Homework 1

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September 18, 2024

1   NE1.1

1.1   Problem a

$$G(s) = \frac{1}{s^2 + 2s + 6}$$

ODE:

$$y''(t) + 2y'(t) + 6y(t) = u(t)$$

let $x_1(t) = y(t), x_2(t) = y'(t)$, then we have
$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -2x_2(t) - 6x_1(t) + u(t) \end{cases}$$
so the state space representation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1}$$

thus $A = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$

1.2   Problem b

let $G_1(s) = \frac{1}{s^2 + 2s + 6} = \frac{W(s)}{U(s)}$, $G_2(s) = s + 3 = \frac{Y(s)}{W(s)}$, then

$$G(s) = G_1(s)G_2(s)$$

ODE:
$$w''(t) + w'(t) + w(t) = u(t)$$
$$y(t) = w'(t) + 3w(t) \tag{2}$$

let $x_1(t) = w(t)$, $x_2(t) = w'(t)$, then we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3}$$

and

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

thus $A = \begin{bmatrix} 0 & 1 \\ -6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$

1.3   Problem c

similar to (a), we have $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -8 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$

1.4   Problem d

similar to (b), we have $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -66 & -44 & -11 & -10 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 6 & 4 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$

2   AE1.11

suppose A is an $m \times m$ matrix, B is an $n \times n$ matrix. to prove

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}$$

, we need to prove that

$$\begin{bmatrix} A & D \\ C & B \end{bmatrix} \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

first, consider the left-top block:

$$A(A^{-1} + E\Delta^{-1}F) + D(-\Delta^{-1}F) = AA^{-1} + AE\Delta^{-1}F - D\Delta^{-1}F$$
$$= I_m + AE\Delta^{-1}F - D\Delta^{-1}F$$

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since $E = A^{-1}D$, we have $AE\Delta^{-1}F = D\Delta^{-1}F$, thus we get I_m.

second, consider the right-top block:

$$
\begin{aligned}
A(-E\Delta^{-1}) + D(\Delta^{-1}) &= -AE\Delta^{-1} + D\Delta^{-1} \\
&= -D\Delta^{-1} + D\Delta^{-1} \\
&= 0
\end{aligned}
$$

third, consider the left-bottom block:

$$
\begin{aligned}
C(A^{-1} + E\Delta^{-1}F) + B(-\Delta^{-1}F) &= CA^{-1} + CE\Delta^{-1}F - B\Delta^{-1}F \\
&= CA^{-1} + (CE - B)\Delta^{-1}F \\
&= CA^{-1} + (B - CA^{-1}D)\Delta^{-1}F \\
&= CA^{-1} - \Delta\Delta^{-1}F \\
&= CA^{-1} - F \\
&= 0
\end{aligned}
$$

since $E = A^{-1}D$, we have $CE\Delta^{-1}F = B\Delta^{-1}F$, thus we get 0.

fourth, consider the right-bottom block:

$$
\begin{aligned}
C(-E\Delta^{-1}) + B(\Delta^{-1}) &= -CA^{-1}D\Delta^{-1} + B\Delta^{-1} \\
&= (B - CA^{-1}D)\Delta^{-1} \\
&= \Delta^{-1}\Delta \\
&= I_n
\end{aligned}
$$

thus we have proved that

$$
\begin{bmatrix} A & D \\ C & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}
$$

Q.E.D.

3   AE1.12

first, consider $\lambda \neq 0$:

notice that the Jordan block is an upper triangular matrix, thus the inverse of a Jordan block is also an upper triangular matrix.

so we suppose the inverse of the k-th Jordan block has the form of:

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} \\
0 & a_{22} & \cdots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{kk}
\end{bmatrix}
$$

from the fact that the multiplication of a Jordan block and its inverse is an identity matrix, we have

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{kk} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

thus we have

$$a_{11}\lambda = 1$$
$$a_{11} + \lambda a_{12} = 0$$
$$a_{12} + \lambda a_{13} = 0$$
$$\vdots$$
$$a_{22}\lambda = 1$$
$$a_{22} + \lambda a_{23} = 0$$
$$a_{23} + \lambda a_{24} = 0$$
$$\vdots$$
$$a_{k-1k-1}\lambda = 1$$
$$a_{k-1k-1} + \lambda a_{k-1k} = 0$$

thus the inverse of the k-th Jordan block is

$$\begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \frac{1}{\lambda^3} & \cdots & (-1)^{k+1}\frac{1}{\lambda^k} \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda^2} & \cdots & (-1)^k\frac{1}{\lambda^{k-1}} \\ 0 & 0 & \frac{1}{\lambda} & \cdots & (-1)^{k+1}\frac{1}{\lambda^{k-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\lambda} \end{bmatrix}$$

if $\lambda = 0$, then the inverse does not exist.

4   CME1.4

the corresponding characteristic equation is

$$w^{(4)}(t) + 6w^{(3)}(t) + 86w''(t) + 176w'(t) + 680w(t) = u(t)$$
$$y(t) = 100w(t) + 20w'(t) + 10w''(t)$$

let $H_1(s) = \frac{W(s)}{U(s)}, H_2(s) = \frac{Y(s)}{W(s)}$, then

$$H_1(s) = \frac{1}{s^4 + 6s^3 + 86s^2 + 176s + 680}$$

4

$$H_2(s) = 10s^2 + 20s + 100$$

so the overal transfer function is

$$H(s) = H_1(s)H_2(2) = \frac{10s^2 + 20s + 100}{s^4 + 6s^3 + 86s^2 + 176s + 680}$$

5 CE1.1

5.1 Problem a

set the positive direction to be "right", according to Newton's second law,

$$u_1(t) = k_1 y_1(t) + k_2(y_1(t) - y_2(t)) + c_1 y_1'(t) + c_2(y_1(t) - y_2(t)) + m_1 y_1''(t)$$
$$u_2(t) = k_2(y_2(t) - y_1(t)) + k_3(y_2(t) - y_3(t)) + c_2(y_2'(t) - y_1'(t)) + c_2(y_2'(t) - y_3'(t)) + m_2 y_2''(t)$$
$$u_3(t) = k_3(y_3(t) - y_2(t)) + k_4(y_3(t) - y_4(t)) + c_3(y_3'(t) - y_2'(t)) + c_4(y_4'(t)) + m_3 y_3''(t)$$

let $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, y'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix}, y''(t) = \begin{bmatrix} y_1''(t) \\ y_2''(t) \\ y_3''(t) \end{bmatrix}$, the matrix-vector form

is

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