Lab Session : 11 K. Surya Prakash

Q1) Euclidean distance matrices

Given $x_{ij} = \|P_i - P_j\|^2$; $i \in \{1, \dots n\}$

$= \|P_i\|^2 + \|P_j\|^2 - 2P_i^T P_j$

$= y_{ii} + y_{jj} - 2y_{ij}$

$P: [P_1 \cdots P_n]$ $d \times n$

where, y_{ij} are elements in Y.

s.t. $y_{ij} = P_i^T P_j \Rightarrow Y = P^T P$

$\therefore$ By comparing x to y ; X can be written as

$$\boxed{X = \operatorname{diag}(Y)\, 1^T + (\operatorname{diag}(Y)\, 1^T)^T - 2Y}$$

Here, Y is a PSD :

Because. Consider any arbitrary u vector

st $u^T Y u = u^T P^T P u$

$= (Pu)^T Pu$

$= \|Pu\|^2 \geq 0$

$\Rightarrow$ $\boxed{Y \text{ is a PSD}}$

(b) we showed that

$$X = \underbrace{\text{diag}(Y)\,1^T + (\text{diag}(Y)\,1^T)^T - 2Y}$$

Linear combination of Y ~~&~~ PSD

∴ we know that set of PSD matrices ~~convex cones~~ is a convex cone.

<u>Proof</u>: Let P be set of all PSD

Let $X, Y \in P$

⇒ $\theta X + (1-\theta)Y$ should belong to P
↳ i.e, shd be PSD

⇒ consider arbitrary vector 'x'

s.t $x^T(\theta X + (1-\theta)Y)x$

$$= \theta \underbrace{x^T X x}_{\geq 0} + (1-\theta) \underbrace{x^T Y x}_{\geq 0}$$

≥ 0

∴ $\theta X + (1-\theta)Y$ is a PSD ∈ convex set

$\Rightarrow$ cone since

$$x^T(\theta X)x = \theta x^T X x \geq 0$$

only if $\theta > 0$

$\Rightarrow$ Cone

$\therefore$ Set of PSD's is a convex cone.

$\therefore$ x is an image of such PSD

$\Rightarrow$ X also belong to a convex cone

(c) Consider $X = \begin{bmatrix} x_{11} & X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}$

$\because$ $x_{ij} = \|p_i - p_j\|^2$

if $i = j \Rightarrow x_{ii} = 0 \quad \forall\, i = \{1 \ldots n\}$

$\therefore$ $\boxed{\text{diag}(X) = 0}$

$\therefore$ Prooving $X_{22} - X_{21}1^T - 1X_{21}^T \leq 0$

Let $P_1 = 0 \Rightarrow p_1^T p_j = 0 \quad \forall j$

Gram Matrix: $Y = \begin{bmatrix} 0 & 0 \\ 0 & Y_{22} \end{bmatrix}$ $\left(Y_{22} \rightarrow n-1 \times n-1\right)$

$\Rightarrow$ We can write that:

$$X_{21} = \|P_1 - P_j\|^2 \quad \forall \; j \in \{2, \cdots n\}$$

$$= \|P_j\|^2$$

$$= P_j^T P_j = \mathrm{diag}(Y_{22})$$

$$\therefore \boxed{X_{21} = \mathrm{diag}(Y_{22})} \rightarrow (1)$$

Similary we can re-write

$$X_{22} \Rightarrow \quad X_{22}(i,j) = \|P_i - P_j\|_2^2 \; ; \; i,j \in \{2, \cdots n\}$$

$$Y_{22}(i,j) = P_i^T P_j \; ; \; i,j \in \{2, \cdots n\}$$

Using result obtained in (1).

$$\Rightarrow \boxed{X_{22} = \mathrm{diag}(Y_{22})\, 1^T - 2(Y_{22}) + 1\, \mathrm{diag}(Y_{22})^T} \rightarrow (2)$$

Replace (1) in (2)

$$\Rightarrow X_{22} = X_{21} 1^T - 2 Y_{22} + 1 X_{21}^T$$

$$\Rightarrow 2Y_{22} = X_{21} 1^T + 1 X_{21}^T - X_{22}$$

$\Rightarrow$ we know that

Y_{22} : ($\cancel{\ }$ $n-1 \times n-1$) Gram matrix

$\Rightarrow Y_{22} \to PSD$

$Y_{22} \succeq 0$

$\Leftrightarrow X_{21} 1^T + 1 X_{21}^T - X_{22} \succeq 0$

$$\Rightarrow \boxed{X_{22} - X_{21} 1^T - 1 X_{21}^T \preceq 0}$$

(d) Prove $\operatorname{diag}(X) = 0$ } Proved earlier

$$\left(I - \frac{1}{n} 1 1^T\right) X \left(I - \frac{1}{n} 1 1^T\right) \preceq 0$$

$\rightarrow$ (i)

Let / Y

we know that

$$X = \operatorname{diag}(Y) 1^T - 2Y + 1 \operatorname{diag}(Y)^T$$

$\Rightarrow$ $\cancel{y}$ Let $y = \operatorname{diag}(Y)$

$$\therefore X1 = (y 1^T + 1 y^T - 2Y) 1$$

$$= ny + (1^T y) 1$$

$$\Rightarrow y = \frac{1}{n}(X1 - (1^T y) 1),$$

$$Y = \frac{-1}{2}\left(X - y1^T - 1y^T\right)$$

$$= \frac{-1}{2}\left(X - \frac{1}{n}X11^T - \frac{1}{n}11^T X + \frac{1}{n^2}(1^T X 1)11^T\right)$$

$$= \frac{-1}{2}\left(I - \frac{1}{n}11^T\right)X\left(I - \frac{1}{n}11^T\right)$$

$\Rightarrow$ we know that, $Y \succeq 0$

$$\Rightarrow \boxed{\left(I - \frac{1}{n}11^T\right)X\left(I - \frac{1}{n}11^T\right) \preceq 0}$$

(e) Need to show: $w_{ij} = e^{-x_{ij}}$

$\rightarrow \boxed{W \succeq 0}$

Hint: $E e^{iz^T x} = e^{-\frac{1}{2}\|x\|_2^2}$; $x \sim N(0, I)$

Consider $z = p_i - p_k$

$$w_{ik} = e^{-\frac{1}{2}\|p_i - p_k\|^2} = E e^{jz^T(p_i - p_k)}$$

$$W = E\begin{bmatrix} e^{jz^Tp_1} \\ \vdots \\ e^{jz^Tp_n} \end{bmatrix}\begin{bmatrix} e^{jz^Tp_1} \\ \vdots \\ e^{jz^Tp_n} \end{bmatrix}^H$$

$$u^TWu = E\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}^T\begin{bmatrix} e^{jz^Tp_1} \\ \vdots \\ e^{jz^Tp_n} \end{bmatrix}\begin{bmatrix} e^{jz^Tp_1} \\ \vdots \\ e^{jz^Tp_n} \end{bmatrix}^H\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$= E\left|\sum_k u_k e^{jz^Tp_k}\right|^2$$

$$\geq 0$$