HW-02

K. Surya Prakash
EE18BTECH11026

Q1. $x + x' = d + \frac{25h^2}{32d}$

Given $h_{tx} = h$; $h_{rx} = h/4$

$\Rightarrow$ for this setting

$$x + x' - l = \frac{2h_t h_r}{d} = \frac{h^2}{2d}$$

$$l = \sqrt{\left(\frac{3h}{4}\right)^2 + d^2} \Rightarrow d\sqrt{1 + \left(\frac{3h}{4d}\right)^2}$$

$$l = d\left(1 + \frac{9h^2}{16d^2} \cdot \frac{1}{2}\right)$$

$$\therefore x + x' \simeq d + \frac{9h^2}{32d} + \frac{h^2}{2d}$$

$$\boxed{x + x' = d + \frac{25h^2}{32d}}$$

$R = -1$, $G_{tx} = G_{rx} = 1$

For 2-ray model:

$$r(t) = R\left\{\left[\sum \alpha_n(t)\, e^{-j\phi_n(t)}\, u(t-\tau_n(t))\right] e^{j2\pi f_c t}\right\}$$

Impulse respons:

$$c(\tau,t) = \alpha_0 e^{-j\phi_0(t)}\, \delta(\tau-\tau_0) + \alpha_1 e^{-j\phi_1(t)}\, \delta(\tau-\tau_1)$$

⇒ Parameters:

$$\alpha_0 = \frac{\lambda G_1}{4\pi d} = \frac{\lambda}{4\pi d} \quad ; \quad G = \sqrt{1.1} = 1$$

$$\alpha_1 = \frac{R\lambda G}{4\pi(x+x')} = \frac{-\lambda}{4\pi\left(d + \frac{25h^2}{32d}\right)}$$

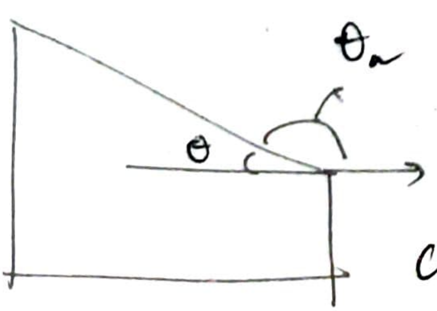
$$* \ \tau_0 = \frac{l}{c} \simeq \frac{d + \frac{9h^2}{32d}}{c}$$

$$* \ \tau_1 = \frac{x+x'}{c} = \frac{d + \frac{25h^2}{32d}}{c}$$

delay

→ Doppler shift.

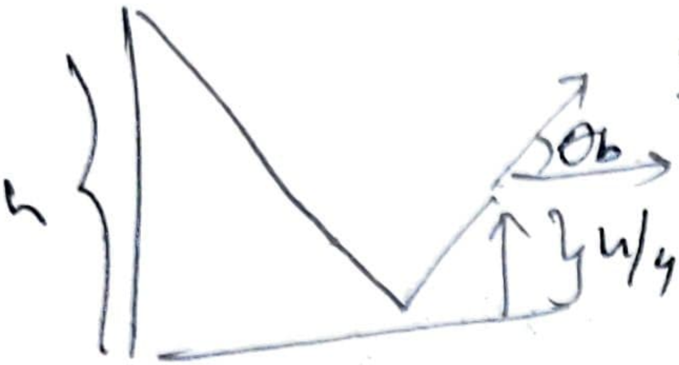
$$f_D = \frac{v}{\lambda}\cos\theta$$



$$\theta_a = \pi - \tan^{-1}\left(\frac{3h}{4d}\right)$$

$$\cos\theta_a = \frac{d}{\sqrt{d^2 + \frac{9h^2}{16}}}$$

* For reflected:



