Differential Equations

Separation of Variables

Let us consider the Helmholtz equation,

$$(\nabla^2 + k^2)\psi = 0,$$

where ∇^2 is the Laplacian operator and k^2 is a constant.

Cartesian coordinates

In cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

$\therefore$ Helmholtz equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2\psi = 0. \quad (1)$$

We may guess that the solution to the above equation will have the following form

$$\psi(x, y, z) = X(x)\, Y(y)\, Z(z) \quad (2)$$

~~Then~~ we get substituting eq. (2) in (1),

$$YZ \frac{d^2X}{dx^2} + XZ \frac{d^2Y}{dy^2} + XY \frac{d^2Z}{dz^2} + k^2 XYZ = 0$$

$f(x) + g(y) + h(z) = \text{const.}$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0.$$

The first three terms in the above

equation depend on x, y and z alone. Since these are independent variables we can expect

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = -n^2 \qquad (2)$$

where l, m, n are constants and are constrained by the equation

$$k^2 = l^2 + m^2 + n^2.$$

By solving the relations in Eq-(2), a solution to the Helmholtz equation is

$$\psi_{lmn}(x,y,z) = X_l(x)\, Y_m(y)\, Z_n(z).$$

The general solution will have the form

$$\psi = \sum_{l,m,n} a_{lmn}\, \psi_{lmn}$$

Circular cylindrical coordinates

In circular cylindrical coordinates, the Laplacian operator is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

The Helmholtz equation is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0.$$

We can guess that the solution to have the form

$$\psi = P(\rho)\, \Phi(\phi)\, Z(z).$$

we get

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{RZ}{\rho^2}\frac{d^2\Phi}{d\phi^2}+R\Phi\frac{d^2Z}{dz^2}$$
$$+k^2 R\Phi Z=0$$

$$\Rightarrow \frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2}$$
$$+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2=0.$$

We can expect

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=l^2$$

Then,

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+(k^2+l^2)\rho^2=0$$

Put $k^2+l^2=n^2$ and $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2$.

Then we get

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left(n^2\rho^2-m^2\right)R=0.$$

The above equation is Bessel's differential equation.

General solution to the Helmholtz equation will have the form

$$\psi(\rho,\phi,z)=\sum_{m,n}a_{mn}R_{mn}(\rho)\Phi_m(\phi)Z_n(z).$$

Spherical polar coordinates

In spherical polar coordinates, the Helmholtz equation becomes into equation depend on x, y and z alone. Since these are independent variables we can expect

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = -n^2 \qquad (3)$$

where l, m, n are constants and are constrained by the equation

$$k^2 = l^2 + m^2 + n^2.$$

By solving the relations in Eq.(3), a solution to the Helmholtz equation is

$$\psi_{lmn}(x, y, z) = X_l(x)\, Y_m(y)\, Z_n(z).$$

The general solution will have the form

$$\psi = \sum_{l,m,n} a_{lmn}\, \psi_{lmn}$$

Circular cylindrical coordinates

In circular cylindrical coordinates, the Laplacian operator is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

The Helmholtz Equation is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0.$$

We can guess that the solution to have the form

$$\psi = P(\rho)\, \Phi(\phi)\, Z(z).$$

we get

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{PZ}{\rho^2}\frac{d^2\Phi}{d\phi^2}+P\Phi\frac{d^2Z}{dz^2}$$
$$+k^2P\Phi Z=0$$

$$\Rightarrow \frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2}$$
$$+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2=0.$$

We can expect

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=l^2$$

Then,

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+(k^2+l^2)\rho^2=0$$

Put $k^2+l^2=n^2$ and $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2$.

Then we get

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+(n^2\rho^2-m^2)P=0.$$

The above equation is Bessel's differential equation.

General solution to the Helmholtz equation will have the form

$$\psi(\rho,\phi,z)=\sum_{m,n}a_{mn}P_{mn}(\rho)\,\Phi_m(\phi)\,Z_n(z).$$

<u>Spherical Polar coordinates</u>

In spherical polar coordinates, the Helmholtz equation becomes into

$$\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+k^2\psi=0.$$

Taking the solution as $\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$, we get

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}+k^2=0.$$

$$\Rightarrow r^2\sin^2\theta\left[\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+k^2\right]+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0$$

After putting $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2$, we get

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+k^2-\frac{m^2}{r^2\sin^2\theta}=0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2r^2+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0$$

The first two terms depends on r and the last two terms depends on θ.

We can take

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2r^2=Q$$

and $$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-Q,$$

Here Q is a constant.

In the above two equations, the first one gives spherical Bessel equation and the second one gives associated Legendre equation.

The general solution to the Helmholtz equation is

$$\psi_{lm}(r,\theta,\phi) = \sum_{l,m} a_{lm} R_l(r)\,\Theta_{lm}(\theta)\,\Phi_m(\phi)$$

<u>Singular points</u>

Consider a linear, second-order, homogeneous differential equation in $y(x)$.

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

If $P(x)$ and $Q(x)$ remain finite at $x = x_0$, then x_0 is an ordinary point. On the other hand, if $P(x)$ or $Q(x)$ diverges as $x \to x_0$, x_0 is a singular point.

A singular point can have two natures.

1. If $P(x)$ or $Q(x)$ diverges, ~~as $x \to x_0$~~, but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remains finite as $x \to x_0$, then x_0 is called a regular or nonessential singular point.

2. If $(x-x_0)P(x)$ or $(x-x_0)^2 Q(x)$ diverges as $x \to x_0$, then x_0 is called a irregular or essential singular point.

To determine the nature of singularity at $x \to \infty$, we do the above procedure by substituting $x = \frac{1}{z}$ and taking $z \to 0$.

Example: The Bessel's equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

Series solution - Frobenius's method

A differential equation of the form

$$y'' + P(x) y' + Q(x) y = 0$$

can have one solution of series expansion, provided the point of expansion is an ordinary or at most a regular singular point.

Consider the linear oscillator equation

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0. \quad -(1)$$

we try with the following series expansion around $x_0 = 0$.

$$y(x) = x^k (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_0 \neq 0$$

we get

$$\frac{d^2 y}{dx^2} = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}.$$

substituting the series expansion in Eq.(1) we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0 \quad -(2)$$

Expand ($\lambda = 0, 1, 2, \dots$)

We have to demand that the coefficient of ~~coeff~~ all powers of x should be zero.
The two lowest powers of x in the above equation are x^{k-2} and x^{k-1}; their coefficients should be zero, hence we have

$$a_0 k(k-1) = 0$$

$$a_1 (k+1) k = 0 \quad -(3).$$

Since $a_0 \neq 0$, we have ~~k=0~~ ~~or k=1~~. $k(k-1)=0$. This is called indicial equation, which is coming from the lowest power of x.
The solution of indicial equation is $k=0$ or $k=1$.
To satisfy Eq.(3), we take $a_1=0$.
From x^{k+2} and higher powers ~~For j≥2~~, we get the following recurrence relation from eq.(2).

