v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3, \ldots, v_n \pm w_n]$$

Dot or Inner Product and Angle between Vectors

It is often desired to know the angle between two vectors, for example to determine if they are perpendicular or not. This can be done using the dot product of two vectors defined as

$$\vec{v}\cdot\vec{w} = \langle\vec{v},\vec{w}\rangle = v_1w_1 + v_2w_2 + v_3w_3 + \cdots + v_nw_n = \sum_{i=1}^{n} v_iw_i$$

With this definition, the angle between the vectors is given by

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

The magnitude of a vector can also be conveniently expressed as a dot product

$$|\vec{v}| = \sqrt{\langle\vec{v},\vec{v}\rangle}$$

Note, if the dot product equals zero then $\theta = \pi/2$ and the vectors are perpendicular.

The Law of Cosines is a Dot Product

The law of cosines can be represented using vectors as depicted in the figure

$$|\vec{v}-\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}|\,|\vec{w}|\cos\theta$$

\begin{eqnarray} (v_1-w_1)^2 + (v_2-w_2)^2 &=& v_1^2 + v_2^2 + w_1^2 + w_2^2 - 2 |\vec{v}| |\vec{w}| \cos\theta \ v_1^2 - 2 v_1 w_1 +w_1^2 + v_2^2 - 2 v_2 w_2 + w_2^2 &=& v_1^2 + v_2^2 + w_1^2 + w_2^2 - 2 |\vec{v}| |\vec{w}| \cos\theta \

- 2 (v_1 w_1 + 2 v_2 w_2) &=& - 2 |\vec{v}| |\vec{w}| \cos\theta \ \vec{v} \cdot \vec{w} &=& |\vec{v}| |\vec{w}| \cos\theta \end{eqnarray}

This shows that the dot product is another form of the Law of Cosines.

Cross Products

The cross product of two vectors is a vector perpendicular to the two vectors. The cross product is only defined for vectors in three dimensions. It is calculated through the determinant

$$\begin{aligned} \vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2 w_3 - v_3 w_2)\hat{i} + (v_3 w_1 - v_1 w_3)\hat{j} + (v_1 w_2 - v_2 w_1)\hat{k} \end{aligned}$$

In vector form, it can be written then

$$\vec{v} \times \vec{w} = \begin{bmatrix} (v_2 w_3 - v_3 w_2) \\ (v_3 w_1 - v_1 w_3) \\ (v_1 w_2 - v_2 w_1) \end{bmatrix}$$

A convenient way to remember the direction the resultant vector points from the cross product is to use the "right-hand rule". Positioning the right hand fingers as shown in the figure, and assigning $\vec{v}$ and $\vec{w}$ to the index and middle fingers imagined to be in a plane, the resultant vector is in the direction the thumb points, perpendicular to the plane formed by the other fingers.

Test your understanding - Vectors

The look vector, $\vec{\ell}$, to a target is given by

$$\vec{\ell} = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$$

then the unit vector, $\hat{\ell}$, pointing in this direction is

A. $\hat{\ell} = \begin{bmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

B. $\hat{\ell} = \begin{bmatrix} 0 \\ \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}$

C. $\hat{\ell} = \begin{bmatrix} 0 \\ \frac{5}{3} \\ \frac{5}{4} \end{bmatrix}$

D. $\hat{\ell} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In [7]:
```
vec1
```

If $\vec{v} = (0, \sin\theta_\ell, -\cos\theta_\ell)$, which of the following vectors is perpendicular to $\vec{v}$?

A. $(0, \cos\theta_\ell, \sin\theta_\ell)$

B. $(0, -\cos\theta_\ell, \sin\theta_\ell)$

C. $(\cos\theta_\ell, 0, \sin\theta_\ell)$

D. None of the above.

In [8]:
```
vec2
```

4.0 Linear or Matrix Algebra

Definition of a Matrix

A matrix is simply an rectangular array of numbers. The numbers may be real or complex.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \color{blue}{a_{21}} & \color{blue}{a_{22}} & \color{blue}{\dots} & \color{blue}{a_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The row index is the first index, running from 1 to m, as shown in blue, $a_{k,i}$, $i = 1, n$ for fixed k. The column index is the second index, running from 1 to n; the column is defined as the numbers arranged vertically $a_{i,k}$, $i = 1, m$ for fixed k.

If $m = n$, the matrix is said to be square. Mostly we will use 2×2 or 3×3 matrices.

Vectors are often represented as matricies with only 1 column, as we saw previously.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Matrices are useful because they can be used to represent linear transformations, for example rotations, or to solve systems of linear equations.

