14.1.1 B.F

(6)

For the product of the generating functions $g(x,t)\,g(x,-t)$ show that

$$1 = [J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$

$n = 1, 2, 3.$

$$\Rightarrow \quad g(x,t)\,g(x,-t) = e^{\left(\frac{x}{2}\right)\left(t - \frac{1}{t} - t + \frac{1}{t}\right)} = 1$$

$$= \sum_{m,n} J_m(x) t^m J_n(x) (-t)^n .$$

has zero coefficient of t^{m+n} for $m = -n \neq 0$,

this yields $\quad 1 = \sum_{n=-\infty}^{\infty} J_n^2(x) = J_0^2(x) + 2\sum_{n\ge 1}^{\infty} J_n^2(x)$

using $(-1)^n J_{-n} = J_n$ for real x the

the inequality follows from here.

14.1.2 ~~The Bessel function generating function satisfies the indicated~~ equation.

(a) Using a generating function $g(x,t) = g(u+v,t)$

$= g(u,t)\,g(v,t)$.

show that,

(a) $J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u) J_{n-s}(v)$

(b) $J_0(u+v) = J_0(u) J_0(v) + 2\sum_{s=1}^{\infty} J_s(u) J_{-s}(v)$

→ The Bessel ~~function~~ generating function

$g(u+v,t) = g(u,t)\, g(v,t)$.

(a) $$\sum_{n=-\infty}^{\infty} J_n(u+v)\, t^n = \sum_{\nu=-\infty}^{\infty} J_\nu(u)\, t^\nu \sum_{\mu=-\infty}^{\infty} J_\mu(v)\, t^\mu$$

equating the co-efficients of t^n on both sides of the eqn, which for the right-hand side involves terms for which $\mu = n-\nu$, so

$$J_n(u+v) = \sum_{\nu=-\infty}^{\infty} J_\nu(u)\, J_{n-\nu}(v)$$

(b) Applying the above formula for $n=0$, note that for $|\nu| \neq 0$, the summation contains the two terms $J_\nu(u) J_{-\nu}(v)$ and $J_{-\nu}(u) J_\nu(v)$. But because for any x, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$, both these terms are equal, with value $(-1)^\nu J_\nu(u) J_\nu(v)$. combining this we got the answer.

14.1.3 Using only the generating function

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

and not the explicit series form of $J_n(x)$, show that $J_n(x)$ has odd or even parity according to wheather n is odd or even, that is,

$$J_n(x) = (-1)^n J_n(-x).$$

⇒ The generating function remains unchanged if we change the signs of both x and t. and therefore

$$\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} J_n(-x)(-t)^n = (-1)^n J_n(-x) t^n$$

for this eqn to be satisfied it is necessary that, for all n, $J_n(x) = (-1)^n J_n(-x)$.

14.1.4 Use the basis recurrance formula to prove the following

ⓐ $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

ⓑ $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$

ⓒ $J_n(x) = J'_{n+1} + \frac{n+1}{x} J_{n+1}(x).$

→ The basic recurrence relation are

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad — (1)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \quad — (2)$$

→ (a) $\frac{d}{dx}\left[x^n J_n(x)\right] = n x^{n-1} J_n(x) + x^n J_n'(x)$

$$= \frac{x^n}{2}\left[\frac{2n}{x} J_n(x) + 2J_n'(x)\right]$$

replace $\left(\frac{2n}{x}\right) J_n(x)$ using (1) and $2J_n'(x)$ using (2),

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{x^n}{2}\left[J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= x^n J_{n-1}(x)$$

(b) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$

$$= \frac{x^{-n}}{2}\left[-\frac{2n}{x} J_n(x) + 2J_n'(x)\right],$$

Replace $-\frac{2n}{x} J_n(x)$ and $2J_n'(x)$ from (1) and (2).

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{x^{-n}}{2}\left[-J_{n-1}(x) - J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= -x^{-n} J_{n+1}(x)$$

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(c) Start from (2) with n replaced by $n+1$, and use (1) to replace $J_{n+2}(x)$ by its equivalent in term of J_{n+1} and J_n

$$2J'_{n+1}(x) = J_n(x) - J_{n+2}(x) = J_n(x) - \frac{2(n+1)}{x} J_{n+1}(x) + J_n(x)$$

$$\Rightarrow \quad J_n(x) = J'_{n+1} + \frac{n+1}{x} J_{n+1}(x)$$

14.1.5 Derive the Jacobi-Anger expansion

$$e^{i\rho \cos\phi} = \sum_{m=-\infty}^{\infty} i^m J_m(\rho) e^{im\phi}.$$

This is an expansion of a plane wave in a series of cylindrical waves.

$$\Rightarrow \quad e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

In the generating function for the J_n as given above, make the substitution $t = ie^{i\phi}$, leading to the formula,

$$e^{i\frac{\rho}{2}\left(ie^{i\phi} - \frac{1}{ie^{i\phi}}\right)} = e^{i\rho\cos\phi} = \sum_{m=-\infty}^{\infty} J_m(\rho)[ie^{i\phi}]^m$$

14.1.6 show that (a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$

(b) $\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$.