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0$$

$$\Rightarrow a_{j+2} = -a_j \frac{\omega^2}{(k+j+2)(k+j+1)} \quad (4)$$

Since we have taken $a_1=0$, all the odd coefficients vanish, i.e.

$$a_3 = a_5 = a_7 \cdots = 0.$$

From ~~the~~ indicial equation we get $k=0$ or $k=1$. We first try $k=0$ in Eq. (4), then

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)}$$

This leads to

$$a_2 = -a_0 \frac{\omega^2}{1\cdot 2} = -\frac{\omega^2}{2!} a_0$$

$$a_4 = -a_2 \frac{\omega^2}{3\cdot 4} = \frac{\omega^4}{4!} a_0$$

$$a_6 = -a_4 \frac{\omega^2}{5\cdot 6} = \frac{-\omega^6}{6!} a_0.$$

By induction, we can show that

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0$$

Our series solution with $k=0$ becomes

$$y(x)\Big|_{k=0} = \sum_{j=0} a_j x^j$$

$$= a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \dots \right]$$

$$= a_0 \cos \omega x.$$

with $k=1$, the recurrence relation, eq(1) becomes into

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}$$

substituting $j=0, 2, 4,$ etc, we get

$$a_2 = -a_0 \frac{\omega^2}{2\cdot 3} = \frac{-\omega^2}{3!} a_0$$

$$a_4 = -a_2 \frac{\omega^2}{4\cdot 5} = \frac{\omega^4}{5!} a_0$$

$$a_6 = -a_4 \frac{\omega^2}{6\cdot 7} = -\frac{\omega^6}{7!} a_0$$

and in general

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0.$$

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~~The~~ The series solution with $k=1$ becomes

$$y(x)_{k=1} = \sum_{j=0} a_j x^{j+1}$$

$$= a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots\right]$$

$$= \frac{a_0}{\omega}\left[\omega x - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots\right]$$

$$= \frac{a_0}{\omega} \sin \omega x$$

Comments about Frobenius's method

1. ~~Sometim~~ In some cases, Frobenius's method give series solution, but this may not be convergent series. This can happen if the point of expansion is an essential singularity.

2. If x_0 is not an essential singularity, we can also try with the following series expansion

$$y(x) = \sum_{\lambda=\lambda}^{k} a_\lambda (x-x_0)^{k+\lambda}, \quad a_0 \neq 0.$$

<u>Limitations</u> of series approach - <u>Bessel's Equation</u>

~~Th~~ The Bessel's differential equation is $x^2 y'' + x y' + (x^2 - n^2) y = 0$.

where n is a real number.

Substituting ~~consider~~ the series expansion

$$y(x) = \sum_{\lambda=0} a_\lambda x^{k+\lambda}.$$

in the above equation, we get

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r}$$

$$+ \sum_{r=0}^{\infty} a_r x^{k+r+2} - \sum_{r=0}^{\infty} a_r n^2 x^{k+r} = 0.$$

For $r=0$, the coefficient of x^k is

$$a_0 \left[k(k-1) + k - n^2 \right] = 0$$

$$\Rightarrow k^2 - n^2 = 0, \quad k = \pm n$$

For $r=1$, the coefficient of x^{k+1} is

$$a_1 \left[(k+1)(k + k+1 - n^2 \right] = 0$$

$$\Rightarrow a_1 (k+1-n)(k+1+n) = 0$$

For $k = \pm n$, we get $a_1 = 0$.

For $r \geq 2$, we get the recurrence relation as

$$a_{j+2} \left[(k+j+2)(k+j+1) + (k+j+2 - n^2 \right] + a_j = 0$$

For $k = n$, we get

$$a_{j+2} = -a_j \frac{1}{(j+2)(2n+j+2)}$$

Since $a_1 = 0$, all the odd coefficients are zero. The even coefficients are

$$a_2 = -a_0 \frac{1}{2(2n+2)} = -\frac{a_0 n!}{2^2 1!(n+1)!}$$

$$a_4 = -a_2 \frac{1}{4(2n+4)} = \frac{a_0 n!}{2^4 2!(n+2)!}$$

$$a_6 = -a_4 \frac{1}{6(2n+6)} = -\frac{a_0 n!}{2^6\, 3!\, (n+3)!}$$

and

$$a_{2p} = (-1)^p \frac{a_0 n!}{2^{2p}\, p!\, (n+p)!}$$

$\therefore$ the series solution for $k=n$, is

$$y(x) = a_0 x^n \left[1 - \frac{n!\, x^2}{2^2\, 1!\, (n+1)!} + \frac{n!\, x^4}{2^4\, 2!\, (n+2)!} - \cdots \right]$$

$$= a_0 \sum_{j=0}^{\infty} (-1)^j \frac{n!\, x^{n+2j}}{2^{2j}\, j!\, (n+j)!}$$

$$= a_0 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!\, (n+j)!} \left(\frac{x}{2}\right)^{n+2j}$$

For $k=-n$ and n not an integer, we get the second distinct solution.

If n is a integer, for $k=-n$, the recurrence relation is

$$a_{j+2} = -a_j \frac{1}{(j+2)(-2n+j+2)}$$

$\therefore$ there is a singularity in a_{j+2} for $j+2 = 2n$. Hence for integer n, we cannot get the second independent solution for Bessel's equation by this series technique.

<u>Fuch's</u> <u>theorem</u>: We can always obtain at least one power-series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point.

Regular and Irregular singularities

Consider the two differential Equations

$$y'' - \frac{6}{x^2} y = 0 \quad -(1)$$

$$y'' - \frac{6}{x^3} y = 0 \quad -(2).$$

$x_0 = 0$ is a regular singular point in Eq(1) and it is an essential singular point for Eq.(2).

Substituting $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$ in Eq.(1) we get

$$\sum_{\lambda=0}^{\infty} \left[(k+\lambda)(k+\lambda-1) - 6\right] a_\lambda x^{k+\lambda-2} = 0$$

coefficient of x^{k-2} is

$$a_0 \left[k(k-1) - 6 \right] = 0 \Rightarrow k = 3, -2$$

$$a_\lambda \left[(k+\lambda)(k+\lambda-1)-6 \right] = 0 \quad \lambda \geq 1$$

$\therefore$ Indicial Equation is $k(k-1) - 6 = 0$

$\Rightarrow k = 3, -2 \Rightarrow \lambda = 5$

there is no recurrence relation for Eq.(1). The series solution to Eq.(1) can be obtained for $a_0 \neq 0$ and $a_\lambda = 0$ for $\lambda \geq 1$.

