9.4.2 Show that the Helmholtz eq^n
$\nabla^2\psi + k^2\psi = 0$ is still seperable in circular cylindrical co-ordinates if k^2 is generalized to
$k^2 + f(\rho) + \frac{1}{\rho^2} g(\phi) + h(z)$.

$\rightarrow$ If $\psi = R(\rho)\Phi(\phi)Z(z)$

then $\left(\frac{1}{R\rho}\frac{d}{d\rho}\rho\frac{dR}{d\rho} + f(\rho) + k^2\right) + \frac{1}{\rho^2}\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + g(\phi)\right)$

$+ \left(\frac{1}{Z}\frac{d^2Z}{dz^2} + h(z)\right) = 0$.

leads to - $\frac{d^2Z}{dz^2} + h(z)Z = n^2 Z$

$\frac{d^2\Phi}{d\phi^2} + g(\phi)\Phi = -m^2\Phi$

$$\rho\frac{d}{d\rho}\rho\frac{dR}{d\rho} + \left[(n^2 + f(\rho) + k^2)\rho^2 - m^2\right]R = 0$$

9.4.3 Seperate variables in the Helmholtz equation in spherical polar co-ordinates, spliting off the radial component first, then show that ~~result~~ the eq^n are same as before.

$\rightarrow$ We ~~ψ~~ $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$L^2 Y(\theta,\phi) = l(l+1)\,Y(\theta,\phi)$

where $L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

we have $(\nabla^2+k^2)\psi(r,\theta,\phi) = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{L^2}{r^2} + k^2\right)R(r)Y(\theta,\phi)$

$$\frac{d}{dr}r^2\frac{dR}{dr} + (k^2r^2 - l(l+1))R = 0$$

the order in which the variables are seperated doesn't matter

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9.4.4 Verify that $\nabla^2\psi(r,\theta,\phi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\phi)\right]\psi(r,\theta,\phi) = 0$

is seperable.

$\Rightarrow \frac{1}{R}\frac{d}{dr}r^2\frac{dR}{dr} + (k^2+f(r))r^2 = \frac{L^2}{r^2} - g(\theta) - \frac{h(\phi)}{\sin^2\theta} = l(l+1)$

$\Rightarrow \frac{d}{dr}r^2\frac{dR}{dr} + [(k^2+f(r))r^2 - l(l+1)]R = 0$,

$$-\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial P}{\partial\theta} - P[g(\theta) + l(l+1)]\sin^2\theta + m^2P = 0,$$

$$\frac{d^2\Phi}{d\phi^2} + h(\phi)\Phi = -m^2\Phi$$

7.4.1 Show that Legendre's equation has regular singularities at $x = -1, 1$ and ∞. ②

$\Rightarrow$ for Legendre's function $P(x) = \frac{2x}{1-x^2}$, $Q(x) = \frac{l(l+1)}{1-x^2}$.

for $y'' + P(x)y' + Q(x)y = 0$

Now $P(x)$ and $Q(x)$ diverges at $x = 1, -1, \infty$.

Now $(x-1)P(x) = (x-1)\cdot\frac{2x}{(1-x)(1+x)} = -\frac{2x}{1+x}\Big|_{x\to 1}$ $\to$ finite at $x = 1$

$(x-1)^2 Q(x) = (x-1)^2 \frac{l(l+1)}{(1-x)(1+x)} = \frac{l(l+1)(x-1)}{(1+x)}$ $\to$ finite at $x = 1$,

similarly $(x+1)P(x)$ and $(x+1)^2 Q(x)$ is finite ~~diverges~~ at $x = -1$.

so $x = \pm 1$ is a regular singular point.

at take $x \to \frac{1}{z}$ ~~and do the same~~

~~need to check $\frac{P(z^{-1})}{z^2}$ at $z \to 0$.~~

~~$= \frac{z^{-1}}{z^2} \sim \frac{1}{z^2}$ diverges more rapidly than $\frac{1}{z}$.~~

~~so ∞ is an irregular singularity.~~

at $z \to 0$ $\quad \dfrac{2z - P(z^{-1})}{z^2} =$

$$= \frac{2z - 2/z}{1 - \frac{1}{z^2}} = 2\left(z + \frac{z}{1-z^2}\right) \text{ is regular}$$

and $\quad \dfrac{Q(z^{-1})}{z^4} = \dfrac{l(l+1)}{z^2(z^2-1)} \sim z^{-2}$ diverges,

so ∞ is a regular singularity.

7.4.2

Show that Laguerre's equation, like the Bessel equation, has a regular singularity at $x=0$ and an irregular singularity at $x = \infty$.

