Reading Asst

* Review of RLS:

given: $\{u(j), d(j)\}\Big|_{j=1}^{i-1}$

$$w(i-1) \Rightarrow \min_{w} \sum_{j=1}^{i-1} \left| d(j) - u(j)^T w \right|^2 \rightarrow (1)$$

$u(j) \Rightarrow$ input at j^{th} instant $\in L \times 1$

$d(j) \Rightarrow$ Desired response

Soln:

let $U(i-1) = [u(1), \ldots, u(i-1)]_{L \times i-1}$

$d(i-1) = [d(1), \ldots d(i-1)]^T \}$ $1 \times i-1$

cost fn

$\Rightarrow$ $J(w) = \| d(i-1) - U^T w \|^2$

$\frac{\partial J}{\partial w} = \cancel{2(d(i-1) - U^T w)}$

$= -2 . U (d(i-1) - U^T w) = 0$

$\Rightarrow$ $\boxed{w = (U U^T)^{-1} U d}$

$\Rightarrow$ $\boxed{w(i-1) = [U(i-1) U^T(i-1)]^{-1} U(i-1) d(i-1)}$

But there's a problem, for every 'i',
$\Rightarrow$ The task become tedious

Hence, we need to come up with a recursive algo.

↳ Hence, $P(i-1) = [U(i-1) U^T(i-1)]^{-1}$
$L \times L$

Need to find a relation
b/w $P(i)$ & $P(i-1)$

As proved in earlier claims

$$\Rightarrow P(i) = \left[P(i-1) - \frac{P(i-1)\,u(i)\,u(i)^T P(i-1)}{1 + u(i)^T P(i-1)\,u(i)} \right]$$

$w(i) = P(i)\,U(i)\,d(i)$

And thus, it's reduced to

$$\Rightarrow w(i) = w(i-1) + \underbrace{\frac{P(i-1)\,u(i)}{1 + u(i)^T P(i-1)\,u(i)}}_{\text{gain}} \underbrace{[d(i) - u(i)^T w(i-1)]}_{\text{posterior error}}$$

Algo of RLS

$\Rightarrow$ **Init:** $w(0) = 0$, $P(0)$.

$\Rightarrow$ **Iteration:** $i \geq 1$:-

$r(i) = 1 + u(i)^T P(i-1)\,u(i)$

gain :- $k(i) = P(i-1)\,u(i) / r(i)$

post. error :- $d(i) - u(i)^T w(i-1) = e(i)$

$\Rightarrow$ ~~$w(i)$~~

- **Weight updation:**

$$w(i) = w(i-1) + k(i)\, e(i)$$

$$P(i) = P(i-1) - (k(i)\, k^T(i)\, r(i))$$

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* Kernel RLS *

* We use Mercer thm:

$$u(i) \longrightarrow \phi(u(i))$$

given; $\{d(1), d(2), \ldots\}$ & $\{\phi(1), \phi(2), \ldots\}$

Here, wt vector,

$$\min_{w} \sum_{j=1}^{i} |d(j) - w^T \phi(j)|^2 + \underbrace{\lambda \|w\|^2}_{\text{Regularisation term}}$$

* Since, $\phi(.) \Rightarrow$ can be of high dimensions we need a regulariser so that weights don't shoot up.

$d(i) = [d(1) \ldots d(i)]^T \} \; i \times 1$

$\phi(i) = [\phi(1), \ldots \phi(i)]$
$L \times i$

\# Similar to regularised least-squares

$$J(w) = \sum_{j=1}^{i} |d(j) - w^T \phi(j)|^2 + \lambda \|w\|^2$$

$$\frac{\partial J}{\partial w} = \cancel{J(w) = \|d(i) - w^T \phi}$$

$w \to L \times 1$

$$\Rightarrow J(w) = \|d(i) - \phi^T w\|^2 + \lambda \|w\|^2$$
i

$$\frac{\partial J}{\partial w_i} = \cancel{-(d(i)} - 2\phi(d(i) - \phi^T w) + 2\lambda w = 0$$