The solutions are x^3 and x^{-2}.

In a similar way, we can obtain indicial equation for Eq.(2) as

$$-6 a_0 = 0.$$

This is contradictory since $a_0 \neq 0$. Hence for Eq.(2), there exists no series solution.

consider the two differential Equations

$$y'' + \frac{1}{x} y' - \frac{a^2}{x^2} y = 0 \quad (3)$$

$$y'' + \frac{1}{x^2} y' - \frac{a^2}{x^2} y = 0 \quad (4)$$

$x=0$ is a regular singular point for Eq. (3) and essential singular point for Eq. (4).

By substituting $y = \sum_{r=0}^{\infty} a_r x^{k+r}$, in Eq. (3) we can find no recurrence relation but the indicial Equation is $k^2 - a^2 = 0$.

For Eq. (3), the solutions are x^a and x^{-a}. in the case of Eq. (4), the indicial Equation is

$$k = 0$$

and the recurrence relation would be

$$a_{j+1} = a_j \frac{a^2 - j(j-1)}{j+1}.$$

we have

$$\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} \frac{j^2}{j} = \infty$$

$\therefore$ Hence the series ~~expansion~~ solution to Eq. (4) would be divergent.

<u>Linear independence of solutions</u>

Consider the functions ϕ_i, $i = 1, 2, \dots, n$.

these functions are linearly independent if

$$\sum_{i=1}^{n} k_i \phi_i = 0 \quad \text{.(1)}$$

has the only solution ~~has only one solution provided~~
$k_i = 0$, $i = 1, 2, \dots, n$.

Differentiating Eq.(1) for $(n-1)$ times we get.

$$\sum_{i=1}^{n} k_i \phi_i' = 0$$

$$\vdots$$

$$\sum_{i=1}^{n} k_i \phi_i^{(n-1)} = 0$$

The above equations can be put into the following matrix form

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = 0$$

write $W = \det \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & & & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}$, which is called the Wronskian.

If ~~the Wronskian is not equal to~~ $W \neq 0$, then all k_i are zero and hence the set of function ϕ_i, $i = 1, \dots, n$ are linearly independent.

<u>A second solution</u>

Let the following differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

has two independent solutions y_1 and y_2 (in the case of Bessel's equation, we know one solution y_1 and we have to find the second one y_2.) let y_1 is known to us.

The Wronskian ~~for~~ of these solutions is

$$W = y_1 y_2' - y_1' y_2$$

$$\Rightarrow W' = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$$

$$= y_1\left(-P(x) y_2' - Q(x) y_2\right)$$

$$- y_2\left(-P(x) y_1' - Q(x) y_1\right)$$

$$= -P(x)\left(y_1 y_2' - y_1' y_2\right)$$

$$= -P(x)\, W$$

$$\Rightarrow \frac{dW}{W} = -P(x)\,dx$$

$$\Rightarrow \int_a^x \frac{dW}{W} = -\int_a^x P(x_1)\,dx_1,$$

where a is a constant

$\therefore$ we get $\ln \dfrac{W(x)}{W(a)} = -\displaystyle\int_a^x P(x_1)\,dx_1$

$$\Rightarrow W(x) = W(a) \exp\left[-\int_a^x P(x_1)\,dx_1\right]$$

we have

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

$\therefore$ we get $\dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right) = W(a)\dfrac{\exp\left[-\int_a^x P(x_1)dx_1\right]}{y_1^2}$

Integrating the above equation from $x_2 = b$ to $x_2 = x$, we get

$$\frac{y_2(x)}{y_1(x)} - \frac{y_2(b)}{y_1(b)} = W(a)\int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1)dx_1\right]}{y_1^2(x_2)}dx_2$$

$$\Rightarrow y_2(x) = y_1(x)\frac{y_2(b)}{y_1(b)} + y_1(x)\,W(a)\int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1)dx_1\right]}{y_1^2(x_2)}dx_2$$

we can drop the term $y_1(x)\dfrac{y_2(b)}{y_1(b)}$, since it doesn't contribute anything new to the general solution.

we can adjust the normalization factor such that $W(a) = 1$.

$\therefore$ The second solution will have the form

$$y_2(x) = y_1(x)\int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1)dx_1\right]}{y_1^2(x_2)}dx_2$$

<u>Nonhomogeneous equation - Green's function</u>

In electrostatic, the potential ϕ due to a charge source ρ satisfies the Poisson's equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \quad -(1)$$

we know that the solution to this equation is

$$\phi(\vec{r}_1) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}d^3x_2 \quad -(2)$$

Suppose we have function $G(\vec{r}_1, \vec{r}_2)$ which satisfies the following equation

$$\nabla^2 G = -\delta^{(3)}(\vec{r}_1 - \vec{r}_2) \qquad -(3)$$

We can interpret G as a potential at $\vec{r}_1$ due to a source at $\vec{r}_2$. G is called the Green's function.
Consider

$$\int (\psi \nabla^2 G - G \nabla^2 \psi)\, d^3 r_2$$

$$= \int \vec{\nabla} \cdot [\psi \vec{\nabla} G - G \vec{\nabla} \psi]\, d^3 r_2$$

$$= \int [\psi \vec{\nabla} G - G \vec{\nabla} \psi] \cdot \hat{n}\, dS$$

$$= \int [\psi \vec{\nabla} G - G \vec{\nabla} \psi] \cdot \hat{n}\, r^2 \sin\theta\, d\theta\, d\phi$$

Assume that the quantity in square bracket falls faster than r^{-2}. Then by taking the volume or enclosing surface to be infinitely large, we get

$$\int (\psi \nabla^2 G - G \nabla^2 \psi)\, d^3 r_2 = 0$$

$$\Rightarrow \int \psi \nabla^2 G\, d^3 r_2 = \int G \nabla^2 \psi\, d^3 r_2$$

Substituting Eqs. (1) and (3), we get

$$-\int \psi(\vec{r}_2)\, \delta^{(3)}(\vec{r}_1 - \vec{r}_2)\, d^3 r_2 = -\int G(\vec{r}_1, \vec{r}_2) \frac{\rho(\vec{r}_2)}{\epsilon_0}\, d^3 r_2$$

$$\Rightarrow \psi(\vec{r}_1) = \frac{1}{\epsilon_0} \int G(\vec{r}_1, \vec{r}_2)\, \rho(\vec{r}_2)\, d^3 r_2 \qquad -(4)$$

To find the Green's function G, consider the following electrostatic problem.

