Assignment-5

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(1)

(1.0) $x(n) = (a^n + a^{-n})u(n)$; a is real.

$\Rightarrow$ Important Result: (I)

z transform for $x(n) = a^n u(n) = \dfrac{1}{1-\frac{a}{z}}$

Proof $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow \sum_{n=0}^{\infty} a^n z^{-n}$ $\because$ $x(n) = \begin{cases} a^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$

$\therefore X(z) = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$; ~~If $\left|\frac{a}{z}\right| < 1$~~

$\rightarrow$ ~~if~~ For $X(z)$ to converge the summation should be $< \infty$.

$\therefore \left|\frac{a}{z}\right| < 1 \Rightarrow$ ROC. $\Rightarrow z > a$

$$\boxed{X(z) = \frac{1}{1-\left(\frac{a}{z}\right)} \Rightarrow \frac{z}{z-a}} \; ; |z| > |a|$$

$\Rightarrow$ Using Result (I)

1.0) $x(n) = a^n u(n) + a^{-n} u(n)$

$$X(z) = \frac{1}{1-\frac{a}{z}} + \frac{1}{1-\frac{1}{az}} = \frac{z(az-1) + az(z-a)}{(az-1)(z-a)}$$

$$= \frac{2az^2 - (a^2+1)z}{(az-1)(z-a)}$$

$\therefore$ Here Region of converge is,

$$\left(\left|\frac{a}{z}\right| < 1 \cup \left|\frac{1}{az}\right| < 1\right)$$

$$= \boxed{|z| > |a| \cup (|z| > 1/|a|)}$$

②

$\therefore$ If $|a| > 1 \Rightarrow ROC = |z| > |a|$

If $|a| < 1 \Rightarrow ROC \Rightarrow |z| > \frac{1}{|a|}$

* <u>Poles:</u> $(az - 1) = 0$ and $z - a = 0$

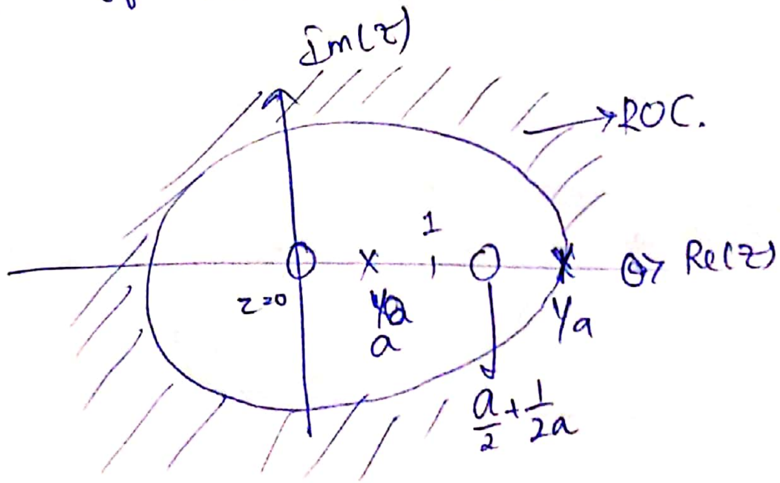
$z = 1/a, \; z = a \qquad \therefore \left(z_p = 1/a, a\right)$

* <u>zeros:</u> $2a z^2 - (a^2+1) z = 0$

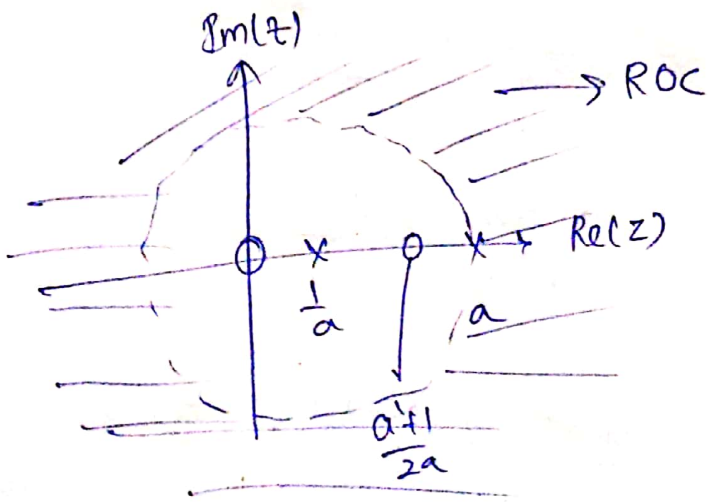
$\Rightarrow z = 0$ and $z = \frac{a^2+1}{2a} \qquad \left(z_z = \frac{a^2+1}{2a}, 0\right)$

<u>Pole-zero plot</u>

* <u>Case-I:</u> If $|a| < 1$



<u>Case-II:</u> If $|a| > 1$



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(1.1)

Sol

$$x(n) = (-1)^n 2^{-n} u(n)$$

$$x(n) = \left(\frac{-1}{2}\right)^n u(n)$$

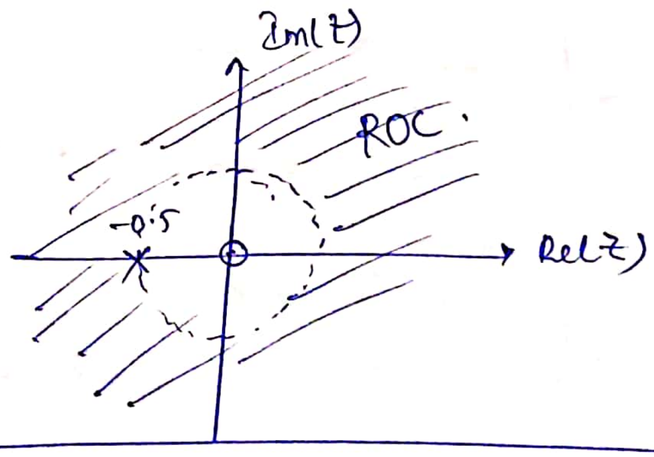
Using Result (1) :(Shown in 1.0) $a = -1/2$

$$X(z) = \frac{1}{1 + \frac{1}{2z}} = \frac{2z}{1+2z}$$

ROC: $\left|\frac{1}{2z}\right| < 1 \Rightarrow |z| > 1/2$

Poles: $1 + 2z = 0 \Rightarrow \boxed{z = -1/2}$; zeros $\Rightarrow \boxed{z = 0}$