$$= (\lambda I + \phi \phi^T) w = \phi d(i)$$

$$\Rightarrow \boxed{w = (\lambda I + \phi \phi^T)^{-1} \phi d(i)}$$

$$\boxed{w(i) = (\lambda I + \phi(i) \phi(i)^T)^{-1} \phi(i) d(i)}$$

* Proving through Matrix inversion lemma *

To prove:

$$(\lambda I + \phi\phi^T)^{-1}\phi = \phi(\lambda I + \phi^T\phi)^{-1}$$

Using

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

$\rightarrow$ Let $A = \lambda I$, $B = \phi$, $C = I$, $D = \phi^T$

LHS $= [\lambda^{-1}I - \lambda^{-2}\phi(I + \lambda^{-1}\phi^T\phi)^{-1}\phi^T]\phi$

$$= \lambda^{-1}\phi - \lambda^{-2}\phi(I + \lambda^{-1}\phi^T\phi)^{-1}\phi^T\phi \rightarrow (1)$$

RHS $=$ $A = \lambda I$, $B = \phi^T$, $C = I$, $D = \phi$

$$\rightarrow \phi[\lambda^{-1}I - \lambda^{-2}\phi^T[I + \lambda^{-1}\phi\phi^T)]^{-1}\phi]$$

$$= \lambda^{-1}\phi - \lambda^{-2}\phi\phi^T[I + \lambda^{-1}\phi\phi^T]^{-1}\phi$$

$\rightarrow (2)$

$\Rightarrow$ Consider (1) - (2)

$\Rightarrow \lambda^{-1}\phi - \lambda^{-2}\phi(I + \lambda^{-1}\phi^T\phi)^{-1}\phi^T\phi - \cancel{\lambda^{-1}\phi}$

$+ \lambda^{-2}\phi\phi^T[I + \lambda^{-1}\phi\phi^T]^{-1}\phi$

$\Rightarrow \lambda^{-2}\phi[\phi^T[I + \lambda^{-1}\phi\phi^T]^{-1} - [I + \lambda^{-1}\phi^T\phi]^{-1}\phi^T]\phi$

$= \lambda^{-1}\phi[\phi^T[\lambda I + \phi\phi^T]^{-1} - [\lambda I + \phi^T\phi]^{-1}\phi^T]\phi \rightarrow (3)$

$\Rightarrow$ Let $(3) \leq T$

$\Rightarrow$ Let $T = (\lambda I + \phi\phi^T)^{-1}\phi - \phi(\lambda I + \phi^T\phi)^{-1}$

$\Rightarrow T^T = \phi^T[\lambda I + \phi\phi^T]^{-1} - [\lambda I + \phi^T\phi]^{-1}\phi^T$

$\Rightarrow$ we have $T = \lambda^{-1}[\phi T^T\phi)$

$\Rightarrow$ this to hold true,

$\boxed{T = 0}$

$\rightarrow$ Hence $(1) - (2) = 0$

$\Rightarrow$ LHS = RHS

we transformed $\phi\phi^T \to \phi^T\phi$

where we can find using the Kernel trick. { done using matrix inversion }

$$\Rightarrow w(i) = \phi(i)\left[\lambda I + \underbrace{\phi(i)^T \phi(i)}_{\text{Kernel trick}}\right]^{-1} d(i)$$

<u>dim</u>

let $w(i) = \phi(i)\, a(i)$ } $L \times 1$

$\Rightarrow a(i) = \left[\lambda I + \phi^T(i)\, \phi(i)\right]^{-1} d(i)$ } $i \times 1$

let $Q(i) : \left[\lambda I + \phi^T(i)\, \phi(i)\right]^{-1}$ } $i \times i$

$\Rightarrow$ $Q(i)$ } $i \times i$ dim

$Q(i-1)$ } $i-1 \times i-1$ dim

$$\Rightarrow [Q(i)]^{-1} = \begin{bmatrix} Q(i-1)^{-1} & h(i) \\ h^T(i) & \lambda + \phi^T(i)\,\phi(i) \end{bmatrix}$$

i.e, we need to add a col' & row from $Q(i)$ to $Q(i-1)$.

