EE3025 INDEPENDENT PROJECT
Assignment : 02

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

Download all the python codes from

https://github.com/Surya291/ACADEMIA/tree/master/IDP_3_2/Asst_02/codes

and latex code from

https://github.com/Surya291/ACADEMIA/tree/master/IDP_3_2/Asst_02/ee18btech11026.tex

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

2.1 The Digital Filter

1. *Tolerances:* The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. *Passband:* The passband of filter number j, j going from 109 to 135 is from $\{3 + 0.6$(j-109)$\}$kHz to $\{3 + 0.6$(j-107)$\}$kHz. Since our filter number is 114, substituting $j = 114$ gives the passband range for our bandpass filter as 6 kHz - 7.2 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 7.2$ kHz and $F_{p2} = 6.0$ kHz. The corresponding normalized digital filter passband frequencies are $\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.3\pi$

and $\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.25\pi$ kHz. The centre frequency is then given by $\omega_c = \frac{\omega_{p1}+\omega_{p2}}{2} = 0.275\pi$.

3. *Stopband:* The *transition band* for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are $F_{s1} = 7.2 + 0.3 = 7.5$ kHz and $F_{s2} = 6.0 - 0.3 = 5.7$ kHz. The corresponding normalized frequencies are $\omega_{s1} = 0.3125\pi$ and $\omega_{s2} = 0.2375\pi$.

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan \frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as $\Omega_{p1} = 0.5095$, $\Omega_{p2} = 0.4142$ and $\Omega_{s1} = 0.5345$, $\Omega_{s2} = 0.3914$ respectively.

3 The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyschev approximation* to design our bandpass IIR filter.

3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{1}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4594$ and $B = \Omega_{p1} - \Omega_{p2} = 0.0953$. The low pass filter has the passband edge at $\Omega_{Lp} = 1$ and stopband edges at $\Omega_{Ls_1} = 1.4653$ and $\Omega_{Ls_2} = -1.5511$. We choose the stopband edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4653$.

2. *The Low Pass Chebyschev Filter Paramters:* The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{2}$$

where $c_N(x) = \cosh(N \cosh^{-1} x)$ and the integer N, which is the order of the filter, and ϵ are design paramters. Since $\Omega_{Lp} = 1$, (2) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{3}$$

Also, the design paramters have the following constraints

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \leq \epsilon \leq \sqrt{D_1},$$
$$N \geq \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \tag{4}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$. After appropriate substitutions, we obtain $N \geq 4$ and $0.3184 \leq \epsilon \leq 0.6197$. In Figure 1, we plot $|H(j\Omega)|$ for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega)|$ decreases in the transition band.

The below code computes the above mentioned quantities and generates Fig 1.

```
./codes/IIR/paraplot.py
```

```
./codes/IIR/para.py
```

We choose $\epsilon = 0.4$ for our IIR filter design.Generated a low pass filter plot for $\epsilon = 0.4$ in Figure 2

The below code generates Figure 2

```
./codes/IIR/lpanalog.py
```

3. *The Low Pass Chebyschev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{5}$$

where

$$c_4(x) = 8x^4 + 8x^2 + 1. \tag{6}$$

The poles of the frequency response in (2) lying in the left half plane are in general obtained as $r_1 \cos\phi_k + jr_2 \sin\phi_k$, where

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \ldots, N-1$$
$$r_1 = \frac{\beta^2 - 1}{2\beta}, r_2 = \frac{\beta^2 + 1}{2\beta}, \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}} \tag{7}$$

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1}(s_L^2 - 2r_1\cos\phi_k s_L + r_1^2\cos^2\phi_k + r_2^2\sin^2\phi_k)} \tag{8}$$

