

A Model With Clean and Dirty Capital Stocks and and R&D in Green Innovation

Assume there are two capital sectors, each with AK production technology ($Y_i = A_i K_i$, $i = d, g$) and each with its own capital stock that evolves with quadratic adjustments costs and Brownian shocks as follows:

$$\begin{aligned} dK_d/K_d &= [\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] dt + \sigma_d dW \\ dK_g/K_g &= [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] dt + \sigma_g dW \end{aligned}$$

We also assume there is R&D investment that leads to an increased arrival rate of a one time jump in Sector 2 productivity. The arrival rate is denoted as t and evolves as follows

$$d\lambda_t/\lambda_t = (\varphi i_\lambda - \alpha_\lambda) dt + \sigma_\lambda dW$$

The key difference between the sectors is that production from Sector 1 generates emissions. As a result, the evolution of atmospheric temperature is give by the Matthews Approximation, so that temperature Y_t and cumulative carbon emissions are given by

$$dY_t = E_t(\beta_f dt + \varsigma dW)$$

where β_f is the Matthews parameter and η is the scaling factor converting Sector d output $A_d K_d$ into emissions such that

$$E_t = \eta A_d K_d$$

Output can be used in for consumption, investment in either capital stock, or for R&D into improving the productivity of Sector g :

$$C = A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda$$

We assume exponential-quadratic damages to preferences so that our utility is augmented when accounting for climate damages. Flow utility is a log function over consumption, assuming perfect substitutability over output from the two sectors so that

$$U(C) = \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N_t$$

where the $\log N_t$ follows from BBH2 as

$$\begin{aligned} \log N_t &= \Gamma(Y) \\ \Gamma(y) &= \gamma_1 y + \frac{\gamma_2}{2} y^2 + \frac{\gamma_3}{2} \mathbf{1}_{y \geq \bar{y}} (y - \bar{y})^2 \end{aligned}$$

Taking these pieces together we get the HJB equation

$$\begin{aligned} \delta V(K_d, K_g, \lambda, Y, \log N) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) - \delta \log N \\ &+ \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} V_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} V_g K_g + \frac{\sigma_d^2 K_d^2}{2} V_{dd} + \frac{\sigma_g^2 K_g^2}{2} V_{gg} \\ &+ \beta_f E_d V_Y + \frac{1}{2} \varsigma^2 E_d^2 V_{YY} + [\{\gamma_1 + \gamma_2 Y_t + \gamma_3 (Y_t - 2) \mathbf{I}\{Y_t > 2\}\} \beta_f E_d \\ &+ \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2} \varsigma^2 E_d^2] V_{\log N} + \frac{\varsigma^2 E_d^2}{2} V_{\log N \log N} \\ &+ (\varphi i_\lambda - \alpha_\lambda) \lambda V_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} V_{\lambda\lambda} + \lambda (V(K_d, K_g, \lambda, Y, \log N; A'_g) - V(K_d, K_g, \lambda, Y, \log N; A_g)) \end{aligned}$$

The FOC for investment and R&D are given by

$$\begin{aligned}
0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\
0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + (1 - \phi_d i_d) V_d \\
0 &= -\delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1} + \varphi V_\lambda
\end{aligned}$$

Using the fact that $\varphi V_\lambda = \delta(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda)^{-1}$, we can simplify to

$$\begin{aligned}
i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d} \\
i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g} \\
i_\lambda &= \frac{1}{\lambda} \left((A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{V_\lambda}{V_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{V_\lambda}{V_g})) K_g - \frac{\delta}{\varphi V_\lambda} \right)
\end{aligned}$$

We can analytically simplify out $\log N_t$ to get a simplified HJB

$$\begin{aligned}
\delta v(K_d, K_g, \lambda, Y) &= \max_{i_g, i_d, i_\lambda} \delta \log(A_d K_d - i_d K_d + A_g K_g - i_g K_g - i_\lambda \lambda) \\
&+ \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\
&+ \beta_f E_d v_Y + \frac{1}{2} \varsigma^2 E_d^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t + \gamma_3(Y_t - 2)\} \mathbf{I}\{Y_t > 2\}] \beta_f E_d + \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2} \varsigma^2 E_d^2 \\
&+ (\varphi i_\lambda - \alpha_\lambda) \lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda (v(K_d, K_g, \lambda, Y; A'_g) - v(K_d, K_g, \lambda, Y; A_g))
\end{aligned}$$

FOC

$$\begin{aligned}
i_d &= \frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d} \\
i_g &= \frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g} \\
i_\lambda &= \frac{1}{\lambda} \left((A_d - (\frac{1}{\phi_d} - \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_d})) K_d + (A_g - (\frac{1}{\phi_g} - \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_g})) K_g - \frac{\delta}{\varphi v_\lambda} \right)
\end{aligned}$$

From this we can layer on different forms of uncertainty.