$\Rightarrow \quad P(x) = \dfrac{1-x}{x}, \quad Q = \dfrac{n}{x}.$

$x = 0$ is a regular singularity

as $x \cdot P(x)$ and $x Q(x)$ is finite.

for $z \to 0$, $\quad \dfrac{2z - P(z^{-1})}{z^2} = \dfrac{z+1}{z^2} \sim \dfrac{1}{z^2}$ diverges more rapidly than $1/z$, so ∞ is an irregular singularity.

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7-4-3 Show that Chebyshev's equation, like the Legendre equation, has regular singularities at $x=-1, 1$ and ∞

③

→ Writing the Chebyshev equation in the form,-

$$y'' + \left(\frac{x}{1-x^2}\right)y' + \left(\frac{x^2}{1-x^2}\right)y = 0 ,$$

we can see that the ~~co-eff~~ $P(x) = \frac{x}{1-x^2}$

$$Q(x) = \frac{x^2}{1-x^2} .$$

$P(x)$ and $Q(x)$ are singular at $x \to \pm 1$, and that each singularity is first order, so the ODE has regular singularities at these points.

for $x \to \infty$ take $x \to 1/z$, at find for $z \to 0$,

$$\frac{2z - P(z^{-1})}{z^2} = \frac{2}{z} - \frac{1/z}{z^2-1}$$

$$\frac{Q(z^{-1})}{z^4} = \frac{x^2}{z^4 - x^2} ,$$

these have at $x=0$, singularities that are respectively of first and second order, indicating that the ODE has a regular singularity at infinity.

7.5.8 For the special case of no azimuthal dependence, the quantum mechanical analysis of the hydrogen molecular ion leads to the eq^n

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du}{d\eta}\right]+\alpha u+\beta\eta^2 u=0.$$

Develop a power-series solution for $u(\eta)$. Evaluate the first three non vanishing coefficient in term of a_0.

$\Rightarrow$ substituting $\sum_{j=0}^{\alpha} a_j \eta^{j+k}$ we obtain

$$a_{j+2}(j+k+2)(j+k+1)-a_j\left[(j+k)(j+k+1)-\alpha\right]+\beta a_{j-2}=0$$

for $j=-2$, $a_{-2}=0$, $=a_{-4}$. by definition, and the indicial equation $k(k-1)a_0=0$ comes out i.e $k=0$ or $k=1$ for $a_0\neq 0$.

for $j=-1$, with $a_{-3}=0=a_{-1}$ we have $a_1 k(k+1)=0$, If $k=1$, then $a_1=0 \Rightarrow a_3=a_5=0$.

for $j=0$, $k=0$, we get $2a_2=-a_0\alpha$ and

$6a_2 = a_0(2-\alpha)$ for $k=1$,

for $j=1, k=0$, we find $6a_3 = 12a_3 = a_1(6-\alpha)$ for $k=1$,
finally for $j=3$, $k=1$, we have

$$20a_4 - (12-\alpha)a_2 + \beta a_0 = 0,$$ giving the

expansion

$$u_{k=1} = a_0\eta\left\{1 + \frac{2-\alpha}{6}\eta^2 + \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20}\right]\eta^4 + \cdots\right].$$

7.5.9 To a good approximation, the interaction of two nucleons may be described by a mesonic potential

$$V = \frac{Ae^{-ax}}{x},$$

attractive for A negative. Show that the resultant Schroedinger wave equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E-V)\psi = 0$$ has the

following series solution through the first three non-vanishing coefficients.

$$\psi = a_0\left\{x + \tfrac{1}{2}A'x^2 + \tfrac{1}{6}\left[\tfrac{1}{2}A'^2 - E' - aA'\right]x^3 + \cdots\right\}.$$

→ substitute $\psi = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$,

and $A' = \frac{2mA}{\hbar^2}$, $E' = \frac{2mE}{\hbar^2}$, $V = \frac{A}{x}e^{-ax}$ $A<0$

$a>0$.

we obtain

$$2a_2 + 6a_3x + \cdots + \left[-A' + (E' + aA')x - \tfrac{1}{2}A'a^2x^2 + \cdots\right] \times (a_1 + a_2x + \cdots) = 0.$$

where the co-efficients of all power of x vanishes. this implies,-

$$a_0 = 0, \quad 2a_2 = A'a_1, \quad 6a_3 + a_1(E' + aA') - A'a_2 = 0$$

$\rightarrow$ putting this you get the given sln.