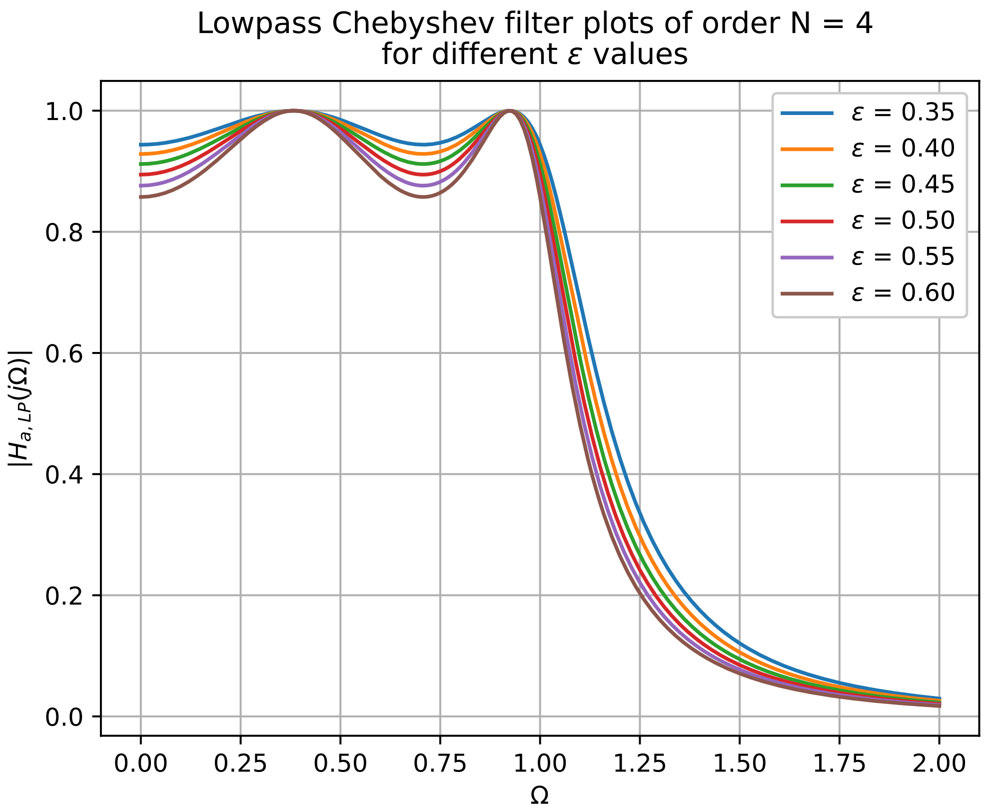

Figure 1: The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

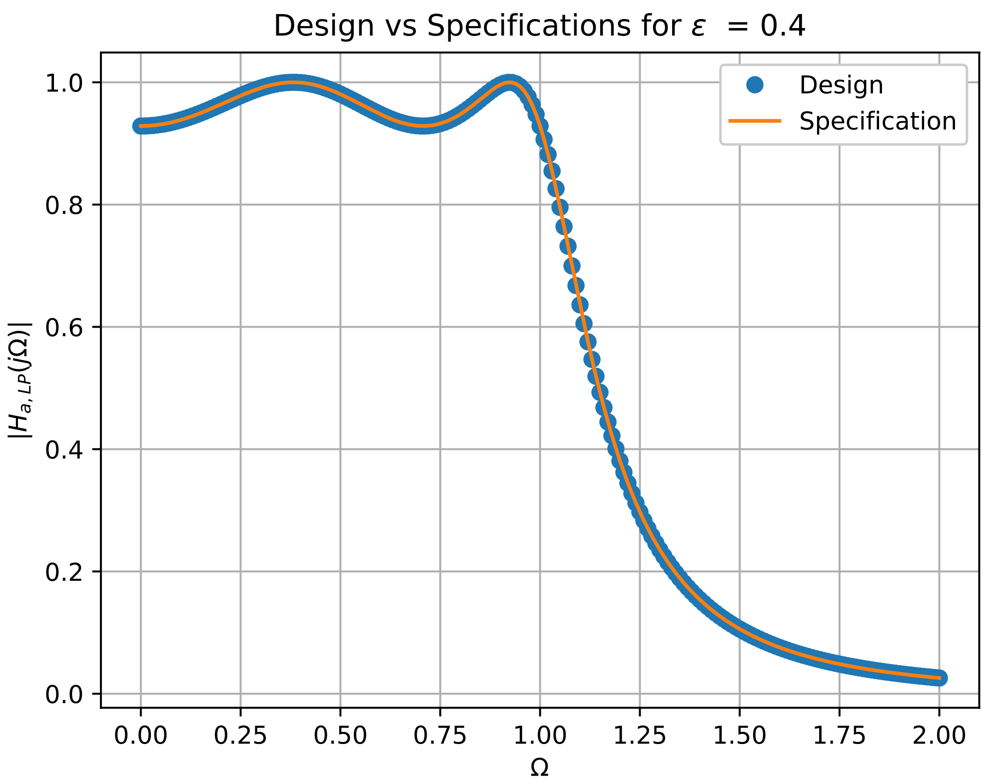

Figure 2: The magnitude response plots from the specifications in Equation 5 and the design in Equation 9

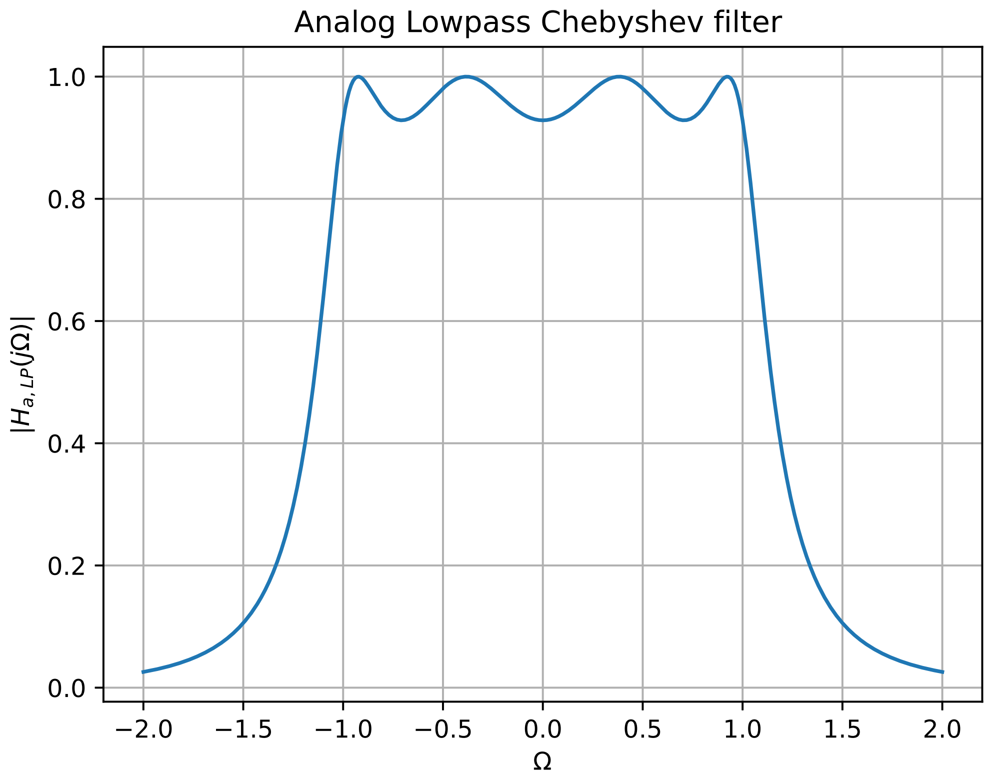

Figure 3: The magnitude response of analog lowpass IIR filter

Substituting $N = 4$, $\epsilon = 0.5$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, from (7) and (8), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{9}$$

The below code computes the above formulations

```
./codes/IIR/lp_stable_cheb.py
```

In Figure 2 we plot $|H(j\Omega)|$ using (5) and (9), thereby verifying that our low-pass Chebyschev filter design meets the specifications. Plotting the magnitude response for analog lowpass filter in Figure 3 which is generated using the below code

