A Model With Clean and Dirty Capital Stocks and and R&D in Green Innovation-

Assume there are two capital sectors, each with AK production technology ($Y_i = A_i K_i,\ i\ = d, g$) and each with its own capital stock that evolves with quadratic adjustments costs and Brownian shocks as follows:

$$\mathrm{d}K_d/K_d = [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2]\,\mathrm{d}t + \sigma_d\,\mathrm{d}W$$
$$\mathrm{d}K_g/K_g = [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2]\,\mathrm{d}t + \sigma_g\,\mathrm{d}W$$

We also assume there is R&D investment that leads to an increased arrival rate of a one time jump in Sector 2 productivity. The arrival rate is denoted as t and evolves as follows

$$\mathrm{d}\lambda_t/\lambda_t = (\varphi i_\lambda - \alpha_\lambda)\,\mathrm{d}t + \sigma_\lambda\,\mathrm{d}W$$

The key difference between the sectors is that production from Sector 1 generates emissions. As a result,the evolution of atmospheric temperature is give by the Matthews Approximation, so that temperature Y_t and cumulative carbon emissions are given by

$$\mathrm{d}Y_t = E_t(\beta_f\,\mathrm{d}t + \varsigma\,\mathrm{d}W)$$

where β_f is the Matthews parameter and η is the scaling factor converting Sector d output $A_d K_d$ into emissions such that

$$E_t = \eta A_d K_d$$

Output can be used in for consumption, investment in either capital stock, or for R&D into improving the productivity of Sector g :

$$C = A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda$$

We assume exponential-quadratic damages to preferences so that our utility is augmented when accounting for climate damages. Flow utility is a log function over consumption, assuming perfect substitutability over output from the two sectors so that

$$U(C) = \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N_t$$

where the $\log N_t$ follows from BBH2 as

$$\log N_t = \Gamma(Y)$$
$$\Gamma(y) = \gamma_1 y + \frac{\gamma_2}{2} y^2 + \frac{\gamma_3}{2} \mathbf{1}_{y \geq \bar{y}} (y - \bar{y})^2$$

Taking these pieces together we get the HJB equation

$$\begin{aligned}
\delta V(K_d, K_g, \lambda, Y, \log N) = \max_{i_g, i_d, i_\lambda} & \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N \\
& + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} V_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} V_g K_g + \frac{\sigma_d^2 K_d^2}{2} V_{dd} + \frac{\sigma_g^2 K_g^2}{2} V_{gg} \\
& + \beta_f E_d V_Y + \frac{1}{2} \varsigma^2 E_d^2 V_{YY} + [\{\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - 2) \mathbf{I}\{Y_t > 2\}\} \beta_f E_d \\
& + \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2} \varsigma^2 E_d^2] V_{\log N} + \frac{\varsigma^2 E_d^2}{2} V_{\log N \log N} \\
& + (\varphi i_\lambda - \alpha_\lambda) \lambda V_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} V_{\lambda\lambda} + \lambda \left(V(K_d, K_g, \lambda, Y, \log N; A_g') - V(K_d, K_g, \lambda, Y, \log N; A_g) \right)
\end{aligned}$$

The FOC for investment and R&D are given by

$$
\begin{aligned}
0 =& -\delta(A_dK_d - i_dK_d + A_gK_g - i_gK_g - i_\lambda\lambda)^{-1} + (1-\phi_d i_d)V_d \\
0 =& -\delta(A_dK_d - i_dK_d + A_gK_g - i_gK_g - i_\lambda\lambda)^{-1} + (1-\phi_d i_d)V_d \\
0 =& -\delta(A_dK_d - i_dK_d + A_gK_g - i_gK_g - i_\lambda\lambda)^{-1} + \varphi V_\lambda
\end{aligned}
$$

Using the fact that $\varphi V_\lambda = \delta(A_dK_d - i_dK_d + A_gK_g - i_gK_g - i_\lambda\lambda)^{-1}$, we can simplify to

$$
\begin{aligned}
i_d =& \frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{V_\lambda}{V_d} \\
i_g =& \frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{V_\lambda}{V_g} \\
i_\lambda =& \frac{1}{\lambda}\left((A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{V_\lambda}{V_d}))K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{V_\lambda}{V_g}))K_g - \frac{\delta}{\varphi V_\lambda}\right)
\end{aligned}
$$