Pole-zero plot:



(1.2)
Sol.

$$x(n) = Ar^n \cos(w_0 n + \phi) \quad ; 0 < r < 1$$

$$= \frac{Ar^n}{2}\left(e^{j(w_0 n + \phi)} + e^{-j(w_0 n + \phi)}\right)$$

$$\Rightarrow \frac{Ae^{j\phi}}{2}\left(re^{jw_0}\right)^n + \frac{Ae^{-j\phi}}{2}\left(re^{-jw_0}\right)^n$$

Using Result (1) : Here,

$$X(z) = \frac{Ae^{j\phi}}{2}\left(\frac{1}{1-\frac{re^{jw_0}}{z}}\right) + \frac{Ae^{-j\phi}}{2}\left(\frac{1}{1-\frac{re^{-jw_0}}{z}}\right)$$

For $X(z)$ to be finite: $\left(\left|\frac{re^{jw_0}}{z}\right| < 1 \cup \left|\frac{re^{-jw_0}}{z}\right| < 1\right)$

(4)

$$\Rightarrow \left(\left\{\left|\frac{\gamma}{z}\right|<1\right\} \cup \left\{\left|\frac{\gamma}{z}\right|<1\right\}\right) \Rightarrow \underline{|z|>|\gamma|}$$

$\therefore$ $\boxed{\text{ROC: } |z|>|\gamma|}$

$$X(z) = \frac{Ae^{j\phi}}{2}\left(\frac{z}{z-\gamma e^{jw_0}}\right) + \frac{Ae^{-j\phi}}{2}\left(\frac{ze^{jw_0}}{ze^{jw_0}-\gamma}\right)$$

$$= \frac{Ae^{j\phi}}{2}\left(\frac{z e^{-jw_0}}{ze^{-jw_0}-\gamma}\right) + \frac{Ae^{-j\phi}}{2}\left(\frac{ze^{jw_0}}{ze^{jw_0}-\gamma}\right)$$

$$\Rightarrow \frac{A}{2}\left(\frac{ze^{-j(w_0-\phi)}}{ze^{-jw_0}-\gamma} + \frac{ze^{j(w_0-\phi)}}{ze^{jw_0}-\gamma}\right)$$

$$= \frac{A}{2}\left(\frac{z^2e^{j\phi} - \gamma z e^{-j(w_0-\phi)} + z^2 e^{-j\phi} - \gamma z e^{j(w_0-\phi)}}{z^2 - \gamma z e^{-jw_0} - \gamma z e^{jw_0} + \gamma^2}\right)$$

$$\Rightarrow \frac{A}{2}\left(\frac{2z^2\cos\phi - 2\gamma z\cos(w_0-\phi)}{z^2+\gamma^2 - 2z\gamma\cos(w_0)}\right)$$

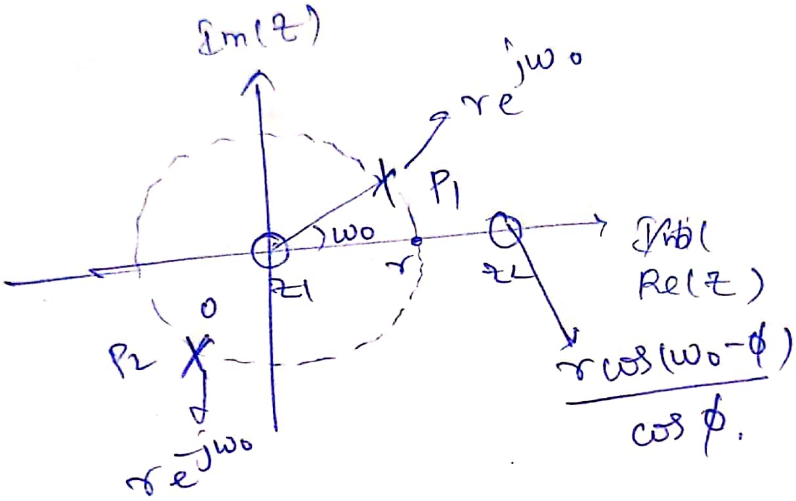
$$\boxed{X(z) = \frac{A(z^2\cos\phi - \gamma z\cos(w_0-\phi))}{z^2+\gamma^2-2z\gamma\cos(w_0)}}$$

ROC: $|z|>|\gamma|$

Poles: $z = \gamma e^{jw_0}$, $z = \gamma e^{-jw_0}$

Zeros: $z=0$; $z = \dfrac{\gamma\cos(w_0-\phi)}{\cos\phi}$

(1.2)
contn :


Im(z)
$re^{j\omega_0}$
P_1
ω_0
r
z_1
z_2
Real
Re(z)
P_2
0
$re^{-j\omega_0}$
$\frac{r\cos(\omega_0-\phi)}{\cos\phi}$.

(1.3 Sol)

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n - 2^n ; & n \ge 0 \\ 0, & n < 0 \end{cases} = \left(\left(\frac{1}{3}\right)^n - 2^n \right) u(n).$$

$$x(n) = \left(\frac{1}{3}\right)^n u(n) - 2^n u(n)$$

Using Result-(I):

$$X(z) = \frac{1}{1 - \frac{1}{3z}} - \frac{1}{1 - \frac{2}{z}} = \frac{3z}{3z-1} - \frac{z}{z-2}$$

$$= \frac{3z(z-2) - z(3z-1)}{(3z-1)(z-2)} = \frac{-5z}{3z^2 - 7z + 2}$$

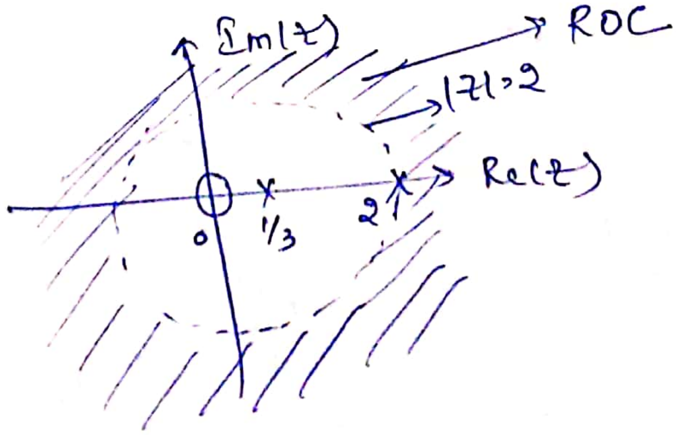
$$\boxed{X(z) = \frac{-5z}{3z^2 - 7z + 2}}$$

