A FINITE VOLUME SCHEME FOR THE CONVECTION-DIFFUSION SYSTEM USING THE POLYNOMIAL RECONSTRUCTION OPERATOR

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INTRODUCTION

We present a new very high-order finite volume scheme for the two-dimensional convection-diffusion problem system based on the local Polynomial Reconstruction Operator (PRO-Scheme). We detail the design of the schemes and provide an example with regular solution. The solution is presented for Patankar scheme and for PRO scheme. In this last case we present the solution using a polynomial reconstruction of degree 1 and 2.

Keywords: CONVECTION-DIFFUSION, FINITE VOLUME, POLYNOMIAL RECONSTRUCTION.

1. THE 2D CONVECTION-DIFFUSION PROBLEM

Convection-diffusion equation is one of the most popular equations in engineering or environmental applications. Producing very efficient numerical schemes to obtain accurate and relevant approximations is a constant challenging objective. Here, we propose a new approach based on two ingredients: a finite volume scheme coupled with the Polynomial Reconstruction Operator to achieve a very high-order algorithm. The method is an extension of the bi dimensional geometry initially proposed by [1].
We introduce the steady-state convection-diffusion equation

$$\nabla.(V\rho - a\nabla\rho) = f \text{ in } \Omega, \quad \rho = \rho_d \text{ on } \Gamma_D, \quad a\nabla\rho.n = g_N \text{ on } \Gamma_N,$$

where ρ is the unknown function, ρ_d the Dirichlet condition on boundary Γ_D, g_N the Neumann condition on boundary Γ_N (see Figure 1). In the following, $V = V(x,y)$ stands for the convection velocity while $a = a(x,y) \geq 0$ represents the diffusion coefficient.

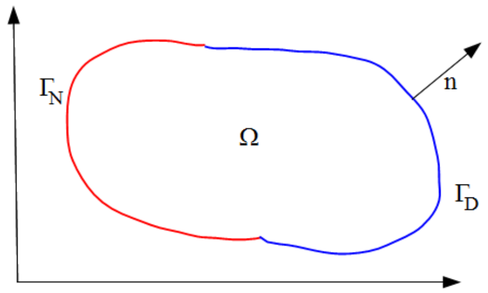

Figure 1 – Computational domain with Neumann and Dirichlet boundary.

2. A HIGH-ORDER FINITE VOLUME SCHEME

We consider a mesh T_h of triangle polyhedral cells K_i and edges $e_{ij} = K_i \cap K_i$ where n_{ij} are the outward normal vectors, and $\nu(i)$ the index set of the neighbor cells of K_i. For each edge, e_{ij}, q_{ij}^r are the associated Gauss points (see Figure 2).

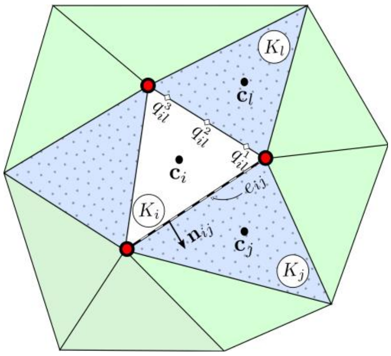

Figure 2 – Mesh notations.

Let us denote by ρ_i an approximation of the mean value of ρ over cell K_i. For any cell K_i, the generic very high-order finite volume schemes writes [2,3]:

$$\mathcal{g}_i = \sum_{j\in\nu(i)} \sum_{r=1,2,3} \omega_r |e_{ij}| \left[-\mathcal{F}^{\mathrm{d}}(\rho_{ij,r}, \rho_{ji,r}, n_{ij}) + \mathcal{F}^{\mathrm{c}}(\rho_{ij,r}, \rho_{ji,r}, n_{ij})\right] - |K_i| f_i,$$

where ω_r are the associated weights for the integration quadrature rule over the edge and $|e_{ij}|$ and $|K_i|$ represent the length and area of edge $|e_{ij}|$and cell K_i, respectively. Functions $\mathcal{F}^d(\rho_{ij,r}, \rho_{ji,r}, n_{ij})$ and $\mathcal{F}^{\mathrm{c}}(\rho_{ij,r}, \rho_{ji,r}, n_{ij})$ represent convective and diffusive contributions of the numerical across the interface evaluated at the Gauss points. The crucial issue is the calculation of very accurate approximations of these fluxes at the Gauss points.