Matrix Transpose

The transpose of an $m \times n$ matrix A, denoted A^t, is an $n \times m$ matrix obtained by interchanging rows and columns: the ijth entry in the transpose is given by

$$A^t_{ij} A_{ji}$$

Written out in array form

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}^t = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \vdots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Arithmetic

Addition and Subtraction

Two matrices of the same dimension can be added or subtracted simply by adding/subtracting the corresponding entries

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

A matrix can be multiplied by a number (real or complex) by multiplying all entries by that number

$$rA = r\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & \vdots & \vdots \\ ra_{m1} & \cdots & ra_{mn} \end{bmatrix}$$

Matrix Multiplication

Two matrices can be multiplied if the number of columns of the first matrix equals the number of rows of the second matrix

$$C_{mn} = A_{mk}B_{kn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix}_{mk} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}_{kn}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i1} & \cdots & c_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}_{mn}$$

The ith row (in blue) of A and jth column (in red) of B are multiplied as a dot product, shown in the box in C.

Matrix multiplication is associative, but {\em not commutative}:

$$A(BC) = (AB)C$$

Matrix Inverse

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A square matrix A is said to be invertible if there exists a matrix B such that

$$AB = BA = I$$

where I is the matrix with all ones on the diagonal and 0 elsewhere (note: $AI = A$)

If such a B exists then it is called the inverse of A and denoted A^{-1}. For a 2×2 matrix defined as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant $\det(A) = ad - bc$, and the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general if $\det(A) \neq 0$, then an inverse exists. There are general formulas for computing the inverse, however they are not needed for this course.

Rotation Matrices

Matrices can be used represent the rotation through an angle about an axis in either two or three dimensions.

In two dimension the rotation matrices are of the form

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation matrices have two very special properties:

- The dot product of every row or column with itself equals 1.
- The inverse of a rotation matrix is equal to its transpose.

$$A^{-1} = A^{t}$$

For the 2×2 rotation matrix

$$R^{-1}(\theta) = R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Euler Angles and Three-dimensional Rotations

Euler angles provide a means of specifying an arbitrary 3-dimensional rotation matrix by specifying three angles and specified sequence of axes about which to rotate.

There are twelve possible right handed Euler angle sequences that may be used to specify any rotation matrix - and conversely every rotation matrix can be decomposed into any of the twelve possible Euler angle sequences.

Euler angle sequences are the sequence of axes about which to rotate, the twelve sequences are

Rotations in three dimensions can be decomposed as the product of 3 elemental rotations (there are 12 right handed possibilities.)

The elemental Rotation matrices are

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.0 Basic Calculus

Derivatives

The derivative of a function provides a measure of how quickly a function is changing as well as to provide a way to approximate a function linearly in a neighborhood of a point: The derivative of a function gives the slope of the tangent line to the graph of the function at a point.

The derivative is defined as

$$\frac{df}{dx}(x) = \dot{f}(x) = f'(x) = \lim_{x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

In region around x_0 a function can be approximated by its tangent line

$$f(x) \approx f(x_o) + f'(x_o)(x - x_o)$$

Product, Quotient and Chain Rules

The following three rules are very useful when computing derivatives

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg}{dx}(x) + g(x)\frac{df}{dx}(x)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df}{dx}(x) - f(x)\frac{dg}{dx}(x)}{g^2(x)}$$

Chain Rule:

$$\frac{d}{dx}f(g(x)) = \frac{df}{dx}(g(x))\frac{dg}{dx}(x)$$

Derivatives of Some Useful Functions

Polynomials

$$\frac{d}{dx}a = 0 \qquad \frac{d}{dx}x^n = nx^{n-1}$$

Trigonometric Functions

$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

Exponential and Log Functions

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

Partial Derivatives and Gradient

Sometimes we have functions that have more than one independent variable, *e.g.*,

$$\rho(h_p, \theta_\ell) = \frac{h_p}{\cos\theta_\ell}$$

The partial derivative of a functions is the derivative with respect to one of the variables while keeping the remining variables constant.

$$\frac{\partial f}{dx_i}(x_1, \cdots, x_n) = \lim_{x \to 0} \frac{f(x_1, \cdots, x_i + \Delta x, \cdots, x_n) - f(x_1, \cdots, x_n)}{\Delta x}$$

The gradient is the vector of partial derivatives with respect to all the variables

$$\vec{\nabla} f(x_1, \cdots, x_n) = \vec{\nabla} f(\vec{x}) = \left[\frac{\partial f}{\partial x_1}(\vec{x}), \cdots, \frac{\partial f}{\partial x_n}(\vec{x})\right]$$

Approximating a Multi-Variable Function

In defining the derivative above in one dimension as the local slope at a point, we saw that we could approximate the function there as:

$$f(x) \approx f(x_o) + f'(x_o)(x - x_o)$$

This can be extended to the multi-variable case, where the gradient can be used to approximate a function locally by the linear function

$$f(\vec{x}) \approx f(\vec{x}_o) + \langle \vec{\nabla} f(\vec{x}_o), \vec{x} - \vec{x}_o \rangle$$

where the notation $\langle , \rangle$ represents the dot product, described in the linear algebra section.

