Homework 1

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1 NE1.1

Problem a

$$G(s) = \frac{1}{s^2 + 2s + 6}$$

ODE:

$$y''(t) + 2y'(t) + 6y(t) = u(t)$$

let $x_1(t) = y(t), x_2(t) = y'(t)$, then we have

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -2x_2(t) - 6x_1(t) + u(t) \end{cases}$$

so the state space representation is

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' &= \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{1}$$

thus $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$

Problem b

let $G_1(s) = \frac{1}{s^2+2s+6} = \frac{W(s)}{U(s)}$, $G_2(s) = s + 3 = \frac{Y(s)}{W(s)}$, then

$$G(s) = G_1(s)G_2(s)$$

ODE:

$$\begin{aligned} w''(t) + w'(t) + w(t) = u(t) \\ y(t) = w'(t) + 3w(t) \end{aligned} \tag{2}$$

let $x_1(t) = w(t), x_2(t) = w'(t)$, then we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3}$$

and

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

thus $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 3 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$

Problem c

similar to (a), we have $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -8 & -4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$

Problem d

similar to (b), we have $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -66 & -44 & -11 & -10 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 6 & 4 & 1 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$

2 AE1.11

suppose $\mathbf{A}$ is an $m \times m$ matrix, $\mathbf{B}$ is an $n \times n$ matrix. to prove

$$\begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{C} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{E}\boldsymbol{\Delta}^{-1}\mathbf{F} & -\mathbf{E}\boldsymbol{\Delta}^{-1} \\ -\boldsymbol{\Delta}^{-1}\mathbf{F} & \boldsymbol{\Delta}^{-1} \end{bmatrix}$$

, we need to prove that

$$\begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{C} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{E}\boldsymbol{\Delta}^{-1}\mathbf{F} & -\mathbf{E}\boldsymbol{\Delta}^{-1} \\ -\boldsymbol{\Delta}^{-1}\mathbf{F} & \boldsymbol{\Delta}^{-1} \end{bmatrix} = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}$$

first, consider the left-top block:

$$\begin{aligned} \mathbf{A}(\mathbf{A}^{-1} + \mathbf{E}\boldsymbol{\Delta}^{-1}\mathbf{F}) + \mathbf{D}(-\boldsymbol{\Delta}^{-1}\mathbf{F}) &= \mathbf{A}\mathbf{A}^{\mathbf{-1}} + \mathbf{A}\mathbf{E}\boldsymbol{\Delta}^{-1}\mathbf{F} - \mathbf{D}\boldsymbol{\Delta}^{-1}\mathbf{F} \\ &= \mathbf{I_m} + \mathbf{A}\mathbf{E}\boldsymbol{\Delta}^{-1}\mathbf{F} - \mathbf{D}\boldsymbol{\Delta}^{-1}\mathbf{F} \end{aligned}$$

since $\mathbf{E} = \mathbf{A}^{-1}\mathbf{D}$, we have $\mathbf{AE\Delta}^{-1}\mathbf{F} = \mathbf{D\Delta}^{-1}\mathbf{F}$, thus we get $\mathbf{I_m}$.

second, consider the right-top block:

$$\begin{aligned}\mathbf{A}(-\mathbf{E\Delta}^{-1}) + \mathbf{D}(\mathbf{\Delta}^{-1}) &= -\mathbf{AE\Delta}^{-1} + \mathbf{D\Delta}^{-1} \\ &= -\mathbf{D\Delta}^{-1} + \mathbf{D\Delta}^{-1} \\ &= \mathbf{0}\end{aligned}$$

third, consider the left-bottom block:

$$\begin{aligned}\mathbf{C}(\mathbf{A}^{-1} + \mathbf{E\Delta}^{-1}\mathbf{F}) + \mathbf{B}(-\mathbf{\Delta}^{-1}\mathbf{F}) &= \mathbf{CA}^{-1} + \mathbf{CE\Delta}^{-1}\mathbf{F} - \mathbf{B\Delta}^{-1}\mathbf{F} \\ &= \mathbf{CA}^{-1} + (\mathbf{CE} - \mathbf{B})\mathbf{\Delta}^{-1}\mathbf{F} \\ &= \mathbf{CA}^{-1} + (\mathbf{B} - \mathbf{CA}^{-1}\mathbf{D})\mathbf{\Delta}^{-1}\mathbf{F} \\ &= \mathbf{CA}^{-1} - \mathbf{\Delta\Delta}^{-1}\mathbf{F} \\ &= \mathbf{CA}^{-1} - \mathbf{F} \\ &= \mathbf{0}\end{aligned}$$