3. THE POLYNOMIAL RECONSTRUCTION OPERATORS

For a given piecewise constant function $(\rho_i)_{K_i\in T_h}$ defined on the mesh cells, we introduce the polynomial function of degree d as proposed by [4, 5]:

$$\rho_i(x,y) = \rho_i + \sum_{1\le\alpha\le d} \Re_\alpha\{(x-c_x)^{\alpha_1}(y-c_y)^{\alpha_2} - M_\alpha\}, \quad M_\alpha = \frac{1}{|K_i|}\int_{K_i} (x-c_x)^{\alpha_1}(y-c_y)^{\alpha_2} dxdy,$$

$\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$. Now we determine the coefficients $\Re = (\Re_\alpha)$ in two different ways depending on the flux approximations. To this end, for any cell K_i we denote by S_i the stencil associated to cell K_i. In the present study, we shall use the whole corona around the cell, i.e. $S_i = \{j;\, S_j \cap S_i \ne \emptyset\}$. Note that $\upsilon(i) \subset S_i$. We first define the functional,

$$\hat{\mathrm{E}}_i(\Re) = \sum_{j\in S_i}\left[\rho_j - \frac{1}{|K_i|}\int_{K_i}\rho_i(x,y)dxdy\right]^2$$

and denote by $\hat{\Re}_\alpha$the coefficients of the polynomial which minimizes $\hat{\mathrm{E}}_i(\Re)$.
We also introduce a second polynomial reconstruction operator which includes the Dirichlet condition. Consider a cell K_i. If cell $K_i \cap \partial\Omega = \emptyset$ then we set $\tilde{E}_i(\Re) = \hat{\mathrm{E}}_i(\Re)$ whereas if we deal with a cell in contact with the boundary $e_{bi} = K_i \cap \partial\Omega$, we set

$$\tilde{E}_i(\Re) = \sum_{j\in S_i}\left[\rho_j - \frac{1}{|K_i|}\int_{K_i}\rho_i(x,y)dxdy\right]^2 + \left[\rho_{ei} - \frac{1}{|e_{bi}|}\int_{e_{bi}}\rho_i(s)ds\right]^2$$

where $\rho_{ei} = \frac{1}{|e_{bi}|}\int_{e_{bi}} \rho_d(s)ds$ is the mean value on edge e_{bi} of the Dirichlet condition and denote by $\widetilde{\mathfrak{R}}_\alpha$the coefficients of the polynomial which minimizes $\tilde{E}_i(\mathfrak{R})$. Note that one has to provide a rich enough stencil, larger than the number of unknowns $\frac{d(d+1)}{2}$ for a polynomial reconstruction of degree d. If the corona does not contain enough elements, one has to pick-up element in the second corona.
Based on the two reconstructions, we then define the numerical convective and diffusive fluxes by:

$$\mathcal{F}^d_{ij,r}(\tilde{\rho}, n_{ij}) = a(q^r_{ij})\frac{\nabla\tilde{\rho}_i(q^r_{ij}) + \nabla\tilde{\rho}_j(q^r_{ij})}{2}.n_{ij},$$
$$\mathcal{F}^c_{ij,r}\left(\hat{\rho}, n_{ij}\right) = [V(q^r_{ij}).n_{ij}]^- \hat{\rho}_j(q^r_{ij}) + [V(q^r_{ij}).n_{ij}]^+ \hat{\rho}_i(q^r_{ij}),$$

with $[V.n]^+$ and $[V.n]^-$ the positive and negative part of the normal velocity.
For an edge e_{bi} on the boundary, we only have one cell K_{bi} with outward normal vector n_{bi}. We slightly modify the scheme as a function of the boundary conditions. For the Neumann condition Γ_N, we set $\mathcal{F}^d_{bi,r}(\tilde{\rho}, n_{bi}) = g_N$ and $\mathcal{F}^c_{bi,r}\left(\hat{\rho}, n_{bi}\right) = 0$ while we prescribe $\mathcal{F}^d_{bi,r}(\tilde{\rho}, n_{bi}) = a(q^r_{bi})\nabla\tilde{\rho}_{bi}(q^r_{bi}).n_{bi}$ and $\mathcal{F}^c_{ij,r}\left(\hat{\rho}, n_{ij}\right) = [V(q^r_{bi}).n_{bi}]^- \rho_d + [V(q^r_{bi}).n_{ij}]^+ \hat{\rho}_{bi}(q^r_{bi})$ for the Dirichlet condition on Γ_D where q^r_{bi} are the Gauss points for the quadrature rule on the edge.

