Chapter 1: Matrix Operations
Section 3: Determinants

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September 11, 2021

Exercises

(1) (a)

$$\det \begin{bmatrix} 1 & i \\ 2-i & 3 \end{bmatrix} = (1)(3) - (i)(2-i) = 3 - 2i + i^2 = 2 - 2i$$

(b)

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = (1)(-1) - (1)(1) = -1 - 1 = -2$$

(c)

$$\det \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 2\det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 0\det \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + 1\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= 2((1)(2) - (0)(0)) + 1((0)(0) - (1)(1)) = 4 - 1 = 3$$

(d)

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 0 & 9 & 7 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 5 & 8 & 0 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= 1\det \begin{bmatrix} 2 & 6 & 9 \\ 0 & 3 & 7 \\ 0 & 0 & 4 \end{bmatrix} = 2\det \begin{bmatrix} 3 & 7 \\ 0 & 4 \end{bmatrix} = 2((3)(4) - (7)(0)) = 24$$

(e)

$$\det\begin{bmatrix}1&4&1&3\\2&3&5&0\\4&1&0&0\\2&0&0&0\end{bmatrix} = -\det\begin{bmatrix}3&4&1&1\\0&3&5&2\\0&1&0&4\\0&0&0&2\end{bmatrix} = \det\begin{bmatrix}3&1&4&1\\0&5&3&2\\0&0&1&4\\0&0&0&2\end{bmatrix}$$

$$= 3\det\begin{bmatrix}5&3&2\\0&1&4\\0&0&2\end{bmatrix} = 15\begin{bmatrix}1&4\\0&2\end{bmatrix} = 15(2-0) = 30$$

(2)

$$\det\begin{bmatrix}1&2&5&6\\3&1&7&7\\0&0&2&3\\4&2&1&5\end{bmatrix} = -\det\begin{bmatrix}2&1&5&6\\1&3&7&7\\0&0&2&3\\2&4&1&5\end{bmatrix}$$

$$= -\det\begin{bmatrix}2&1&5&6-5\\1&3&7&7-7\\0&0&2&3-2\\2&4&1&5-1\end{bmatrix} = -\det\begin{bmatrix}2&1&5&1\\1&3&7&0\\0&0&2&1\\2&4&1&4\end{bmatrix}$$

(3)

$$\det A = (2)(4) - (1)(3) = 5$$

$$\det B = (1)(-2) - (5)(1) = -7$$

$$\det AB = \det\begin{bmatrix}17&-4\\21&-7\end{bmatrix} = (17)(-7) - (21)(-4) = -119 + 84 = -35$$

$$\det AB = -35 = (5)(-7) = (\det A)(\det B)$$

(4) (a) Let A be an $n \times n$ matrix, where

$$A = \begin{bmatrix} & & & & 1\\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$$

I claim that $\det A = 1$.

Suppose $n = 1$. Then clearly $\det A = 1$. Suppose for $n = k - 1$, that $\det A = 1$. Then, if $n = k$,

$$A = \left[\begin{array}{c|c} & A' \\ \hline 1 & \end{array}\right]$$

Where A' has dimensions $k - 1 \times k - 1$. Note that $\det A' = 1$, from the inductive hypothesis. Then

$$\det A = 1 \det A' = 1$$

(b) Let A be an $n \times n$ matrix, where

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

I claim that $\det A = n + 1$.

Suppose $n = 1$. Then clearly $\det A = 2$. Suppose for $n = i$, where $i < k$, that $\det A = i + 1$. Then, if $n = k$,

$$A = \left[\begin{array}{c|ccc} 2 & -1 & 0 & \cdots \\ \hline -1 & & & \\ 0 & & A' & \\ \vdots & & & \end{array}\right]$$

Where A' has dimensions $k - 1 \times k - 1$. In particular,

$$A' = \left[\begin{array}{c|ccc} 2 & -1 & 0 & \cdots \\ \hline -1 & & & \\ 0 & & A" & \\ \vdots & & & \end{array}\right]$$

