

## A Model With Clean and Dirty Capital Stocks and R&D in Green Innovation-

Assume there are two capital sectors, each with AK production technology ( $Y_i = A_i K_i$ ,  $i = d, g$ ) and each with its own capital stock that evolves with quadratic adjustments costs and Brownian shocks as follows:

$$\begin{aligned} dK_d/K_d &= [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] dt + \sigma_d dW \\ dK_g/K_g &= [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] dt + \sigma_g dW \end{aligned}$$

We also assume there is R&D investment that leads to an increased arrival rate of a one time jump in Sector 2 productivity. The arrival rate is denoted as  $\lambda$  and evolves as follows

$$d\lambda_t/\lambda_t = (\varphi i_\lambda - \alpha_\lambda) dt + \sigma_\lambda dW$$

The key difference between the sectors is that production from Sector 1 generates emissions. As a result, the evolution of atmospheric temperature is given by the Matthews Approximation, so that temperature  $Y_t$  and cumulative carbon emissions are given by

$$dY_t = E_t(\beta_f dt + \varsigma dW)$$

where  $\beta_f$  is the Matthews parameter and  $\eta$  is the scaling factor converting Sector  $d$  output  $A_d K_d$  into emissions such that

$$E_t = \eta A_d K_d$$

Output can be used in for consumption, investment in either capital stock, or for R&D into improving the productivity of Sector  $g$  :

$$C = A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda$$

We assume exponential-quadratic damages to preferences so that our utility is augmented when accounting for climate damages. Flow utility is a log function over consumption, assuming perfect substitutability over output from the two sectors so that

$$U(C) = \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N_t$$

where the  $\log N_t$  follows from BBH2 as

$$\begin{aligned} \log N_t &= \Gamma(Y) \\ \Gamma(y) &= \gamma_1 y + \frac{\gamma_2}{2} y^2 + \frac{\gamma_3}{2} \mathbf{1}_{y \geq \bar{y}} (y - \bar{y})^2 \end{aligned}$$

Taking these pieces together we get the HJB equation

$$\begin{aligned} \delta V(K_d, K_g, \lambda, Y, \log N) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N \\ &+ \left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} V_d K_d + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} V_g K_g + \frac{\sigma_d^2 K_d^2}{2} V_{dd} + \frac{\sigma_g^2 K_g^2}{2} V_{gg} \\ &+ \beta_f E_d V_Y + \frac{1}{2} \varsigma^2 E_d^2 V_{YY} + [\{\gamma_1 + \gamma_2 Y_t\} \beta_f E_d + \frac{\gamma_2}{2} \varsigma^2 E_d^2] V_{\log N} + \frac{\varsigma^2 E_d^2}{2} V_{\log N \log N} \\ &+ (\varphi i_\lambda - \alpha_\lambda) \lambda V_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} V_{\lambda\lambda} + \lambda (V(K_d, K_g, \lambda, Y, \log N; A'_g) - V(K_d, K_g, \lambda, Y, \log N; A_g)) \end{aligned}$$

The FOC for investment and R&D are given by

$$\begin{aligned} 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\ 0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \varphi V_\lambda \end{aligned}$$

Using the fact that  $\varphi V_\lambda = \delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1}$ , we can simplify to

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d} \\ i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g} \\ i_\lambda &= \frac{1}{\lambda} \left( (A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g})) K_g - \frac{\delta}{\varphi V_\lambda} \right) \end{aligned}$$

We can analytically simplify out  $\log Nt$  to get a simplified HJB

$$\begin{aligned} \delta v(K_d, K_g, \lambda, Y) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) \\ &\quad + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\ &\quad + \beta_f E_d v_Y + \frac{1}{2} \zeta^2 E_d^2 v_{YY} - [\{\gamma_1 + \gamma_2 Y_t\} \beta_f E_d + \frac{\gamma_2}{2} \zeta^2 E_d^2] \\ &\quad + (-\alpha_\lambda + \varphi i_\lambda) \lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda (v(K_d, K_g, \lambda, Y; A'_g) - v(K_d, K_g, \lambda, Y; A_g)) \end{aligned}$$

FOC

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d} \\ i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g} \\ i_\lambda &= \frac{1}{\lambda} \left( (A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g})) K_g - \frac{\delta}{\varphi v_\lambda} \right) \end{aligned}$$

From this we can layer on different forms of uncertainty.