We can analytically simplify out $\log N_t$ to get a simplified HJB

$$
\begin{aligned}
\delta v(K_d, K_g, \lambda, Y) = & \max_{i_g, i_d, i_\lambda} \delta \log(A_dK_d - i_dK_d + A_gK_g - i_gK_g - i_\lambda\lambda) \\
& + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\
& + \beta_f E_d v_Y + \frac{1}{2}\varsigma^2 E_d^2 v_{YY} \\
& - [\{\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - 2)\mathbf{I}\{Y_t > 2\}\}\beta_f E_d + \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2}\varsigma^2 E_d^2] \\
& + (\varphi i_\lambda - \alpha_\lambda)\lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda\left(v(K_d, K_g, \lambda, Y; A_g') - v(K_d, K_g, \lambda, Y; A_g)\right)
\end{aligned}
$$

FOC

$$
\begin{aligned}
i_d =& \frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{v_\lambda}{v_d} \\
i_g =& \frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{v_\lambda}{v_g} \\
i_\lambda =& \frac{1}{\lambda}\left((A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d}\frac{v_\lambda}{v_d}))K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g}\frac{v_\lambda}{v_g}))K_g - \frac{\delta}{\varphi v_\lambda}\right)
\end{aligned}
$$

From this we can layer on different forms of uncertainty.

1 Post jump

1.1 Desired model

Replace E_d with $\eta A_d K_d$. We solve post jump HJB:

$$
\begin{aligned}
\delta v(K_d, K_g, Y; A_g') = & \max_{i_g, i_d} \delta \log(A_dK_d - i_dK_d + A_g'K_g - i_gK_g) \\
& + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\
& + \beta_f(\eta A_d K_d) v_Y + \frac{1}{2}\varsigma^2 (\eta A_d K_d)^2 v_{YY} - [\{\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - 2)\mathbf{I}\{Y_t > 2\}\}\beta_f(\eta A_d K_d)] \\
& - \frac{\gamma_2 + +\gamma_3 \mathbf{I}\{Y_t > 2\}}{2}[\varsigma^2 (\eta A_d K_d)^2]
\end{aligned}
$$

denote

$$mc = \delta(A_d K_d - i_d K_d + A'_g K_g - i_g K_g)^{-1}$$

FOC

$$\begin{aligned}
i_d =& \frac{1}{\phi_d} - \frac{mc}{\phi_d v_d} \\
i_g =& \frac{1}{\phi_g} - \frac{mc}{\phi_g v_g}
\end{aligned}$$

1.2 Test HJB, possible transformation of state variables

1.2.1 $\log K, L, Y$

$$X = [\log K, L, Y]', \quad \log K = \log(K_d + K_g), \quad L = \log K_g - \log K_d$$
$$R = \frac{K_g}{K_d + K_g} = \frac{\exp(L)}{1 + \exp(L)}$$

$$\begin{aligned}
\mathrm{d}(K_d + K_g) &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] K_d + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] K_g \right) \mathrm{d}t + (\sigma_d K_d + \sigma_g K_g)\, \mathrm{d}W \\
\mathrm{d}K/K &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2](1-R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R \right) \mathrm{d}t + (\sigma_d(1-R) + \sigma_g R)\, \mathrm{d}W \\
\mathrm{dlog}\, K &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2](1-R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R - \frac{\mid \sigma_d(1-R) + \sigma_g R \mid^2}{2} \right) \mathrm{d}t \\
&+ (\sigma_d(1-R) + \sigma_g R)\, \mathrm{d}W
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}L = \mathrm{d}(\log K_g - \log K_d) &= \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 - \frac{|\sigma_g|^2}{2}) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 - \frac{|\sigma_d|^2}{2}) \right) \mathrm{d}t + (\sigma_g - \sigma_d)\, \mathrm{d}W \\
\mathrm{d}\exp(L) = \exp(L)\, (\mathrm{d}L) &+ \frac{\exp(L)}{2} |\sigma_d - \sigma_g|^2\, \mathrm{d}t \\
\mathrm{d}R =&
\end{aligned}$$

$$\mathrm{d}Y = \eta A_d (1-R) K (\beta_f\, \mathrm{d}t + \varsigma\, \mathrm{d}W)$$