ROC: $\left\{ \left|\frac{1}{3z}\right| < 1 \cup \left|\frac{2}{z}\right| < 1 \right\}$

$= \{ |z| > 2 \}$

Poles: $z = 1/3, 2$; zeros: $z = 0$

Pole-zero plot:



(1.4) Convolution; $x(n) = x_1(n) * x_2(n)$

Sol

$x(n) \longleftrightarrow X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) z^{-(k)} z^{-(n-k)}$$

keeping k constant.

$$= \sum_{k=-\infty}^{\infty} \cancel{\sum_{n=-\infty}^{\infty}} (x_1(k)) z^{-k} \underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k) z^{-(n-k)}}_{\text{By } \cancel{\text{keeping}}\ n-k=b.}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} X_2(z)$$

$$= X_1(z) X_2(z)$$

$$\therefore \boxed{X(z) = X_1(z) \cdot X_2(z)}$$

$\therefore$ Given; $x_1(n) = \left(\frac{1}{2}\right)^n u(n) \longleftrightarrow X_1(z) = \dfrac{1}{1 - \frac{1}{2z}}$

$x_2(n) = \cos(\pi n)\, u(n) = (-1)^n u(n) \longleftrightarrow X_2(n) = \dfrac{1}{1 + \frac{1}{z}}$

$\therefore x(n) = x_1(n) * x_2(n)$

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$\therefore \quad X(z) = X_1(z) \cdot X_2(z)$

$$= \left(\frac{1}{1-\frac{1}{2z}}\right)\left(\frac{1}{1+\frac{1}{z}}\right) \Rightarrow X(z) = \frac{2z^2}{(2z-1)(z+1)}$$

$\therefore$ ROC: $\left|\frac{1}{2z}\right| < 1 \cup \left|\frac{-1}{z}\right| < 1$

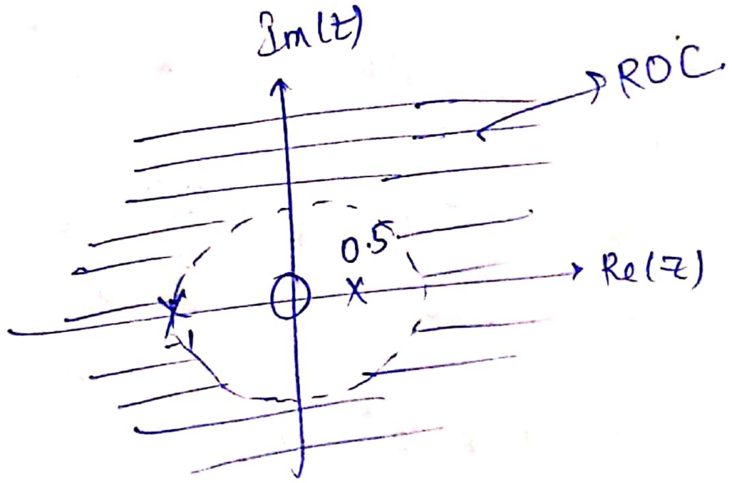
$\Rightarrow |z| > \frac{1}{2} \cup |z| > 1$

$\boxed{\text{ROC}: \ |z| > 1}$

<u>Poles:</u> $z = 1/2$ and $z = -1$

<u>zeros:</u> $\Rightarrow z = 0$

<u>Pole-zero plot:</u>



(1.5) $X(z) = \dfrac{z^{-6} + z^{-7}}{1 - z^{-1}}$

Sol.

Result - II : $x(n) \longleftrightarrow X(z)$

$x(n - n_0) \longleftrightarrow z^{-n_0} X(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$z^{-k} X(z) = z^{-k} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$; let k is an integer

$= \sum_{n=-\infty}^{\infty} x(n) z^{-(n+k)} = \sum_{n=-\infty}^{\infty} x(n+k-k) z^{-(n+k)}$

let substitute $n+k$ with 'm'

$z^{-k} X(z) = \sum_{m=-\infty}^{\infty} x(m-k) z^{-m} = \sum_{m=-\infty}^{\infty} y(m) z^{-m}$.

Here $y(m) = x(m-k)$.

$\therefore \boxed{x(n-k) \longleftrightarrow z^{-k} X(z)}$

Using Result - II :

$X(z) = \dfrac{z^{-6}}{1 - \frac{1}{z}} + z^{-7} \left(\dfrac{1}{1 - \frac{1}{z}} \right)$

we know that $u(n) \longleftrightarrow \dfrac{1}{1 - \frac{1}{z}}$. : since it's causal.

$\therefore$ $u(n-6) \longleftrightarrow z^{-6} \left(\dfrac{1}{1 - 1/z} \right)$

$\therefore x(n) = u(n-6) + u(n-7)$

1.6) $X(z) = \frac{1}{4}\left(\frac{1+6z^{-1}+z^{-2}}{(1-2z^{-1}+2z^{-2})(1-\frac{1}{2}z^{-1})}\right)$

Sol.

Dividing them into partial fractions.

~~$A(1-z^{-1})$~~ $\quad X(z) = \frac{A}{1-\frac{z^{-1}}{2}} + \frac{B+Cz^{-1}}{1-2z^{-1}+2z^{-2}}$

Finding coefficients:

$\Rightarrow A+B = 1/4 \; ; \; -2A + B/_2 + C = 3/_2 \; ;$

$2A - C/_2 = 1/4 \qquad 2A + B/_2 - C = -3/_2$

$A + B = 2A - C/_2 \qquad \Rightarrow \boxed{B = -3/5}$

$B + C/_2 = A$

$A - 3/5 = 1/4$

$\boxed{A = 17/20}$

$-3/5 + C/_2 = 17/20$

$\boxed{C = \frac{29}{20}}$

$\therefore \; X(z) = \frac{17/20}{1-0.5z^{-1}} - \frac{3/5(1-z^{-1})}{1-2z^{-1}+2z^{-2}} + \frac{23/10 \; z^{-1}}{1-2z^{-1}+2z^{-2}}$

Result-3:

$a^n \sin(\omega_0 n)u(n) \leftrightarrow \frac{a\sin\omega_0 \, z^{-1}}{1-2az^{-1}\cos\omega_0 + a^2z^{-2}}$

