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Feyn Calc – Computer-algebraic calculation of Feynman amplitudes

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We present the computer program *Feyn Calc* for automatic algebraic calculation of Feynman amplitudes. The purpose of the package is to facilitate the process of calculating radiative corrections in renormalizable gauge theories. We have used the program to calculate tree level diagrams as well as 1- and 2-loop corrections in the Standard Model. The package is written in the computer algebra languages *Macsyma* as well as *Mathematica* allowing the necessary degree of generality.

1. Introduction

In order to investigate in more detail the presently very successful Standard Model (SM) of electroweak interactions perturbative evaluation is indispensable. For calculating radiative corrections always the same procedure has to be done: application of Feynman rules, calculation and summation of Feynman diagrams and finally feeding of the analytic results into a numerical program which calculates the cross section. In principle this is straightforward, but as one approaches more complicated processes in 1-loop order with hundreds of diagrams (e.g. $e^+e^- \to W^+W^-$) or very complicated ones (e.g. $gg \to t\bar{t}$ or $W^+W^- \to W^+W^-$) the work becomes quite cumbersome and boring. Therefore the need of computeralgebraic programs has been felt for a long time. The realization of this idea was done in two different directions: In the sixties M. Veltman wrote the special purpose program *Schoonschip* [1] which is "action based", i.e. the user has to specify the next command at each step. Schoonschip is written in *Assembler* resulting in a high speed. At about the same time *Reduce* [2] was developed by A.C. Hearn. Here the approach was more general: written in *Lisp* more mathematical knowledge has been incorporated; e.g. factoring algorithms and integration routines. But the price for a higher degree of generality is always slower performance of the program.

In the last decades more complicated theories like QCD, the Glashow–Salam–Weinberg-model, extended electroweak models, SUSY and GUTs have been developed and investigated perturbatively. Applying the traditional methods (hand calculations, Schoonschip, Reduce) to some of these models, the desire for a higher degree of automation arose.

The first step of automation, the automatic Feynman rule application and diagram drawing, is done by the program *Feyn Arts* [3]. As input of *Feyn Arts*, just the incoming and outgoing particles and the considered model have to be specified. The output consists of Feynman graph drawings and the corresponding analytical expressions which result from the application of the Feynman rules. Now the algebraic calculation of the amplitude has to be done, yielding a result which is suitable for further numerical evaluation. For this task *Feyn Calc* was designed and developed.

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The design goal of the program is the applicability to general renormalizable gauge theories for an arbitrary process up to the n-loop level. The status quo of *Feyn Calc* is that the calculation of processes up to four external particles at the 1-loop level and 2-loop self-energies in the SM are handled well. Checks have been performed by comparison of results with hand calculations for $e^+e^- \to W^+W^-$, $e^+e^- \to Z\,H$, $gg \to t\bar{t}$ and 2-loop photon self-energy diagrams.

The package *Feyn Calc* is realized as two versions in the general purpose languages *Macsyma* [4] and *Mathematica* [5]. The program is generically slower than the optimized Schoonschip, but the time loss is acceptable; e.g. the calculation of a typical box diagram for $gg \to t\bar{t}$ or a 2-loop photon self-energy graph takes a few minutes with the Mathematica version of *Feyn Calc* on a 386 PC (4MB). Since this turns out to be approximately an order of magnitude slower than Schoonschip or the modernized successor *Form* it seems a real drawback. But this disadvantage is more than compensated by the fact that all diagrams are calculated with the same program automatically, i.e. once the standard matrix elements and the kinematical substitutions are specified, no further interactive work has to be done. Furthermore the advent of fast workstations (> 20 MIPS) helps to overcome this CPU-time problem.

2. The structure of *Feyn Calc*

The program *Feyn Calc* performs an automatic algebraic calculation of Feynman diagrams leading to a polynomial multiplying predefined (process dependent) standard matrix elements with Mandelstam variables, parameters of the model and invariant integrals. The resulting expression for the invariant amplitude can be subsequently translated into *Fortran* or *C*. Adding the relevant counter term diagrams, squaring of amplitudes and the numerical evaluation of the remaining integrals still have to be done by an additional program in Fortran or in C.

Apart from the automatic algebraic calculation, one may of course use only parts of the provided package, e.g. for trace calculations (in 4 or D dimensions) or manipulations of the D-dimensional *Dirac matrices* and Lorentz vectors, just like Schoonschip. For subsequent algebraic or analytic manipulations one may use the usual powerful capabilities of Mathematica or Macsyma.

The input of *Feyn Calc* may consist of the *Feyn Arts* output or a suitable hand typed file containing the analytical expressions of the Feynman diagrams. Additionally for each complete process certain specifications for standard matrix elements, simplifications and substitutions (e.g. for introducing kinematical variables) should be given.

2.1. Relations used

We list the used definitions and the implemented relations. The basic relation defining the D-dimensional Dirac-algebra for the matrices γ^μ is:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbf{1}, \quad \mu, \nu = (0, \cdots, D-1); \tag{2.1}$$

consequently

$$\gamma^\mu \gamma_\mu = g^\mu_\mu \cdot \mathbf{1} = D \cdot \mathbf{1}. \tag{2.2}$$

We define the matrix γ_5 in four dimensions as

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{2.3}$$

where $\epsilon^{\mu\nu\rho\sigma}$ denotes the total antisymmetric Levi–Civita-tensor with $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}$ and $\epsilon^{0123} = +1$.

This γ_5 satisfies

$$\{\gamma_5, \gamma^\nu\} = 0, \qquad \gamma_5^2 = \mathbf{1}, \quad \nu = (0, \cdots, 3). \tag{2.4}$$

We will use an anticommuting γ^5 also in D dimensions. For the discussion on this subtle subject see ref. [6] and references therein.