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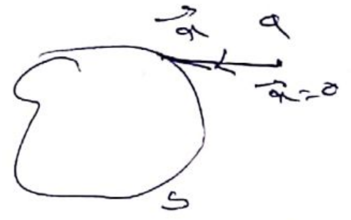

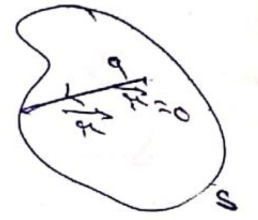

For a point charge q, we have

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}.$$

From Gauss's law,

$$\oint_S \vec{E}\cdot d\vec{S} = \begin{cases} 0, & \text{if origin is outside the volume} \\ \dfrac{q}{\epsilon_0}, & \text{if origin is within the volume} \end{cases}$$

But, $\oint_S \vec{E}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \int \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d^3x$

~~∴ we have~~

$$\cancel{\int \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d^3x = \Big\{ 0, \text{ if origin}}$$

One can show that $\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$.

$$\therefore \oint_S \vec{E}\cdot d\vec{S} = \frac{-q}{4\pi\epsilon_0}\int \nabla^2\left(\frac{1}{r}\right) d^3x.$$

∴ Gauss's law becomes

$$\int \nabla^2\left(\frac{1}{r}\right) d^3x = \begin{cases} 0, & \text{if origin is outside the volume} \\ -4\pi, & \text{if origin is within the volume.} \end{cases}$$

We can express this as

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^{(3)}(\vec{r})$$

If the point charge lies at $\vec{r}_2$, the above relation modifies to

$$\nabla^2\frac{1}{|\vec{r}_1-\vec{r}_2|} = -4\pi\,\delta^{(3)}(\vec{r}_1-\vec{r}_2)$$

comparing the above equation with Eq.(3), we can write the Green's function as

$$G(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi|\vec{r}_1 - \vec{r}_2|}$$

substituting this in Eq.(4) we get

$$\phi(\vec{r}_1) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} d^3r_2$$

This equation matches with Eq.(2).

Suppose that a second-order, linear differential operator $\mathcal{L}$ satisfies the following nonhomogeneous equation

$$\mathcal{L}\, y(\vec{r}_1) = -f(\vec{r}_1) \quad -(5)$$

If we find a Green's function of the form

$$\mathcal{L}\, G(\vec{r}_1, \vec{r}_2) = -\delta^{(3)}(\vec{r}_1 - \vec{r}_2).$$

then the particular solution to the nonhomogeneous equation is

$$y(\vec{r}_1) = \int G(\vec{r}_1, \vec{r}_2) f(\vec{r}_2) d^3r_2.$$

<u>Quantum Mechanical Scattering</u>

Suppose a beam of particles are scattered by a potential $V(\vec{r})$,

The wave function for this system satisfies the time-independent Schrodinger equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\Rightarrow \left(\nabla^2 + k^2\right)\psi(\vec{r}) = -\left[-\frac{2m}{\hbar}V(\vec{r})\,\psi(\vec{r})\right] \quad \text{---(6)}$$

where $k^2 = \frac{2mE}{\hbar^2}$.

In the asymptotic limit, we expect the wave function to have the form

$$\psi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + f_k(\theta,\phi)\frac{e^{ikr}}{r}, \quad k=|\vec{k}|$$

Eq. (6) has the form of Eq. (5), with the ~~oper~~ differential operator as $L = \nabla^2 + k^2$.

$\therefore$ The particular solution to Eq. (6) is

$$\psi_p(\vec{r}_1) = -\int \frac{2m}{\hbar^2} V(\vec{r}_2)\,\psi(\vec{r}_2)\,G(\vec{r}_1,\vec{r}_2)\,d^3r_2$$

where $G(\vec{r}_1,\vec{r}_2)$ is the Green's function of $\nabla^2 + k^2$.

the general sol to Eq (6) is $\varphi = c_1 e^{i\vec{k}\cdot\vec{r}} + c_2 e^{-ik\cdot r} + \psi_p$

Since we expect $\psi(\vec{r}_1)$ to have the above asymptotic form, we can write

$$\psi(\vec{r}_1) = e^{i\vec{k}\cdot\vec{r}} - \int \frac{2m}{\hbar^2} V(\vec{r}_2)\,\psi(\vec{r}_2)\,G(\vec{r}_1,\vec{r}_2)\,d^3r_2.$$

The Green's function is given by

$$G(\vec{r}_1,\vec{r}_2) = \frac{e^{ik|\vec{r}_1-\vec{r}_2|}}{4\pi\,|\vec{r}_1-\vec{r}_2|}.$$

$$\therefore\ \psi(\vec{r}_1) = e^{i\vec{k}\cdot\vec{r}} - \int \frac{2m}{\hbar^2} V(\vec{r}_2)\,\psi(\vec{r}_2)\,\frac{e^{ik|\vec{r}_1-\vec{r}_2|}}{4\pi\,|\vec{r}_1-\vec{r}_2|}\,d^3r_2 \quad \text{---(7)}$$

In the limit $V(\vec{r}) \to 0$, the wave function is

$$\psi_0(\vec{r}_1) = e^{i\vec{k}\cdot\vec{r}_1}$$

* ~~For a weak potential~~ the first correction to the wave function is

$$\phi_1(\vec{r}_1) = e^{i\vec{k}\cdot\vec{r}_1} - \int \frac{2m}{\hbar^2} V(\vec{r}_2) \frac{e^{ik|\vec{r}_1 - \vec{r}_2|}}{4\pi|\vec{r}_1 - \vec{r}_2|} e^{i\vec{k}\cdot\vec{r}_2} d\tau_2$$

This is called the Born approximation.

~~Substituting $\phi_0(\vec{r}_1)$ in Eq. (7), we~~

For a weak potential, we can substitute $\phi_0(\vec{r})$ in Eq(7), which gives us approximate wave function.

Bessel Functions

We define Bessel function through a generating function of the following form

$$g(x,t) = e^{\left(\frac{x}{2}\right)\left(t - \frac{1}{t}\right)}$$

By expanding the generating function we obtain

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n .$$

$J_n(x)$ is called the Bessel function of first kind of integral order n.

We have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{xt/2}\; e^{-\frac{x}{2t}}$$

$$= \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{r!\, s!} \left(\frac{x}{2}\right)^{r+s} \cancel{\frac{1}{r!\,s!}}\; t^{r-s} \quad (1)$$

For $n \geq 0$, we can put $r - s = n$.