$a^n \cos(\omega_0 n)u(n) \leftrightarrow \frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$

Proof: $u(n) \leftrightarrow \frac{1}{1-\frac{1}{z}} \; ; \; e^{j\omega_0 n} \leftrightarrow \frac{1}{1-\frac{e^{j\omega_0}}{z}}$

$\therefore \; 2a^n\cos(\omega_0 n)u(n) \leftrightarrow \frac{1}{2}\left(\frac{z}{z-ae^{j\omega_0}} + \frac{z}{z-ae^{-j\omega_0}}\right) = \frac{2z^2-2az\cos(\omega_0)}{z^2+a^2-2az\cos(\omega_0)}$

$\leftrightarrow \frac{1-az^{-1}\cos(\omega_0)}{1+a^2z^{-2}-2az^{-1}\cos(\omega_0)}$

$a^n\sin(\omega_0 n)u(n) \leftrightarrow \frac{1}{2j}\left(\frac{z}{z-ae^{j\omega_0}} - \frac{z}{z-ae^{-j\omega_0}}\right) = \frac{2az\sin(\omega_0)}{z^2+a^2-2az\cos(\omega_0)}$

(8)

$$X(z) = \frac{17/20}{1-0.5z^{-1}} - \frac{3/5\,(1-z^{-1})}{\underbrace{1-2z^{-1}+2z^{-2}}} + \frac{\frac{23}{10}(z^{-1})}{\underbrace{1-2z^{-1}+2z^{-2}}}$$

Using Result - 3 : $\boxed{\begin{matrix} w_0 = \cancel{\pi/2}\ 0 \\ a = 1 \end{matrix}}$ $\quad w_0 = \pi/2,\ a = 1$

$\cancel{x(n) = \frac{17}{20}(0.5)^n u(n) - \frac{3}{5}}$

For Second term : $a\cos w_0 = 1$, $a^2 = 2$ $\Rightarrow$ $\boxed{\begin{matrix} a = \sqrt{2} \quad \cos w_0 = 1/\sqrt{2} \\ w_0 = \pi/4 \end{matrix}}$

Third term : $a\sin w_0 = 1 \Rightarrow a^2 = 2 \Rightarrow \boxed{a = \sqrt{2},\ w_0 = \pi/4}$

$$\therefore \boxed{x(n) = \left[\frac{17}{20}\left(\frac{1}{2}\right)^n - \frac{3}{5}(\sqrt{2})^n \cos\left(\frac{\pi n}{4}\right) + \frac{23}{10}\left(\sin\left(\frac{\pi n}{4}\right)(\sqrt{2})^n\right)\right]u(n)}$$

$\because$ Causal ; $x(n) = 0 ;\ \forall\ n < 0.$

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1.7)
Sol

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})} = \frac{5z^{-1}}{} \frac{A}{1-2z^{-1}} + \frac{B}{3-z^{-1}}$$

$\Rightarrow A(3-z^{-1}) + B(1-2z^{-1}) = 5z^{-1} \Rightarrow$ $3A+B=0 \Rightarrow 6A+2B=0$

$A+2B=-5 \Rightarrow A+2B=-5$

$\boxed{A=1}$ $\boxed{B=-3}$

$\boxed{\therefore A=1 \;;\; B=-3}$

$$X(z) = \frac{1}{1-2z^{-1}} - \frac{3}{3-z^{-1}} \Rightarrow \frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{z^{-1}}{3}}$$

$\therefore$ There are 3 cases; here based on 3 ROC's

(i) $|z|>2$; ~~$|z|>3$~~ $|z|>1/3$

$$X(z) = \frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{z^{-1}}{3}} \Rightarrow (2)^n u(n) - \left(\frac{1}{3}\right)^n u(n).$$

(ii) for $\{|z|<2\} \cup \{|z|>1/3\}$

$$X(z) = \frac{z}{z-2} - \frac{1}{1-\frac{z^{-1}}{3}} \Rightarrow \frac{-z}{1-\frac{z}{2}} - \frac{1}{1-\frac{z^{-1}}{3}}$$

(9)

$$X(z) = \frac{-z/2}{1-\frac{z}{2}} - \frac{1}{1-\frac{z^{-1}}{3}}$$

$$= -\frac{z}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right) - \frac{1}{1-\frac{z^{-1}}{3}}$$

$$= -\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{1-z^{-1}/3}$$

$$= -\sum_{n=-1}^{-\infty}\left(\frac{1}{2}\right)^{-n}(z)^{-n} - \frac{1}{1-z^{-1}/3}$$

$$= -\sum_{n=-\infty}^{\infty} 2^n z^{-n} u(-n-1) - \frac{1}{1-z^{-1}/3}$$

$$\therefore \boxed{x(n) = -2^n u(-n-1) - \left(\frac{1}{3}\right)^n u(n)}$$

* Case-III: $|z| < 1/3$

Like before ; Result-III ; $\frac{1}{1-\alpha z^{-1}} \longleftrightarrow -\alpha^n u[-n-1]$ for $|z| < |\alpha|$

Proved aboved for 1st term in RHS.

$$\therefore X(z) = \frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{z^{-1}}{3}}$$

$$x(n) = -2^n u(-n-1) + \left(\frac{1}{3}\right)^n u(-n-1)$$

$$\boxed{x(n) = \left(\left(\frac{1}{3}\right)^n - 2^n\right) u(-n-1)}$$

1.8)

Sol

$$X(z) = \frac{1 - 2z^{-1} + 2z^{-2} - z^{-3}}{(1-z^{-1})(1-0.5z^{-1})(1-0.2z^{-1})} = \cancel{\frac{A}{1-z^{-1}}} + \cancel{\frac{B}{1-0.5z^{-1}}} + \cancel{\frac{C}{1-0.2z^{-1}}}$$

$\rightarrow$ By root method $\Rightarrow$ $A = 0$, $B =$

$$\Rightarrow \frac{(1-z^{-1})(1-z^{-1}+z^{-2})}{(1-z^{-1})(1-0.5z^{-1})(1-0.2z^{-1})} = \frac{1-z^{-1}+z^{-2}}{(1-0.5z^{-1})(1-0.2z^{-1})}$$

$$X(z) = 10 + \frac{5}{1-0.5z^{-1}} - \frac{14}{1-0.2z^{-1}}$$

$\therefore$ ROC: $|z| > 0.5$

$|z| > 0.2$

$$\boxed{x(n) = 10\delta(n) + 5\left(\frac{1}{2}\right)^n u(n) - 14\left(\frac{1}{5}\right)^n u(n)}$$