Defining the chiral projectors

$$\gamma_6 = \tfrac{1}{2}(1 + \gamma_5), \ \gamma_7 = \tfrac{1}{2}(1 - \gamma_5), \tag{2.5}$$

with

$$\gamma_6^2 = \gamma_6, \qquad \gamma_7^2 = \gamma_7, \qquad \gamma_6\gamma_7 = 0 = \gamma_7\gamma_6, \tag{2.6}$$

the following relations are obtained from (2.4):

$$\gamma_6\gamma^\mu = \gamma^\mu\gamma_7, \qquad \gamma_7\gamma^\mu = \gamma^\mu\gamma_6. \tag{2.7}$$

For *Dirac chains* we use the notation

- $\Gamma^{(l)} = \gamma_{\mu_1}\gamma_{\mu_2} \cdots \gamma_{\mu_{l-1}}\gamma_{\mu_l}$,
- $\Gamma_R^{(l)} = \gamma_{\mu_l}\gamma_{\mu_{l-1}} \cdots \gamma_{\mu_2}\gamma_{\mu_1}$,
- $\Gamma_i^{(l)} = \gamma_{\mu_1} \cdots \gamma_{\mu_{i-1}}\gamma_{\mu_{i+1}} \cdots \gamma_{\mu_l}$,
- $\Gamma_{ij}^{(l)} = \gamma_{\mu_1} \cdots \gamma_{\mu_{i-1}}\gamma_{\mu_{i+1}} \cdots \gamma_{\mu_{j-1}}\gamma_{\mu_{j+1}} \cdots \gamma_{\mu_l}$.

If $\Gamma^{(l)}$ consists of an odd number of γ-matrices, the following relations are valid in four dimensions:

$$\gamma^\mu\Gamma^{(l)}\gamma_\mu = -2\Gamma_R^{(l)}, \qquad \gamma^\mu\Gamma^{(l)}\gamma_\nu\gamma_\mu = 2\Gamma_R^{(l)}\gamma_\nu + 2\gamma_\nu\Gamma^{(l)}. \tag{2.8}$$

The D-dimensional case (for even or odd l) is treated with the very useful and – to our knowledge – most compact contraction formula:

$$\gamma^\mu\Gamma^{(l)}\gamma_\mu = (-1)^l\left\{(D - 2l)\Gamma^{(l)} - 4\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}(-1)^{j-i}\Gamma_{ij}^{(l)}g_{\mu_i\mu_j}\right\}. \tag{2.9}$$

Furthermore we use

$$\not{q}\Gamma^{(l)}\not{q} = (-1)^l q^2\Gamma^{(l)} + 2\sum_{i=1}^{l}(-1)^{i+1}\Gamma_i^{(l)}\not{q}q_{\mu_i}. \tag{2.10}$$

The *Dirac equation* for the fermion spinors $u(p, m)$, $v(p, m)$ and their adjoints $\bar{u}$, $\bar{v}$ reads

$$(\not{p} - m)u(p, m) = 0, \qquad (\not{p} + m)v(p, m) = 0, \tag{2.11}$$

$$\bar{u}(p, m)(\not{p} - m) = 0, \qquad \bar{v}(p, m)(\not{p} + m) = 0. \tag{2.12}$$

2.2. *The algorithm of Feyn Calc*

The algorithm for the calculation of 1-loop diagrams consists of the following steps which are run through automatically, some of them several times. The result is always a polynomial in masses, kinematical variables (Mandelstam variables) and matrix elements containing polarization vectors and/or Dirac chains, depending on the process.

1. Contraction of Lorentz indices, but not yet using eqs. (2.9) and (2.10).
2. Use of transverality of polarization vectors of vector particles: $k^\mu\epsilon_\mu(k) = 0$.

3. Application of energy momentum conservation (if specified).
4. Partial controlled expansion; no expansion of sums of Dirac matrices yet, i.e. sums of Dirac matrices are treated as a single variable in order to minimize the intermediate expression swell; e.g.: $\bar{v}(\not{p}+\not{q}+m)\not{\epsilon}u$ will be expanded to $v(\not{p}+\not{q})\not{\epsilon}u+m\bar{v}\not{\epsilon}u$.
5. Anticommutation of all γ_5, γ_6, γ_7 to the "right end" of each product of Dirac matrices (enclosed by spinors or occuring in Dirac traces).
6. Contraction of all $\gamma^\mu \not{\epsilon} \not{p} \cdots \gamma_\mu$ and $\not{q}\not{\epsilon}\not{p}\cdots\not{q}$, i.e. using eqs. (2.9) and (2.10).
7. Full expansion of products of sums of γ-matrices.
8. Application of the Dirac equation (after moving the relevant terms through the Dirac chain adjacent to the spinors).
9. Trace evaluation, if necessary (see section 2.3).
10. Canonical ordering of slashed γ's for combining terms and minimizing the number of terms.
11. Optional: Treatment of q^2 in the numerator via $q^2=(q^2-m^2)+m^2$ and cancellation of q^2-m^2 with factors in the denominator (resulting in lower-order invariant integrals).
12. Collecting terms according to the number of q_μ's (rank of tensor) in each summand.
13. Tensor integral decomposition, i.e. using eqs. (2.22) and (2.23); collecting coefficients of invariant Passarino–Veltman integrals.
14. Taking the limit for $D\to 4-\epsilon$, i.e.: $\epsilon A_0(m)=2m^2$, $\epsilon B_0=2$, $\epsilon B_1=-1$, $\epsilon B_0=\frac{1}{2}(-\frac{1}{3}p^2+m_1^2+m_2^2)$, $\epsilon B_{11}=\frac{2}{3}$, $\epsilon C_{00}=\frac{1}{2}$, $\epsilon C_{001}=\epsilon C_{002}=-\frac{1}{6}$, $\epsilon D_{0000}=-\frac{1}{12}$, (see eqs. (2.22), (2.23)).
15. Repeating steps 1, 2, 5, 6, 7, 8.
16. Canonical ordering of (slashed) γ's in order to match the predefined matrix elements.
17. Insertion of simplifications like Mandelstam variables for scalar products, abbreviations for standard matrix elements, etc.
18. Structured output (factoring out standard matrix elements or invariant integrals).
19. Optional translation into Fortran or C code.

2.3. Trace calculation

The algorithm for the calculation of traces is fairly straightforward. The procedure is easy and therefore reliable. Subsequently we list the necessary relations in order to calculate any arbitrary * D- or 4-dimensional trace.
The well known formulas for the recursive calculation of traces are:

$$\mathrm{Tr}(\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}})=0, \tag{2.13}$$

and

$$\mathrm{Tr}(\gamma_{\mu_1}\cdots\gamma_{\mu_{2n}})=g_{\mu_1\mu_2}\,\mathrm{Tr}(\gamma_{\mu_3}\cdots\gamma_{\mu_{2n}})-g_{\mu_1\mu_3}\,\mathrm{Tr}(\gamma_{\mu_2}\gamma_{\mu_4}\cdots\gamma_{\mu_{2n}})$$
$$+\cdots+g_{\mu_1\mu_{2n}}\,\mathrm{Tr}(\gamma_{\mu_2}\cdots\gamma_{\mu_{2n-1}}). \tag{2.14}$$

* The arbitrariness is only limited by the available memory of the used hardware: while a PC is usually already quite busy with 10^3 terms a good workstation will handle 10^4 safely and 10^5–10^6 terms should be in the realm of the most powerful machines running computer algebra software.