$\tan\theta = \frac{5h}{4d}$

$\theta_b' = \tan^{-1}\left(\frac{5h}{4d}\right)$

$$\boxed{\theta_b = \pi - \tan^{-1}\left(\frac{5h}{4d}\right)}$$

$$\phi_{D0} = \int 2\pi f_{D0}(t)\,dt = \int 2\pi \cdot \frac{v}{\lambda} \cos(\theta_0(t))\,dt$$

We know that $\boxed{d = vt}$

$$\cos(\theta_0(t)) = \frac{d}{\sqrt{d^2 + \frac{9h^2}{16}}}$$

$$\Rightarrow \phi_{D0} = \frac{-2\pi v}{\lambda} \int_t \frac{4vt}{\sqrt{16v^2t^2+9h^2}}\, dt$$

$$\boxed{\phi_{D0} = -\frac{2\pi v}{\lambda}\left(\sqrt{t^2+\left(\frac{3h}{4v}\right)^2}\right)}$$

$$\phi_{D1} = \int \frac{2\pi v}{\lambda} \cos(\theta_1(t))\, dt$$

$$= \frac{-2\pi v}{\lambda} \int_t \frac{4vt}{\sqrt{16v^2t^2+25h^2}}\, dt$$

$$= \frac{-2\pi v}{\lambda}\sqrt{t^2+\left(\frac{5h}{4v}\right)^2}$$

$\therefore$ channel impulse response.

$$c(\tau,t) = \alpha_0(t)\, e^{-j(2\pi f_c \tau_0(t) - \phi_{D0})}\, \delta(\tau-\tau_0(t))$$

$$+ \alpha_1(t)\, e^{-j(2\pi f_c \tau_1(t) - \phi_{D1})}\, \delta(\tau-\tau_1(t))$$

where $\alpha_0(t) = \frac{\lambda}{4\pi vt}$, $\alpha_1(t) = \frac{\lambda}{4\pi\left(d+\frac{25h^2}{32d}\right)}$

$$\tau_0 = \frac{d+\frac{9h^2}{32d}}{c} \qquad \tau_1 = \frac{d+\frac{25h^2}{32d}}{c}$$

$$\phi_{D0} = \frac{-2\pi v}{\lambda}\sqrt{t^2+\left(\frac{3h}{4d}\right)^2} \qquad \phi_{D1} = \frac{-2\pi v}{\lambda}\sqrt{t^2+\left(\frac{5h}{4v}\right)^2}$$

Q2) $X: N(0,\sigma^2)$
$Y: N(0,\sigma^2)$ } independent

Required to show:

$Z = |X + jY| = \sqrt{X^2 + Y^2}$: Rayleigh RV

$Z^2 = X^2 + Y^2$: Exponential RV

* $F_Z(z) = Pr(x^2 + y^2 \le z^2)$

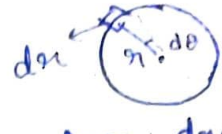
$= \int_{x^2+y^2\le z^2} f_{x,y}(x,y)\, dx\, dy$ } Joint distribution

$= \iint_{x^2+y^2\le z^2} f_X(x) f_Y(y)\, dx\, dy$ } Since independ.

$= \iint_{x^2+y^2\le z^2} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2 \left\{ e^{-\frac{(x^2)}{2\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}} \right\} dx\, dy$

$= \iint_{x^2+y^2\le z^2} \left(\frac{1}{2\pi\sigma^2}\right) \cdot e^{-\frac{(x^2+y^2)}{2\sigma^2}}\, dx\, dy$

Change of variable:



let $x^2+y^2 = r^2$

$x = r\cos\theta$, $y = r\sin\theta$

∴ Area: $dx\,dy = (r\,dr)\,d\theta$

$$\Rightarrow F_Z(z) = \int_0^{2\pi}\int_0^{z} \left(\frac{1}{2\pi\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} \cdot r\,dr\,d\theta$$

$$= \int_0^{z} \left(\frac{1}{2\pi\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} r\,dr\,(\theta)\Big|_0^{2\pi}$$

$$= \int_0^{z} \left(\frac{1}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} r\,dr$$; let $r^2 = t$

