Ay. F

(1)

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$$Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y = f(x)$$

over the range $a \leq x \leq b$ subject to homogeneous boundary condition at $x = a$ and $x = b$.

Our Green's function needs to satisfy the b.c

$$L\,G(x,t) = \delta(x-t)$$

so that $y(x)$ the soalution

$$y(x) = \int_a^b G(x,t)\, f(t)\, dt$$

$$L\,y(x) = \int_a^b L\,G(x,t)\, f(t)\, dt = \int_a^b \delta(x-t)\, f(t)\, dt$$

$$= \underline{f(x)}.$$

We continue with the study of problems on an interval (a,b) with one homogeneous boundary condition at ~~each~~ end point of the interval.

Given a value of t, it is necessary for x in the range $a \leq x < t$ that $G(x,t)$ have an x dependence $y_1(x)$ that is a solution to the homogeneous eqn $L = 0$ and that also satisfies the boundary cond. at $x = a$

the most general $G(x,t)$ satisfying these condition must have the form

$$G(x,t) = y_1(x)\, h_1(t) \qquad x < t$$

~~conver~~ conversely $t \leq x \leq b$

$$G(x,t) = y_2(x)\, h_2(t) \qquad x > t$$

where y_2 is a solution of $L = 0$ that satisfies the boundary condition at $x = b$.

$$G(x,t) = G(t,x)^*$$

from this we get $h_2^* = A y_1$, $h_1^* = A y_2$

A is a constant to be determined.

$$G(x,t) = \begin{cases} A\, y_1(x)\, y_2(t) & x < t \\ A\, y_2(x)\, y_1(t) & x > t \end{cases}$$

A is determined by

$$A\left[y_2'(t)\, y_1(t) - y_1'(t)\, y_2(t) \right] = \frac{1}{p(t)}.$$

$$y(x) = A\, y_2(x) \int_a^x y_1(t)\, f(t)\, dt + A\, y_1(x) \int_x^b y_2(t) f(t)\, dt$$

Example

Consider the ODE $-y'' = f(x)$

with boundary condition $y(0) = 0 = y(1)$.

the corresponding homogeneous eq $-y'' = 0$

has general solution $y_0 = c_0 + c_1 x$

$y_1 = x$ that satisfied $y_1(0) = 0$.

$y_2 = 1 - x$ —— $y_2(1) = 0$

For this ODE, the co-efficient $p(t) = -1$

$y_1'(x) = 1$, $y_2'(x) = -1$,

$$A = [(-1)[(-1)(x) - (1)(1-x)]]^{-1} = 1.$$

Our Green's function is therefore

$$G(x,t) = \begin{cases} x(1-t) & 0 \le x < t \\ t(1-x) & t < x \le 1. \end{cases}$$

let $f(x) = \sin \pi x$,

$$y_p(x) = \int_0^1 G(x,t) \sin \pi t \, dt$$

$$= (1-x)\int_0^x t \sin \pi t \, dt + x \int_x^1 (1-t) \sin \pi t \, dt$$

$$= \frac{1}{\pi^2} \sin \pi x$$

10.1.2 $\mathcal{L}y = \frac{d^2y}{dx^2} + y = f(x)$.

with the initial conditions $y(0)=0$ and $y'(0)=0$

this operator $\mathcal{L}$ has $p(x)=1$.

We start by noting the homogeneous eq^n $\mathcal{L}y=0$ has two linearly independent solutions

$y_1 = \sin x$ and $y_2 = \cos x$.