The calculation of traces is done recursively with eq. (2.14) down to the special cases:

$$\mathrm{Tr}\mathbf{1} = 4,\ \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{2.15}$$

It follows (in four dimensions):

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^\mu\gamma^5) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5) = 0. \tag{2.16}$$

Using the identity

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}\gamma^\sigma\gamma^5, \tag{2.17}$$

the following recursion formula for traces involving γ_5's (in four dimensions) is derived:

$$\begin{aligned}\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}\gamma^5) = {} & g^{\mu_{2n-2}\mu_{2n-1}}\,\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}\gamma^5) - g^{\mu_{2n-2}\mu_{2n}}\,\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n-1}}\gamma^5)\\ & + g^{\mu_{2n-1}\mu_{2n}}\,\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n-2}}\gamma^5) + \mathrm{i}\epsilon^{\mu_{2n-2}\mu_{2n-1}\sigma}\,\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n-3}}\gamma^\sigma).\end{aligned} \tag{2.18}$$

Special cases are

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}, \tag{2.19}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\tau\gamma^5) = -4\mathrm{i}(g^{\mu\nu}\epsilon^{\rho\sigma\lambda\tau} - g^{\mu\rho}\epsilon^{\nu\sigma\lambda\tau} + g^{\nu\rho}\epsilon^{\mu\sigma\lambda\tau} + g^{\sigma\lambda}\epsilon^{\mu\nu\rho\tau} - g^{\sigma\tau}\epsilon^{\mu\nu\rho\lambda} + g^{\lambda\tau}\epsilon^{\mu\nu\rho\sigma}). \tag{2.20}$$

If necessary the formulas above may be extended in their applicability to the case where within the framework of D-dimensional γ-matrices as four-dimensional γ_5 is used. Calculating products of traces one might get products of ϵ's, which are eliminated by:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'} = -\begin{vmatrix} g^{\mu\mu'} & g^{\mu\nu'} & g^{\mu\rho'} & g^{\mu\sigma'} \\ g^{\nu\mu'} & g^{\nu\nu'} & g^{\nu\rho'} & g^{\nu\sigma'} \\ g^{\rho\mu'} & g^{\rho\nu'} & g^{\rho\rho'} & g^{\rho\sigma'} \\ g^{\sigma\mu'} & g^{\sigma\nu'} & g^{\sigma\rho'} & g^{\sigma\sigma'} \end{vmatrix}. \tag{2.21}$$

The basic strategy for the trace calculations consists in simplifying the expressions *before* actually calculating the trace.

The steps for the calculation of Dirac traces (which are all done automatically) are:

1. Expand all arguments of the traces and use linearity of the trace.
2. Move all γ_5, γ_6 and γ_7 to the right using eq. (2.7).
3. Bring all traces to standard form where the γ-matrix with the smallest index becomes the first one; using the cyclic property of the trace.
4. Eliminate γ's with equal indices (and slashed objects) using the contraction formulas (2.9) and (2.10).
5. Insert the explicit expression (2.5) for γ_6 and γ_7.
6. Calculate the traces using eqs. (2.13)–(2.16) and (2.18)–(2.20).
7. Contract Lorentz indices, if possible.
8. Simplify the result, i.e. factorization and eventually substitution of Mandelstam variables in such a way, that the resulting expression becomes as short as possible.

2.4. Tensor integral decomposition

The scalar and tensorial integral are defined in the following way (note that we use slightly different conventions than ref. [7]):

$$
\begin{aligned}
A_0 &= \frac{\mu^{D-4}}{\mathrm{i}\pi^2}\int \frac{\mathrm{d}^D q}{(2\pi)^{D-4}}\frac{1}{\left[q^2-m_1^2\right]},\\
\{B_0;\ B_\mu;\ B_{\mu\nu}\} &= \frac{\mu^{D-4}}{\mathrm{i}\pi^2}\int \frac{\mathrm{d}^D q}{(2\pi)^{D-4}}\frac{\{1;\ q_\mu;\ q_\mu q_\nu\}}{\left[q^2-m_1^2\right]\left[(q-p_1)^2-m_2^2\right]},\\
\{C_0;\ C_\mu;\ C_{\mu\nu};\ C_{\mu\nu\rho}\} &= \frac{\mu^{D-4}}{\mathrm{i}\pi^2}\int \frac{\mathrm{d}^D q}{(2\pi)^{D-4}}\frac{\{1;\ q_\nu;\ q_\mu q_\nu;\ q_\nu q_\rho q_\rho\}}{\left[q^2-m_1^2\right]\prod_{j=1}^{2}\left[(q-p_j)^2-m_{j+1}^2\right]},\\
\{D_0;\ D_\mu;\ D_{\mu\nu};\ D_{\mu\nu\rho};\ D_{\mu\nu\rho\sigma}\} &= \frac{\mu^{D-4}}{\mathrm{i}\pi^2}\int \frac{\mathrm{d}^D q}{(2\pi)^{D-4}}\frac{\{1;\ q_\mu;\ q_\mu q_\nu;\ q_\mu q_\nu q;\ q_\mu q_\nu q_\rho q_\sigma\}}{\left[q^2-m_1^2\right]\prod_{j=1}^{3}\left[(q-p_j)^2-m_{j+1}^2\right]}.
\end{aligned}
\tag{2.22}
$$