1 Post jump

1.1 Desired model

Replace E_d with $\eta A_d K_d$. We solve post jump HJB:

$$\begin{aligned}
\delta v(K_d, K_g, Y; A'_g) &= \max_{i_g, i_d} \delta \log(A_d K_d - i_d K_d + A'_g K_g - i_g K_g) \\
&+ \{\alpha_d + i_d - \frac{\phi_d}{2} i_d^2\} v_d K_d + \{\alpha_g + i_g - \frac{\phi_g}{2} i_g^2\} v_g K_g + \frac{\sigma_d^2 K_d^2}{2} v_{dd} + \frac{\sigma_g^2 K_g^2}{2} v_{gg} \\
&+ \beta_f (\eta A_d K_d) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d K_d)^2 v_{YY} - [\{\gamma_1 + \gamma_2 Y_t + \gamma_3(Y_t - 2)\} \mathbf{I}\{Y_t > 2\}] \beta_f (\eta A_d K_d) \\
&- \frac{\gamma_2 + \gamma_3 \mathbf{I}\{Y_t > 2\}}{2} [\varsigma^2 (\eta A_d K_d)^2]
\end{aligned}$$

denote

$$mc = \delta(A_d K_d - i_d K_d + A'_g K_g - i_g K_g)^{-1}$$

FOC

$$i_d = \frac{1}{\phi_d} - \frac{mc}{\phi_d v_d}$$

$$i_g = \frac{1}{\phi_g} - \frac{mc}{\phi_g v_g}$$

1.2 Test HJB, possible transformation of state variables

1.2.1 $\log K, L, Y$

$$X = [\log K, L, Y]', \quad \log K = \log(K_d + K_g), \quad L = \log K_g - \log K_d$$

$$R = \frac{K_g}{K_d + K_g} = \frac{\exp(L)}{1 + \exp(L)}$$

$$\begin{aligned} d(K_d + K_g) &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] K_d + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] K_g \right) dt + (\sigma_d K_d + \sigma_g K_g) dW \\ dK/K &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R \right) dt + (\sigma_d (1 - R) + \sigma_g R) dW \\ d\log K &= \left([\alpha_d + i_d - \frac{\phi_d}{2} i_d^2] (1 - R) + [\alpha_g + i_g - \frac{\phi_g}{2} i_g^2] R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) dt \\ &\quad + (\sigma_d (1 - R) + \sigma_g R) dW \end{aligned}$$

$$dL = d(\log K_g - \log K_d) = \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 - \frac{|\sigma_g|^2}{2}) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 - \frac{|\sigma_d|^2}{2}) \right) dt + (\sigma_g - \sigma_d) dW$$

$$d\exp(L) = \exp(L) (dL) + \frac{\exp(L)}{2} |\sigma_d - \sigma_g|^2 dt$$

$$dR =$$

$$dY = \eta A_d (1 - R) K (\beta_f dt + \varsigma dW)$$

$$\begin{aligned} \delta v(\log K, L, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A'_g - i_g)R) + \delta \log K \\ &\quad + \left(\left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} (1 - R) + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} R - \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} \right) v_k \\ &\quad + \frac{|\sigma_d (1 - R) + \sigma_g R|^2}{2} v_{kk} \\ &\quad + \left((\alpha_g + i_g - \frac{\phi_g}{2} i_g^2) - (\alpha_d + i_d - \frac{\phi_d}{2} i_d^2) \right) v_l \\ &\quad + \beta_f (\eta A_d (1 - R) K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d (1 - R) K)^2 v_{YY} \\ &\quad - [\{\gamma_1 + \gamma_2 Y_t\} \beta_f (\eta A_d (1 - R) K) + \frac{\gamma_2^2}{2} \varsigma^2 (\eta A_d (1 - R) K)^2] \end{aligned}$$

1.2.2 $\log K, R, Y$

$\log K \in [4., 8.5]$, $R \in [0.14, 0.99]$ and $Y \in [0., 3.]$

$$\begin{aligned}
\delta v(\log K, R, Y; A'_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A'_g - i_g)R) + \delta \log K \\
&+ \left(\left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} (1 - R) + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} R - \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} \right) v_k \\
&+ \frac{|\sigma_d(1 - R) + \sigma_g R|^2}{2} v_{kk} \\
&+ \left(\left(\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right) - \left(\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right) \right) [R(1 - R)] v_r \\
&+ \beta_f(\eta A_d(1 - R)K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d(1 - R)K)^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t\} \beta_f(\eta A_d(1 - R)K) + \frac{\gamma_2^2}{2} \varsigma^2 (\eta A_d(1 - R)K)^2]
\end{aligned}$$