$dt = 2r\,dr$

lim: $0 \to z^2$

$$= \int_0^{z^2} \left(\frac{1}{2\sigma^2}\right) e^{-t/2\sigma^2}\,dt$$

$$\Rightarrow \oint \frac{e^{-t/2\sigma^2}}{\frac{1}{2\sigma^2}} \cdot \left(\frac{1}{2\sigma^2}\right)\Bigg|_0^{z^2} \Rightarrow 1 - e^{-z^2/2\sigma^2}$$

$$F_Z(z) = 1 - e^{-z^2/2\sigma^2} \quad (z \geq 0)$$

$$\therefore f_Z(z) = \frac{d\,F_Z(z)}{dz} = \underbrace{\frac{z}{\sigma^2} e^{-z^2/2\sigma^2}}_{\text{Rayleigh distribution}}$$

b) Prooving z^2 is exponential

$$F_{Z^2}(z) = P(Z^2 \leq z) = P(Z \leq \sqrt{z})$$

$$= P(X^2 + Y^2 \leq \sqrt{z})$$

* Prooved earlier: replacing z^2 with z

$$\therefore F_{Z^2}(z) = 1 - e^{-z/2\sigma^2}$$

$$f_{Z^2}(z) = \frac{d}{dz}(1 - e^{-z/2\sigma^2}) = \underbrace{\frac{1}{2\sigma^2} e^{-z/2\sigma^2}}_{\text{Exponential}}$$

$$\therefore f_Z(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \quad \Big\}\ \text{Rayleigh RV } (Z)$$

$$f_{Z^2}(z) = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} \quad \Big\}\ \text{Exponential RV } (Z^2)$$

Q3

$$P_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+(y-\mu)^2)}{2\sigma^2}}$$

$X : N(0,\sigma^2)$; $Y : N(\mu,\sigma^2)$

$\Rightarrow$ finding in terms of z, ϕ

$$P_{Z,\phi}(z,\phi) = P_{X,Y}(x,y) \cdot |J_{XY}|$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+(y-\mu)^2)}{2\sigma^2}} \cdot z$$

$$\Rightarrow \left(\frac{z}{2\pi\sigma^2}\right) \exp\left\{-\frac{((x^2+y^2)+\mu^2-2\mu y)}{2\sigma^2}\right\}$$

$$= \frac{z}{\sigma^2} \exp\left\{-\frac{(z^2+\mu^2)}{2\sigma^2}\right\} \cdot \left\{\frac{1}{2\pi} \exp\left\{\frac{z\mu\sin\phi}{\sigma^2}\right\}\right\}$$

$$* \Rightarrow P_Z(z) = \int_{-\pi}^{\pi} P_{Z,\phi}(z,\phi) \cdot d\phi$$

$$= \frac{z}{\sigma^2} \exp\left\{-\left(\frac{z^2+\mu^2}{2\sigma^2}\right)\right\} \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{\frac{z\mu\sin\phi}{\sigma^2}} d\phi$$

Defining $I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\phi}\, d\phi$

↳ Modified Bessel fn.

$\Rightarrow I_0(X) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{X\sin\phi}\, d\phi$

$\Rightarrow$ Given $\int_{-\pi}^{\pi} \frac{1}{2\pi} e\left\{\frac{z\mu}{\sigma^2}\sin\phi\right\} d\phi = I_0\left(\frac{z\mu}{\sigma^2}\right)$

$$\therefore P_Z(z) = \underbrace{\frac{z}{\sigma^2}\exp\left\{-\frac{(z^2+\mu^2)}{2\sigma^2}\right\} I_0\left(\frac{z\mu}{\sigma^2}\right)}_{\text{Rician distribution}}$$

For $P_{Z^2}(z)$

$$= \underbrace{\frac{1}{2\sigma^2}\exp\left\{-\frac{(z+\mu^2)}{2\sigma^2}\right\} I_0\left(\frac{\sqrt{z}\mu}{\sigma^2}\right)}_{\text{Not a defined distribution}}$$

4Q) $Pr(P \le P_0) = 0.05$

Power : exponential RV.