Decomposing the tensorial integrals on account of Lorentz invariance one gets:

$$
\begin{aligned}
&B_\mu = p_{1\mu}B_1, \qquad B_{\mu\nu} = g_{\mu\nu}B_{00} + p_{1\mu}p_{1\nu}B_{11},\\
&C_\mu = p_{1\mu}C_1 + p_{2\mu}C_2 = \sum_{i=1}^{2} p_{i\mu}C_i, \qquad C_{\mu\nu} = g_{\mu\nu}C_{00} + \sum_{i\le j=1}^{2} p_{\langle i\mu}p_{j\nu\rangle}C_{ij},\\
&C_{\mu\nu\rho} = \sum_{i=1}^{2} g_{\langle\mu\nu}p_{i\rho\rangle}C_{00i} + \sum_{i\le j\le k=1}^{2} p_{\langle i\mu}p_{j\nu}p_{k\rho\rangle}C_{ijk},\\
&D_\mu = p_{1\mu}D_1 + p_{2\mu}D_2 + p_{3\mu}D_3 = \sum_{i=1}^{3} p_{i\mu}D_i,\\
&D_{\mu\nu} = g_{\mu\nu}D_{00} + \sum_{i\le j=1}^{3} p_{\langle i\mu}p_{j\nu\rangle}D_{ij},\\
&D_{\mu\nu\rho} = \sum_{i=1}^{3} g_{\langle\mu\nu}p_{i\rho\rangle}D_{00i} + \sum_{i\le j\le k=1}^{3} p_{\langle i\mu}p_{j\nu}p_{k\rho\rangle}D_{ijk},\\
&D_{\mu\nu\rho\sigma} = g_{\langle\mu\nu}g_{\rho\sigma\rangle}D_{0000} + \sum_{i\le j=1}^{3} g_{\langle\mu\nu}p_{i\rho}p_{j\sigma\rangle}D_{00ij} + \sum_{i\le j\le k\le l=1}^{3} p_{\langle i\mu}p_{j\nu}p_{ip}p_{l\sigma\rangle}D_{ijkl},
\end{aligned}
\tag{2.23}
$$

where with "$\langle\cdots\rangle$" in $p_{\langle i_\mu}\cdots p_{j_\sigma\rangle}$ the sum of those permutations of the Lorentz indices is meant which yield different terms e.g.:

$$
p_{\langle 1\mu}p_{2\nu\rangle} = p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}, \qquad p_{\langle 1\mu}p_{1\nu\rangle} = p_{1\mu}p_{1\nu}. \tag{2.24}
$$

The invariant Passarino–Veltman integrals can be optionally reduced to scalar integrals (A_0, B_0, C_0, D_0) either algebraically or numerically. The default of the program is *not* to do the algebraic reduction, since it may become quite time and memory consuming.

3. Sample calculation of a box graph

For illustration of the abilities of the program we will consider the calculation of a box graph for the process $e^+e^- \to W^+W^-$. This diagram – containing a fermion and three gauge bosons as inner lines – has already been calculated by hand (and by Schoonship) in ref. [8]. Of course *Feyn Calc* is capable to perform also more complicated calculations (like keeping the W-masses different or the fermion masses finite), but in order to avoid confusion by too complicated examples we use this relatively simple graph for illustration.

The 4-momenta of the incoming fermions are p_1 are p_2, those of the outgoing bosons are denoted by k_1 and k_2, the helicities of the leptons are σ_+, σ_-, the polarizations of the W-bosons λ_+, λ_-. We neglect the electron mass and consequently have $\sigma = \sigma_- = -\sigma_+$. The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{f\alpha^2}{4s}\sqrt{1 - \frac{4M_W^2}{s}}\,|\mathcal{M}|^2, \tag{3.1}$$

where s is the square of CMS energy and α the fine structure constant. The general invariant matrix element $\mathcal{M}$ can be decomposed in form factors $F_i^{\pm}(s, t)$ and the following 7 standard matrix elements:

$$\mathcal{M}(\sigma, \lambda_+, \lambda_-, s, t) = \sum_{i=1}^{7} \mathcal{M}_i^{\sigma} F_i^{\sigma}(s, t), \tag{3.2}$$

with

$$\begin{aligned}
\mathcal{M}_1^{\sigma} &= \bar{v}(p_1, 0)\not{\epsilon}_1(\not{k}_1 - \not{p}_1)\not{\epsilon}_2\omega_{\sigma}u(p_2, 0),\\
\mathcal{M}_2^{\sigma} &= \bar{v}(p_1, 0)\not{k}_1(\epsilon_1\epsilon_2)\omega_{\sigma}u(p_2, 0),\\
\mathcal{M}_3^{\sigma} &= \bar{v}(p_1, 0)\left[\not{\epsilon}_1(\epsilon_2 k_1) - \not{\epsilon}_2(\epsilon_1 k_2)\right]\omega_{\sigma}u(p_2, 0),\\
\mathcal{M}_4^{\sigma} &= \bar{v}(p_1, 0)\left[\not{\epsilon}_1(\epsilon_2 p_2) - \not{\epsilon}_2(\epsilon_1 p_1)\right]\omega_{\sigma}u(p_2, 0),\\
\mathcal{M}_5^{\sigma} &= \bar{v}(p_1, 0)\not{k}_1\omega_{\sigma}u(p_2, 0)(\epsilon_1 k_2)(\epsilon_2 k_1),\\
\mathcal{M}_6^{\sigma} &= \bar{v}(p_1, 0)\not{k}_1\omega_{\sigma}u(p_2, 0)(\epsilon_1 p_1)(\epsilon_2 p_2),\\
\mathcal{M}_7^{\sigma} &= \bar{v}(p_1, 0)\not{k}_1\omega_{\sigma}u(p_2, 0)\left[(\epsilon_1 k_2)(\epsilon_2 p_2) + (\epsilon_1 p_1)(\epsilon_2 k_1)\right],
\end{aligned} \tag{3.3}$$