4. *The Band Pass Chebyschev Filter:* The analog bandpass filter is obtained

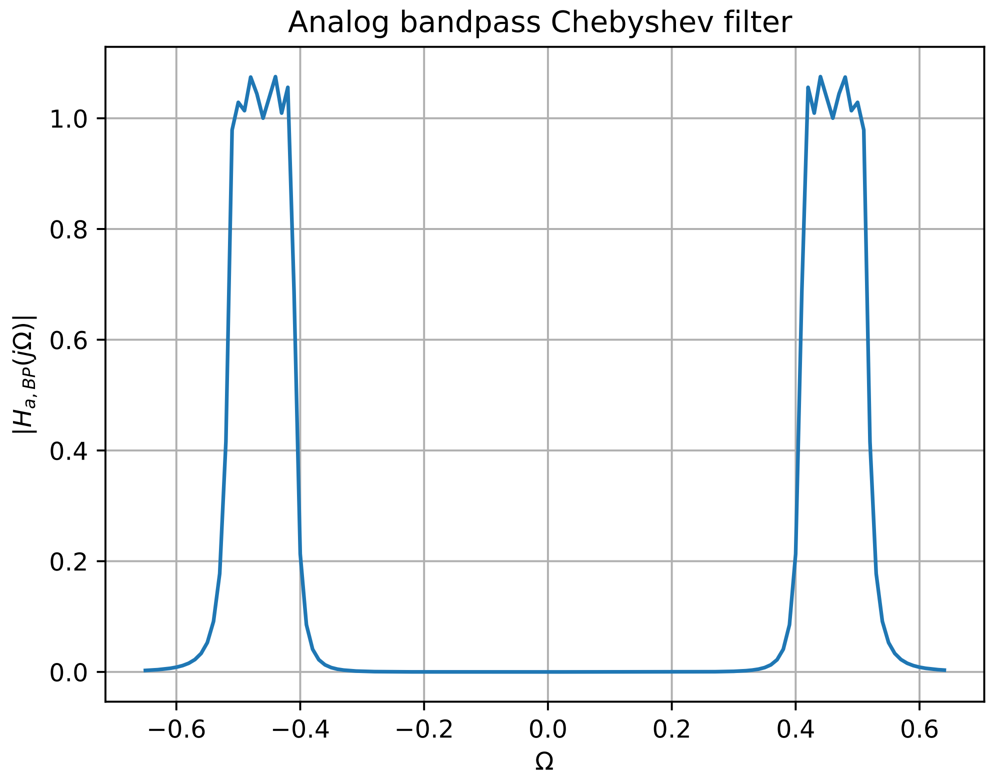

Figure 4: The analog bandpass magnitude response plot from Equation 11

from (9) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{10}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{2.7776 \times 10^{-5} s^4}{s^8 + 0.1055s^7 + 0.8589s^6 + 0.0676s^5 + 0.2735s^4 + 0.0143s^3 + 0.0383s^2 + 0.001s + 0.002} \tag{11}$$

In Figure 4, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{12}$$

where G is the gain of the digital filter. From (11) and (12), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{13}$$

where $G = 2.7776 \times 10^{-5}$,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{14}$$

and

$$\begin{aligned} D(z) = 2.3609 - 12.0002z^{-1} + 31.8772z^{-2} - 53.7495z^{-3} + 62.8086z^{-4} \\ -51.4634z^{-5} + 29.2231z^{-6} - 10.5329z^{-7} + 1.9842z^{-8} \end{aligned} \tag{15}$$

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Figure 5. Again we find that the passband and stopband frequencies meet the specifications well enough.

The below code generates Figures 3, 4, 5

```
./codes/IIR/iir_final.py
```

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1. The *passband frequency* ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

2. *The impulse response* $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi}w(n), \tag{16}$$

where $w(n)$ is the Kaiser window obtained from the design specifications.