$\Rightarrow$ As derived in Q2

$\Rightarrow Pr(P \le P_0) = 1 - e^{-P_0/2\sigma^2} = 0.05$

$2\sigma^2$: Avg. power P_{avg}

$\Rightarrow 1 - e^{-P_0/P_{avg}} = 0.05$

$\Rightarrow e^{-P_0/P_{avg}} = 0.95$

$\Rightarrow +P_0/P_{avg} = \ln\left(\frac{1}{0.95}\right)$

$\Rightarrow 10\log(P_0) - 10\log(P_{avg}) = -12.9$

$\Rightarrow P_0(dB) - P_{avg}(dB) = -12.9$

$\Rightarrow P_{avg}(dB) = +12.9 + (-70-30)_{dB}$

$= -87.1$ dB

$P_{avg}(dB) = -87.1$ dB

$P_{avg}(dBm) = -57.1$ dBm

Q5)

a) W : Average received power (considering only path loss)

Z_1 : Shadowing in path-1

Z_2 : Shadowing in path-2

(b) Show $P_{out} = [Q(\frac{\Delta}{\sigma})^2]$

$P_{out} = Pr((P_{r,1} < T) \cap (P_{r,2} < T])$

Since, Z_1, Z_2 are independent

$\Rightarrow$ P_{r_1}, P_{r_2} are independent

$= Pr[P_{r,1} < T] \cdot Pr[P_{r_2} < T]$

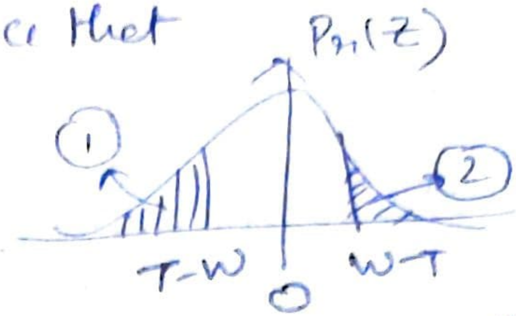
$Pr[P_{r_1} < T] = ?$

Since, $P_{r_1} = W + Z_1 \sim N(0, \sigma^2)$

For, $P_{r_1} < T \Rightarrow Z_1 < T - W$

$\therefore Pr(P_{r_1} < T) = Pr(Z_1 < T - W) = 1 - \Phi\left(\frac{(T-W)-0}{\sigma}\right)$

→ From the pdf we can see that



$$\int_{-\infty}^{T-W} f_z(z) = \int_{W-T}^{\infty} f_z(z)$$

→ $1-Q\left(\frac{T-W}{\sigma}\right) = Q\left(\frac{W-T}{\sigma}\right)$ } Due to ~~odd~~ even nature of pdf

∴ $P_r(P_{r_1} < T) = Q\left(\frac{W-T}{\sigma}\right)$

Similarly

∴ $P_r(P_{r_2} < T) = Q\left(\frac{W-T}{\sigma}\right)$; } $P_r\{Z_2 < T-W)$

∴ $P_{out} = \left[Q\left(\frac{W-T}{\sigma}\right)\right]^2$

$$\boxed{P_{out} = \left[Q\left(\Delta/\sigma\right)\right]^2 ,\quad \Delta = W-T}$$ //

Q6) Considering Z_1, Z_2 are corelated by a corelation coeff: 'b'

$$P_{out} = Pr(P_{r_1} \le T \cap P_{r_2} \le T)$$

$$= Pr(Z_1 \le T-W \cap Z_2 \le T-W)$$

$$= F_{Z_1 Z_2}(T-W, T-W)$$

$$\Rightarrow F_{Z_1 Z_2}(T-W, T-W) = \int_{-\infty}^{T-W} \int_{-\infty}^{T-W} f_{Z_1 Z_2}(z_1, z_2)\, dz_1\, dz_2$$

$$= \int_{-\infty}^{T-W} \int_{-\infty}^{T-W} f_{Z_2/Z_1}\left(z_2 / Z_1 = z_1\right) \cdot f_{Z_1}(z_1)\, dz_1\, dz_2$$