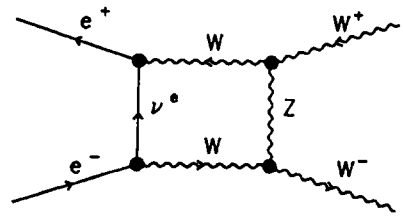

Fig. 1.

```
Loadfe[feyncalc, speceeww]                  Load the program and the specifications
Finte[                                      the function calculating the FeynmanINTEgr
eeWWboxwithnWWZ,                            the name of the graph
(* prefactor *)                             for overall scalar prefactors
alpha cw^2/( 8  Pi sw^4 ),                  αc_w^2/(8π)s_w^4
(* numerator *)                             all other terms in the numerator
SpC[                                        put the non-commutative objects into "SpC"
    SpI[ p1,0 ].                            v̄(p1,0) ·
    Gm[mu].Gs[q+k1-p1].Gm[nu].ga[7].         γ^μ(q̸ + k̸1 − p̸1)γ^ν γ7 ·
    SpI[ p2,0]                              u(p2,0)
    ](   g[la,mu] Lv[2 q+k1, ro]            ( g_λμ(2q + k1)_ρ
       + g[mu,ro] Lv[-q-2 k1, la]           + g_μρ(−q − 2k1)_λ
       + g[ro,la] Lv[k1-q, mu]              + g_ρλ(k1 − q)_μ
      )                                     )
    * ( g[la,si] Lv[-k2-q, nu]              ·(g_λσ(−k2 − q)_ν
      + g[si,nu] Lv[2 k2-q, la]             + g_σν(2k2 − q)_λ
      + g[nu,la] Lv[2 q-k2, si]             + g_νλ(2q − k2)_σ
      )  Lv[Eps[k1],ro] Lv[Eps[k2],si],     ) ε1^ρ ε2^σ
(* denominator *)                           codification of the denominator
Dn[ Df[q,       MZ], Df[q+k1,MW],           1/([q^2 − MZ^2][(q + k1)^2 − MW^2])
    Df[q+k1-p1,0],  Df[q-k2,MW]               [(q + k1 − p1)^2][(q − k2)^2 − MW^2])
  ],    Mandel, Specsimp, Endsimp           Simplifications (defined in speceeww.m)
        ]
```

Fig. 2. *Feyn Calc* input file for calculating the graph of fig. 1.

where $\sigma = \pm$, $\omega_+ = \gamma_6$, $\omega_- = \gamma_7$ and s, t, u are the usual Mandelstam variables. Applying the Feynman rules (by hand or by the program *Feyn Arts*) we get for the box of fig. 1

$$\delta\mathcal{M}^- = \frac{\alpha c_w^2}{(4\pi)2s_w^4}\int\frac{d^4q}{i\pi^2}\frac{\bar{v}(p_1,0)\gamma^\mu(\not{q}+\not{k}_1-\not{p}_1)\gamma^\nu\omega_-\mu(p_2,0)\Gamma_{\lambda\mu\rho}\Gamma^\lambda_{\sigma\nu}\epsilon_1^\rho\epsilon_2^\sigma}{\left[q^2-M_Z^2\right]\left[(q+k_1)^2-M_W^2\right](q+k_1-p_1)^2\left[(q-k_2)^2-M_W^2\right]}, \tag{3.4}$$

with

$$\Gamma_{\lambda\mu\rho} = g_{\lambda\mu}(2q+k_1)_\rho + g_{\mu\rho}(-q-2k_1)_\lambda + g_{\rho\lambda}(k_1-q)_\mu,$$
$$\Gamma_{\lambda\sigma\nu} = g_{\lambda\sigma}(-k_2-q)_\nu + g_{\sigma\nu}(2k_2-q)_\lambda + g_{\nu\lambda}(2q-k_2)_\sigma.$$

In order to ease the readability of the computeralgebraic in- and output we added a TEX-translation on the right part of the following figures. The Mathematica input file for the box is shown in fig. 2. Note that we use "." in Mathematica for non-commutative products. The specification file for $e^+e^- \to W^+W^-$, *speceeww.m*, is listed in fig. 3. The result for the box graph is given in fig. 4.

4. Graphs for the 2-loop γ-self-energy

As a demonstration of the capabilities of *Feyn Calc* for 2-loop problems we will present some algebraic calculations on the fermion loop-diagrams of fig. 5 with Z as inner particle.

```
(* Abbreviations *)
e1 = ep[k1]                                   W+-polarization vector  ε1
e2 = ep[k2]                                   W--polarization vector  ε2
C[x__]:=SpC[SpI[p1,0].x.om.SpI[p2,0]]         C(...) := v̄(p1,0)...ω_σ u(p2,0)


(* Energy momentum conservation *)
Enmomcon =  k2 :> p1+p2-k1                    k2 = p1 + p2 - k1  as rule

(* Substitutions *)
Specsimp = {
S[ p1,p1 ] :> 0                               p1^2 = 0
S[ p2,p2 ] :> 0                               p2^2 = 0
S[ k1,k1 ] :> MW^2                            k1^2 = MW^2
S[ p1,p2 ] :> S/2                             p1 · p2 = s/2
S[ k1,p2 ] :> ( T + S - MW^2 )/2,             k1 · p2 = (t + s - MW^2)/2
S[ k1,p1 ] :> ( MW^2 - T )/2,                 k1 · p1 = (MW^2 - t)/2
S[ e2,p2 ] :> S[e2,k1]-S[e2,p1]  }            ε2 · p2 = ε2 · k1 - ε2 · p1

Endsimp = {                                   For final substitutions
MW^2 :> MW2, MZ^2 :> MZ2 }                    MW^2 → MW2, MZ^2 → MZ2

(* Mandelstam variables *)
Mandel = {S,T,U,2 MW^2}                       i. e.: s + t + u = 2 MW^2

(* Standard matrix elements *)
For[ i=1, i<=2, i++,
     If[ i==1, om=Ga[6]; P="+"];              σ = +; ω_σ = ω+ = γ6
     If[ i==2, om=Ga[7]; P="-"];              σ = -; ω_σ = ω- = γ7
    ];
Matrixelements[                               Defining M_i^σ; see (3.3)
C[Gs[e1].Gs[k1-p1].Gs[e2]],          MA[P,1],        M_1^σ
C[Gs[k1] S[e1,e2]],                  MA[P,2],        M_2^σ
C[Gs[e1] S[e2,k1]-Gs[e2] S[e1,k2]], MA[P,3],        M_3^σ
C[Gs[e1] S[e2,p2]-Gs[e2] S[e1,p1]], MA[P,4],        M_4^σ
C[Gs[k1]]  S[e1,k2] S[e2,k1],        MA[P,5],        M_5^σ
C[Gs[k1]]  S[e1,p1] S[e2,p2],        MA[P,6],        M_6^σ
C[Gs[k1]] (S[e1,k2] S[e2,p2]
          +S[e1,p1] S[e2,k1]),       MA[P,7]         M_7^σ
  ]           ]
```