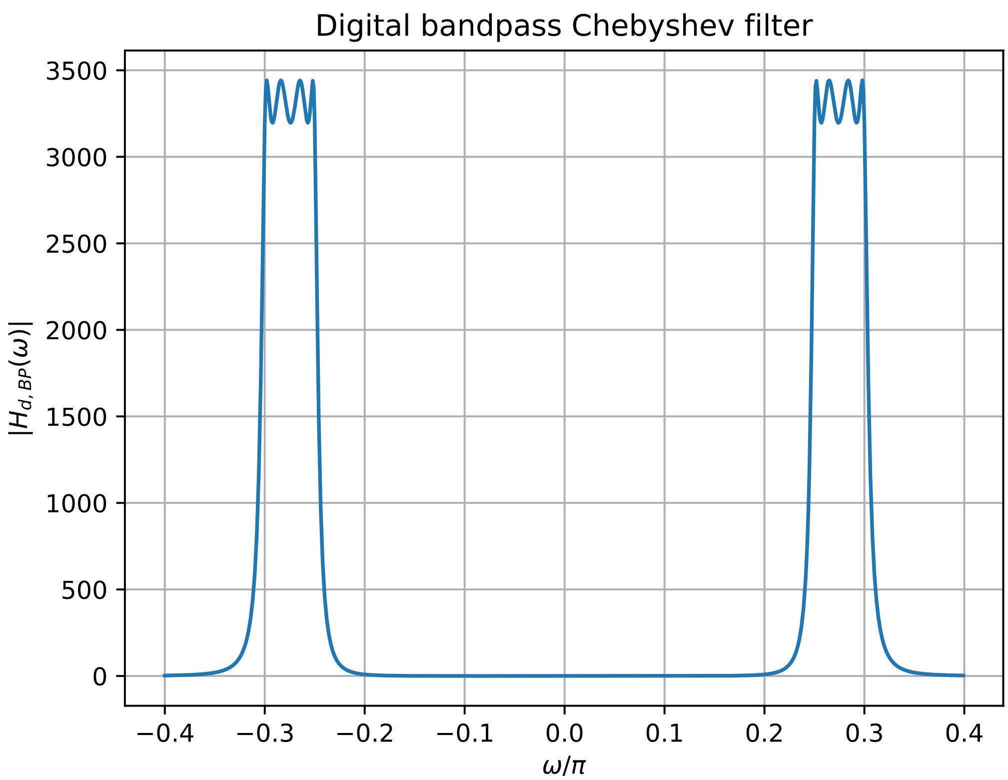

Figure 5: The magnitude response of the bandpass digital filter designed to meet the given specifications

4.2 The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \le n \le N, \qquad \beta > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}, \end{aligned} \tag{17}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to

$$N \ge \frac{A-8}{4.57\Delta\omega}, \tag{18}$$

   where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \ge 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \tag{19}$$

   In our design, we have $A = 16.4782 < 21$. Hence, from (19) we obtain $\beta = 0$.

3. We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (17) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \le n \le 100 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{20}$$

From (16) and (20), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -100 \le n \le 100 \\ &= 0, \qquad\qquad \text{otherwise} \end{aligned} \tag{21}$$

The magnitude response of the filter in (21) is shown in Figure 6.

4.3 Converting into a causal FIR Bandpass Filter

$$\omega_c = 0.275\pi \tag{22}$$

A low pass filter's impulse response can be converted into a band pass using the below transformation

