final_exam_prog

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[2]:
```
import numpy as np
import cvxpy as cp
```

1 Final exam

1.1 Question 03

1.1.1 Koidala Surya Prakash

1.1.2 EE18btech11026

1.1.3 3. a Convexity ?

We can reformulate the problem in the form :

objective := x.T(Q)x + (c.T)x constraints := Ax <= b

where : x = [x1,x2]

```
Q = [[1,-0.5]
     [-0.5, 2] ]

 c = [-1, 0 ]

 A = [[1,-2]
     ,[1,4],
      [5,-76]]

 b = [u1,
      u2,
      1]
```

Since the obj is in quadratic form and constraints are in linear form

This is a QP problem

We know that all QP problems are convex optimisation problems . Hence this problem is convex.

1.1.4 3. b : Finding optimal variables

[3]:
```
### Constants
u1 = -2
u2=  -3

b = np.array([[u1,u2,1]]).T

Q = np.array([[1,-0.5],[-0.5, 2]])
c = np.array([[-1,0]]).T

A = np.array([[1,-2],[1,4],[5,-76]])
```

[4]:
```
'''
Objective : A QP problem of the form (x.T)Q(x) + c.T*x
with affine constraints : Ax <= b

Therefore this is a convex problem
'''

x = cp.Variable((2,1))
obj = cp.Minimize( cp.quad_form(x, Q) + c.T@x )
constraints = [A@x <= b]

prob = cp.Problem(obj, constraints)
prob.solve()

lam = constraints[0].dual_value

print("The following are the optimal primal variables : \n")

print('x1* = %.4f'%x.value[0][0])
print('x2* = %.4f'%x.value[1][0])
print('\n')

print("The following are the optimal dual variables : \n")
print('lamda_1* = %.4f' %lam[0])
print('lamda_2* = %.4f' %lam[1])
print('lamda_3* = %.4f' %lam[2])
```

```
The following are the optimal primal variables :

x1* = -2.3333
x2* = -0.1667


The following are the optimal dual variables :
```

```
lamda_1* = 2.8645
lamda_2* = 2.2980
lamda_3* = 0.0675
```

1.2 3.c Verifying KKT conditions

The primal variables and dual optimal variables satisfy the below mentioned kkt conditions.

1. fi(x*) <= 0 ==> Ax-b <= 0
2. lam >=0
3. lam*fi(x)* == 0
4. dfo(x)/dx + sum(lam_i* dfi(x)/dx) == 0

[ ]:
```python
'''
Verifying KKT conditions..
'''
'''
1. fi(x*) <= 0 ==> Ax-b <= 0
'''

print('Condition 1 :: obeys all constraints')
print('____________')
print('Ax* - b =  \n', np.round(A@x.value - b, 3))

# 2. lam >=0
print("\nCondition 2 :: all lamda values are positive")
print('____________')
print('lam_i = \n',lam)

# 3. lam*fi(x*) == 0

print("\nCondition 3 :: since constraint func are zero")
print('____________')
print("lam_i * x* = \n",np.round(lam*(A@x.value - b),3) )


# 4. dfo(x)/dx + sum(lam_i* dfi(x)/dx  ) == 0
'''
dfo(x)/dx = Qx + c
dfi(x)/dx  = a == > sum(lam_i* dfi(x)/dx) = A.T*lam
'''
print("\nCondition 4 :: dfo(x)/dx + sum(lam_i* dfi(x)/dx  ) == 0")
print('____________')
print(np.round(2*Q@x.value + c + A.T@lam,3))
```

```
Condition 1 :: obeys all constraints
____________
```

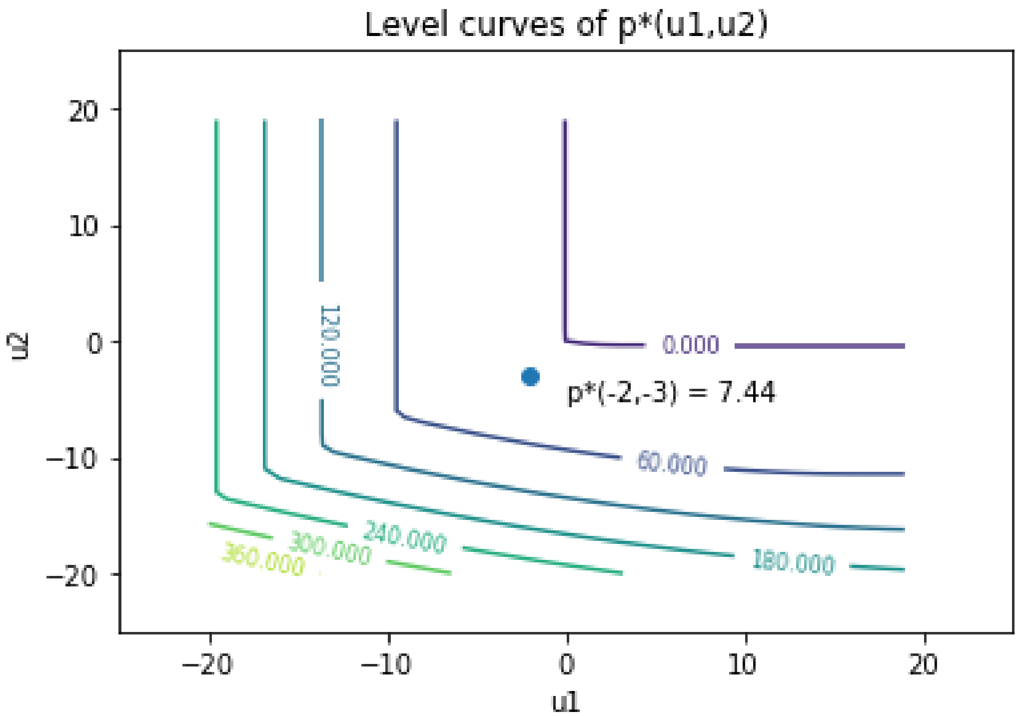
```
Ax* - b =
 [[0.]
 [0.]
 [0.]]

Condition 2 :: all lamda values are positive
------------
lam_i =
 [[2.86447804]
 [2.29803246]
 [0.0674979 ]]

Condition 3 :: since constraint func are zero
------------
lam_i * x* =
 [[0.]
 [0.]
 [0.]]

Condition 4 :: dfo(x)/dx + sum(lam_i* dfi(x)/dx  ) == 0
------------
[[ 0.]
 [-0.]]
```