$$Q(i)^{-1} = \lambda I + \phi^T(i)\phi(i)$$

$$Q(i-1) = \lambda I + \phi^T(i-1)\phi(i-1)$$

$$\phi(i-1) = \begin{bmatrix} \phi(1) & \cdots & \phi(i-1) \end{bmatrix}_{L \times i-1}$$

$$\Rightarrow \phi^T\phi = \begin{bmatrix} \phi(1) \\ \vdots \\ \phi(i-1) \end{bmatrix} \begin{bmatrix} \phi(1) & \cdots & \phi(i-1) \end{bmatrix}$$

If another data point adds i.e.,

To find $\phi^T(i)\phi(i) \Rightarrow \begin{bmatrix} \phi^T_{(i-1)}\phi_{(i-1)} & \phi^T(i-1)\phi(i) \\ \phi^T(i)\phi(i-1) & \|\phi(i)\|^2 \end{bmatrix}$

$$\Rightarrow [Q(i)]^{-1} = \begin{bmatrix} \lambda I_{(i-1 \times i-1)} + \phi^T_{(i-1)}\phi_{(i-1)} & \phi^T(i-1)\phi(i) \\ \phi^T_{(i)}\phi(i-1) & \lambda + \|\phi(i)\|^2 \end{bmatrix}$$

$$\Rightarrow [Q(i)]^{-1} = \begin{bmatrix} Q(i-1)^{-1} & h(i) \\ h^T(i) & \lambda + \phi^T(i)\phi(i) \end{bmatrix} \rightarrow (1)$$

Here, $h(i) = \phi^T(i-1)\phi(i)$

* But we need to find a relation b/w $Q(i)$ & $Q(i-1)$

→ **Block matrix inversion method:**

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

From ①:

$A \to Q(i-1)^{-1}$

$B \to h(i)$

$C \to h(i)^T$

$D \to \lambda + \phi(i)^T \phi(i)$

$$\Rightarrow \left[(Q(i))^{-1}\right]^{-1} = Q(i) = r(i)^{-1} \begin{bmatrix} Q(i-1)\,r(i) + z(i)z(i)^T & -z(i) \\ -z(i)^T & 1 \end{bmatrix}$$

$r(i) = \lambda + \phi(i)^T \phi(i) - z(i)^T h(i)$

$z(i) = Q(i-1)\,h(i)$

$a(i) = Q(i)\, d(i)$ $\rightarrow$ $d(i) = \begin{bmatrix} d(i-1) \\ d(i) \end{bmatrix}$

$$a(i) = \begin{bmatrix} a(i-1) - z(i)\, r(i)^{-1} e(i) \\ r(i)^{-1} e(i) \end{bmatrix}$$

$e(i)$ is the posterior error

$\rightarrow$ $\boxed{w(i) = \phi(i)\, a(i)}$

$\Rightarrow$ We now have a recursive relation

$\hookrightarrow$ $e(i) = d(i) - f_{i-1}(u(i))$

$= d(i) - h(i)^T a(i-1)$

Points to remember

$\hookrightarrow$ Unlike RLS, as we iterate, our the dimensions of $a(i)$, $Q(i)$ increases,

$\rightarrow$ this is due to our transformation from $\phi\phi^T \rightarrow \phi^T\phi$, to use kernel trick,

* Diff. b/w KLMS, KRLS :

→ KLMS & KRLS is similar

But, for every iter.

→ we add $u(i)$ with coeff $r(i)^{-1} e(i)$

→ KRLS updates by $-z(i) r(i)^{-1} e(i)$

But KLMS, never updates prev. coeff.