$$h_{bp}(n) = h_{lp}(n) * 2cos(n\omega_c) \tag{23}$$

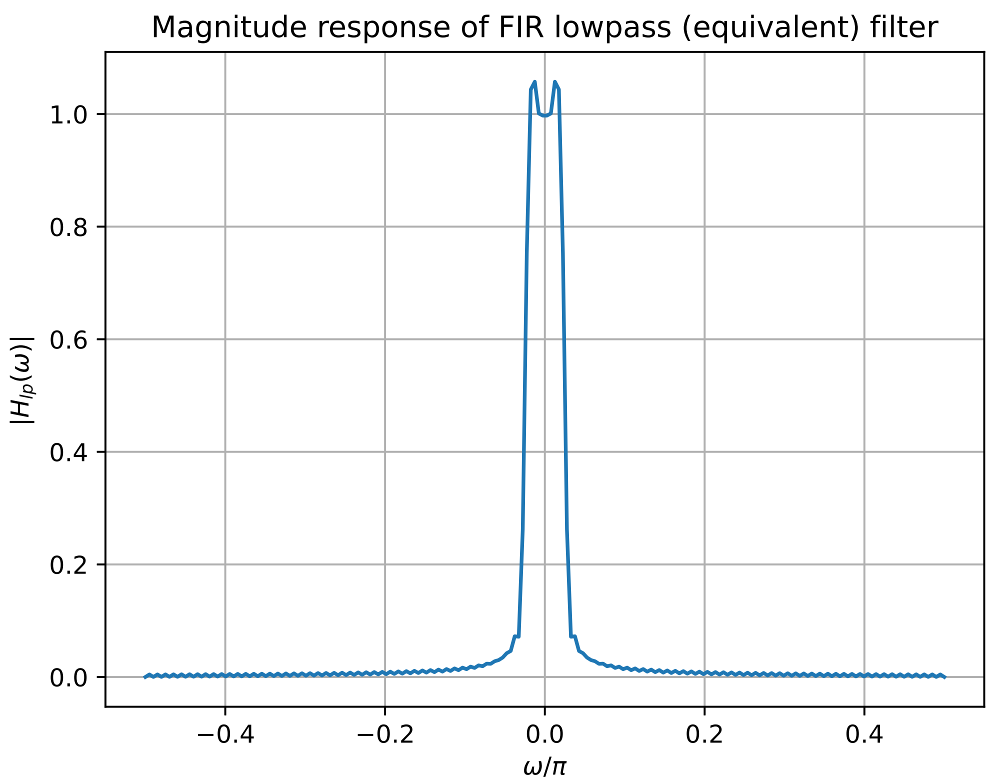

Figure 6: Magnitude response of the FIR lowpass digital filter

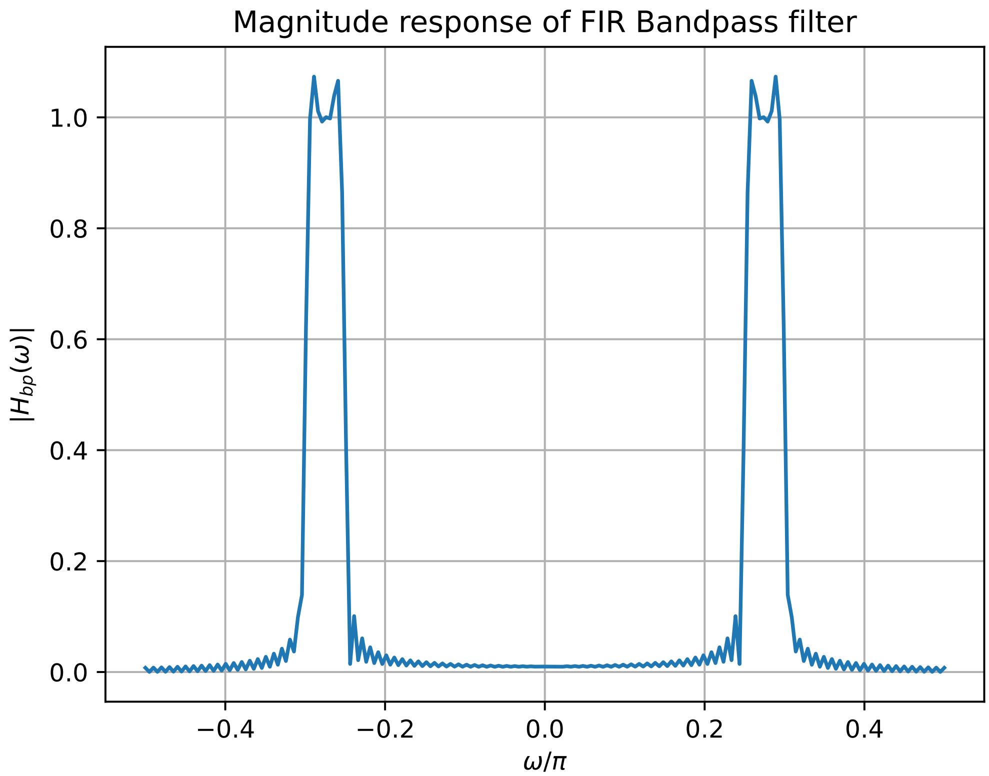

Figure 7: Magnitude response of the FIR bandpass digital filter obtained using (**??**)

The magnitude response of the FIR bandpass filter designed to meet the given specifications is plotted in Figure 7.

The below code computes the FIR filter specifications and generates Figures 6,7

```
./codes/FIR/test.py
```