$\because$ $|z| > 2$ satisfies both

1.9)

Sol

$$X(z) = \frac{3}{1-\frac{10}{3}z^{-1}+z^{-2}} = \frac{9/8}{z^{-1}-3} - \frac{9/8}{z^{-1}-1/3}$$

$$= 9/8\left(\frac{-1}{3}\left(\frac{1}{1-\frac{z^{-1}}{3}}\right) + \frac{3}{1-3z^{-1}}\right)$$

$$\Rightarrow \frac{27}{8}\left(\frac{1}{1-3z^{-1}}\right) - \left(\frac{3}{8}\right)\left(\frac{1}{1-\frac{z^{-1}}{3}}\right)$$

Here There can be 3 ROC's

i) $|z|>3$, (ii) $|z|<3 \cup |z| > 1/3$ (iii) $|z| < 1/3$

$\because$ Given ROC is $|z|=1$

$\therefore$ we need to consider Region (ii) $|z|<3$ & $|z|>1/3$

$$\therefore X(z) = \frac{27}{8}\left(\frac{1}{1-3z^{-1}}\right) - \frac{3}{8}\left(\frac{1}{1-\frac{z^{-1}}{3}}\right)$$

(11) From Result -3: we know that

$$\frac{1}{1-\alpha z^{-1}} \longleftrightarrow -\alpha^n u[-n-1] \quad ; \text{ for } |z|<|\alpha|$$

∴ This satisfies for 1st term.

$$x(n) = \frac{27}{8}\left(-(3)^n u[-n-1]\right) - \frac{3}{8}\left(\left(\frac{1}{3}\right)^n u(n)\right)$$

$$\boxed{x(n) = -\frac{27}{8}(3)^n u[-n-1) - \frac{3}{8}\left(\frac{1}{3}\right)^n u(n)}$$

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2.1) Solu) * "For a sinusoid signal, location of zeros affects only their phase"

From the Result (3) proved before in prob:(1.2).

$$\cos(\omega_0 n+\phi) \longleftrightarrow \frac{z^2\cos\phi - z\cos(\omega_0-\phi)}{z^2+1-2z\cos(\omega_0)}$$

Consider the zeros, here; $z^2\cos\phi - z\cos(\omega_0-\phi)=0$

$$\Rightarrow z=0, \boxed{z=\frac{\cos(\omega_0-\phi)}{\cos\phi}}$$

For $\sin(\omega_0 n+\phi) \longleftrightarrow \dfrac{z\sin(\omega_0-\phi)+z^2\sin\phi}{z^2+1-2z\cos(\omega_0)}$

Zeros; $z=0$, $z=\dfrac{-\sin(\omega_0-\phi)}{\sin\phi}$.

* Since both the denominator's of sinusoid's signal's z-transform does not have ϕ term,

→ 'ϕ' is present only in numerator, thus present in zeros of z-transform ; while no affect on poles.

→ Thus, the statement is true

∴ By changing phase changes only roots not poles

2.2) "A LTI system is BIBO, stable iff ROC of system function includes unit circle"

Sol.

→ we know that the system is stable iff

$\sum_{n=-\infty}^{\infty}|h(n)| < \infty$, where $h(n)$ is impulse response.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \leq \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

$$\therefore \quad |H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|\,|z^{-n}|$$

If the ROC Includes unit circle $|z| = 1$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| \rightarrow \text{Thus,}$$

We know that $|H(z)| < \infty$; since, for system lies in ROC.

$$\therefore \sum_{n=-\infty}^{\infty} h(n) < \infty \Rightarrow \text{Stable system}$$

⇒ Eg; $h(n) = \left(\frac{1}{2}\right)^n u(n)$ ⇒ $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$

⇒ ∴ Bounded valued output ⇒ BIBO stable

→ If $h(n) = (2)^n u(n)$ ⇒ $\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} (2)^n = \infty$. ⇒ Not stable

$h(n) = \left(\frac{1}{2}\right)^n$

→ $H(z) = \dfrac{1}{1 - \frac{1}{2z}}$

$\left|\frac{1}{2z}\right| < 1$

$\boxed{z > \frac{1}{2}}$

∴ since $|z| = 1$ is included it's stable

$h(n) = (2)^n$

$H(z) = \dfrac{1}{1 - \frac{2}{z}}$

$\frac{2}{z} < 1 \Rightarrow \boxed{z > 2}$

Since, $|z| > 2$ does not contain $|z| = 1$, ⇒ we can say that it's not stable

30)

3.1) Sol.

$$y(n) = 0.5\, y(n-1) + x(n)$$

$$x(n) = 10 \cos\left(\frac{\pi n}{4}\right) u(n)$$

z-transform: $Y(z) = 0.5 z^{-1} Y(z) + X(z)$

$\Rightarrow$ $Y(z)\left(1 - 0.5 z^{-1}\right) = X(z)$

$$H(z) = \frac{1}{1 - 0.5 z^{-1}}$$

As proved earlier in prob: 1.2: Result-II: 3:

$$\cos(w_0 n)\, u(n) \longleftrightarrow \cancel{z^2 - z\cos(w_0 n)} \quad \frac{1 - a z^{-1}\cos(w_0)}{1 - 2z^{-1}\cos(w_0) + z^{-2}}$$

$\therefore$ $Y(z) = H(z) \cdot X(z)$, $w_0 = \pi/4 \Rightarrow \cos(w_0) = 1/\sqrt{2}$

$$= \left(\frac{1}{1 - 0.5 z^{-1}}\right) \cdot \left(\frac{1 - z^{-1}/\sqrt{2}}{1 - \sqrt{2} z^{-1} + z^{-2}}\right)$$

$$= \frac{A}{1 - 0.5 z^{-1}} + \frac{B z^{-1} + C}{1 - \sqrt{2} z^{-1} + z^{-2}}$$

By solving in ~~Equa~~ MATLAB:

$$Y(z) = \frac{-1.86}{1 - 0.5 z^{-1}} + \frac{6.78 - 29.09}{1 - e^{j\pi/4} z^{-1}} +$$

$$Y(z) = \frac{A}{1 - 0.5 z^{-1}} + \frac{B}{1 - e^{j\pi/4} z^{-1}} + \frac{C}{1 - e^{-j\pi/4} z^{-1}}$$