This can be written in long form as:

$$f(x_1, \cdots, x_n) \approx f(x_{1_o}, \cdots, x_{n_o}) + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x_1, \cdots, x_n)(x_k - x_{k_o})$$

Integral Calculus

Integration is the inverse operation to differentiation, and can be used to find the area under the graph of a function.

The anti-derivative, $F(x)$, of a function $f(x)$ is the function whose derivative is equal to $f(x)$

$$F(x) = \int f(x) dx \qquad \frac{d}{dx} F(x) = f(x)$$

The definite integral relates the signed area under a curve with the anti-derivative

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integrals of Some Useful Functions

Polynomials

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

Trigonometric Functions

$$\int \cos(x)dx = \sin(x) \qquad \int \sin(x)dx = -\cos(x)$$

Exponential and Log Functions

$$\int e^{ax}dx = \frac{1}{a}e^{ax} \qquad \int \frac{1}{x}dx = \ln x$$

Test your understanding - Calculus

Consider the formula relating the interferometric phase, ϕ, to the baseline length, b, look angle θ_ℓ and baseline orientation angle α given by

$$\phi = \frac{4\pi}{\lambda} b\sin(\theta_\ell - \alpha)$$

what is the partial derivative of the phase with respect to baseline orientation angle?

A. $\frac{\partial\phi}{\partial\alpha} = -\frac{4\pi}{\lambda} b\cos(\theta_\ell - \alpha)$

B. $\frac{\partial\phi}{\partial\alpha} = \frac{4\pi}{\lambda} b\sin(\theta_\ell - \alpha)$

C. $\frac{\partial\phi}{\partial\alpha} = \frac{4\pi}{\lambda} \sin(\theta_\ell - \alpha)$

D. $\frac{\partial\phi}{\partial\alpha} = \frac{4\pi}{\lambda} b\sin^2(\theta_\ell - \alpha)$

In [9]:
```
calc
```

6.0 Fourier Transforms

In this module, we give a quick introduction to Fourier Transforms, the mathematical means of examining the spectrum of a time function. We start with the basics of sine functions.

Sinusoidal Functions

The sine function or sinusoid is a mathematical function that describes a smooth repetitive oscillation. It occurs often in pure mathematics, as well as physics, signal processing, electrical engineering and many other fields. The most basic functional form is:

$$\begin{aligned} s(t) &= A\sin(\omega t + \phi) \\ &= A\sin(2\pi f t + \phi) \end{aligned}$$

The following basic parameters describing the sinusoid

- A, the amplitude, is the peak deviation of the function from 0.
- ω, the angular frequency, specifies how many oscillations occur in a unit time interval, in radians per second
    - The angular frequency, ω, period, T, and frequency, f, are related by

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- ϕ, the phase, specifies where in its cycle the oscillation begins at $t = 0$.

When $\phi \neq 0$, the entire waveform appears to be shifted in time by the amount ϕ / ω seconds. A negative value represents a delay, and a positive value represents an advance, "head-start".

In [10]:
```
def s(A,f,phi,t):
    return A*np.sin(2*np.pi()*f*t + phi)
```

Examples of sinusoids

Here are some examples of sinusoids and how they change according to changes in amplitude, phase, and frequency.

Phasors

Euler’s Identity showed a relationship between sines, cosines and complex exponentials. It should not be a surprise then that sine functions can be views in terms of complex exponentials, which are called {\em phasors}.

A phasor is a complex function of the form

$$z(t) = Ae^{i\omega t + \phi_0} = A\cos(\omega t + \phi_0) + iA\sin(\omega t + \phi_0)$$

where ϕ_0 is a constant phase offset.

The figure on the right shows the phasor representation in the complex plane. As t increases, for $\omega > 0$ the phasor rotates counterclockwise. For $\omega < 0$, it rotates clockwise. In both cases, the phasor makes one rotation in T units of time.

Summing Sinusoids with Different Frequencies

By combining waves of different frequencies and amplitude we can generate interesting signals. The curves below show the function

$$f_N(t) = \frac{1}{2} + \sum_{k=1}^{N} \frac{2}{(2k-1)\pi} \sin((2k-1)t)$$

for a number of values of N. As higher frequency sinusoids are added, it is possible to represent functions with faster variations in time.

Frequency Content of a Signal

We just showed how we could generate interesting signals by combining sinusoids of different amplitude, frequency and phase.

The inverse problem is the subject of Fourier analysis. That is: given a signal, what are the component frequencies present in the signal?