since $\mathbf{E} = \mathbf{A}^{-1}\mathbf{D}$, we have $\mathbf{CE\Delta}^{-1}\mathbf{F} = \mathbf{B\Delta}^{-1}\mathbf{F}$, thus we get $\mathbf{0}$.

fourth, consider the right-bottom block:

$$\begin{aligned}\mathbf{C}(-\mathbf{E\Delta}^{-1}) + \mathbf{B}(\mathbf{\Delta}^{-1}) &= -\mathbf{CA}^{-1}\mathbf{D\Delta}^{-1} + \mathbf{B\Delta}^{-1} \\ &= (\mathbf{B} - \mathbf{CA}^{-1}\mathbf{D})\mathbf{\Delta}^{-1} \\ &= \mathbf{\Delta}^{-1}\mathbf{\Delta} \\ &= \mathbf{I_n}\end{aligned}$$

thus we have proved that

$$\begin{bmatrix}\mathbf{A} & \mathbf{D} \\ \mathbf{C} & \mathbf{B}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{A}^{-1} + \mathbf{E\Delta}^{-1}\mathbf{F} & -\mathbf{E\Delta}^{-1} \\ -\mathbf{\Delta}^{-1}\mathbf{F} & \mathbf{\Delta}^{-1}\end{bmatrix}$$

Q.E.D.

3 AE1.12

first, consider $\lambda \neq 0$:

notice that the Jordan block is an upper triangular matrix, thus the inverse of a Jordan block is also an upper triangular matrix.

so we suppose the inverse of the k-th Jordan block has the form of:

$$\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{kk}\end{bmatrix}$$

from the fact that the multiplication of a Jordan block and its inverse is an identity matrix, we have

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{kk} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

thus we have

$$\begin{aligned} a_{11}\lambda &= 1 \\ a_{11} + \lambda a_{12} &= 0 \\ a_{12} + \lambda a_{13} &= 0 \\ &\vdots \\ a_{22}\lambda &= 1 \\ a_{22} + \lambda a_{23} &= 0 \\ a_{23} + \lambda a_{24} &= 0 \\ &\vdots \\ a_{k-1k-1}\lambda &= 1 \\ a_{k-1k-1} + \lambda a_{k-1k} &= 0 \end{aligned}$$

thus the inverse of the k-th Jordan block is

$$\begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \frac{1}{\lambda^3} & \cdots & (-1)^{k+1}\frac{1}{\lambda^k} \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \cdots & (-1)^{k}\frac{1}{\lambda^{k-1}} \\ 0 & 0 & \frac{1}{\lambda} & \cdots & (-1)^{k+1}\frac{1}{\lambda^{k-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\lambda} \end{bmatrix}$$

if $\lambda = 0$, then the inverse does not exist.

4 CME1.4

the corresponding characteristic equation is

$$\begin{aligned} w^{(4)}(t) + 6w^{(3)}(t) + 86w''(t) + 176w'(t) + 680w(t) &= u(t) \\ y(t) &= 100w(t) + 20w'(t) + 10w''(t) \end{aligned}$$

let$H_1(s) = \frac{W(s)}{U(s)}$, $H_2(s) = \frac{Y(s)}{W(s)}$, then

$$H_1(s) = \frac{1}{s^4 + 6s^3 + 86s^2 + 176s + 680}$$

$$H_2(s) = 10s^2 + 20s + 100$$

so the overal transfer function is

$$H(s) = H_1(s)H_2(2) = \frac{10s^2 + 20s + 100}{s^4 + 6s^3 + 86s^2 + 176s + 680}$$