Using partial fractions:

$\Rightarrow$ $A + B + C = 10$

$A + \frac{B}{2} e^{-j\pi/4} + \frac{C}{2} e^{j\pi/4} = 0$

$B e^{-j\pi/4} + C e^{j\pi/4} = -2A$

$A = -1.86$

$B = 5.93 - 3.3j$

$C = 5.93 + 3.3j$

$\therefore\ Y(z) = \dfrac{-1.86}{1-0.5z^{-1}} + \dfrac{6.78\,z\angle{-29.09}}{1-e^{j\pi/4}z^{-1}} + \dfrac{6.78\,\angle{29.09}}{1-e^{-j\pi/4}z^{-1}}$

$\therefore\ Y(z) = \dfrac{-1.86}{1-0.5z^{-1}} + (13.56)\left(\dfrac{\cos(29.09) - z\cos(\frac{\pi}{4}-29.09)}{z^2+1-2z\cos(\pi/4)}\right)$

$$\boxed{y(n) = -1.86\left(\frac{1}{2}\right)^n + 13.56\cdot\cos\left(\frac{\pi n}{4} - 29.09^\circ\right)u(n)}$$

$\because$ We got a sinusoid ~~in~~ and and exponential term in o/p.

transient: Which ~~over~~ gets to zero as ~~t~~ $n \to \infty$

$$\boxed{y_t(n) = -1.86\left(\frac{1}{2}\right)^n}$$

Steady state:- Which lasts at ~~t~~ $n \to \infty$

$$\boxed{y_s(n) = 13.56\cos\left(\frac{\pi n}{4} - 29.09^\circ\right)u(n)}$$

3) (3.2) $y(n) = 2.5\,y(n-1) - y(n-2) + x(n) - 5x(n-1) + 6x(n-2)$

Sol.

Sol.

Converting into z-transform.

$$Y(z) = 2.5\,z^{-1}Y(z) - z^{-2}Y(z) + X(z) - 5z^{-1}X(z) + 6z^{-2}X(z)$$

$$Y(z)\left[1 - 2.5z^{-1} - z^{-2}\right] = X(z)\left(1 - 5z^{-1} + 6z^{-2}\right)$$

For unit sample response $x(n) = \delta(n) \leftrightarrow X(z) = 1$

$$\therefore \quad H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} - z^{-2}} = 6 + \frac{10z^{-1} + 5}{z^{-2} - 2.5z^{-1} + 1}$$

$$= 6 + \frac{5(2z^{-1} - 1)}{(2z^{-1} - 1)\left(\frac{1}{2}z^{-1} - 1\right)} = 6 + \frac{5}{\frac{1}{2}z^{-1} - 1} = 6 - \frac{5}{1 - \frac{1}{2}z^{-1}}$$

$$H(z) = 6 - \frac{5}{1 - \frac{z^{-1}}{2}}$$

Two ROC's: $|z| > \frac{1}{2} \Rightarrow \boxed{h(n) = 6\delta(n) - 5\left(\frac{1}{2}\right)^n u(n)}$

for $|z| < \frac{1}{2} \Rightarrow h(n) = 6\delta(n) + 5\left(\frac{1}{2}\right)^n u(-n-1)$

QUES 4)

UNDERSTANDING THE PRACTICAL USE OF Z-TRANSFORMS.

- The practical use of z transforms is that we can make a different type of system by just varying its poles and zeros in the z domain.
- These changes affect the relationship between the input and output in the discrete-time domain and this can be easily computed in a computer. Thus proving the essence of DSP.
- For example, I implemented a low pass filter, which lowers the amplitude of high-frequency waveforms while barely affecting the low-frequency components.
- I came up with the filter's frequency response in the z domain just by knowing its zeros and poles.
- Here if we consider the normalized frequency which ranges between $[0, \pi]$ in the circular domain.
- Here 0 refers to the low-frequency components while π referring to high-frequency ones.
- Let z be the point moving along the unit circle $|z| = 1$, where frequency refer to the angle of a point .

- So we need to lower the strength of the components as we approach π i.e z = -1. Therefore z =-1 is the zero of the impulse response of the filter.
- While z = 0 can be the pole of it, since it does not contribute to the amplitude of the transfer function as the distance between the moving point and origin is always unity.
- Therefore the impulse response is $H(z) = (z+1) \div 2z$
- This gives us the relation of y[n] = (x[n] + x[n-1])/2
  Which is nothing but a moving average operation.
  The amplitude of the impulse is nothing but the euclidean distance between z on the unit circle and the poles and the zeros.

❖ AN INTUITIVE EXPLANATION OF WHY DOES A MOVING AVERAGE SYSTEM WOULD WORK?

ANS : let us take x[n] = [1,100,1,100,...]
This can be decomposed into [50,50,50,50...] ***(low freq component***)+[-49,50,-49,50...] ***(high freq component***) since its changing more frequently.
If we apply a moving average operation to it.
We see low freq = [50,50,50,50]

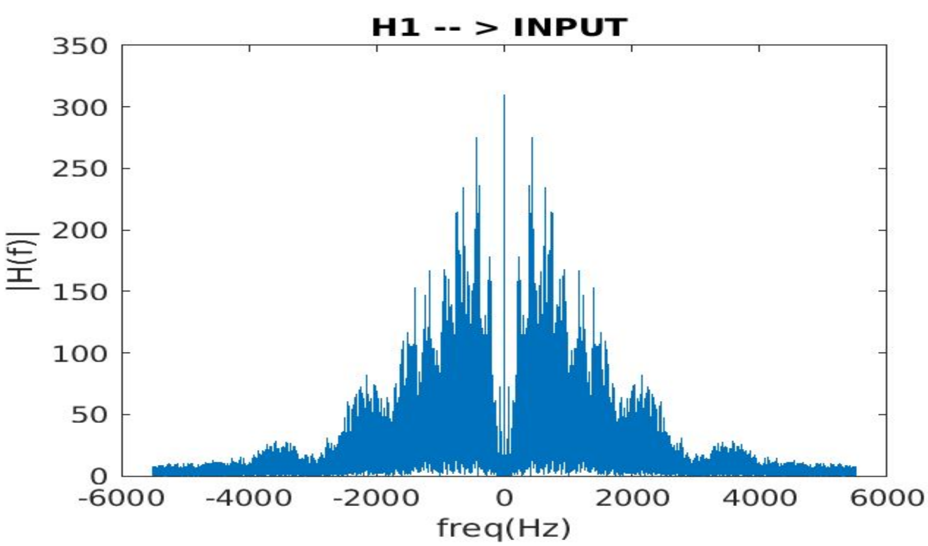
But high freq = [0.5,0.5,0.5,0.5]components are reduced in magnitude and now are barely changing .
Thus we can see that the moving average operation lowers the magnitude of the high-frequency components.