## 1 Post jump

### 1.1 Desired model

Replace  $E_d$  with  $\eta A_d K_d$ . We solve post jump HJB:

$$\begin{aligned} \delta v(K_d, K_g, Y; A'_g) &= \max_{i_g, i_d} \delta \log(A_d K_d - i_d K_d + A'_g K_g - i_g K_g) \\ &\quad + \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\ &\quad + \beta_f (\eta A_d K_d) v_Y + \frac{1}{2} \zeta^2 (\eta A_d K_d)^2 v_{YY} - [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d K_d) + \frac{\gamma_2}{2} \zeta^2 (\eta A_d K_d)^2] \end{aligned}$$

denote

$$mc = \delta(A_d K_d - i_d K_d + A'_g K_g - i_g K_g)^{-1}$$

FOC

$$\begin{aligned} i_d &= \frac{1}{\phi_d} - \frac{mc}{\phi_d v_d} \\ i_g &= \frac{1}{\phi_g} - \frac{mc}{\phi_g v_g} \end{aligned}$$

## 1.2 Test HJB, possible transformation of state variables

### 1.2.1 $\log K, L, Y$

$$X = [\log K, L, Y]', \quad \log K = \log(K_d + K_g), \quad L = \log K_g - \log K_d$$

$$R = \frac{K_g}{K_d + K_g} = \frac{\exp(L)}{1 + \exp(L)}$$

$$\begin{aligned} d(K_d + K_g) &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] K_d + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] K_g \right) dt + (\sigma_d K_d + \sigma_g K_g) dW \\ dK/K &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R \right) dt + (\sigma_d (1 - R) + \sigma_g R) dW \\ d\log K &= \left( [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) dt \\ &\quad + (\sigma_d (1 - R) + \sigma_g R) dW \end{aligned}$$

$$dL = d(\log K_g - \log K_d) = \left( (\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) \right) dt + (\sigma_g - \sigma_d) dW$$

$$dY = \eta A_d (1 - R) K (\beta_f dt + \varsigma dW)$$

$$\begin{aligned} \delta v(\log K, L, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A'_g - i_g)R) + \delta \log K \\ &\quad + \left( \left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} (1 - R) + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) v_k \\ &\quad + \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} v_{kk} \\ &\quad + \left( (\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) \right) v_l \\ &\quad + \beta_f (\eta A_d (1 - R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1 - R) K)^2 v_{YY} \\ &\quad - [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d (1 - R) K) + \frac{\gamma_2^2}{2} \varsigma^2 (\eta A_d (1 - R) K)^2] \end{aligned}$$

### 1.2.2 $\log K, R, Y$

$$\begin{aligned}
\delta v(\log K, R, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A'_g - i_g)R) + \delta \log K \\
&+ \left( \{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \} (1 - R) + \{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \} R - \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} \right) v_k \\
&+ \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} v_{kk} \\
&+ \left( (\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) \right) [R(1 - R)] v_r \\
&+ \beta_f(\eta A_d(1 - R)K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d(1 - R)K)^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t\} \beta_f(\eta A_d(1 - R)K) + \frac{\gamma_2}{2} \varsigma^2 (\eta A_d(1 - R)K)^2]
\end{aligned}$$

FOC

$$mc = \delta((A_d - i_d)(1 - R) + (A'_g - i_g)R)^{-1}$$

$$\begin{aligned}
mc &= (1 - \phi_d i_d) (v_k - R v_r) & \Rightarrow i_d &= \frac{1}{\phi_d} \left[ 1 - \frac{mc}{v_k - R v_r} \right] \\
mc &= (1 - \phi_g i_g) (v_k + (1 - R) v_l) & \Rightarrow i_g &= \frac{1}{\phi_g} \left[ 1 - \frac{mc}{v_k + (1 - R) v_r} \right]
\end{aligned}$$