$$\begin{aligned}
\delta v(\log K, L, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1-R) + (A'_g - i_g)R) + \delta \log K \\
&+ \left(\{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\}(1-R) + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} R - \frac{\mid \sigma_d(1-R) + \sigma_g R \mid^2}{2} \right) v_k \\
&+ \frac{\mid \sigma_d(1-R) + \sigma_g R \mid^2}{2} v_{kk} \\
&+ \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha + i_d - \frac{\phi_d}{2} i_d^2) \right) v_l \\
&+ \beta_f (\eta A_d (1-R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1-R) K)^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d (1-R) K) + \frac{\gamma_2}{2} \varsigma^2 (\eta A_d (1-R) K)^2]
\end{aligned}$$

1.2.2 $\log K, R, Y$

$\log K \in [4., 8.5]$, $R \in [0.14, 0.99]$ and $Y \in [0., 3.]$

$$
\begin{aligned}
\delta v(\log K, R, Y; A_g') = &\max_{i_g, i_d} \delta \log((A_d - i_d)(1-R) + (A_g' - i_g)R) + \delta \log K \\
&+ \left(\{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\}(1-R) + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\}R - \frac{\mid \sigma_d(1-R) + \sigma_g R \mid^2}{2} \right) v_k \\
&+ \frac{\mid \sigma_d(1-R) + \sigma_g R \mid^2}{2} v_{kk} \\
&+ \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha + i_d - \frac{\phi_d}{2} i_d^2) \right) [R(1-R)]\, v_r \\
&+ \beta_f (\eta A_d (1-R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1-R) K)^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d (1-R) K) + \frac{\gamma_2}{2} \varsigma^2 (\eta A_d (1-R) K)^2]
\end{aligned}
$$

FOC

$$
mc = \delta((A_d - i_d)(1-R) + (A_g' - i_g)R)^{-1}
$$

$$
\begin{aligned}
mc =& (1 - \phi_d i_d)\left((v_k - R v_r)\right) && \Rightarrow i_d = \frac{1}{\phi_d}\left[1 - \frac{mc}{v_k - R v_r}\right] \\
mc =& (1 - \phi_g i_g)\left(v_k + (1-R) v_l\right) && \Rightarrow i_g = \frac{1}{\phi_g}\left[1 - \frac{mc}{v_k + (1-R) v_r}\right]
\end{aligned}
$$