Where $A"$ has dimensions $k - 2 \times k - 2$. Note that $\det A' = k$, and $\det A" = k - 1$, from the inductive hypothesis. Then

$$\det A = 2 \det A' - (-1) \det \left[\begin{array}{c|cc} -1 & 0 & \cdots \\ \hline -1 & & \\ 0 & & A" \\ \vdots & & \end{array}\right]$$

$$= 2k + \det \left[\begin{array}{c|ccc} -1 & -1 & 0 & \cdots \\ \hline 0 & & & \\ 0 & & A"^\top & \\ \vdots & & & \end{array}\right] = 2k - \det A"^\top$$

$$= 2k - \det A" = 2k - (k-1) = k+1$$

(5) Lemma: Let A be a $n \times n$ upper triangular matrix. Then $\det A = a_{11}a_{22}...a_{nn}$. Proof: If A is a 1×1 matrix, then trivially $\det A = a_{11}$. Suppose the statement is true for all $k-1 \times k-1$ matrices. Suppose A is a $k \times k$ matrix, ie.

$$A = \left[\begin{array}{c|c} a_{11} & * \\ & A' \end{array}\right]$$

where A' is a $k-1 \times k-1$ matrix, and $*$ is a $1 \times k-1$ matrix. Then $\det A = a_{11} \det A' = a_{11}a_{22}...a_{nn}$.

Thus, via the Lemma,

$$\det \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 2 & 3 & & \vdots \\ 3 & 3 & 3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ n & \cdots & \cdots & \cdots & n \end{bmatrix} = \det \begin{bmatrix} -1 & 2 & 3 & \cdots & n \\ 0 & 2 & 3 & & \vdots \\ 0 & 3 & 3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & n & \cdots & \cdots & n \end{bmatrix}$$

$$= ... = \det \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 & n \\ 0 & -1 & -1 & \cdots & -1 & n \\ 0 & 0 & -1 & \cdots & -1 & n \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & n \end{bmatrix} = (-1)^{n-1}n$$

(6)

$$
\det\begin{bmatrix}
2 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
 & 1 & 2 & 1 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
= \frac{1}{2}\det\begin{bmatrix}
2 & 1 & & & & & & \\
2 & 4 & 2 & & & & & \\
 & 1 & 2 & 1 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
$$

$$
= \frac{1}{2}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & 1 & 2 & 1 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
= \frac{1}{6}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & 3 & 6 & 3 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
$$

$$
= \frac{1}{6}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & & 4 & 3 & & & & \\
 & & 1 & 2 & 1 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
= \frac{1}{24}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & & 4 & 3 & & & & \\
 & & 4 & 8 & 4 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
$$

$$
= \frac{1}{24}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & & 4 & 3 & & & & \\
 & & & 5 & 4 & & & \\
 & & & 1 & 2 & 1 & & 1 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
= \frac{1}{120}\det\begin{bmatrix}
2 & 1 & & & & & & \\
 & 3 & 2 & & & & & \\
 & & 4 & 3 & & & & \\
 & & & 5 & 4 & & & \\
 & & & 5 & 10 & 5 & & 5 \\
 & & & & 1 & 2 & 1 & \\
 & & & & & 1 & 2 & \\
 & & & & 1 & & & 2
\end{bmatrix}
$$

$$
= \frac{1}{120} \det \begin{bmatrix}
2 & 1 & & & & & & \\
& 3 & 2 & & & & & \\
& & 4 & 3 & & & & \\
& & & 5 & 4 & & & \\
& & & & 6 & 5 & & 5 \\
& & & & 1 & 2 & 1 & \\
& & & & & 1 & 2 & \\
& & & & 1 & & & 2
\end{bmatrix}
$$