### 1.3 test results

#### 1.3.1 Preliminary results: try $i_d = i_g = 0$ , no optimization

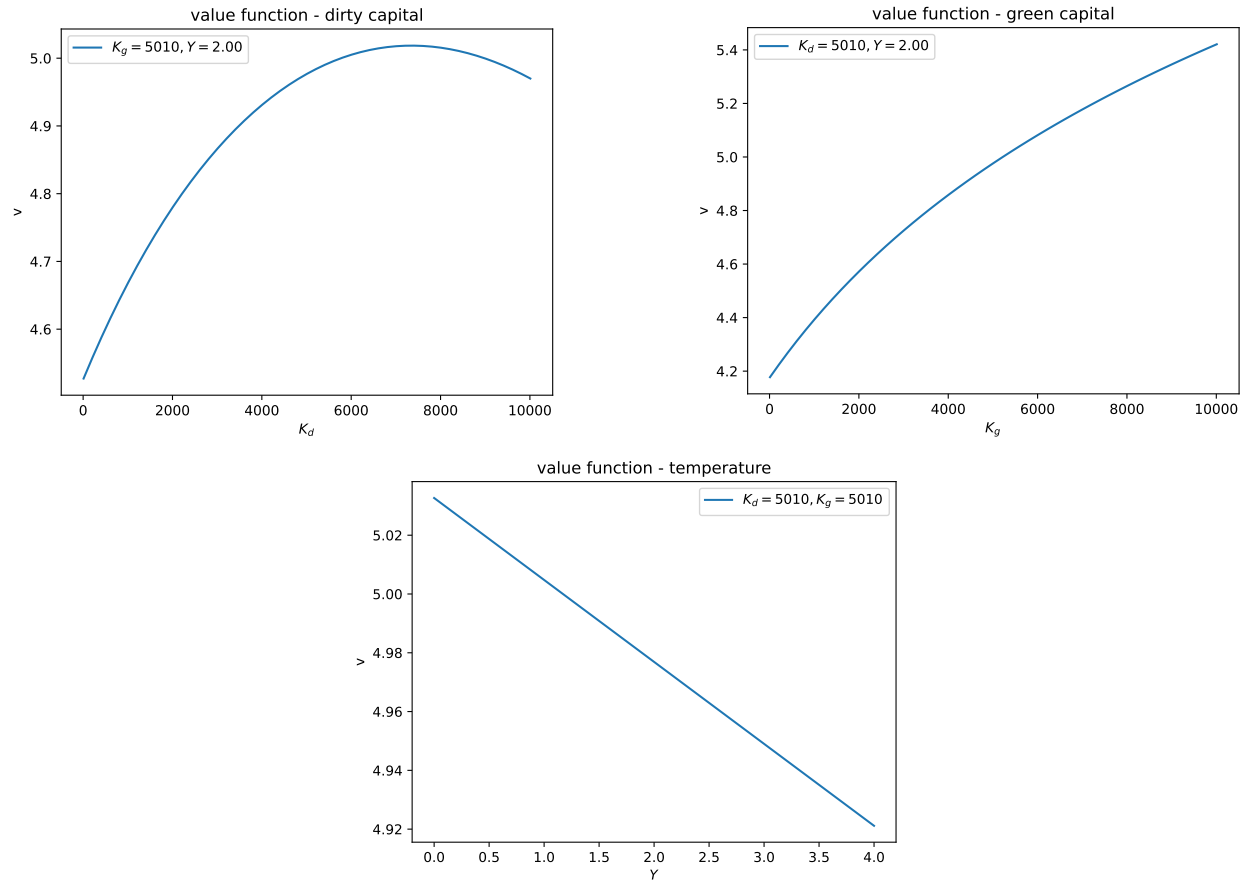


Figure 1: Results for value function,  $i_d = 0, i_g = 0$

#### 1.3.2 steady state

try set

$$\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 = 0 \Rightarrow i_d^* = 0.022$$

$$\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 = 0 \Rightarrow i_g^* = 0.022$$

Figure 2: Results for value function,  $i_d^* = 0.022, i_g^* = 0.022$

## 2 Pre jump

### A State variables

Capital:

$$K_d \in [0, 10,000]$$

$$K_g \in [0, 10,000]$$

$$\lambda \in [0, 0.1]$$

$$Y \in [0, 4]$$

### B Parameters

Economy

Parameters	values
$\delta$	0.01
$(\alpha_d, \phi_d, \sigma_d)$	(-0.02, 8, 0.016)
$(\alpha_g, \phi_g, \sigma_g)$	(-0.02, 8, 0.016)
$(\alpha_\lambda, \varphi, \sigma_\lambda)$	(0, 0.1, 0.016)
$A_d$	0.12
$(A_g, A'_g)$	(0.10, 0.15)

Temperature and damage

Parameters	values
$\beta_f$	1.86 / 1000
$\varsigma$	1.2 * 1.86 / 1000
$\gamma_1$	0.00017675
$\gamma_2$	2 * 0.0022
$\gamma_3$	0
$\bar{y}$	2

$\eta$  are determined as follows:

- Initial capital:  $K_0 = 85/0.115 \approx 739$
- $K_{d,0} = K_0 \times \frac{2}{3} \approx 493$
- Choose  $\eta$  such that  $10 = \eta \times 0.12 \times 493 \Rightarrow \eta \approx 0.17$