Let us denote by $= \rho(\rho_1, \dots, \rho_I)$ the vector of unknowns with I the number of cells. Then, we define an affine operator from $\mathbb{R}^I$ into $\mathbb{R}^I$ given component by component by

$$\mathcal{g}_i(\rho) = \sum_{j\in\nu(i)} \sum_{r=1,2,3} \omega_r |e_{ij}| \left[-\mathcal{F}^d_{ij,r}(\tilde{\rho}, n_{ij}) + \mathcal{F}^c_{ij,r}\left(\hat{\rho}, n_{ij}\right)\right] - |K_i| f_i.,$$

with $\tilde{\rho} = \tilde{\rho}(\rho)$ and $\hat{\rho} = \hat{\rho}(\rho)$ the two polynomial reconstruction operators associated to the diffusion and the convection respectively. For the boundary edges, one has to join the contribution due to the Dirichlet or Neumann conditions.
To provide the approximation to the convection diffusion equation, we have to solve the affine problem $\mathcal{g}(\rho) = (0, \dots, 0)$ in $\mathbb{R}^I$. We rewrite the system under the form $A\rho = b$ setting $b = \mathcal{g}(0, \dots, 0)$ and the column $A_j = \mathcal{g}(\varepsilon^j) - b$, with ε^j, the j-canonical vector. We solve the linear system using a classical $A = LU$ decomposition. Note that an iterative technique such as Krylov method should be more efficient but usually requires a preconditioning matrix to improve the convergence.

4. NUMERICAL SIMULATIONS

Numerical simulations are carried out to highlight the performance of the numerical scheme to achieve very high-order approximations even with large Peclet number. We consider two Delaunay meshes M1 and M2 with 870 and 1804 triangular cells, respectively. In the example, we solve a 2D steady convection-diffusion problem, on a unit square domain Ω :

$$\nabla.(V\rho - a\nabla\rho) = f \text{ in } \Omega, \quad \rho = 0 \text{ on } \Gamma$$

with *V=(3,3)* , *a=1*, and *f=1*. In the example presented, we use the first corona cells to define our stencil. This is achieved considering all cells that have one vertex in common with the reference cell (see Figure 2).
In Figures 3, 4, and 5 we plot the ϱ function using respectively Patankar [6], PRO 1 (polynomial reconstruction operator of degree 1), and PRO 2 (polynomial reconstruction operator of degree 2) using mesh M1. The three figures are very similar but we notice we violate the Maximum Principle with PRO 1, while PRO2 recovers this property.

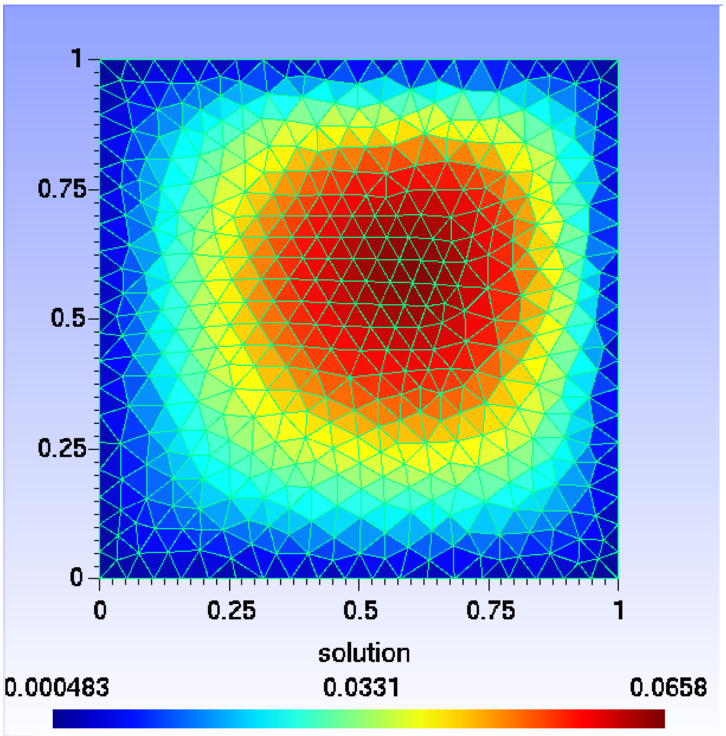

Figure 3 – Results for Patankar scheme with mesh M1.