$$
= \frac{1}{4320} \det \begin{bmatrix}
2 & 1 & & & & & & \\
& 3 & 2 & & & & & \\
& & 4 & 3 & & & & \\
& & & 5 & 4 & & & \\
& & & & 6 & 5 & & 5 \\
& & & & 6 & 12 & 6 & \\
& & & & & 1 & 2 & \\
& & & & 6 & & & 12
\end{bmatrix}
= \frac{1}{4320} \det \begin{bmatrix}
2 & 1 & & & & & & \\
& 3 & 2 & & & & & \\
& & 4 & 3 & & & & \\
& & & 5 & 4 & & & \\
& & & & 6 & 5 & & 5 \\
& & & & & 7 & 6 & -5 \\
& & & & & 1 & 2 & \\
& & & & & -5 & & 7
\end{bmatrix}
$$

$$
= \frac{1}{5292000} \det \begin{bmatrix}
2 & 1 & & & & & & \\
& 3 & 2 & & & & & \\
& & 4 & 3 & & & & \\
& & & 5 & 4 & & & \\
& & & & 6 & 5 & & 5 \\
& & & & & 35 & 30 & -25 \\
& & & & & 35 & 70 & \\
& & & & & -35 & & 49
\end{bmatrix}
$$

$$
= \frac{1}{5292000} \det \begin{bmatrix}
2 & 1 & & & & & & \\
& 3 & 2 & & & & & \\
& & 4 & 3 & & & & \\
& & & 5 & 4 & & & \\
& & & & 6 & 5 & & 5 \\
& & & & & 35 & 30 & -25 \\
& & & & & & 40 & 25 \\
& & & & & & 30 & 24
\end{bmatrix}
$$

$$= \frac{1}{63504000} \det \begin{bmatrix} 2 & 1 & & & & & & \\ & 3 & 2 & & & & & \\ & & 4 & 3 & & & & \\ & & & 5 & 4 & & & \\ & & & & 6 & 5 & & 5 \\ & & & & & 35 & 30 & -25 \\ & & & & & & 120 & 75 \\ & & & & & & 120 & 96 \end{bmatrix}$$

$$= \frac{1}{63504000} \det \begin{bmatrix} 2 & 1 & & & & & & \\ & 3 & 2 & & & & & \\ & & 4 & 3 & & & & \\ & & & 5 & 4 & & & \\ & & & & 6 & 5 & & 5 \\ & & & & & 35 & 30 & -25 \\ & & & & & & 120 & 75 \\ & & & & & & & 21 \end{bmatrix}$$

From the Lemma of the previous exercise, then we have

$$\frac{1}{63504000}(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 35 \cdot 120 \cdot 21) = 1$$

(7) Suppose A is a 1×1 matrix. Then for an arbitrary 1×1 matrix B, $\det(A + B) = a_{11} + b_{11} = \det A + \det B$. Furthermore, if $C = [ca_{11}]$, then $\det C = ca_{11} = c \det A$.

Suppose for all $n-1 \times n-1$ matrices, the determinant operates linearly on rows. Consider the $n \times n$ matrices A, B, C, where for some k, $c_{kj} = a_{kj} + b_{kj}$. Otherwise, if $i \neq k$, $a_{ij} = b_{ij} = c_{ij}$. In particular, $A_{i1} = B_{i1} = C_{i1}$, and by the inductive hypothesis $\det C_{k1} = \det A_{k1} + \det B_{k1}$. Then

$$\det C = \sum_{i=1}^{n} (-1)^{i+1} c_{i1} \det C_{i1}$$

$$= (-1)^{k+1}(a_{k1} + b_{k1}) \det A_{k1} + \sum_{i \neq k} (-1)^{i+1} a_{i1}(\det A_{i1} + \det B_{i1})$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1} + \sum_{i=1}^{n} (-1)^{i+1} b_{i1} \det B_{i1} = \det A + \det B$$