Then,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!\, s!} \left(\frac{x}{2}\right)^{n+2s} t^n$$

∴ the ~~Bessel function~~ coefficient of t^n is

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s} \quad -(2)$$

For $n<0$, we can put ~~n~~ $-s=-n$ in Eq. (1). The Bessel function is

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(s-n)!} \left(\frac{x}{2}\right)^{2s-n} \quad \text{—(3)}$$

For integer n, there is a singularity in $(s-n)!$ for $s=0,1,\dots,n-1$. We avoid these singularities by starting the series from $s=n$. Replacing s by $s+n$ in Eq. (3), we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{s!\,(s+n)!} \left(\frac{x}{2}\right)^{2s+n}$$

$$= (-1)^n J_n(x).$$

In Eqs. (2) & (3), we replace n with a real number ν. We then get $J_\nu(x)$ and $J_{-\nu}(x)$. These are Bessel functions of non-integral order of ν.

Recurrence Relations

We have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \quad \text{—(4)}$$

Taking differentiation with 't' gives

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n J_n(x)\, t^{n-1}$$

$$\Rightarrow \frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} = \sum_{n=-\infty}^{\infty} \frac{2n}{x} J_n(x)\, t^{n}$$

$$\Rightarrow \quad J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad -(5)$$

Differentiating Eq.(4) w.r.t. x, we get

$$\frac{1}{2}\left(t - \frac{1}{t}\right) e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

$$\Rightarrow \cancel{\frac{1}{2}\left(t - \frac{1}{t}\right)\sum_{n=-\infty}^{\infty}}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} = \sum_{n=-\infty}^{\infty} 2 J_n'(x)\, t^n$$

$$\Rightarrow \quad J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x). \quad -(6)$$

Adding Eqs (5) & (6), we get

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x) \quad -(7)$$

Multiplying the above with x^n, we get

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

Subtracting Eqs. (5) & (6), we get

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x) \quad -(8)$$

Multiplying the above with x^{-n}, we get

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Bessel's differential Equation

Suppose the function $Z_\nu(x)$ satisfies the recurrence relations, stated previously, by replacing $J_n(x)$ by $Z_\nu(x)$.

From Eq.(7), we can write

$$x Z'_\nu(x) = x Z_{\nu-1}(x) - \nu Z_\nu(x) \quad -(9)$$

$$\Rightarrow \frac{d}{dx}\left[x Z'_\nu(x)\right] = \frac{d}{dx}\left[x Z_{\nu-1}(x) - \nu Z_\nu(x)\right]$$

$$\Rightarrow x Z''_\nu(x) + (\nu+1) Z'_\nu(x) - x Z'_{\nu-1}(x) - Z_{\nu-1}(x) = 0 \quad -(10)$$

After doing $x \times (10) - \nu \times (9)$, we get

$$x^2 Z''_\nu(x) + x Z'_\nu(x) - \nu^2 Z_\nu(x) + x\left[(\nu-1) Z_{\nu-1}(x) - x Z'_{\nu-1}(x)\right] = 0.$$

In eq.(8) replace n with $\nu-1$, we get

$$x Z_\nu(x) \cancel{x Z'_{\nu-1}(x)} = (\nu-1) Z_{\nu-1}(x) - x Z'_{\nu-1}(x)$$

Using this equation, we get

$$x^2 Z''_\nu(x) + x Z'_\nu(x) + (x^2 - \nu^2) Z_\nu(x) = 0.$$

This is called Bessel's differential equation.

Put $x = k\rho$, k is a constant.

$$\frac{d}{dx} = \frac{d\rho}{dx}\frac{d}{d\rho} = \frac{1}{k}\frac{d}{d\rho}$$

$$\frac{d^2}{dx^2} = \frac{1}{k^2}\frac{d^2}{d\rho^2}$$

$$x\frac{d}{dx} = k\rho \, \frac{1}{k}\frac{d}{d\rho} = \rho\frac{d}{d\rho}$$

$$x^2\frac{d^2}{dx^2} = k^2\rho^2 \, \frac{1}{k^2}\frac{d^2}{d\rho^2} = \rho^2\frac{d^2}{d\rho^2}.$$

$\therefore$ Bessel's differential equation becomes into

$$\rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + \left(k^2\rho^2 - \nu^2\right) Z_\nu(k\rho) = 0$$

$$\Rightarrow \rho \frac{d}{d\rho}\left(\rho Z_\nu\right) + \left(k^2\rho^2 - \nu^2\right) Z_\nu = 0$$

This is the equation we came across after separating variables of Helmholtz equation in circular cylindrical coordinates.

In spherical polar coordinates, from Helmholtz equation, we have got

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + k^2 r^2 = Q.$$

put $Q = n(n+1)$, then we get

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left[k^2r^2 - n(n+1)\right]R = 0 \quad (11)$$

we can convert this into Bessel's differential equation by substituting

$$R = \frac{Z}{(kr)^{1/2}}.$$

then, we get

$$r^2 \frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - \left(n+\tfrac{1}{2}\right)^2\right]Z = 0$$

$\therefore$ the solution to Eq. (11) is

$$R(r) = \frac{Z_{n+\frac{1}{2}}(kr)}{(kr)^{1/2}}.$$

Integral Representation

we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

put $t = e^{i\theta}$, then

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta} .$$

Consider

$$J_1(x)e^{i\theta} + J_{-1}(x)e^{-i\theta} = J_1(x)\left(e^{i\theta} - e^{-i\theta}\right)$$
$$= 2iJ_1(x)\sin\theta$$

$$J_2(x)e^{2i\theta} + J_{-2}(x)e^{-2i\theta} = J_2(x)\left(e^{2i\theta} + e^{-2i\theta}\right)$$
$$= 2J_2(x)\cos 2\theta .$$

∴ we can write

$$e^{ix\sin\theta} = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] + 2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right]$$

$$\Rightarrow \cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin\left[(2n-1)\theta\right]$$

we have the following orthogonality relations

$$\int_0^{\pi} \cos n\theta \cos m\theta \, d\theta = \frac{\pi}{2}\delta_{nm}$$

$$\int_0^{\pi} \sin n\theta \sin m\theta \, d\theta = \frac{\pi}{2}\delta_{nm} .$$

∴ we get

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = \begin{cases} J_n(x), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = \begin{cases} 0, & n \text{ even} \\ J_n(x), & n \text{ odd.} \end{cases}$$

* Combining the above two relations,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta.$$

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Legendre Function

The generating function for Legendre polynomials is

$$g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \qquad |t|<1$$

$P_n(x)$ are ~~Legendre~~ polynomials.

