Q15) Part -b) Maxima of $f(x) = x^{a-1}e^{-x}$

$\frac{df}{dx} = 0$ at inflection points $\Rightarrow$ $\frac{d}{dx}\left(x^{a-1}e^{-x}\right) = (a-1)x^{a-2}e^{-x} - x^{a-1}e^{-x} = 0$

$\Rightarrow \left(\frac{a-1}{x} - 1\right)e^{-x}x^{a-1} = 0$

So the $x = a-1$ is an inflection point (For $a>1$, $x=0$ is also an inflection point)

Checking $\left.\frac{d^2f}{dx^2}\right|_{x=a-1}$ $\Rightarrow (a-1)(a-2)x^{a-3}e^{-x} - 2(a-1)x^{a-2}e^{-x} + x^{a-1}e^{-x}$

$\Rightarrow x^{a-3}e^{-x}\left((a-1)(a-2) - 2(a-1)x + x^2\right)$

At $x = a-1 \Rightarrow (a-1)(a-2) - 2(a-1)^2 + (a-1)^2 = 1-a < 0$ for $a>1$

As $\left.\frac{d^2f}{dx^2}\right|_{x=a-1} < 0 \Rightarrow$ $x = a-1$ is a local Maxima!

In order to check if its a global maxima in $x \in [0,\infty)$ consider the boundry and inflextion points

$f(0) = 0$

$f(x\to\infty) = 0$

$f(a-1) = (a-1)^{a-1}e^{1-a} > 0 \quad \forall a>1$

Hence $x = a-1$ is the Global Maxima in $x \in [0,\infty)$

Part - c) Considering $z = \frac{x}{c+x}$ s.t maxima falls at $z = \frac{1}{2}$

$\Rightarrow \frac{1}{2} = \frac{a-1}{c+(a-1)} \Rightarrow c + (a-1) = 2a-2 \Rightarrow \boxed{c = a-1}$

If we fix $c = a-1$, the peak will fall at $z = \frac{1}{2}$!

Part - d) For large values of x, x^{a-1} might cause overflow and e^{-x} causes underflow. So writing the expression as $e^{(a-1)\ln x - x}$, the exponent

$\Rightarrow (a-1)\ln x - x$

Will be within overflow / underflow limit if x is in admissiable region.

Hence using $e^{(a-1)\ln x - x}$ to compute $x^{a-1}e^{-x}$ will avoid the issues

Part -e) Consider $z = \frac{x}{(a-1)+x} \Rightarrow x = \frac{(a-1)z}{1-z} \Rightarrow dx = \frac{(a-1)}{(1-z)^2}dz$

Then the integrand becomes $e^{(a-1)\ln\left(\frac{(a-1)z}{1-z}\right) - \frac{(a-1)z}{1-z}} \cdot \frac{(a-1)}{(1-z)^2}$