- **SIMULATION RESULTS :**

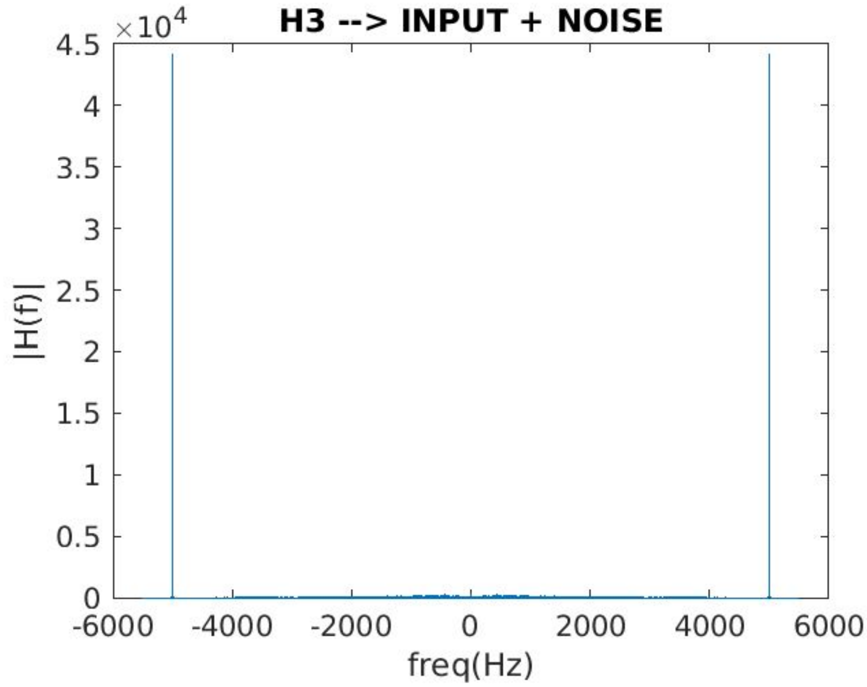
  I took a .wav file (sampled at Fs = 11025 Hz) from the internet, added a sinusoid noise $cos(2 * pi * 5000 * t)$ and tried to pass it through the moving average system and the notch filter. The below are the results .

- For notch filter :

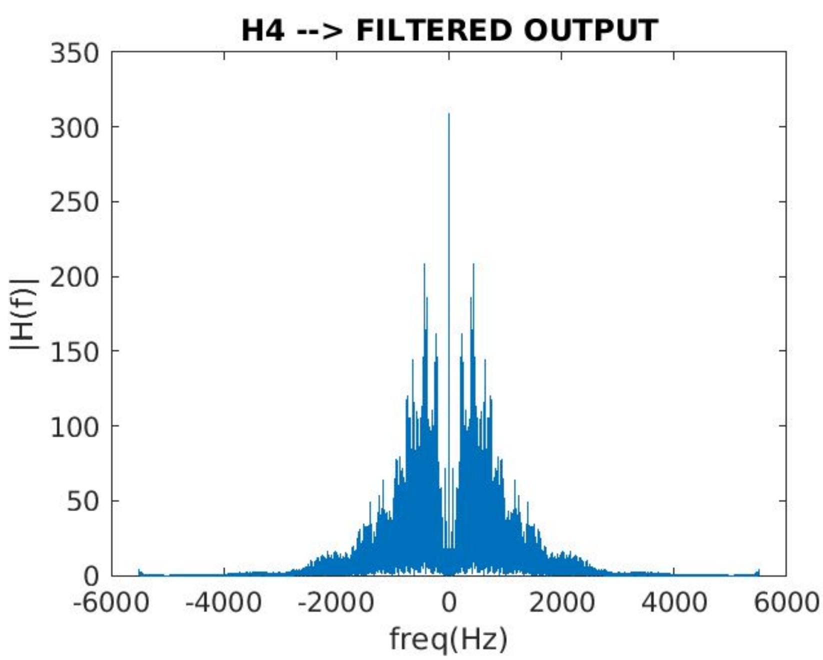
  1) Input frequency spectrum.



2) Input + noise frequency spectrum



Filtered outputs frequency spectrum.