However the only linear combination that satisfies the b.c at $x=0$ is $y=0$, so

$G(x,t)=0$ for $x<t$,

for $x>t$ there are no boundary condition

so $G(x,t) = c_1(t)\sin x + c_2(t)\cos x$ $x>t$

Impose the requirement

$$G(t_-,t) = G(t_+,t)$$

$$\Rightarrow 0 = c_1(t)\sin t + c_2(t)\cos t$$

$$\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} = 1$$

$$c_1(t)\cos t - c_2(t)\sin t = 1.$$

$$\Rightarrow c_1(t) = \cos t, \quad c_2(t) = -\sin t$$

$$G(x,t) = \cos t\sin x - \sin t\cos x = \sin(x-t) \text{ for } x>t$$

So

$$G(x,t) = \begin{cases} 0 & x < t \\ \sin(x-t) & x > t \end{cases}$$

$$y(x) = \int_0^{\infty} G(x,t) f(t)\, dt$$

$$= \int_0^{x} \sin(x-t) f(t)\, dt$$

10.1.3 $\left(\frac{d^2}{dx^2} + k^2\right) \psi(x) = g(x)$,

the general solution with $g=0$ is spanned by two function: $y_1 = e^{-ikx}$ and $y_2 = e^{ikx}$

for large positive x, we must have solution y_2, while for large negative x the solution is y_1

$$G(x,x') = \begin{cases} A\, y_1(x')\, y_2(x) & x > x' \\ A\, y_2(x')\, y_1(x) & x < x' \end{cases}$$

$p(x) = 1$. $A = \dfrac{1}{y_2'(x) y_1(x) - y_1'(x) y_2(x)} = -\dfrac{i}{2k}$.

$$G(x,t) = -\frac{i}{2k} \exp\left(i|x-x'|\right)$$

Legendre Polynomial

15.1.1 Derive the Legendre ODE by manipulating the Legendre polynomial recurrence relations

$\Rightarrow$ $(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$

$\Rightarrow$ differentiating

$$(1-x^2)P_n''(x) - 2xP_n'(x) = nP_{n-1}'(x) - nP_n(x) - nxP_n'(x).$$

Now we also have $P_{n-1}'(x) = -nP_n(x) + xP_n'(x)$,

replacing $P_{n-1}'(x)$, we get the required term

15.1.2 Derive the following closed formula for the Legendre polynomial $P_n(x)$,

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

Where $[n/2]$ stands for the integer part of $n/2$.

$\Rightarrow$ Expand $(1-2xt+t^2)^n$ in

$$g(x,t) = \sum_{n=0}^{\infty} \binom{-1/2}{n} \sum_{j=0}^{n} \binom{n}{j} t^{2j} (-2xt)^{n-j}$$

Now change the summation variable n to $m = n+j$ the range of m will be from zero to infinity but the range of j will now only include

~~no larger than m/2.~~

(8)

15.1.4 The shifted Legendre polynomials, designated by the symbol $P_n^*(x)$ are orthogonal with unit weight $[0,1]$, with normalization integral

$\langle P_n^* | P_n^* \rangle = 1/2n+1$,

(a) find the recurrance relation satisfied by $P_n^*(x)$,

(b) show that all the co-efficients of the P_n^* are integers.

$\Rightarrow$ If we set $P_n^*(x) = P(y)$ where $y = 2x-1$, then

$$\int_0^1 P_n^*(x) P_m^*(x)\, dx = \frac{1}{2}\int_{-1}^{1} P_n(y) P_m(y)\, dy = \frac{1}{2}\,\frac{2}{2n+1}\,\delta_{nm}$$

this equation confirms the orthogonality and normalization of the $P_n^*(x)$.

(a) Replacing x by $2x-1$ in the P_n recurrance formula, we find,

$$(n+1)\, P_{n+1}^*(x) - (2n+1)(2x-1)\, P_n^*(x) + n P_{n-1}^*(x) = 0,$$

(b) By examination of the first few P_n^*, we guess that they are given by the general formula

$$P_n^*(x) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} x^k.$$

this formula is easily proved by mathematical induction, using the recurrance formula

$$P^*_{n+1}(x) = \frac{1}{n+1}(2n+1)(2x-1)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}x^k$$

$$-\frac{n}{n+1}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k-1}{k}(-1)^{n-1-k}x^k.$$