Now consider the $n \times n$ matrices A', B', where for some k, $b_{kj} = ca_{kj}$, Otherwise, if $i \neq k$, $a_{ij} = b_{ij}$. In particular, $A_{i1} = B_{i1}$, and by the inductive hypothesis $\det B_{k1} = c \det A_{k1}$. Then

$$\det B = \sum_{i=1}^{n} (-1)^{i+1} b_{i1} \det B_{i1}$$

$$= (-1)^{k+1} ca_{kj} \det A_{k1} + \sum_{i \neq k} (-1)^{i+1} a_{i1} (c \det A_{k1})$$

$$= c \sum_{i=1}^{n} (-1)^{k+1} a_{i1} \det A_{i1} = c \det A$$

(8)

$$\det(-A) = \det \begin{bmatrix} -A_1 \\ \hline -A_2 \\ \hline \vdots \\ \hline -A_n \end{bmatrix} = (-1)^n \det \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{bmatrix} = (-1)^n \det A$$

(9) Lemma: Let E be an elementary matrix. Then $\det E = \det E^\top$. Proof: If $E = I + ae_{ij}$ is an elementary matrix of the first kind, then $E^\top = I + ae_{ji}$. Clearly, $E^\top$ is also an elementary matrix of the first kind, so $\det E = \det E^\top = 1$. If $E = I + e_{ij} + e_{ji} - e_{ii} - e_{jj}$ is an elementary matrix of the second kinda, then $E^\top = I + e_{ji} + e_{ij} - e_{ii} - e_{jj} = E$, so $\det E = \det E^\top$. If $E = I + (c-1)e_{ii}$ is an elementary matrix of the third kind, then $E^\top = I + (c-1)e_{ii} = E$, so $\det E = \det E^\top$.

Suppose A is not invertible. Then $A^\top$ is also not invertible, and therefore $\det A = \det A^\top = 0$.

Suppose A is invertible. Then $A^\top$ is also invertible, and for some elementary matrices $E_1, ..., E_p$, $\det E_p ... \det E_1 \det A = \det A^\top \det E_1^\top ... \det E_p^\top = \det I = 1$. Note that from the Lemma, $\det E_i = \det E_i^\top$. Therefore, $\det A = \det A^\top$.

(10)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

$$ad \det \begin{bmatrix} 1 & 0 \\ c/d & 1 \end{bmatrix} - \det \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = ad \det \begin{bmatrix} 1 & 0 \\ c/d & 1 \end{bmatrix} - cb \det \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$$

$$= ad\left(\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \det\begin{bmatrix}1 & 0\\ c/d & 0\end{bmatrix}\right) - bc\left(\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \det\begin{bmatrix}0 & d/c\\ 0 & 1\end{bmatrix}\right)$$

$$= ad\left(1 + \frac{c}{d}\det\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\right) - bc\left(1 + \frac{d}{c}\det\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right) = ad - bc$$

(11)

$$\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$$

(12) Suppose A is a 1×1 submatrix. Then trivially

$$\det\begin{bmatrix}A & B\\ 0 & D\end{bmatrix} = (\det A)(\det D)$$

Suppose the statement is true for all $k-1 \times k-1$ submatrices A. Let A' be a $k \times k$ submatrix. Let

$$X = \begin{bmatrix}A' & B'\\ 0 & D'\end{bmatrix}$$

Then

$$\det X = \sum_{i=1}^{k}(-1)^{i+1}a'_{i1}\det X_{i1} = \sum_{i=1}^{k}(-1)^{i+1}a'_{i1}(\det A'_{i1})(\det D)$$

$$= (\det D)\sum_{i=1}^{k}(-1)^{i+1}a'_{i1}(\det A'_{i1}) = (\det D)(\det A')$$

(13) Let

$$X = \begin{bmatrix}I_n & \\ -C & A\end{bmatrix}, Y = \begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

Then

$$XY = \begin{bmatrix}A & B\\ -CA + AC & -CB + AD\end{bmatrix} = \begin{bmatrix}A & B\\ & AD - CB\end{bmatrix}$$

Note that $\det X = \det X^{\top} = \det A$ and $(\det X)(\det Y) = \det XY = (\det A)(\det(AD - CB))$. Thus $\det Y = \det(AD - CB)$.

Let

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, B = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, C = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, D = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$$

Note that $BC \neq CB$.

$$AD - CB = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that $\det Y = 0$, but $\det(AD - CB) = 1$. Thus the formula does not hold.