FOC

$$mc = \delta((A_d - i_d)(1 - R) + (A'_g - i_g)R)^{-1}$$

$$\begin{aligned}
mc &= (1 - \phi_d i_d) ((v_k - R v_r)) & \Rightarrow i_d &= \frac{1}{\phi_d} \left[1 - \frac{mc}{v_k - R v_r} \right] \\
mc &= (1 - \phi_g i_g) (v_k + (1 - R) v_l) & \Rightarrow i_g &= \frac{1}{\phi_g} \left[1 - \frac{mc}{v_k + (1 - R) v_r} \right]
\end{aligned}$$

2 Pre jump

State variables $\log K, R, Y, \lambda$, HJB, controls i_d, i_g and $i_\lambda/K(\tilde{i}_\lambda)$:

$$\begin{aligned}
\delta v(\log K, R, Y, \lambda; A_g) &= \max_{i_g, i_d} \delta \log((A_d - i_d)(1 - R) + (A_g - i_g)R - \tilde{i}_\lambda \lambda) + \delta \log K \\
&+ \left(\left\{ \alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right\} (1 - R) + \left\{ \alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right\} R - \frac{|\sigma_d(1 - R)|^2 + |\sigma_g R|^2}{2} \right) v_k \\
&+ \frac{|\sigma_d(1 - R)|^2 + |\sigma_g R|^2}{2} v_{kk} \\
&+ \left(\left(\alpha_g + i_g - \frac{\phi_g}{2} i_g^2 \right) - \left(\alpha_d + i_d - \frac{\phi_d}{2} i_d^2 \right) \right) [R(1 - R)] v_r \\
&+ \frac{1}{2} R^2 (1 - R)^2 [\sigma_d^2 + \sigma_g^2] v_{rr} \\
&+ \beta_f(\eta A_d(1 - R)K) v_Y + \frac{1}{2} \varsigma^2 (\eta A_d(1 - R)K)^2 v_{YY} \\
&- [\{\gamma_1 + \gamma_2 Y_t\} \beta_f(\eta A_d(1 - R)K) + \frac{\gamma_2^2}{2} \varsigma^2 (\eta A_d(1 - R)K)^2] \\
&+ (\varphi \tilde{i}_\lambda K - \alpha_\lambda) \lambda v_\lambda + \frac{(\sigma_\lambda \lambda)^2}{2} v_{\lambda\lambda} + \lambda (v^{\text{post}}(\log K, R, Y; A'_g) - v(\log K, R, Y, \lambda; A_g))
\end{aligned}$$

FOC

$$\begin{aligned}
mc &= \delta((A_d - i_d)(1 - R) + (A_g - i_g)R - \tilde{i}_\lambda \lambda)^{-1} \\
mc &= K \varphi v_\lambda
\end{aligned}$$

$$\begin{aligned}
mc &= (1 - \phi_d i_d) (v_k - Rv_r) & \Rightarrow i_d &= \frac{1}{\phi_d} \left[1 - \frac{K\varphi v_\lambda}{v_k - Rv_r} \right] \\
mc &= (1 - \phi_g i_g) (v_k + (1 - R)v_l) & \Rightarrow i_g &= \frac{1}{\phi_g} \left[1 - \frac{K\varphi v_\lambda}{v_k + (1 - R)v_r} \right]
\end{aligned}$$

$$\tilde{i}_\lambda = \frac{1}{\lambda} \left((A_d - (\frac{1}{\phi_d} - K \frac{\varphi}{\phi_d} \frac{v_\lambda}{v_k - Rv_r})) K_d + (A_g - (\frac{1}{\phi_g} - K \frac{\varphi}{\phi_g} \frac{v_\lambda}{v_k + (1 - R)v_r})) K_g - \frac{\delta}{K\varphi v_\lambda} \right)$$

A State variables

Capital:

$$\begin{aligned}
K_d &\in [0, 10,000] \\
K_g &\in [0, 10,000] \\
Y &\in [0, 4] \\
\lambda &\in [0, 0.1]
\end{aligned}$$

B Parameters

Economy

Parameters	values
δ	0.01
$(\alpha_d, \phi_d, \sigma_d)$	(-0.02, 8, 0.016)
$(\alpha_g, \phi_g, \sigma_g)$	(-0.02, 8, 0.016)
$(\alpha_\lambda, \varphi, \sigma_\lambda)$	(0, 0.1, 0.016)
A_d	0.12
(A_g, A'_g)	(0.10, 0.15)

Temperature and damage

Parameters	values
β_f	1.86 / 1000
ς	1.2 * 1.86 / 1000
γ_1	0.00017675
γ_2	2 * 0.0022
γ_3	0
\bar{y}	2

η are determined as follows:

- Initial capital: $K_0 = 85/0.115 \approx 739$
- $K_{d,0} = K_0 \times \frac{2}{3} \approx 493$
- Choose η such that $10 = \eta \times 0.12 \times 493 \Rightarrow \eta \approx 0.17$