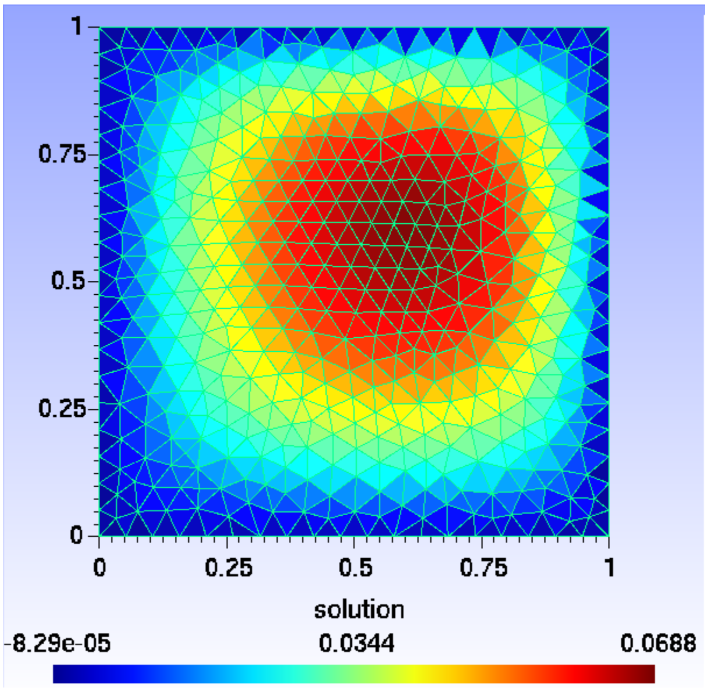

Figure 4 – Results for PRO 1 scheme with mesh M1.

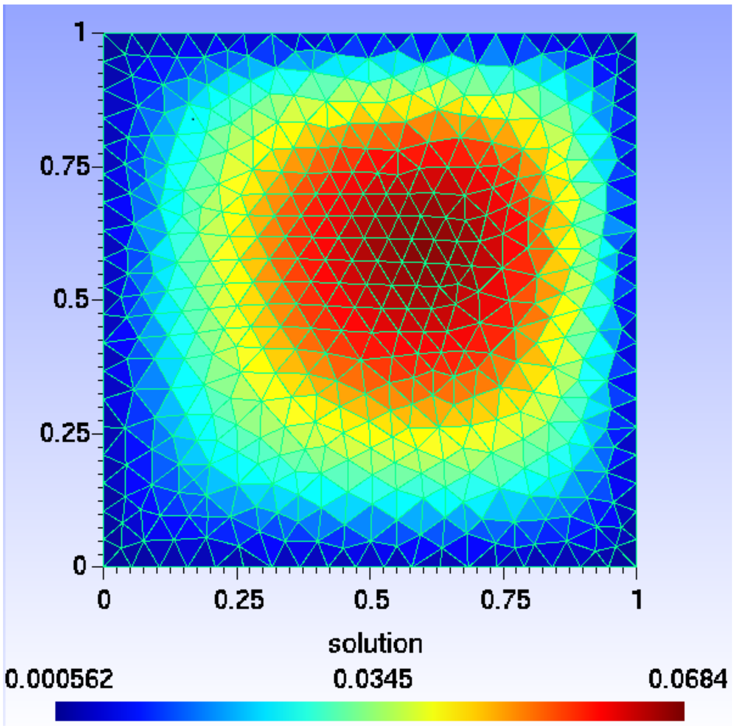

Figure 5 – Results for PRO 2 scheme with mesh M1.

