Math 239 - Lecture 11 - Concatenation - (Block) Decomposition

Review:

Concatenation:

$AB = \{ab : a \in A, bE \in B\}$ set! (not multiset) unambiguous if
$\nexists a_1, a_2 \in A, b_1, b_2 \in B \; a_1 b_1 = a_2 b_2$

Star:

$A* = E \cup (A^1) \cup (A^2) \cup (A^3) \cup \ldots$

Unambiguous

- A^k unambiguous $\forall k \leq 0$
- all pairwise disjoint

Product Lemma

If AB is unambiguous then $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$.

Star Lemma

If $A*$ is unambiguous, then $\Phi_{A*}(x) = \dfrac{1}{1 - \Phi_A(x)}$

Note: we say an expression for binary strings is unambiguous if all products and stars unambiguous and unions disjoint

Expressions for the set of all binary strings:

Let B be the set of all binary strings;

#1: $B = \{0,1\}*$ is **unambiguous**, why?

- every string is uniquely reconstructable
- alternatively, the exp. comes from a "decomposition rule"

A **decomposition rule** is a rule to decompose a string into "parts" **uniquely**. Here the rule is: decompose where each part is a single digit

Ex: $[0][1][0][1]$

each part is in $0,1$, each string is formed from any # of these parts. So we get $0,1*$ as the expression

Remark: decomposing after every 2nd digit $00,01,10,11* \in, 0,1$ is another unique rule (we won't use). however decomposing where each part is length 1 or 2 is not unique (and hence is ambiguous)

Useful unambiguous for expressions for B:

Decomposition Rule	Expression
After each 0	$(\{1\}*\{0\})*, \{1\}*$
After each 1	$(\{0\}*\{1\})*\{0\}*$
After each block of 0 s	$\{0\}*(\{1\}\{1\}*\{0\}\{0\}*)*\{1\}*$
1 s	$\{1\}*(\{0\}\{0\}*\{1\}\{1\}*)*\{1\}*$

Why?

Main parts of Decomp (used any # of times)	Final part
$\{0,10,110,1110,\ldots\}=\{1\}^*\{0\}$	$\{\in,1,11,111,\ldots\}=\{1\}^*$
$\{0\}^*\{1\}$	$\{0\}^*$

Beginning part	Main part	Final part
(possibly empty) block of 0 s: $\{0\}^*$	non empty block of 1s followed by a non empty block of 0 s: $\{1,11,111,\ldots\}\{0,00,000,\ldots\}$	(possibly empty) block of 1s: $\{1\}^*$

Finding Gen series from Block Decompositions:

the 1st rule $B=0,1^*$ is really only useful to find Φ_B. By Star Lemma

$$\Phi_B(x)=\frac{1}{1-\Phi_{\{0,1\}(x)}}=\frac{1}{1-2x}$$

One can check the other rules also give $\Phi_B(x)=\dfrac{1}{1-2x}$

Using the block decomposition rules, we can find unambiguous expresions for sets of binary strings w/ restriction parts and from there use sum/product/star lemmas to find phi_S

Ex. No blocks of exactly 4 1's: We choose a decomposition where the restriction can be used on the parts.

Let's use after each block of 1's: $B=\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$

We modify the exp. by modifying the acceptable parts

	Before(all)	After (number of blocks of ex. 4 1's)
Beg part	$\{1\}*$	$\{\in,1,11,111,11111,\ldots\}$ (no 4 1's)
Main part	$\{0\}\{0\}*\{1\}\{1\}*$	$\{0,00,\ldots\}\{1,11,111,11111,\ldots\}$
Final part	$\{0\}*$	$\{0\}*$

$\{\in,1,11,111,11111,\ldots\}(\{0\}\{0\}*\{1\}\{\in,1,11,1111,\ldots\})*\{0\}*=(\{\in,1,11,111\}\cup\{11111\}\{1\}*)(\{0\}\{0\}*\{1\}(\{\in,1,11\}\cup\{1111\}\{1\}*)*\{0\}*$

Exp $\left(1+x+x^2+x^3+\frac{x^5}{1-x}\right)\left(\frac{1}{1-x\cdot\frac{1}{1-x}\cdot x\cdot\left(1+x+x^2+\frac{x^4}{1-x}\right)}\right)\left(\frac{1}{1-x}\right)$

Reminder main part is starred

so it will show as $\frac{1}{1-\Phi_{\text{Main}}}$ in the middle You final expression is, $\Phi_{\text{Beg}}\left(\frac{1}{1-\Phi_{\text{Main}}}\right)\Phi_{\text{end}}$

- it may be useful to calculate Φ_{Beg}, Φ_{Main}, and Φ_{end} first