15.1.6 By differentiating the generating function $g(x,t)$ w.r.t t, multiplying by $2t$, and then adding $g(x,t)$, show that

$$\frac{1-t^2}{(1-2tx+t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n$$

$$\Rightarrow\quad 2t\frac{\partial g(x,t)}{\partial t} + g = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}$$

$$= \sum_{n=0}^{\infty}\left(2n P_n t^n + P_n t^n\right)$$

$$= \sum_{n=0}^{\infty}(2n+1)P_n t^n$$

15.1.7 (a) Derive $(1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x)$.

$\Rightarrow$ We have $nP_{n-1}(x) = (2n+1)xP_n(x) - (n+1)P_{n+1}(x)$ --- (1)

and $(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$ --- (2)

put $nP_{n-1}(x)$ ~~from~~ from (1) into (2), we get

$(1-x^2)P_n'(x) = (2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nxP_n(x)$

$= (n+1)P_n(x) - (n+1)P_{n+1}(x)$,

15.1.8 Prove that $P_n'(1) = \frac{d}{dx}P_n(x)\Big|_{x=1} = \frac{1}{2}n(n+1)$.

$\Rightarrow$ For $n=1$, we establish $P_1'(1) = 1\cdot 2/2 = 1$ as the first step of a proof by mathematical induction. Now assuming $P_n'(1) = \frac{n(n+1)}{2}$ and using eq^n. $P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x)$,

$P_{n+1}'(1) = (n+1)P_n(1) + P_n'(1)$

$= (n+1) + \frac{n}{2}(n+1) = \frac{1}{2}(n+1)(n+2)$,

which proves our assumed formula for $n+1$

15.19 Show that $P_n(\cos\theta) = (-1)^n P_n(-\cos\theta)$ by use of the recurrence relation relating P_n, P_{n+1} and P_{n-1} and your knowledge of P_0 and P_1.

(2)

→ For a proof by mathematical induction we start by verifying that $P_0(-x) = P_0(x) = 1$ and then $P_1(-x) = -P_1(x) = -x$, We need to show that if $P_m(-x) = (-1)^m P_m(x)$ for $m = n-1$ and $m = n$, the relationship holds for $m = n+1$,

Now we have,–

$$(2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \quad — (1)$$

x replaced by $(-x)$,–

$$-(2n+1)x P_n(-x) = (n+1) P_{n+1}(-x) + n P_{n-1}(-x) \quad — (2)$$

$$(-1)^{n+1}(2n+1)x P_n(x) = (n+1) P_{n+1}(-x) + (-1)^{n-1} n P_{n-1}(x) \quad — (2)$$

Here relationship for P_n and P_{n-1} are used.

Comparing (2) with (1), we get

$$P_{n+1}(-x) = (-1)^{n+1} P_{n+1}(x),$$

15.1.14. Show that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$ when $m<n$

$\Rightarrow$ $x^m = \sum_{l \leq m} a_l P_l(x)$.

Now orthogonality gives $\int_{-1}^{1} x^m P_n(x)\,dx = 0$

$m<n$

15.1.15 Show that $\int_{-1}^{1} x^n P_n(x)\,dx = \frac{2n!}{(2n+1)!!}$

$\Rightarrow$ following the directions in the exercise to use Rodrigues formula and perform integration by parts,

$$F_n = \int_{-1}^{1} x^n P_n(x) = \frac{1}{2^n n!} \int_{-1}^{1} x^n \left(\frac{d}{dx}\right)^n (x^2-1)^n dx$$

$$= \frac{1}{2^n n!} \left[\cancel{x^n \left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n \Big|_{-1}^{1}}^{\,0} - \int_{-1}^{1} n x^{n-1} \left(\frac{d}{dx}\right)^{n-1} (x^2-1)^n dx \right.$$