1.3 test results

1.3.1 Preliminary results: try $i_d = i_g = 0$, no optimization

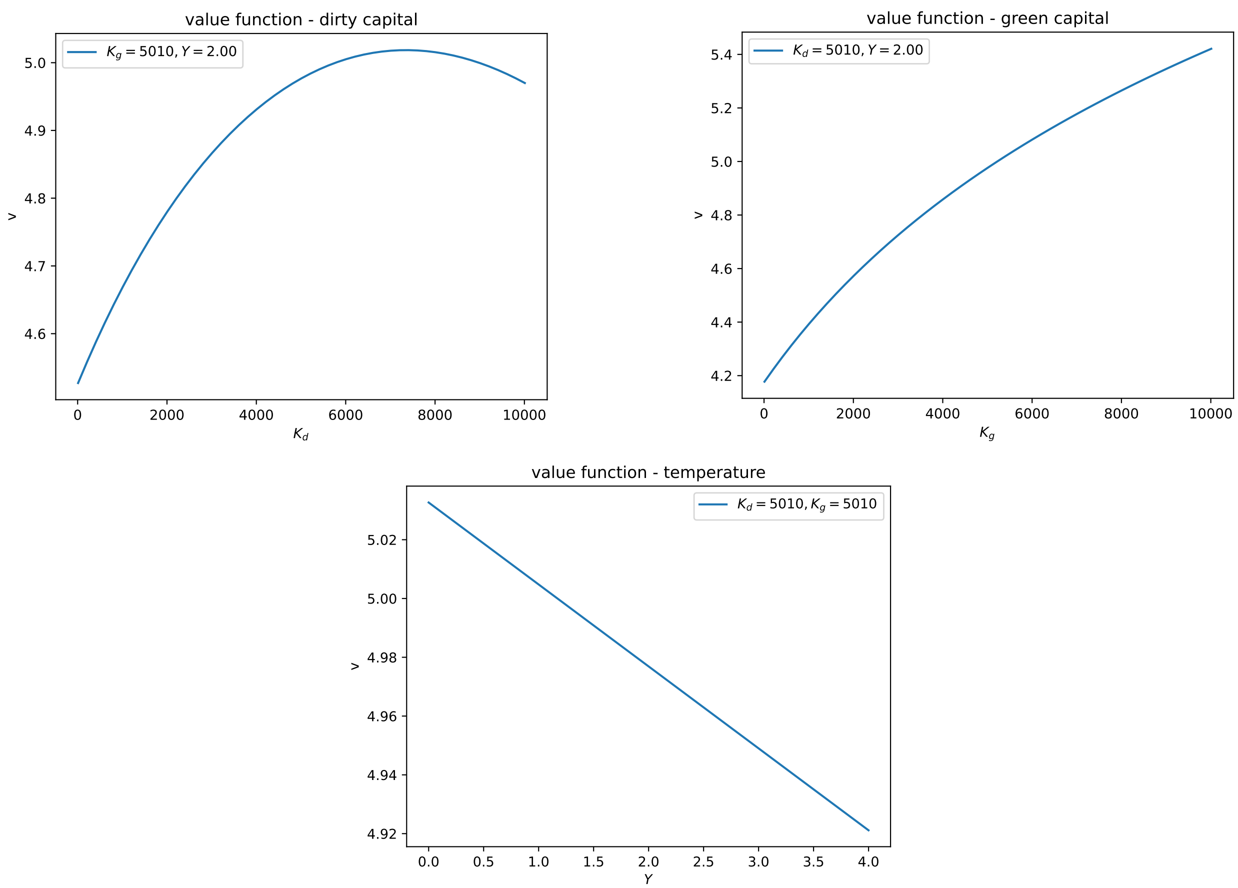

Figure 1: Results for value function, $i_d = 0, i_g = 0$

1.3.2 steady state

try set

$$\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 = 0 \Rightarrow i_d^* = 0.022$$

$$\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 = 0 \Rightarrow i_g^* = 0.022$$