In Figures 6 and 7 we plot the solution on mesh M2 using Patankar and PRO 2 schemes. As expected, the results are very similar, but PRO 2 provides higher maximum, since this scheme has a lower numerical diffusion.

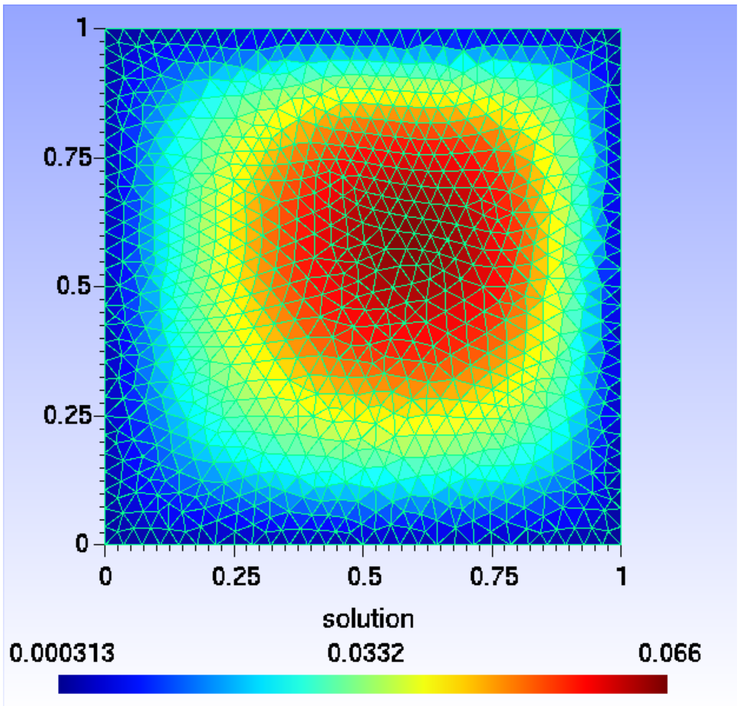

Figure 6 – Results for Patankar scheme with mesh M2.

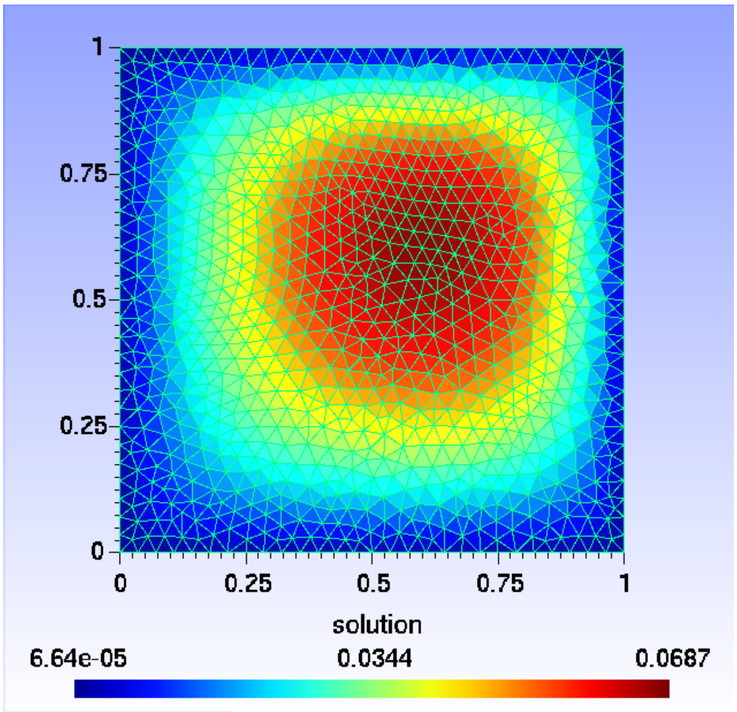

Figure 7 – Results for PRO 2 scheme with mesh M2.

5. CONCLUSION

Preliminary results presented in this study show the feasibility of the approach since the results obtained from Patankar and PRO schemes are relevant.

In order to consider solutions using higher order polynomials one needs to increase the stencil, namely introducing the second corona of cells around the reference cell.

We plan to perform convergence study to check the method order using a PRO *d* reconstruction.

6. REFERENCES

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