1.3 3.d Plotting level curves

[ ]:
```python
### Function finding p*(u1,u2)
def p(u1,u2):
    b = np.array([[u1,u2,1]]).T
    Q = np.array([[1,-0.5],[-0.5, 2]])
    c = np.array([[-1,0]]).T
    A = np.array([[1,-2],[1,4],[5,-76]])

    x = cp.Variable((2,1))
    obj = cp.Minimize( cp.quad_form(x, Q) + c.T@x )
    constraints = [A@x <= b]

    prob = cp.Problem(obj, constraints).solve()
    return obj.value
```

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
import pylab

### Here u1 == X , u2 == Y , p*(u1,u2) = Z
```

```python
# List of points in x axis
XPoints     = []

# List of points in y axis
YPoints     = []

# X and Y points are from -20 to +20 varying in steps of .5
lis = np.arange(-20,20,1)
for val in lis:
    XPoints.append(val)
    YPoints.append(val)

# Z values as a matrix
ZPoints = np.ndarray((len(XPoints),len(YPoints)))


for x in range(0, len(XPoints)):
    for y in range(0, len(YPoints)):
        ZPoints[x][y] = p(XPoints[x],YPoints[y])

# Set the x axis and y axis limits
pylab.xlim([-25,25])
pylab.ylim([-25,25])

# Provide a title for the contour plot
plt.title('Level curves of p*(u1,u2)')

# Set x axis label for the contour plot
plt.xlabel('u1')

# Set y axis label for the contour plot
plt.ylabel('u2')

# Create contour lines or level curves using matplotlib.pyplot module
contours = plt.contour(XPoints, YPoints, ZPoints)

# Display z values on contour lines
plt.clabel(contours, inline=1, fontsize=8)

# Display the contour plot

plt.scatter([-2], [-3])

plt.annotate("p*(-2,-3) = 7.44 ", (0,-5))

plt.show()
```



1.3.1 3.e Convexity of p* from level curves

We can see that the gradient of the slopes are increasing as we move towards right part of the curve. This is seen in both the directions

===> Thus p* is convex

since $d^{2(p)/du1}2 >= 0$ $d^{2(p)/du2}2 >= 0$

1.3.2 3.f Numerical verification

(3f)

$$p(x_1, x_2) = x_1^2 + 2x_2^2 - x_1 x_2 - x_1$$
$$+ \lambda_1 (x_1 - 2x_2 - u_1)$$
$$+ \lambda_2 (x_1 + 4x_2 - u_2)$$
$$+ \lambda_3 (5x_1 - 76x_2 - 1)$$

$\frac{\partial p}{\partial x_1} =$ Objective can be rewritten as

$$Q = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \\ 5 & -76 \end{pmatrix} \quad b = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

$$p(\underline{x}) = x^T Q x + c^T x + \lambda(Ax - b)$$

$\frac{\partial \mathcal{L}(x,\lambda)}{\partial x} = 2Qx + c + A^T\lambda$

Equating $\frac{\partial \mathcal{L}(x,\lambda)}{\partial x} = 0$ to find primal & optimal dual variable

$$\Rightarrow 2\begin{pmatrix} 1 & -0.5 \\ -0.5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 5 \\ -2 & 4 & -76 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x_1 - x_2 - 1 + \lambda_1 + \lambda_2 + 5\lambda_3 = 0 \rightarrow (1)$$

$$\Rightarrow -x_1 + 4x_2 - 2\lambda_1 + 4\lambda_2 - 76\lambda_3 = 0 \ (2)$$

* Values obtained using cvxpy

$x_1^* = -2.33$

$x_2^* = -0.1667$

$\lambda_1^* = 2.8645$

$\lambda_2^* = 2.2980$

$\lambda_3^* = 0.0675$

To verify the relationship

Consider (1):

$$2x_1 - x_2 + 1 = -\lambda_1 - \lambda_2 - 5\lambda_3$$

* Computing LHS:

$$2x_1^* - x_2^* + 1 = -4.66 + 0.167 + 1$$
$$= -5.49$$

* Computing RHS:

$$-\lambda_1^* - \lambda_2^* - 5\lambda_3^* = -5.4975$$

∴ We can see that LHS = RHS

Hence the values are correct
