

# Math 239 - Lecture 11 - Concatenation - (Block) Decomposition

## Review:

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### Concatenation:

$AB = \{ab : a \in A, b \in B\}$  set! (not multiset) unambiguous if  
 $\nexists a_1, a_2 \in A, b_1, b_2 \in B \text{ } a_1b_1 = a_2b_2$

### Star:

$$A^* = E \cup (A^1) \cup (A^2) \cup (A^3) \cup \dots$$

### Unambiguous

- $A^k$  unambiguous  $\forall k \geq 0$
- all pairwise disjoint

### Product Lemma

If  $AB$  is unambiguous then  $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$ .

### Star Lemma

If  $A^*$  is unambiguous, then  $\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$

**Note:** we say an expression for binary strings is unambiguous if all products and stars unambiguous and unions disjoint

## Expressions for the set of all binary strings:

Let  $B$  be the set of all binary strings;

**#1:**  $B = \{0,1\}^*$  is **unambiguous**, why?

- every string is uniquely reconstructable
- alternatively, the exp. comes from a "decomposition rule"

A **decomposition rule** is a rule to decompose a string into "parts" **uniquely**. Here the rule is: decompose where each part is a single digit

Ex:  $[0][1][0][1]$

each part is in  $0,1$ , each string is formed from any # of these parts. So we get  $0,1^*$  as the expression

**Remark:** decomposing after every 2nd digit  $00,01,10,11^* \in 0,1$  is another unique rule (we won't use). however decomposing where each part is length 1 or 2 is not unique (and hence is ambiguous)

### Useful unambiguous for expressions for $B$ :

Decomposition Rule	Expression
After each 0	$(\{1\}^* \{0\})^*, \{1\}^*$
After each 1	$(\{0\}^* \{1\})^* \{0\}^*$
After each block of 0s	$\{0\}^* (\{1\} \{1\}^* \{0\} \{0\}^*)^* \{1\}^*$
1s	$\{1\}^* (\{0\} \{0\}^* \{1\} \{1\}^*)^* \{1\}^*$

## Why?

Main parts of Decomp (used any # of times)		Final part
$\{0,10,110,1110,\dots\} = \{1\}^* \{0\}$		$\{\epsilon,1,11,111,\dots\} = \{1\}^*$
$\{0\}^* \{1\}$		$\{0\}^*$
Beginning part	Main part	Final part
(possibly empty) block of 0 s: $\{0\}^*$	non empty block of 1 s followed by a non empty block of 0 s: $\{1,11,111,\dots\} \{0,00,000,\dots\}$	(possibly empty) block of 1 s: $\{1\}^*$

## Finding Gen series from Block Decompositions:

the 1st rule  $B = 0,1^*$  is really only useful to find  $\Phi_B$ . By Star Lemma

$$\Phi_B(x) = \frac{1}{1 - \Phi_{\{0,1\}(x)}} = \frac{1}{1 - 2x}$$

One can check the other rules also give  $\Phi_B(x) = \frac{1}{1 - 2x}$

Using the block decomposition rules, we can find unambiguous expressions for sets of binary strings w/ restriction parts and from there use sum/product/star lemmas to find  $\Phi_S$

**Ex.** No blocks of exactly 4 1's: We choose a decomposition where the restriction can be used on the parts.

Let's use after each block of 1's:  $B = \{1\}^* (\{0\} \{0\}^* \{1\} \{1\}^*)^* \{0\}^*$

We modify the exp. by modifying the acceptable parts

	Before(all)	After (number of blocks of ex. 4 1's)
Beg part	$\{1\}^*$	$\{\epsilon, 1, 11, 111, 1111, \dots\}$ (no 4 1's)
Main part	$\{0\}\{0\}^*\{1\}\{1\}^*$	$\{0, 00, \dots\}\{1, 11, 111, 1111, \dots\}$
Final part	$\{0\}^*$	$\{0\}^*$

$$\{\epsilon, 1, 11, 111, 1111, \dots\}(\{0\}\{0\}^*\{1\}\{\epsilon, 1, 11, 111, \dots\})^*\{0\}^* = (\{\epsilon, 1, 11, 111\} \cup \{1111\}\{1\}^*)(\{0\}\{0\}^*\{1\}(\{\epsilon, 1, 11\} \cup \{1111\}\{1\}^*))^*\{0\}^*$$

$$\text{Exp} \left( 1 + x + x^2 + x^3 + \frac{x^5}{1-x} \right) \left( \frac{1}{1-x \cdot \frac{1}{1-x} \cdot x \cdot \left( 1 + x + x^2 + \frac{x^4}{1-x} \right)} \right) \left( \frac{1}{1-x} \right)$$

## Reminder main part is starred

so it will show as  $\frac{1}{1-\Phi_{\text{Main}}}$  in the middle You final expression is,  $\Phi_{\text{Beg}} \left( \frac{1}{1-\Phi_{\text{Main}}} \right) \Phi_{\text{end}}$

- it may be useful to calculate  $\Phi_{\text{Beg}}$ ,  $\Phi_{\text{Main}}$ , and  $\Phi_{\text{end}}$  first