Need to model:

$f_{Z_2/Z_1}\left(z_2 / Z_1 = z_1\right)$ and $f_{Z_1}(z_1)$

$\Rightarrow$ we know that if $Z_1, Z_2 \sim N(0, \sigma^2)$

$f_{Z_1}(z_1) = N(0, \sigma^2)$

* Conditional pdf of normal distribution

$f_{Y/X} : N(\mu_{Y/X}, \sigma^2_{Y/X})$

where, $Y, X \sim N(\mu_x, \sigma_x^2)$, $N(\mu_y, \sigma_y^2)$

$\Rightarrow f_{Y|X}$ is also a normal distribution

with $\mu_{Y|X} = \mu_y + \rho\left(\frac{\sigma_y}{\sigma_x}\right)(x - \mu_x)$

$$\sigma_{Y|X}^2 = \sigma_y^2(1-\rho^2)$$

where, ρ: Corelation coeff. $= \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

$$f_{Y|X} = N(\mu_{Y|X}, \sigma_{Y|X}^2)$$

$$P_{out} = \int_{-\infty}^{T-w} \int_{-\infty}^{T-w} \frac{1}{\sqrt{2\pi\sigma_{z_2|z_1}^2}} e^{-\frac{(z_2 - \mu_{z_2|z_1})^2}{2\sigma_{z_2|z_1}^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z_1 - 0)^2}{2\sigma^2}}$$

Here; $\mu_{z_2|z_1} = \rho\left(\frac{\sigma}{\sigma}\right)(z_1 - 0) = \rho z_1$

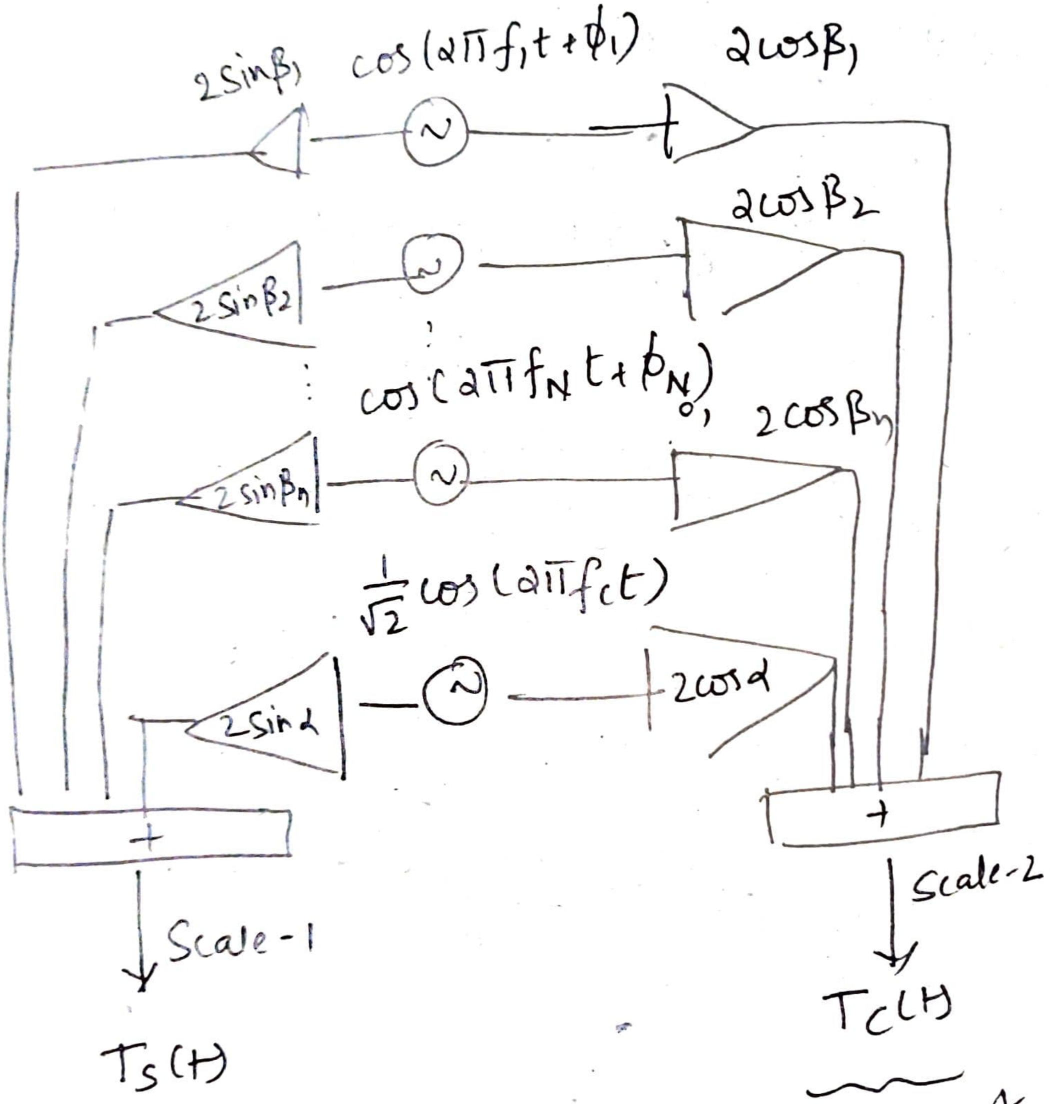
$$\sigma_{z_2|z_1}^2 = \sigma^2(1-\rho^2)$$