$\Rightarrow$ Set $\phi = 0$ in the plane wave expansion of above and seperate into real and imaginary parts this yields,-

(a) $e^{ix} = \sum_{m=-\infty}^{\infty} i^m J_m(x)$, $\cos x = J_0(x) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(x)$,

(b) $\sin x = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(x)$.

using $J_{-2m-1} = -J_{2m+1}$, $i^{-2m-1} = -(-1)^m i$

14.1.7 To hellp remove the generating function from the realm of magic, show that it can be derived from reccurance relation $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$.

$\Rightarrow$ $\sum_{n=-\infty}^{\infty} t^n J_{n-1}(x) + \sum_{n=-\infty}^{\infty} t^n J_{n+1}(x) = \sum_{n=-\infty}^{\infty} \frac{2n}{x} t^n J_n(x)$

$\Rightarrow$ $t\sum_{n=-\infty}^{\infty} t^{n-1} J_{n-1}(x) + t^{-1}\sum_{n=-\infty}^{\infty} t^{n+1} J_n(x) = \sum_{n=-\infty}^{\infty} \frac{2n}{x} t^n J_n(x)$,

writing as in part (c) $g(x,t) = \sum_{m=-\infty}^{\infty} J_m(x) t^m$,

$\frac{dg}{g} = \frac{x}{2}(1 - t^{-2})\, dt$

$\ln y = \frac{x}{2}\left(t - \frac{1}{t}\right) + C_0(x) \longrightarrow y = C(x) e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$, (9)

where $C(x) = \exp(C_0(x)) \rightarrow$ I.C.

The co-efficient of t^0 can be found by expanding $e^{xt/2}$ and $e^{-xt/2}$ separately, multiplying the expansion together, and extract t^0 term

$$e^{\frac{xt}{2}} e^{-xt/2} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \frac{t^n}{n!} \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^m \frac{t^{-m}}{m!}$$

$$\rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,n!} \left(\frac{x}{2}\right)^{2n} t^0 + \text{---}$$

This is the series expansion of $J_0(x)$, some set $C(x) = 1$

14.1.10 ~~To use Mathematical induction~~, assume

Rerive $J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$.

$\Rightarrow$ To use ~~mathematical~~ mathematical induction assume for the formula for J_n is valid for index n and then verify that, under the assumption, it is also valid for index value $n+1$.

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$$

$$\Rightarrow -x^{-n} J_{n+1}(x) = \frac{d}{dx}\left[x^{-n} J_n(x)\right]$$

$$= (-1)^n \frac{d}{dx}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)\right]$$

$$= (-1)^n x \left(\frac{1}{x}\frac{d}{dx}\right)\cdot\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

This equation rearranges to

$$J_{n+1}(x) = (-1)^{n+1} x^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x)$$

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(11)

14.1.13 The differential cross-section in a nuclear scattering experiment is given by $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$. An approximated treatment leads to

$$f(\theta) = -\frac{ik}{2\pi}\int_0^{2\pi}\int_0^R \exp[ik\rho \sin\theta \sin\phi]\,\rho\, d\rho\, d\phi$$

where θ is the angle through which the scattered particle is scattered. R is the nuclear radius. Show that

$$\frac{d\sigma}{d\Omega} = (\pi R^2)\cdot\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2$$

$$\Rightarrow f(\theta) = -\frac{ik}{2\pi}\int_0^R \rho\, d\rho \int_0^{2\pi} d\phi \left[\cos(k\rho\sin\theta\sin\phi) + i\sin(k\rho\sin\theta\sin\phi)\right]$$

Now we have $J_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta - n\theta)\,d\theta$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta$$

and $\int_0^{2\pi}\sin(x\sin\theta - n\theta)\,d\theta = 0$,

put $n=0$ in above two eq^n, the integral of the cosine has value $2\pi J_0(k\rho\sin\theta)$ and the sine integral vanishes. We now make a change of variables from ρ to $x = k\rho\sin\theta$ and then use $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$, that $x J_0(x) = [x J_1(x)]'$, so. $f(\theta) = -\frac{i}{k\sin^2\theta}\int_0^{kR\sin\theta} xJ_0(x)\,dx$

$= -\frac{i}{k\sin^2\theta}\left[xJ_1(x)\right]_0^{kR\sin\theta} = -\frac{iR}{\sin\theta}J_1(kR\sin\theta)$

$|f(\theta)|^2 \longrightarrow$ given the result