Using ~~binomial theorem~~, we have

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(2xt-t^2)^n \quad (1)$$

The first three terms in this series are

$$\frac{0!}{2^0(0!)^2}(2xt-t^2)^0 + \frac{2!}{2^2(1!)^2}(2xt-t^2)^1 + \frac{4!}{2^4(2!)^2}(2xt-t^2)^2$$

$$= 1t^0 + xt^1 + \left(\frac{3}{2}x^2 - \frac{1}{2}\right)t^2 + O(t^3)$$

$\therefore$ The first three Legendre functions are

$P_0(x)=1, \quad P_1(x)=x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$

Using binomial Expansion,

$$(2xt-t^2)^n = t^n \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(2x)^{n-k} t^k$$

$\therefore$ Eq. (1) becomes into

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^k \frac{(2n)!}{2^{2n}\, n!\, k!\,(n-k)!}(2x)^{n-k} t^{n+k}.$$

For infinite series, we have the result

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{k,n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} a_{k,n-2k}$$

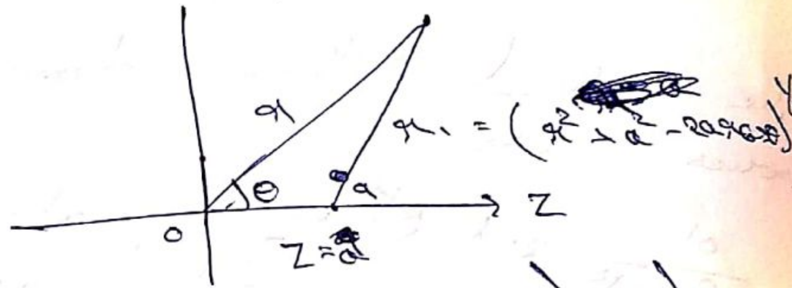
$$\left[\frac{n}{2}\right] = \begin{cases} n/2 & \text{for } n \text{ even} \\ \frac{n-1}{2} & \text{for } n \text{ odd.} \end{cases}$$

Using this result, we get

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^{2n-2k}\, k!\,(n-k)!\,(n-2k)!} (2x)^{n-2k}\, t^n$$

$$\Rightarrow P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, x^{n-2k}$$

Electrostatic potential ~~due to a~~



Potential due to point charge q at a radial distance r_1 is

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r_1}$$

$$= \frac{q}{4\pi\epsilon_0} \left(r^2 + a^2 - 2ar\cos\theta\right)^{-1/2}$$

$$= \frac{q}{4\pi\epsilon_0 r} \left(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2}$$

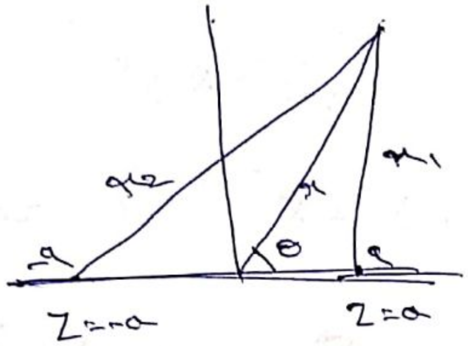
For $r > a$, we identify $\cos\theta = x$ and

we then get

$$\phi = \frac{q}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{a}{r}\right)^n.$$

<u>Electric multipoles</u> → expand

Consider two charges q and −q which are kept at Z = a and −a on Z



$$r_2 = (r^2 + 2ar\cos\theta + a^2)^{1/2} = r\left(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{1/2}$$

The potential at a radial distance r from origin is

$$\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{q}{4\pi\epsilon_0 r}\left\{\left[1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right]^{-1/2} - \left[1 - 2\left(-\frac{a}{r}\right)\cos\theta + \left(\frac{a^2}{r^2}\right)\right]^{-1/2}\right\}$$

$$= \frac{q}{4\pi\epsilon_0 r}\left[\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{-a}{r}\right)^n\right]$$

$$= \frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty}\left[1 - (-1)^n\right] P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

$$= \frac{2q}{4\pi\epsilon_0 r}\left[P_1(\cos\theta)\frac{a}{r} + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \cdots\right]$$

The first term is

$$\phi = \frac{2aq}{4\pi\epsilon_0}\,\frac{P_1(\cos\theta)}{r^2}$$

$2aq$ is the dipole moment and the above expression denotes potential due to ~~the~~ electric dipole.

<u>Recurrence Relations</u>

We have

$$g(t,x) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

$$\Rightarrow \frac{\partial g(t,x)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n t^{n-1}$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} P_n t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n t^{n-1}$$

$$\Rightarrow x\sum_{n=0}^{\infty} P_n t^n - \sum_{n=0}^{\infty} P_n t^{n+1}$$

$$= \sum_{n=0}^{\infty} n P_n t^{n-1} - 2x\sum_{n=0}^{\infty} n P_n t^n + \sum_{n=0}^{\infty} n P_n t^{n+1}$$

Equating t^n-coefficient, we get

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2n x P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \quad (1)$$

we have

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x) t^n$$

$$\Rightarrow t\sum_{n=0}^{\infty} P_n t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n' t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n t^{n+1} = \sum_{n=0}^{\infty} P_n' t^n - 2x\sum_{n=0}^{\infty} P_n' t^{n+1} + \sum_{n=0}^{\infty} P_n' t^{n+2}$$

Equating t^{n+1}-coefficient, we get

$$P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$$

$\Rightarrow P'_{n+1}(x) + P'_{n-1}(x) = 2x P'_n(x) + P_n(x)$ — (2)

Differentiating Eq. (1), we get

$(2n+1)x P'_n(x) + (2n+1)P_n(x) = (n+1) P'_{n+1}(x) + n P'_{n-1}(x)$ (3)

After doing $(2n+1)$ Eq.(2) $- 2 \times$ Eq.(3),

we get

$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$ — (4)

From $\frac{1}{2}$ (Eq.(2) + Eq.(4)), we get

$P'_{n+1}(x) = (n+1) P_n(x) + x P'_n(x)$ — (5)

From $\frac{1}{2}$ (Eq.(2) − Eq.(4)), we get

$P'_{n-1}(x) = -n P_n(x) + x P'_n(x)$ — (6)

Substitute n with $n-1$ in Eq.(5), we get

$P'_n(x) = n P_{n-1}(x) + x P'_{n-1}(x)$

Add to the above equation with

$x \times$ Eq.(6), we get

$(1-x^2) P'_n(x) = n P_{n-1}(x) - n x P_n(x)$

Differentiating the above equation, we get

$(1-x^2) P''_n(x) - 2x P'_n(x) = n P'_{n-1}(x) - n P_n(x) - n x P'_n(x)$

Using Eq(6), ~~and this~~ we get

$(1-x^2) P''_n(x) - 2x P'_n(x) = n[-n P_n(x) + x P'_n(x)] - n P_n(x) - n x P'_n(x)$

$\Rightarrow (1-x^2) P''_n(x) - 2x P'_n(x) + n(n+1) P_n(x) = 0$ (7)

This is the differential equation for Legendre polynomials.