7.5.10 $\quad y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0,$

From the indicial eqn and recurrance relation

$$y = \sum_{j=0}^{\infty} a_j x^{j+k}, \quad y' = \sum_{j=0}^{\infty} a_j (j+k) x^{j+k-1},$$

$$y'' = \sum_{j=0}^{\infty} a_j (j+k)(j+k+1) = 0$$

for $j=-1$, $a_1 = 0$ by definition, so $k=0$. for $a_0 \neq 0$, is the indicial equation, ~~for $j=0$~~

For $j=0$, $-2a_0 + a_1 = 0$, and for $j=1$, $-2a_1 + 2a_2 = 0$

while $j=2$ yields $a_3 = 0$ etc. Hence our solution is

$$y = a_0(1 + 2x + 2x^2)$$

7.6.3 Using the Wronskian determinant, show that the set of functions $\left\{1, \frac{x^n}{n} (n=1,2,\ldots N)\right\}$ is linearly independent

Using $y_n = \frac{x^n}{n!}$, $y_n' = \frac{x^{n-1}}{(n-1)!}$ for $n = 0, 1, \dots N$ we get. (5)

$$W_1 = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1, \quad W_2 = \begin{vmatrix} 1 & x & x^2/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

$= W_1 = 1$

and continuing $W_2 = \dots = W_N = 1$

7.6.4 If the Wronskian of two functions y_1 and y_2 is indentically zero, show by direct integration that $y_1 = c y_2$.

$\Rightarrow$ If $W = y_1 y_2' - y_1' y_2$ then $\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$ as $W = 0$

Integrating we get $\ln y_1 = \ln y_2 + \ln c$

$$y_1 = c y_2 .$$

7.6.9 Legendre's differential eqn

$(1-x^2) y'' - 2xy' + n(n+1) y = 0$ has a regular solution $P_n(x)$ and an irregular soln. $Q_n(x)$. Show that the Wronskian of P_n and Q_n is given by.

$$P_n(x) Q_n'(x) - P_n'(x) Q_n(x) = A_n / 1-x^2$$

$\Rightarrow$

$$P_n(x) Q_n'(x) - Q_n P_n' = W(x) = A_n e^{-\int^x P dt}$$

$$-\int^x P\,dt = \int^x \frac{2t}{1-t^2}\,dt = -\ln(1-x^2)$$

$$P_n(x) Q_n'(x) - Q_n P_n' = A_n / (1-x^2)$$

7.6.11 Show the following when the linear second-order differential equation $py'' + qy' + ry = 0$ is expressed in self-adjoint form:

(a) The Wronskian is equal to a constant divided by p,

$$W(x) = \frac{C}{p(x)},$$

(b) A second solution $y_2(x)$ is obtained from a first solution $y_1(x)$ as

$$y_2(x) = C y_1(x) \int^x \frac{dt}{p(t)[y_1(t)]^2},$$

$\Rightarrow$ from $\left(\frac{d}{dx} p(x) \frac{d}{dx} + q(x)\right) u = 0$, we have,

(a) $$\int \frac{dW}{W} = -\int^x \frac{p'}{p}\,dx = \ln\frac{1}{p} + \ln C = \ln W$$

$$W = \frac{W(a)}{p(x)} \quad \text{with } W(a) = C.$$

(b) ~~A second solution~~

$$W(y_1, y_2) = y_1^2(x) \frac{d}{dx} \frac{y_2(x)}{y_1(x)}.$$

Integr $$y_2(x) = W(a) y_1(x) \int^x \frac{dt}{p(t)[y_1(t)]^2},$$

7.6.12 Transform our linear second-order ODE (6)

$y'' + P(x) y' + Q(x) y = 0$ by the substitution

$y = z \exp\left[-\frac{1}{2}\int^x P(t)\, dt\right]$.

and show that the resulting differential eq^n for z is $z'' + q(x) z = 0$ where $q(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x)$.

→ $y = zE$ where $E = e^{-\frac{1}{2}\int^x P\, dt}$

$y' = z'E - \frac{1}{2} z P E$,

$y'' = z''E - Pz'E - \frac{z}{2}P'E + \frac{z}{4}P^2 E$,

we obtain,—

$y'' + Py' + Q = E\left[z'' - \frac{z}{2}P' - \frac{z}{4}P^2 + Qz\right] = 0$

7.6.14 By direct differentiation and substitution show that

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^s P(t)\, dt\right]}{[y_1(s)]^2}\, ds.$$

$y_1(x)$ satisfies $y_1'' + P(x) y_1' + Q(x) y_1 = 0$,

⇒ Defining $E_1 = \int^x \frac{e^{-\int^s P dt}}{y_1(s)^2}\, ds$, $E(x) = e^{-\int^s P dt}$

and using $y_2 = y_1(x) E_1$, $y_2' = y_1' E_1 + \frac{E}{y_1}$,

$y_2'' = y_1'' E_1 - \frac{PE}{y_1}$