Let f : estimate of i/p - o/p map.

$$f_i = f_{i-1} + r(i)^{-1} \left[k(u(i), \cdot) - \sum_{j=1}^{i-1} z_j(i)\, k(u(j), \cdot) \right]$$

⇒

Algorithm

Init:

$$Q(1) = (\lambda + K(u(1), u(1)))^{-1}$$

$$a(1) = Q(1)\, d(1)$$

⇒ Computation:

$i > 1$:

$$h(i) = [K(u(i), u(1)), \ldots K(u(i), u(i-1))]^T$$

↳ distance vec

$$z(i) = Q(i-1)\, h(i)$$

$$r(i) = \lambda + K(u(i), u(i)) - z^T(i)\, h(i)$$

$$Q(i) = r(i)^{-1} \begin{bmatrix} Q(i-1)\, r(i) + z(i) z^T(i) & -z(i) \\ -z^T(i) & 1 \end{bmatrix}$$

$$e(i) = d(i) - h^T(i)\, a(i-1)$$

$$a(i) = \begin{bmatrix} a(i-1) - z(i)\, r^{-1}(i)\, e(i) \\ r(i)^{-1}\, e(i) \end{bmatrix}$$

⇒ At iter i:, a test input u_*

$$\boxed{f(u_*) = \sum_{j=1}^{i} a_j(i)\, K(u(j), u_*)}$$

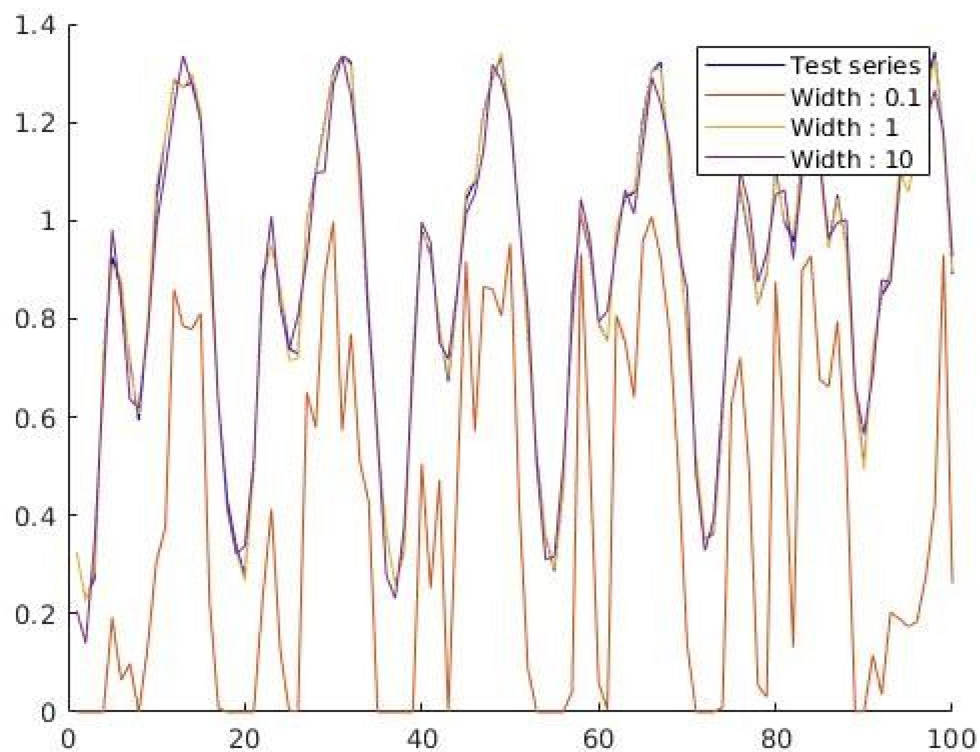
Implementing KRLS in Mackey-Glass dataset : With gaussian kernel