5 CE1.1

Problem a

set the positive direction to be "right", according to Newton's second law,

$$u_1(t) = k_1y_1(t) + k_2(y_1(t) - y_2(t)) + c_1y_1'(t) + c_2(y_1'(t) - y_2'(t)) + m_1y_1''(t)$$
$$u_2(t) = k_2(y_2(t) - y_1(t)) + k_3(y_2(t) - y_3(t)) + c_2(y_2'(t) - y_1'(t)) + c_3(y_2'(t) - y_3'(t)) + m_2y_2''(t)$$
$$u_3(t) = k_3(y_3(t) - y_2(t)) + k_4y_3(t) + c_3(y_3'(t) - y_2'(t)) + c_4y_3'(t) + m_3y_3''(t)$$

the free-body diagram is shown in the figure below:

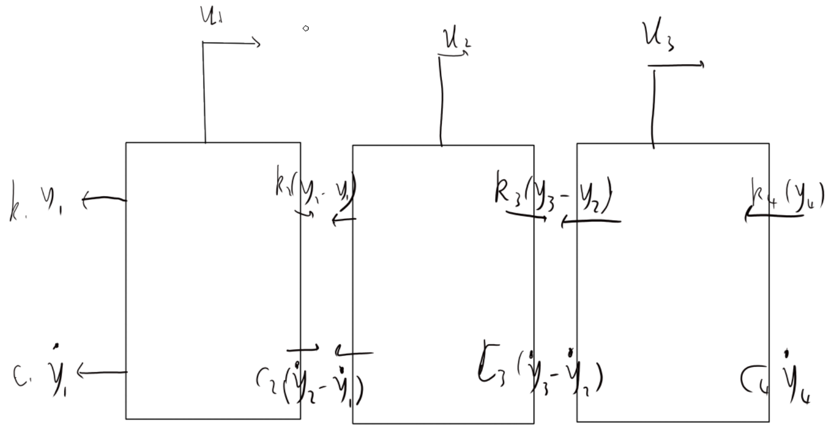

Figure 1: free-body diagram

let $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$, the matrix-vector form is

$$\begin{aligned} &\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} y_1''(t) \\ y_2''(t) \\ y_3''(t) \end{bmatrix} + \\ &\begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} + \\ &\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \\ &\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \end{aligned} \tag{4}$$

thus $\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$

Problem b

let

$$\begin{aligned} x_1(t) &= y_1(t) \\ x_2(t) &= y_2(t) \\ x_3(t) &= y_3(t) \\ x_4(t) &= y_1'(t) \\ x_5(t) &= y_2'(t) \\ x_6(t) &= y_3'(t) \end{aligned}$$

i

$\mathbf{u(t)} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$, $\mathbf{y(t)} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$, $\mathbf{x(t)} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$, so the state-space realization is

$$\mathbf{x'(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix} \mathbf{x(t)} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u(t)}$$

, and

$$\mathbf{y(t)} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x(t)} + \mathbf{0u(t)}$$

thus, $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$, $\mathbf{D} = \mathbf{0}$

ii

we only care $u_1(t)$ and $u_3(t)$, that is to say, we apply a linear transformation $L : R^2 \to R^3$ which takes $\begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix}$ as input and outputs $\begin{bmatrix} u_1(t) \\ 0 \\ u_3(t) \end{bmatrix}$, with $\mathbf{y(t)} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$, the state-space realization is

$$\mathbf{x'(t)} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix} \mathbf{x(t)} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u(t)}$$

, and

$$\mathbf{y(t)} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x(t)} + \mathbf{0u(t)}$$

thus $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$,$\mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$, $\mathbf{D} = \mathbf{0}$

iii

similar to (ii) , the state-space realization is

$$\mathbf{x}'(\mathbf{t}) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix} \mathbf{x}(\mathbf{t}) + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u}(\mathbf{t})$$

, and

$$\mathbf{y}(\mathbf{t}) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x}(\mathbf{t}) + \mathbf{0}\mathbf{u}(\mathbf{t}) \right)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M^{-1}K} & -\mathbf{M^{-1}C} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{D} = \mathbf{0}$$