$$= \int_{-\infty}^{T-w} \int_{-\infty}^{T-w} \left(\frac{1}{2\pi\sigma^2}\right)\left(\frac{1}{\sqrt{1-\rho^2}}\right) \cdot \exp\left\{-\left(\frac{(z_2 - \rho z_1)^2}{2\sigma^2(1-\rho^2)} + \frac{(z_1)^2}{2\sigma^2}\right)\right.$$

$$P_{out} = \int_{-\infty}^{T-w}\int_{-\infty}^{T-w} \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot \exp\left\{-\left(\frac{\zeta_{q_1}^2+\zeta_{q_2}^2-2\rho\,\zeta_{q_1}\zeta_{q_2}}{2\sigma^2(1-\rho^2)}\right)\right\} d\zeta_{q_1}\, d\zeta_{q_2}$$

Q7 Jakes method :-

* Trying to implement fading in a computer



Imag. part

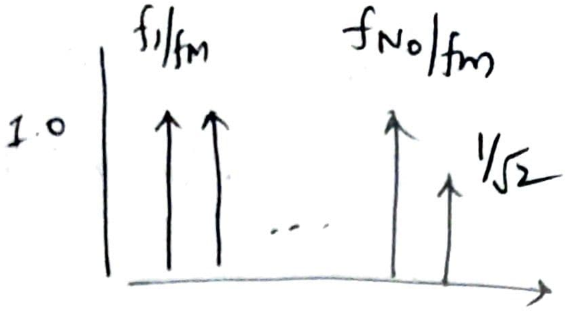
$Z_i(t)$

Real part

$Z_r(t)$

Description:

* consists of $N_o + 1$ oscillators



Oscillator freq f_n/f_m

where, $N = \underbrace{4N_o + 2}_{\text{even but not a multiple of 4}}$

where $N_o \geq 15$

$$\boxed{N_o = \frac{N}{4} - \frac{1}{2}}$$

doppler freq @ each block:

$$f_n = f_D \cos\left(\frac{2\pi}{N} n\right) \quad ; n = 1, \cdots N_o$$

$$\Rightarrow \quad f_1 = f_D \cos\left(\frac{2\pi}{N}\right)$$

$$\vdots$$

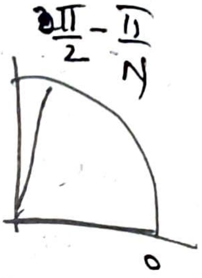
$$f_{N_o} = f_D \cos\left(\frac{2\pi}{N} N_o\right) = f_D \cos\left(\frac{\pi}{2} - \frac{\pi}{N}\right)$$

$$Z_r(t) = 2 \sum_{n=1}^{N_o} \cos\beta_n \cos(2\pi f_n t + \phi_n) + \sqrt{2}\cos\alpha \cos(2\pi f_D t)$$

$$Z_i(t) = \frac{1}{\text{scale}} \left[2 \sum_{n=1}^{N_o} \sin\beta_n \cos(2\pi f_n t + \phi_n) + \sqrt{2} \sin\alpha \cos(2\pi f_D t) \right]$$

* The freq are chosen so that they come from the 1st ~~part~~ Quadrant of the circle

Equally space angle of arrivals



↳ Parameters of interest:

$E[Z_r(t)] = \langle \quad \rangle$ time avg → $Z_r(t)$, Sum of sinusoids

$\langle Z_r(t) \rangle = \langle Z_i(t) \rangle = 0$ } zero-mean

* β, α are scaling factors so that they meet the parameters

Need to have

$$\boxed{E[Z_r^2(t)] = E[Z_i^2(t)] = 1/2}$$

$$\boxed{E(Z(t)\, Z^*(t+\Delta t)) = C \cdot J_0(2\pi f_D \Delta t)}$$

$\Rightarrow$

$\langle z_r^2(t)\rangle = \langle 2\sum_{n=1}^{N_0} \cos^2\beta_n + \cos^2\alpha\rangle$

$\langle z_i^2(t)\rangle = \sum_{n=1}^{N_0} (1+\cos 2\beta_n) + \cos^2\alpha$

$\langle z_r^2(t)\rangle = N_0 + \sum_{n=1}^{N_0} \cos 2\beta_n + \cos^2\alpha \rightarrow (1)$

$\langle z_i^2(t)\rangle = N_0 - \sum_{n=1}^{N_0} \cos 2\beta_n + \sin^2\alpha \rightarrow (2)$

Choice of α and β:

$\alpha = 0$; $\beta_n = \left(\frac{\pi}{N_0+1}\right)\cdot n$; $n = 1, \ldots N_0$

why...?

$\sum_{n=1}^{N_0} e^{j2\beta_n} = \sum_{n=0}^{N_0} e^{j2\beta_n} - 1 = \underbrace{\frac{1 - e^{-j\left(\frac{2\pi}{N_0+1}\right)(N_0+1)}}{1 - e^{j2\beta_n}}}_{0} - 1$

$= 0 = -1$

$\therefore \sum_{n=1}^{N_0} \cos 2\beta_n + j\sin 2\beta_n = -1$

$\Rightarrow \sum_{n=1}^{N_0} \sin 2\beta_n = 0$

$$\Rightarrow \sum_{n=1}^{N_0} \cos 2\beta n = -1$$

$\Rightarrow$

From (1) & (2)