Training iterations: 500

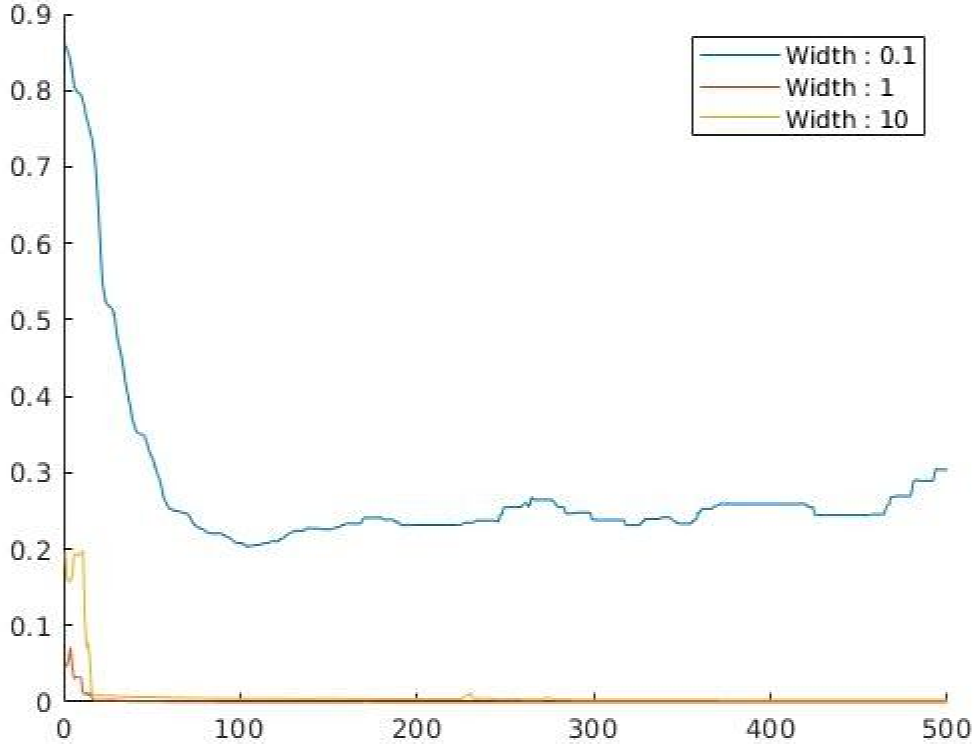
Testing points: 100

Observation of varying width of the gaussian kernel.

Predicted series



Error during training iterations



Comments :

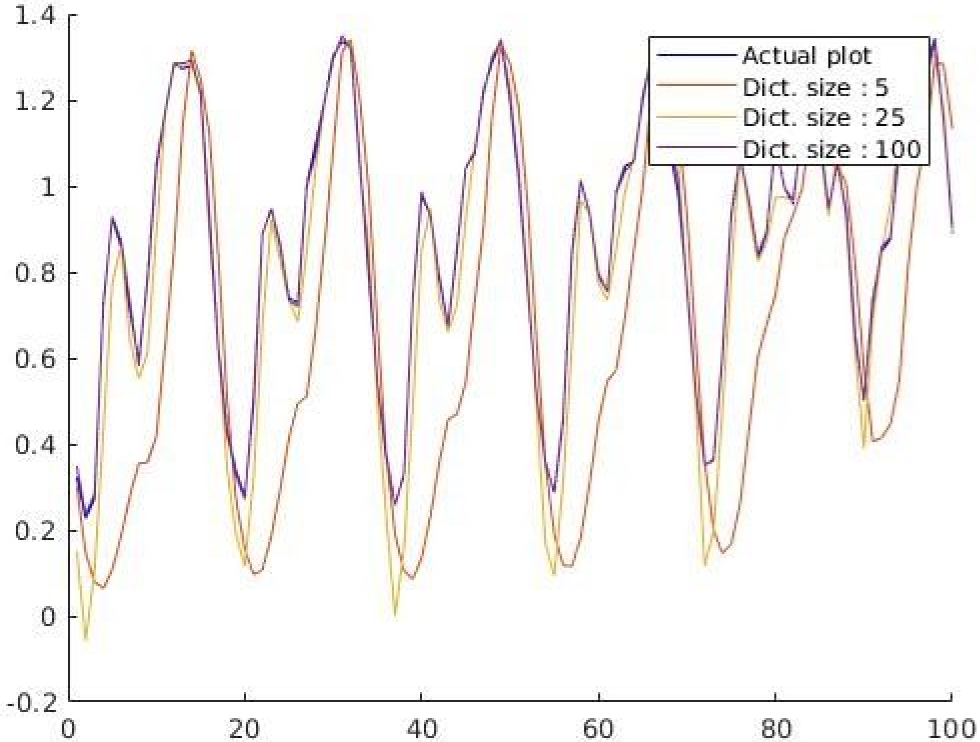
1. We can see that the model with **width: 1** predicts the series perfectly, while 0.1 is very far from actual values, **the width of 10** also works fine but not as good as **the width of 1.**