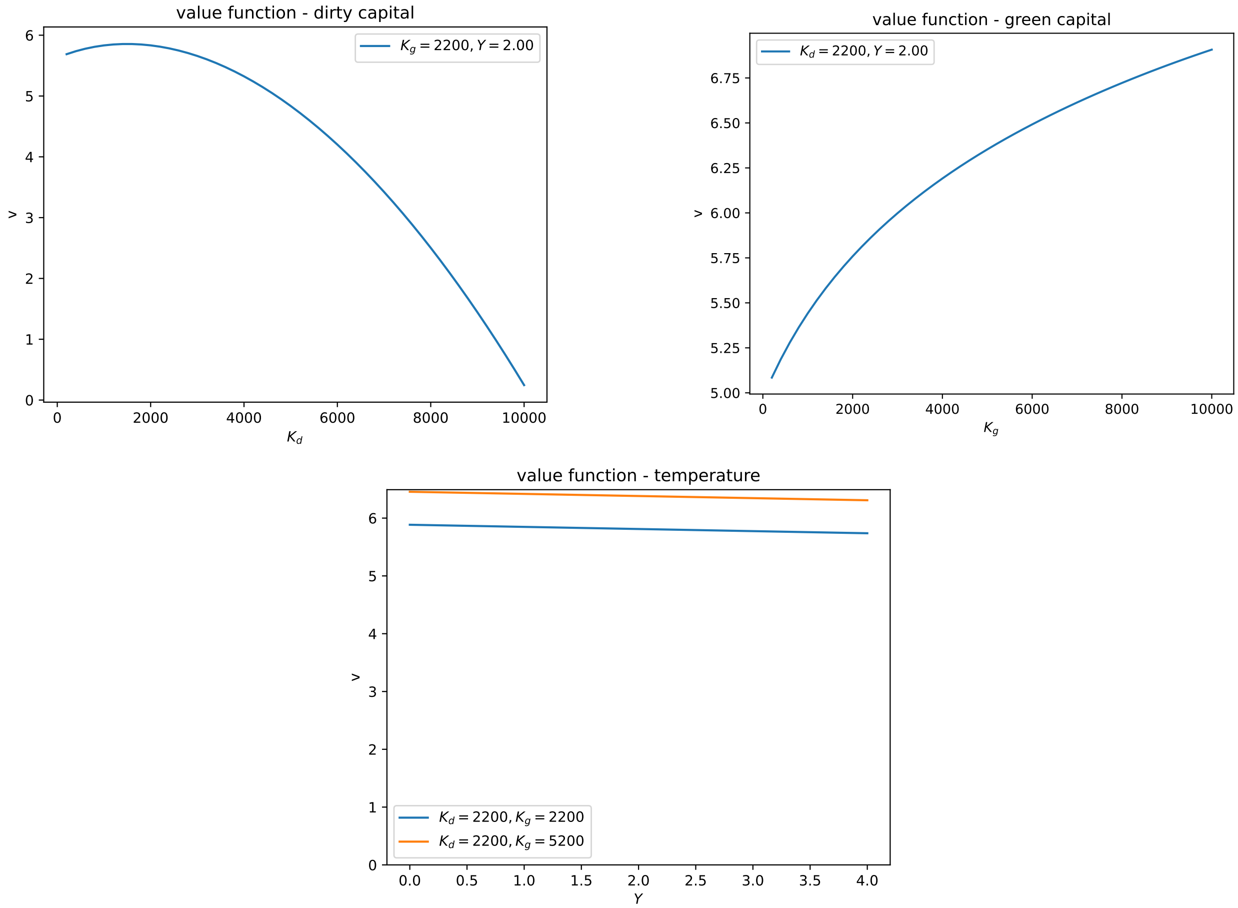

Figure 2: Results for value function, $i_d^* = 0.022, i_g^* = 0.022$

1.3.3 Results with $\gamma_3 = 0$

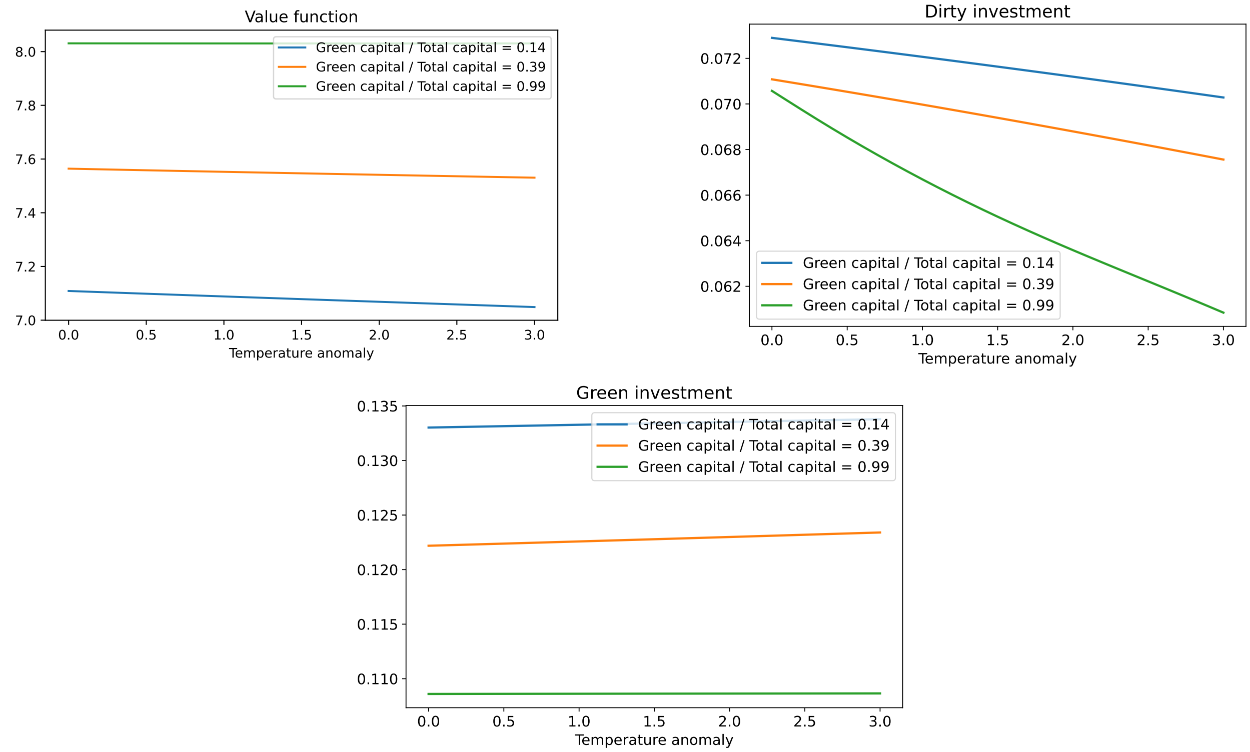

Figure 3: Results for value function, dirty investment and green investment

2 Pre jump

State variables $\log K, R, Y, \lambda$, HJB, controls i_d, i_g and $i_\lambda / K (\tilde{i_\lambda})$:

$$
\begin{aligned}
\delta v(\log K, R, Y, \lambda; A_g) = & \max_{i_g, i_d} \delta \log((A_d - i_d)(1-R) + (A_g - i_g)R - \tilde{i}_\lambda \lambda) + \delta \log K \\
& + \left(\{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\}(1-R) + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\}R - \frac{\mid \sigma_d (1-R) \mid^2 + \mid \sigma_g R \mid^2}{2} \right) v_k \\
& + \frac{\mid \sigma_d (1-R) \mid^2 + \mid \sigma_g R \mid^2}{2} v_{kk} \\
& + \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) \right) [R(1-R)] \, v_r \\
& + \frac{1}{2} R^2 (1-R)^2 \left[\sigma_d^2 + \sigma_g^2 \right] v_{rr} \\
& + \beta_f (\eta A_d (1-R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1-R) K)^2 v_{YY} \\
& - [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d (1-R) K) + \frac{\gamma_2}{2} \varsigma^2 (\eta A_d (1-R) K)^2] \\
& + (\varphi \tilde{i}_\lambda K - \alpha_\lambda) \lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda \left(v^{\text{post}}(\log K, R, Y; A_g') - v(\log K, R, Y, \lambda; A_g) \right)
\end{aligned}
$$

FOC

$$
\begin{aligned}
mc =& \delta((A_d - i_d)(1-R) + (A_g - i_g)R - \tilde{i}_\lambda \lambda)^{-1} \\
mc =& K\varphi v_\lambda
\end{aligned}
$$

$$
\begin{aligned}
mc =& (1-\phi_d i_d)\left((v_k - R v_r\right) & \Rightarrow i_d = \frac{1}{\phi_d}\left[1 - \frac{K\varphi v_\lambda}{v_k - R v_r}\right] \\
mc =& (1-\phi_g i_g)\left(v_k + (1-R)v_l\right) & \Rightarrow i_g = \frac{1}{\phi_g}\left[1 - \frac{K\varphi v_\lambda}{v_k + (1-R) v_r}\right]
\end{aligned}
$$

$$
\tilde{i}_\lambda = \frac{1}{\lambda}\left((A_d - (\frac{1}{\phi_d} - K\frac{\varphi}{\phi_d}\frac{v_\lambda}{v_k - R v_r}))K_d + (A_g - (\frac{1}{\phi_g} - K\frac{\varphi}{\phi_g}\frac{v_\lambda}{v_k + (1-R)v_r}))K_g - \frac{\delta}{K\varphi v_\lambda}\right)
$$

A State variables

Capital:

$$
\begin{aligned}
K_d \in& [0, 10,000] \\
K_g \in& [0, 10,000] \\
\lambda \in& [0, 0.1] \\
Y \in& [0, 4]
\end{aligned}
$$

B Parameters

Economy

Parameters	values
δ	0.01
$(\alpha_d, \phi_d, \sigma_d)$	(-0.02, 8, 0.016)
$(\alpha_g, \phi_g, \sigma_g)$	(-0.02, 8, 0.016)
$(\alpha_\lambda, \varphi, \sigma_\lambda)$	(0, 0.1, 0.016)
A_d	0.12
(A_g, A'_g)	(0.10, 0.15)

Temperature and damage

Parameters	values
β_f	1.86 / 1000
ς	1.2 * 1.86 / 1000
γ_1	0.00017675
γ_2	2 * 0.0022
γ_3	0
$\bar{y}$	2

η are determined as follows:

- Initial capital: $K_0 = 85/0.115 \approx 739$
- $K_{d,0} = K_0 \times \frac{2}{3} \approx 493$
- Choose η such that $10 = \eta \times 0.12 \times 493 \Rightarrow \eta \approx 0.17$