Put $x = \cos\theta$. then

$$\frac{d}{dx} = \frac{d\theta}{dx}\frac{d}{d\theta} = \frac{-1}{\sin\theta}\frac{d}{d\theta}$$

$$\frac{d^2}{dx^2} = \frac{-1}{\sin\theta}\frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\frac{d}{d\theta}\right)$$

$$= \frac{1}{\sin\theta}\left[\frac{-1}{\sin^2\theta}\cos\theta\frac{d}{d\theta} + \frac{1}{\sin\theta}\frac{d^2}{d\theta^2}\right]$$

Using this, Eq.(*) becomes into

$$\frac{\cancel{\theta}}{\sin\theta}\left[\frac{-1}{\sin^2\theta}\cos\theta\frac{dP_n(\cos\theta)}{d\theta} + \frac{1}{\sin\theta}\frac{d^2P_n(\cos\theta)}{d\theta^2}\right]$$

$$+ 2\cos\theta\frac{1}{\sin\theta}\frac{dP_n(\cos\theta)}{d\theta} + n(n+1)P_n(\cos\theta) =$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n(\cos\theta)}{d\theta}\right) + n(n+1)P_n(\cos\theta) =$$

Special values and Parity

we have

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

put $x=1$, then

$$\frac{1}{(1-2t+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(1)t^n$$

$$\Rightarrow \frac{1}{1-t} = \sum_{n=0}^{\infty} P_n(1)t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$$

$$\Rightarrow P_n(1) = 1$$

If we put $x=-1$ in $g(t,x)$, we would get

$$P_n(-1) = (-1)^n$$

Put $x=0$ in $g(t,x)$, we get

$$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\, t^n$$

$\cancel{\Rightarrow 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \dots + (-1)^n \dots 1\cdot 3 \dots}$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} t^{2n} = \sum_{n=0}^{\infty} P_n(0)\, t^n$$

$$\Rightarrow P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$$

$$\text{and } P_{2n+1}(0) = 0$$

we have

$$g(t,x) = (1-2xt+t^2)^{-1/2}$$
$$= \left[1 - 2(-x)(-t) + (-t)^2\right]^{-1/2}$$
$$= g(-t,-x)$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(-x)(-t)^n$$

$$\Rightarrow P_n(x) = (-1)^n P_n(-x)$$

$$\text{or } P_n(-x) = (-1)^n P_n(x)$$

Associated Legendre Functions

When the Helmholtz equation, $(\nabla^2 + k^2)\psi = 0$, is separated in spherical polar coordinates, we get the following equation in Θ-coordinates.

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -Q$$

Put the separation constant as

$Q = n(n+1)$, then

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta$$

This is the diff. eq. for the associated Legendre ~~differ~~ function; with $x = \cos\theta$, the above ~~d~~ equation becomes into

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]$$

The Legendre Equation is

$$(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n(x) = 0.$$

Differentiating the above Equation by m-times, we get

$$\frac{d^m}{dx^m}\left[(1-x^2)P_n''(x)\right] - 2\frac{d^m}{dx^m}\left(x P_n'(x)\right) + n(n+1)\frac{d^m}{dx^m}P_n(x) = 0. \quad (8)$$

we define $u(x) = \frac{d^m}{dx^m}P_n(x)$.

Using Leibniz's formula, we have

$$\frac{d^m}{dx^m}\left[(1-x^2)P_n''(x)\right] = \sum_{s=0}^{m}\frac{m!}{(m-s)!\,s!}\frac{d^{m-s}}{dx^{m-s}}P_n''(x)\frac{d^s}{dx^s}(1-x^2)$$

$$= \frac{m!}{m!\,0!}\frac{d^m}{dx^m}\left(P_n''(x)\right)(1-x^2) + \frac{m!}{(m-1)!\,1!}\frac{d^{m-1}}{dx^{m-1}}\left(P_n''(x)\right)\frac{d}{dx}(1-x^2)$$

$$+ \frac{m!}{(m-2)!\,2!}\frac{d^{m-2}}{dx^{m-2}}\left(P_n''(x)\right)\frac{d^2}{dx^2}(1-x^2)$$

$$= (1-x^2)\frac{d^m}{dx^m}P_n''(x) + m(-2x)\frac{d^{m-1}}{dx^{m-1}}\left(P_n''(x)\right)$$

$$+ \frac{m(m-1)}{2}(-2)\frac{d^{m-2}}{dx^{m-2}}\left(P_n''(x)\right)$$

$$= (1-x^2)u'' - 2mx\,u' - m(m-1)u$$

Similarly, we can show that

$$\frac{d^m}{dx^m}\left(x P_n'(x)\right) = x u' + m u$$

Substituting these in Eq. (8), we get

$$(1-x^2)u'' - 2mxu' - m(m-1)u$$
$$- 2[xu' + mu] + n(n+1)u = 0$$

$$\Rightarrow (1-x^2)u'' - 2x(m+1)u' + (n-m)(n+m+1)u = 0. \quad -(9)$$

Put $\quad \vartheta(x) = (1-x^2)^{m/2}\, u(x)$

or $\quad u(x) = (1-x^2)^{-m/2}\, \vartheta(x)$

we get

$$u' = \left[\vartheta' + \frac{mx\vartheta}{1-x^2}\right](1-x^2)^{-m/2}$$

$$u'' = \left[\vartheta'' + \frac{2mx\vartheta'}{1-x^2} + \frac{m\vartheta}{1-x^2} + \frac{m(m+2)x^2\vartheta}{(1-x^2)^2}\right](1-x^2)^{-m/2}.$$

Substituting these in Eq. (9), we get

$$(1-x^2)\vartheta'' - 2x\vartheta' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]\vartheta = 0.$$

We denote the solution to the above equation as

$$\vartheta = P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

<u>Spherical Harmonics</u>

When the Helmholtz equation, $(\nabla^2 + k^2)\phi = 0$, is separated in spherical polar coordinates, in the θ- and ϕ-coordinates we get the following solutions

$$P_n^m(\cos\theta), \quad e^{im\phi}$$

Product of these two functions is defined to be spherical harmonics

$$Y_n^m(\theta,\phi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\theta)\, e^{i m\phi}$$

These spherical harmonics satisfy the following orthogonality relation

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{n_1}^{m_1 *}(\theta,\phi)\, Y_{n_2}^{m_2}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{n_1,n_2}\,\delta_{m_1,m_2}.$$

The functions $Y_n^m(\theta,\phi)$ have completeness in the θ- and ϕ-coordinates.