Fig. 3. Specification file for $e^+e^- \to W^+W^-$.

```
eeWWboxwithnWWZ =

          2
 ( alpha cw )
--------------- [
          4
   8 Pi sw

MA[-, 1] { 20 D00 +
  (4 D0 + 16 D1 + 8 D11 + 12 D12 +
  8 D13 + 6 D2 + 2 D22) MW2 +
  (D0 + 2 D1 + D2) MZ2 +
  (-2 D0 - 6 D1 - 2 D11 - 2 D12 -
  2 D13 - 2 D2) S +
  (4 D12 + 2 D2 + 2 D22) T } +
MA[-, 2] { -4 C0 + 10 D00 -
  16 D001 - 8 D002 +
  (2 D1 + 2 D11 + 2 D12 + 2 D13)
  MW2 + (-3 D0 + 2 D1 + D2) MZ2 +
  (D0 - 4 D1 + 2 D11 + 6 D12 +
  2 D13 - D2 + 2 D22) T } +
MA[-, 3] { 2 C0 - 3 C1 - 8 D00 -
  8 D001 + (-4 D1 - 3 D11 - 4 D12 -
  11 D13) MW2 +
  (2 D0 - 3 D1) MZ2 + 4 D13 S +
  (-2 D0 + D1 - 3 D11 - 2 D12 +
  5 D13) T } +
MA[-, 4] { 3 C0 + 4 C1 - 26 D00 -
  8 D002 + (-8 D1 - 8 D11 - 18 D12 -
  8 D13 - 4 D2 - 4 D22) MW2 +
  (-2 D1 - 3 D2) MZ2 +
  (4 D1 + 2 D11 + 4 D12 + 2 D13) S +
  (-2 D12 + D2 - 2 D22) T } +
MA[-, 5] { 16 D113 + 8 D123 - 8 D13 } +
MA[-, 6] { 32 D12 + 16 D122 +
  16 D2 + 24 D22 + 8 D222 } +
MA[-, 7] { 8 D112 + 8 D12 + 8 D122 +
  8 D123 + 16 D13 } ]
```

$\alpha c_w^2/(8\pi s_w^4)\cdot$ [

$\mathcal{M}_1^-\{20D_{00}+$
$(4D_0+16D_1+8D_{11}+12D_{12}+$
$8D_{13}+6D_2+2D_{22})M_W^2+$
$(D_0+2D_1+D_2)M_Z^2+$
$(-2D_0-6D_1-2D_{11}-2D_{12}-$
$2D_{13}-2D_2)s+$
$(4D_{12}+2D_2+2D_{22})t\}+$
$\mathcal{M}_2^-\{-4C_0+10D_{00}-$
$16D_{001}-8D_{002}+$
$(2D_1+2D_{11}+2D_{12}+2D_{13})$
$M_W^2+(-3D_0+2D_1+D_2)M_Z^2+$
$(D_0-4D_1+2D_{11}+6D_{12}+$
$2D_{13}-D_2+2D_{22})t\}+$
$\mathcal{M}_3^-\{2C_0-3C_1-8D_{00}-$
$8D_{001}+(-4D_1-3D_{11}-4D_{12}-$
$11D_{13})M_W^2+$
$(2D_0-3D_1)M_Z^2+4D_{13}s+$
$(-2D_0+D_1-3D_{11}-2D_{12}+$
$5D_{13})t\}+$
$\mathcal{M}_4^-\{3C_0+4C_1-26D_{00}-$
$8D_{002}+(-8D_1-8D_{11}-18D_{12}-$
$8D_{13}-4D_2-4D_{22})M_W^2+$
$(-2D_1-3D_2)M_Z^2+$
$(4D_1+2D_{11}+4D_{12}+2D_{13})s+$
$(-2D_{12}+D_2-2D_{22})t\}+$
$\mathcal{M}_5^-\{16D_{113}+8D_{123}-8D_{13}\}+$
$\mathcal{M}_6^-\{32D_{12}+16D_{122}+$
$16D_2+24D_{22}+8D_{222}\}+$
$\mathcal{M}_7^-\{8D_{112}+8D_{12}+8D_{122}+$
$8D_{123}+16D_{13}\}]$

```
The arguments are:
C( p1^2, (p2-p1)^2, p2^2, 0, MW, MW) =
C( 0, S, 0, 0, MW, MW),
D( k1^2, p1^2, p2^2, p3^2, (k2-k1)^2, (k1-p1)^2) =
D( MW2, 0, 0, MW2, S, T, MZ, MW, 0, MW).
```

Fig. 4. Results of *Feyn Calc* for the box of fig. 1.

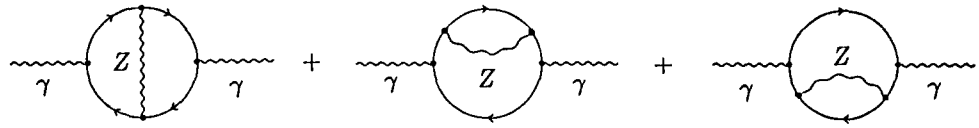

Fig. 5. Gauge invariant set of 2-loop self-energy graphs.

$$\Sigma_{\mu\nu} =$$
$$= \left\langle\left\langle \frac{\mathrm{Tr}\left[\gamma_\mu(\not q_1 + m)\Gamma(\not q_2 + m)\gamma_\nu(\not q_2 + \not p + m)\Gamma(\not q_1 + \not p + m)\right]}{\left[q_1^2 - m^2\right]\left[(q_1+p)^2 - m^2\right]\left[(q_2 - q_1)^2 - M^2\right]\left[q_2^2 - m^2\right]\left[(q_2+p)^2 - m^2\right]} \right\rangle\right\rangle$$
$$+ \left\langle\left\langle \frac{\mathrm{Tr}\left[\gamma_\mu(\not q_1 + m)\Gamma(\not q_2 + m)\Gamma(\not q_1 + m)\gamma_\nu(\not q_1 + \not p + m)\right]}{\left[q_1^2 - m^2\right]^2\left[(q_1+p)^2 - m^2\right]\left[(q_2 - q_1)^2 - M^2\right]\left[q_2^2 - m^2\right]} \right\rangle\right\rangle$$
$$+ \left\langle\left\langle \frac{\mathrm{Tr}\left[\gamma_\mu(\not q_1 + m)\gamma_\nu(\not q_1 + \not p + m)\Gamma(\not q_2 + \not p + m)\Gamma(\not q_1 + \not p + m)\right]}{\left[q_1^2 - m^2\right]\left[(q_1+p)^2 - m^2\right]^2\left[(q_2 - q_1)^2 - M^2\right]\left[(q_2+p)^2 - m^2\right]} \right\rangle\right\rangle, \tag{4.1}$$

where

$$\langle\langle \cdots \rangle\rangle = e^4 \cdot \int \frac{\mathrm{d}^D q_1}{(2\pi)^D} \frac{\mathrm{d}^D q_2}{(2\pi)^D} \cdots . \tag{4.2}$$