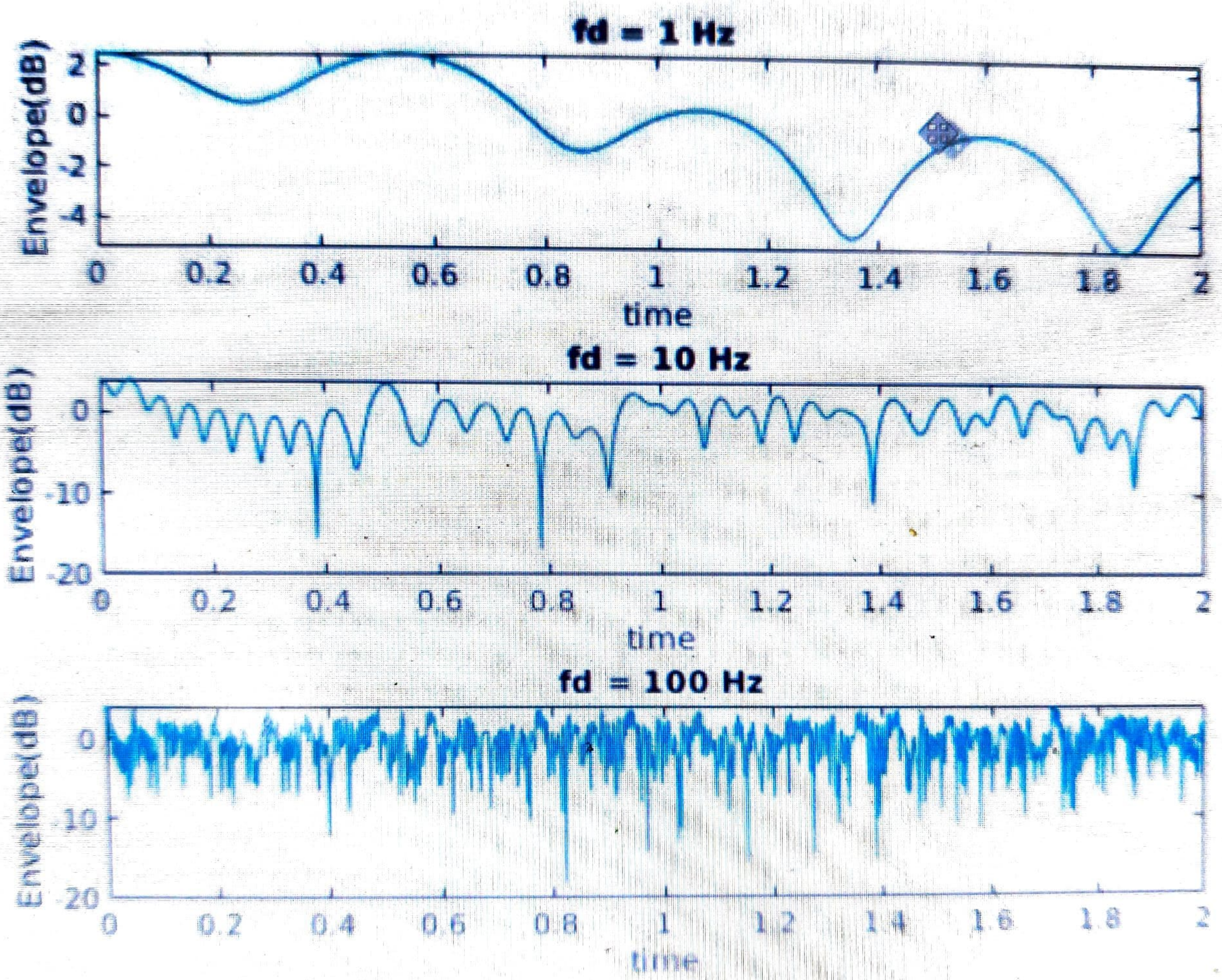
$\langle z_r^2(t)\rangle = N_0 - 1 + 1 = N_0$ $\Rightarrow$ Scale 2: $\cancel{\frac{1}{\sqrt{N_0}}}$

$\langle z_i^2(t)\rangle = N_0 + 1$ $\Rightarrow$ Scale-1 $\cancel{\frac{1}{2\sqrt{N_0+1}}}$

$$\Rightarrow \text{Scale-2: } \frac{1}{\sqrt{2N_0}};$$

$$\text{Scale-1} = \frac{1}{\sqrt{2(N_0+1)}}$$

Plots :


fd = 1 Hz
Envelope(dB)
time
fd = 10 Hz
Envelope(dB)
time
fd = 100 Hz
Envelope(dB)
time