Hermite Functions

The Hermite polynomials, $H_n(x)$, are defined by the following generating function

$$g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

we can show that $H_0(x)=1$, $H_1(x) = 2x$. Differentiating $g(x,t)$ w.r.t. t and x, we get the following recurrence relations

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$$

$$H_n'(x) = 2n H_{n-1}(x).$$

From these recurrence relations, we can show the following differential equation for Hermite functions.

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$$

Putting $x=0$ in $g(x,t)$ we can show the following relations

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$$

$$H_{2n+1}(0) = 0.$$

Using the fact that $g(-x,-t) = g(x,t)$, we can show the parity relation

$$H_n(x) = (-1)^n H_n(-x).$$

Quantum Mechanical Simple Harmonic oscillator

consider a particle in one-dimension for the potential energy $U = \frac{1}{2} k x^2$

$= \frac{1}{2} m \omega^2 z^2$.

The schrodinger wave ~~equation~~ function for the is

$$-\frac{\hbar^2}{2m} \frac{d^2}{dz^2}\psi(z) + \frac{1}{2} k z^2 \psi(z) = E\psi(z).$$

Rescale z as $x = \alpha z$ and define the following relations

$$\alpha^4 = \frac{mk}{\hbar^2}, \quad \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2} = \frac{2E}{\hbar\omega}$$

Then, the schrodinger Equation becomes into

$$\frac{d^2\psi(x)}{dx^2} + (\lambda - x^2)\psi(x) = 0. \quad (1)$$

Now, consider the differential equation ~~recurrence relation~~ for Hermite polynomials

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0.$$

Substitute $\phi_n(x) = e^{-x^2/2} H_n(x)$ in the above equation, we get

$$\phi_n''(x) + (2n+1 - x^2)\phi_n(x) = 0.$$

comparing the above equation with Eq. (1), we get

$$\lambda = \frac{2E}{\hbar\omega} = 2n+1$$

$$\Rightarrow E = \left(n+\frac{1}{2}\right)\hbar\omega.$$

Also, the solution to Eq. (1) can be written as

$$\psi_n(x) = 2^{-n/2}\pi^{-1/4}(n!)^{-1/2} e^{-x^2/2} H_n(x)$$

(normalized)

Raising and Lowering operators

For the quantum mechanical oscillator problem we define operators $\hat{a}$ and $\hat{a}^+$ such that ~~$\hat{a}|0\rangle = 0$ and~~ $[\hat{a}, \hat{a}^+] = 1$. We have $\hat{a}^+|n\rangle = (n+1)^{1/2}|n+1\rangle$ and $\hat{a}|n\rangle = n^{1/2}|n-1\rangle$, $\hat{a}|0\rangle = 0$.

We can represent $\hat{a}$ and $\hat{a}^+$ with the following differential operators.

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$$

and ~~the~~ states $|n\rangle = \psi_n(x)$

~~we have~~ the relations in eq (2) ~~$\hat{a}|0\rangle = 0$~~ in position space becomes into

the

$$\hat{a}^+\psi_n(x) = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\psi_n(x) = (n+1)^{1/2}\psi_{n+1}(x)$$

$$\hat{a}\,\psi_n(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\psi_n(x) = n^{1/2}\psi_{n-1}(x)$$

$$\hat{a}\,\psi_0(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\psi_0(x) = 0.$$

Let us verify these relations using

$$\psi_n(x) = e^{-x^2/2}\frac{H_n(x)}{(2^n n!\,\pi^{1/2})^{1/2}}$$

$$\psi_0(x) = e^{-x^2/2}\frac{H_0(x)}{(2^0\,0!\,\pi^{1/2})^{1/2}} = \pi^{-1/4}e^{-x^2/2}.$$

Now, $\hat{a}\,\psi_0(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\pi^{-1/4}e^{-x^2/2}$

$$= \frac{\cancel{\pi^{-1/4}}}{\sqrt{2}}\left(\pi^{-1/4} x e^{-x^2/2} - \pi^{-1/4} x e^{-x^2/2}\right) = 0$$

Consider,

$$\hat{a}\,\psi_n(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) e^{-x^2/2} \frac{H_n(x)}{(2^n n!\, \pi^{1/2})^{1/2}}$$

$$\cancel{= \frac{1}{\sqrt{2}}\, x\, e^{-x^2/2} \frac{H_n(x)}{(2^n n!\, \pi^{1/2})^{1/2}}}$$

$$= \frac{1}{\sqrt{2}(2^n n!\, \pi^{1/2})^{1/2}}\left[x e^{-x^2/2} H_n(x) - x e^{-x^2/2} H_n(x) + e^{-x^2/2} H_n'(x)\right]$$

$$= \frac{1}{\sqrt{2}} \frac{e^{-x^2/2}\, 2n\, H_{n-1}(x)}{(2^n n!\, \pi^{1/2})^{1/2}}$$

But $\psi_{n-1}(x) = \dfrac{e^{-x^2/2} H_{n-1}(x)}{(2^{n-1}(n-1)!\, \pi^{1/2})^{1/2}}$

$$= \sqrt{2n}\, \frac{e^{-x^2/2} H_{n-1}(x)}{(2^n n!\, \pi^{1/2})^{1/2}}$$

Using this relation, we can write

$$\hat{a}\,\psi_n(x) = n^{1/2}\,\psi_{n-1}(x).$$

Similarly, we can show

$$\hat{a}^{+}\psi_n(x) = (n+1)^{1/2}\psi_{n+1}(x)$$

$$\hat{a}^{+}\psi_n(x) = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) e^{-x^2/2} \frac{H_n(x)}{(2^n n!\, \pi^{1/2})^{1/2}}$$

$$= \frac{1}{\sqrt{2}(2^n n!\, \pi^{1/2})^{1/2}}\left[x e^{-x^2/2} H_n(x) + x e^{-x^2/2} H_n(x) - e^{-x^2/2} H_n'(x)\right]$$

$$= \frac{1}{\sqrt{2}\left(2^n n! \pi^{1/2}\right)^{1/2}} \left[e^{-x^2/2} \left(2x H_n(x) - 2n H_{n-1}(x)\right) \right]$$

$$= \frac{1}{\sqrt{2}} e^{-x^2/2} \frac{H_{n+1}}{\left(2^n n! \pi^{1/2}\right)^{1/2}}$$

we have

$$\psi_{n+1}(x) = e^{-x^2/2} \frac{H_{n+1}(x)}{\left(2^{n+1} (n+1)! \pi^{1/2}\right)^{1/2}}$$

$$= \frac{1}{\sqrt{2(n+1)}} e^{-x^2/2} \frac{H_{n+1}(x)}{\left(2^n n! \pi^{1/2}\right)^{1/2}}$$

Using this relation, we get

$$\hat{a}^{+} \psi_n(x) = (n+1)^{1/2} \psi_{n+1}(x)$$

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