All fermion masses m are equal, since no flavour change for the neutral gauge boson interaction exists. The mass of the boson is labeled with M. The coupling of the inner bosons is denoted by Γ:

$$\Gamma = \gamma_\mu(v + a\gamma_5). \tag{4.3}$$

We decompose $\Sigma_{\mu\nu}$ into its transverse and longitudinal parts:

$$\Sigma_{\mu\nu} = \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\right)\Sigma_{\mathrm{T}} + p_\mu p_\nu \Sigma_{\mathrm{L}}. \tag{4.4}$$

The sum of the three graphs is gauge invariant. This implies

$$p^\mu p^\nu \Sigma_{\mu\nu} = 0 \tag{4.5}$$

and consequently

$$\Sigma_{\mathrm{L}} = 0. \tag{4.6}$$

Using eq. (4.6) and contracting eq. (4.4) with $g^{\mu\nu}$ on both sides yields

$$\Sigma_{\mathrm{T}} = \frac{1}{1-D}\Sigma_\mu^\mu. \tag{4.7}$$

The gauge invariance condition (4.5) may be checked by hand without having to evaluate the traces explicitly. Working out Σ_{T} turns out to be more complicated. In order to handle the calculation with *Feyn Calc* in an elegant way, we introduce the following notation:

$$k_1 = q_1, \qquad k_2 = q_1 + p, \qquad k_3 = q_2 - q_1, \qquad k_4 = q_2, \qquad k_5 = q_2 + p, \tag{4.8}$$

$$T_{ijl\ldots} := \left\langle\left\langle \frac{1}{\left[k_i^2 - m_i^2\right]\left[k_j^2 - m_j^2\right]\left[k_l^2 - m_l^2\right]\cdots} \right\rangle\right\rangle, \tag{4.9}$$

and, if k_i^2 occur in the numerator but not in the denominator, i.e. direct cancellation is not possible, we define

$$Y^{i\cdots}{}_{jl\cdots} := \left\langle\left\langle \frac{k_i^2 \cdots}{\left[k_j^2 - m_j^2\right]\left[k_l^2 - m_l^2\right] \cdots} \right\rangle\right\rangle, \quad i \neq j,\ l. \tag{4.10}$$

In dimensional regularization we may set $\int \mathrm{d}^D q_i 1 = 0$, consequently:

$$T_i = 0 = T_{12}, \quad i = 1, 2, 3, 4, 5. \tag{4.11}$$

In some special cases the 2-loop scalar T-integrals factorize into easier ones. Using translational invariance of the integration momenta and suitable substitutions one gets:

$$\begin{aligned}
T_{1i} &= A_0(m_1) A_0(m_i),\\
T_{11i} &= A_0(m_i) B_0(0, m_1, m_1),\\
T_{12i} &= A_0(m_i) B_0(p^2, m_1, m_2),\\
T_{112i} &= A_0(m_i) \bar{T}_{112},\\
T_{1245} &= B_0(p^2, m_1, m_2) B_0(p^2, m_4, m_5),\\
Y^1_{2j} &= (p^2 + m_2^2) A_0(m_2) A_0(m_j),\\
Y^1_{235} &= A_0(m_3) A_0(m_5) + (p^2 + m_2^2) T_{235},\\
Y^1_{22j} &= \left\{(p^2 + m_2^2) B_0(0, m_2, m_2) + A_0(m_2)\right\} A_0(m_j),\\
Y^1_{2235} &= (p^2 + m_2^2) T_{2235} + T_{235},\\
Y^1_{2345} &= A_0(m_3) B_0(p^2, m_4, m_5) + (p^2 + m_2^2) T_{2345}\\
&\quad - \frac{1}{2m_5^2}\left\{\left[A_0(m_5) + (m_4^2 - m_5^2 - p^2) B_0(p^2, m_4, m_5)\right.\right.\\
&\quad \left.\left. - (m_4^2 - p^2) B_0(p^2, m_4, 0)\right]\left[A_0(m_2) - A_0(m_3)\right] + (m_3^2 - m_2^2 - m_5^2) T_{235} + m_5^2 T_{234}\right.\\
&\quad \left. - (m_3^2 - m_2^2) T_{235'} + (m_3^2 - m_2^2 - m_5^2)(m_4^2 - m_5^2 - p^2) T_{2345} - (m_4^2 - p^2)(m_3^2 - m_2^2) T_{2345'}\right\},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
Y^3_{1245} &= \tfrac{1}{2}\left\{\left[A_0(m_1) + A_0(m_2)\right] B_0(p^2, m_4, m_5) + \left[A_0(m_4) + A_0(m_5)\right] B_0(p^2, m_1, m_2)\right.\\
&\quad - \frac{1}{p^2}\left[\left[A_0(m_1) - A_0(m_2) - (m_1^2 - m_2^2) B_0(p^2, m_1, m_2)\right]\right.\\
&\quad \times \left[A_0(m_4) - A_0(m_5) - (m_4^2 - m_5^2) B_0(p^2, m_4, m_5)\right]\\
&\quad \left.\left. - (m_1^2 + m_2^2 + m_4^2 + m_5^2 - p^2) p^2 B_0(p^2, m_1, m_2) B_0(p^2, m_4, m_5)\right]\right\},
\end{aligned}$$

where $i = 3, 4, 5$ and $j = 3, 5$. With $\bar{T}_{112}$ we abbreviate the derivative of the 1-loop integral:

$$\bar{T}_{112} = \frac{\partial B_0(p^2, m_1, m_2)}{\partial m_1^2}. \tag{4.13}$$

```
(*  Sigmamumu.m     calculating  Sigma[mu mu]  (2-loop selfenergy) *)
    Loadfe[feyncalc];
    Q1 = Gs[ Q[1] ]; Q2 = Gs[ Q[2] ];
    VA = v+a Ga[5];

Tucalc[

  tr[Gm[nu].(Q1+m).Gm[mu].VA.(Q2+m).Gm[nu].(Q2+Gs[p]+m).
     Gm[mu].VA.(Q1+Gs[p]+m)
    ] Dn[  Df[k1,m] Df[k2,m] Df[k3,M] Df[k4,m] Df[k5,m]  ]
+ tr[Gm[nu].(Q1+m).Gm[mu].VA.(Q2+m).Gm[mu].VA.(Q1+m).Gm[nu].(Q1+Gs[p]+m)
    ] Dn[  Df[k1,m] Df[k1,m] Df[k2,m] Df[k3,M] Df[k4,m]  ]
+ tr[Gm[nu].(Q1+m).Gm[nu].(Q1+Gs[p]+m).Gm[mu].VA.(Q2+Gs[p]+m).
     Gm[mu].VA.(Q1+Gs[p]+m)
    ] Dn[  Df[k1,m] Df[k2,m] Df[k2,m] Df[k3,M] Df[k5,m]  ],

   {m^n_Integer?Positive:>m2 m^(n-2), M^n_Integer?Positive:>M2 M^(n-2),
    m^n_Integer?Negative:>1/m2 m^(n+2), a^2:>a2,v^2:>v2 }
     ]
```

Fig. 6. Input file for Σ_μ^μ.

Using the symmetries of the graphs more relations may be derived from relations eqs. (4.12). After trace evaluation the scalar products of integration variables are replaced by:

$$q_1^2 = k_1^2, \qquad q_2^2 = k_4^2, \qquad q_1 \cdot q_2 = -\tfrac{1}{2}\left(k_3^2 - k_1^2 - k_4^2\right),$$
$$p \cdot q_1 = \tfrac{1}{2}\left(k_2^2 - k_1^2 - p^2\right), \qquad p \cdot q_2 = \tfrac{1}{2}\left(k_5^2 - k_4^2 - p^2\right). \tag{4.14}$$

Subsequently all possible cancellations of $k_i^2 - m_i^2$ occuring in the numerators and denominators are done. After that relations (4.12) are applied. Further details of the calculation of 2-loops self-energies like the implementation of the symmetries of the graphs into the program will be presented elsewhere.

The important aspect here is the use of *Feyn Calc* for trace calculations and for actually performing non-trivial tasks like checking gauge invariance. To verify eq. (4.5) the first step is to calculate the three traces. This results in 178 terms which indeed cancel completely after performing the procedure above and using relations (4.12). For the calculation of Σ_μ^μ the Mathematica input (see eq. (4.1)) is shown in fig. 6, the result in fig. 7.

5. Conclusion and outlook

We developed the computeralgebra program *Feyn Calc* as a powerful support for the investigation of interesting processes in high energy physics which would otherwise hardly be feasible. The main feature is the *automatic* algebraic calculation of 1-loop (up to four external particles) and 2-loop (self energy) Feynman graphs, i.e. provided special simplifications are given all graphs of a whole process are calculated with the same program. The result may be translated by an additional program into Fortran or C.

We have checked the Macsyma and Mathematica implementations against each other and against hand calculations for the processes $e^+e^- \to W^+W^-$, $e^+e^- \to ZH$, $gg \to t\bar{t}$ and for several tree-level and self-energy diagrams.

```
                                                                 2
            2     -2 (a2 + v2) A[M] A[m]   2 (a2 + v2) A[m]
(D - 2)    {     ---------------------- + ----------------- +
                           m2                    m2

 2 (m2 - p2) (a2 + v2) (A[M] - A[m]) B0[p2, 0, m]
 ------------------------------------------------ +
                        m2

 2 (2 m2 + p2) (a2 + v2) (A[M] - A[m]) B0[p2, m, m]
 -------------------------------------------------- +
                        m2

                                   2
 2 p2 (a2 + v2) B0[p2, m, m]   +

 2 (-M2 + m2) (m2 - p2) (a2 + v2) T[1'234]
 ----------------------------------------- +
                    m2

                                              2 M2 p2 (a2 + v2) T[1234]
 4 p2 (-4 a2 m2 + M2 (a2 + v2)) T[11234] - ------------------------- +
                                                       m2

 2 p2 (-4 a2 m2 + M2 (a2 + v2)) T[12345] -

                                          2 M2 (a2 + v2) T[134]
 4 (-4 a2 m2 + M2 (a2 + v2)) T[1334] + --------------------- -
                                                 m2

 2 (M2 - m2) (a2 + v2) T[134']
 ----------------------------- - 2 (a2 + v2) T[234] -
              m2

 4 p2 (a2 + v2) (A[M] - A[m]) Tb[112] }

+  (D - 2) {  -8 (a2 + v2) (A[M] - A[m]) B0[p2, m, m] -

                                                                  2
 2 (-8 a2 m2 + 2 M2 (a2 + v2) + 5 p2 (a2 + v2)) B0[p2, m, m]   -

 16 m2 (4 a2 m2 + p2 (a2 - v2) - M2 (a2 + v2)) T[11234] +

 16 (-4 a2 m2 + M2 (a2 + v2) + p2 (a2 + v2)) T[1234] -

            2      2                                      2
 4 (8 a2 m2  + M2   (a2 + v2) + 3 M2 p2 (a2 + v2) + p2   (a2 + v2) -

    2 M2 m2 (3 a2 + v2) - 2 m2 p2 (3 a2 + v2)) T[12345] +

 16 m2 (a2 - v2) T[1334] - 8 (a2 + v2) T[134] + 8 (a2 + v2) T[234] -

 16 m2 (a2 + v2) (A[M] - A[m]) Tb[112] }

+          {  -16 m2 (a2 - v2)

                  2
 (-B0[p2, m, m]   + 4 m2 T[11234] + 4 T[1234] - M2 T[12345] +

   2 m2 T[12345] - p2 T[12345]) }
```

Fig. 7. Results of Σ_{μ}^{μ}, where T[ijk...] = $T_{ijk\ldots}$, a prime means zero mass and Tb denotes $\overline{T}$.

The future line of development lies in adjusting *Feyn Calc* to more external particles and higher orders. We will put more emphasis on the Mathematica version, hoping to construct a user friendly program for distribution in the near future.

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