- A Fourier transform is sort of like a prism that separates white light into the colors of the rainbow. Each color is a different frequency or wavelength of light.

In Synthetic Aperture Radar, Fourier Transforms play a key role in forming images. Fourier Transforms are also important for understanding volumetric correlation in radar interferometry.

The Fourier Transform

The Fourier Transform is the mathematical tool that determines the frequency content of a time function. It transforms a function from the time domain to the frequency domain. The basic idea is very simple, and we use the analogy of a film reel to illustrate heuristically. Imagine we create needle rotating in about an axis like a phasor rotating in the complex plain - for example a pressure gauge that rotates as pressure changes.

If we were to film that rotating needle, phasors rotating at the same frequency as the file frame rate would be captured at the same orientation from frame to frame. If we add up all the frames, this would be large signal.

Phasors rotating at a different frequency from the film frame rate would be captured at random moments, and would not add up to much if the frames were added. For each angular frequency ω that is of interest, we change the frame rate to match that frequency and record the magnitude of the added frames. The figure illustrates this idea.

Summing frames is mathematically represented in the continuous domain by integration. The definition of the Fourier Transform is given by

$$\mathcal{F}(s(t))(\omega) = S(\omega) \equiv \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt$$

If $s(t) = e^{i\omega t}$, the integration would be over all time a constant value of 1; hence the integration is infinite. For other values of ω it can be shown that the integral is 0. This is the mathematical expression of the heuristic example above. For an arbitrary function, the Fourier Transform sorts the information into the spectrum of frequency content in the function.

The sinc Function

The Fourier transform of a rectangular pulse leads to the classical {\em sinc} function and shows the relationship between resolution and bandwidth. The time domain function is

$$
\begin{aligned}
\mathcal{F}(\mathrm{rect}(t))(\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-i\omega t} dt \\
&= \int_{-\tau/2}^{\tau/2} e^{-i\omega t} dt \\
&= -\frac{1}{i\omega} e^{-i\omega t} \Big|_{-\tau/2}^{\tau/2} \\
&= -\frac{1}{i\omega} (e^{-i\omega\tau/2} - e^{i\omega\tau/2}) \\
&= \frac{2}{\omega} \frac{(e^{-i\omega\tau/2} - e^{i\omega\tau/2})}{2i} \\
&= \frac{2}{\omega} \sin\left(\frac{\omega\tau}{2}\right) \\
&= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \tau\, \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$

Note that as the pulse becomes narrower, i.e., as τ becomes smaller and there is finer time resolution, the spectrum becomes broader. This illustrates a general principle that to have finer spatial resolution requires a broader band signal.

Useful Fourier Transform Relationships

Here are a few Fourier transform properties that are useful.

Inverse Transform: $\mathcal{F}^{-1}(S(\omega))(t) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} d\omega$

Linearity: $\mathcal{F}(as(t) + bw(t)) = a\mathcal{F}(s(t)) + b\mathcal{F}(w(t))$

Shift Property: $\mathcal{F}(s(t \pm t_0)) = S(\omega)e^{i \pm \omega t_0}$

Time Reversal: $\mathcal{F}(s(-t)) = S(-\omega)$

Scaling: $\mathcal{F}(s(at)) = \frac{1}{a}S(\frac{\omega}{a})$

Conjugation: $\mathcal{F}(s^*(t)) = S^*(-\omega)$

Parseval's Theorem of energy conservation: $\int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega$

7.0 Decibels

What is a dB?

In engineering and physics, we often encounter quantities that can vary by many orders of magnitude.

It turns out to be convenient to express such quantities in terms of decibels or dBs.

A decibel is a logarithmic unit used to describe a ratio. For example, power is often measured in dB relative to a reference of 1 Watt.

$$P_{\mathrm{dB}} = 10\log_{10}P$$

Since logarithms convert multiplications into additions, doubling a quantity increases it by 3 dB

$$10\log_{10}2P = 10\log_{10}P + 10\log_{10}2 \approx 10\log_{10}P + 3\mathrm{dB}$$

Sometimes a 20 is used instead of a 10 in converting to dB. This is because the quantity needs to be squared before converting.

$$P_{\mathrm{dB}} = 10\log_{10}V^2 = 20\log_{10}V$$

This is usually encountered when considering power ratios based on signal amplitudes or voltages.

Test your understanding - deciBels

An amplifier increases the power of a returned radar echo by a factor of 15. By how many dB has the power increased?

A. $20\log_{10}(15)$ dB

B. 3 dB

C. 15 dB

D. $10\log_{10}(15)$ dB

In [11]:
```
dB
```

Statement of comfort level

I am very comfortable with the lessons in this notebook (4)

In [ ]:
```
```