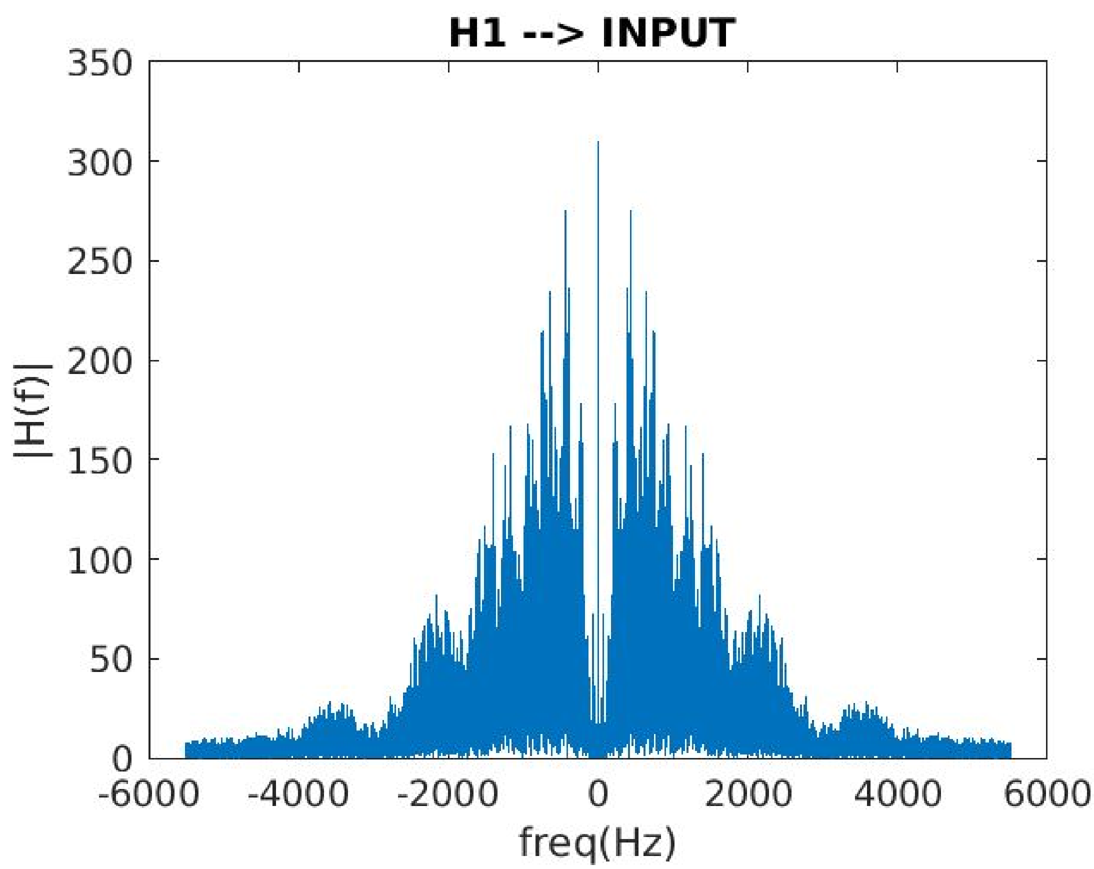


OBSERVATIONS :

We can see that when we use the notch filter we could remove the added noise at 5000Hz completely. We can observe the spike at 5000Hz in the input + noise fft completely vanish when passed through the filter.

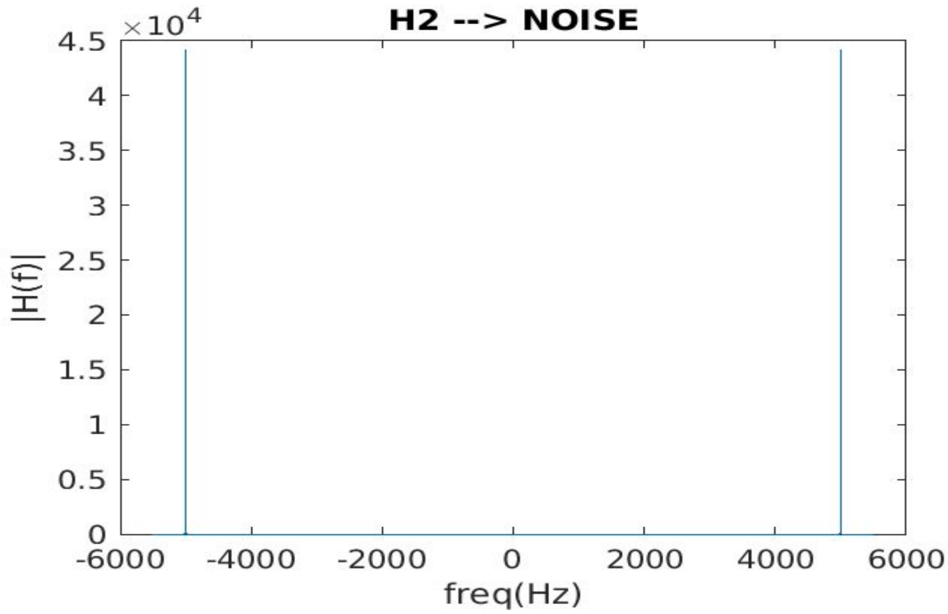
- For Moving Average System :

1) Input frequency spectrum.

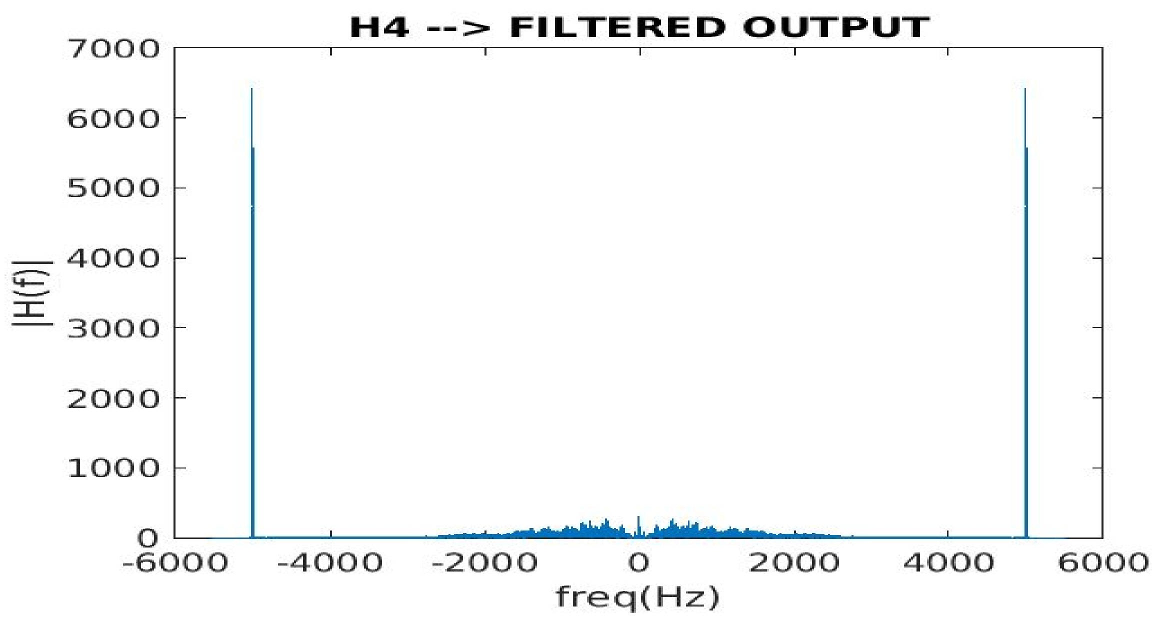


-

2) With input + noise (The input spectrum is not visible due to high magnitude of sinusoids fft)

➢ 

➢ 3) Output after applying the moving avereage operation over discrete time domain.



OBSERVATIONS :

Although the operation did not completely wipe out the noise, it lowered the amplitude of the noise from 450000 to 7000 which is great filtering in terms of frequency.