2. Error on training :
   We can see that **width: 1,** reaches the optimum(zero error) very fast, while **width: 10** reaches slowly, and **width of 0.1** never achieves the zero error.

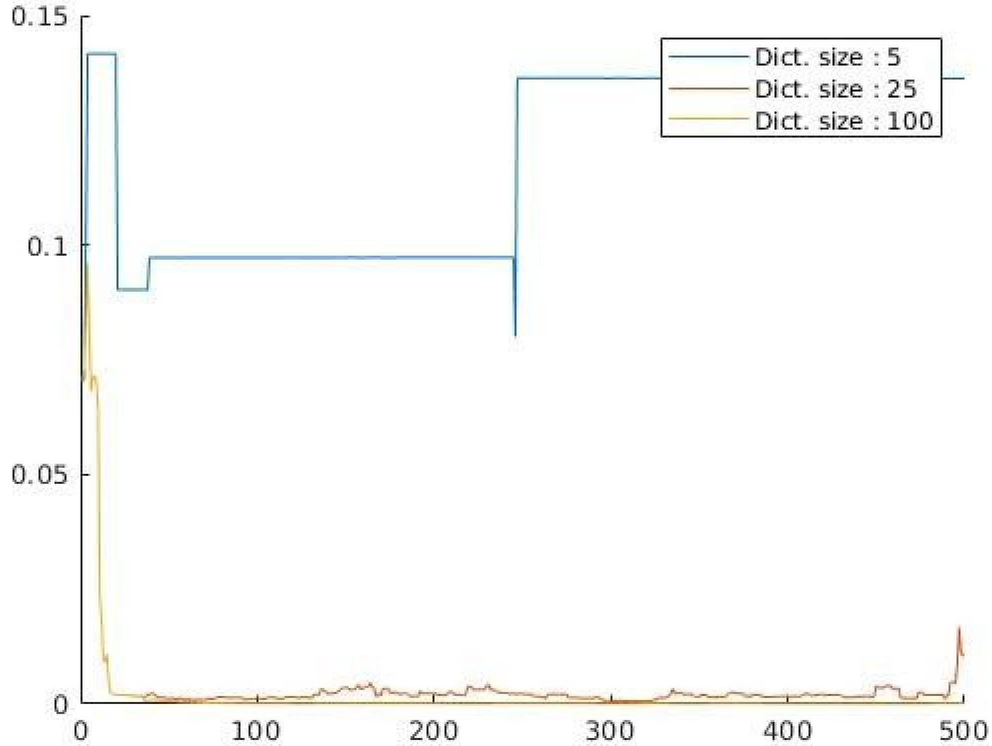
3. By this, we can say that a **width: 1** model is the best pick in terms of speed of convergence and accuracy.

Observations w.r.t the Dictionary size during training

Predicted series



<u>Error during training iterations</u>



Comments

1. We can see that as dictionary size increases, the model gets more closer to true predictions.
2. Hence the higher the dictionary size the better the model will become. This occurs as we will have more samples , more data ⇒ better learning.