A second integration by parts yields

$$F_n = \frac{1}{2^n n!} \int_{-1}^{1} n(n-1)\, x^{n-2} \left(\frac{d}{dx}\right)^{n-2} (x^2-1)^n dx \quad \left[\text{first term vanish}\right]$$

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Further integrations by parts until the differentiation within the integral has been completely removed lead to,-

$$F_n = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} n!\,(x^2-1)^n dx$$

$$= 2^{-n}\left[\underbrace{2\int_0^1 (1-x^2)^n dx}_{\text{Beta function}}\right]$$

$$B\left(\tfrac{1}{2}, n+1\right)$$

$$= \frac{2^{n+1}\, n!}{(2n+1)!!}$$

(5.2.) Using Rodrigues formula, show that $P_n(x)$ are orthogonal and that,-

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}.$$

$\Rightarrow$ for ~~p~~ $n > m$

$$\int_{-1}^{1} P_m P_n\, dx = \frac{1}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\left(\frac{d}{dx}\right)^m (x^2-1)^m \left(\frac{d}{dx}\right)^n (x^2-1)^n dx$$

$$= 0$$

because $\left(\frac{d}{dx}\right)^{m+n}(x^2-1)^m = 0$

for $n=m$, the repeated integrated by parts yields

$$\frac{(-1)^n}{2^{2n} n! n!} \int_{-1}^{1} (x^2-1)^n \left(\frac{d}{dx}\right)^{2n} (x^2-1)^n \, dx = \frac{(2n)!}{2^{2n} n! n!} \int_{-1}^{1} (1-x^2)^n \, dx \;;$$

$$= \frac{2n!}{2^{2n} n! n!} B\left(\tfrac{1}{2}, n+1\right) = \frac{2n!}{2^{2n} n! n!} \frac{2^{n+1} n!}{(2n+1)!!}$$

$$= \frac{2}{2n+1}$$

15.2-4 Verify the Dirac delta function expansion

⇒ ~~Insert the expansion to be~~

$$\delta(1-x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x),$$

$$\delta(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{2} P_n(x).$$

⇒ Insert the expansion to be verified, and then note that the expansion of f(x) in Legendre polynomials takes the form,

$$f(x) = \sum_n a_n P_n(x) \quad , \quad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx$$

$$\int_{-1}^{1} f(x) \delta(1-x) \, dx = \sum_{n=0}^{\infty} \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx$$

$$= \sum_{n=0}^{\infty} a_n .$$

$$= \sum_{n=0}^{\infty} a_n P_n(1) = f(1).$$

$$\int_{-1}^{1} f(x)\,\delta(1+x)\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx$$

(5)

$$= \sum_{n=0}^{\infty} (-1)^n a_n .$$

$$= \sum_{n=0}^{\infty} a_n P_n(-1) = \underline{f(-1)}$$

<u>15.25</u> Neutrons are being scattered by a nucleus of mass A (A>1). In the centre-of mass system the scattering is isotropic, then in the laboratory system the average of the cosine of ~~as~~ the angle of deflection of the neutron is

$$\langle \cos\psi \rangle = \frac{1}{2} \int_0^{\pi} \frac{A\cos\theta + 1}{(A^2 + 2A\cos\theta + 1)^{1/2}} \sin\theta\, d\theta$$

Show by expansion of the denominator, that

$$\langle \cos\psi \rangle = 2/3A$$

$$\Rightarrow (A^2 + 2A\cos\theta + 1)^{-1/2} = \frac{1}{A}\left(1 + \frac{2}{A}\cos\theta + \frac{1}{A^2}\right)^{-1/2}$$

$$= \frac{1}{A} \sum_{n=0}^{\infty} P_n(\cos\theta)\left(-\frac{1}{A}\right)^n ;$$

$$\langle \cos\psi \rangle = \frac{1}{2A} \sum_{n=0}^{\infty} \left(-\frac{1}{A}\right)^n \int_0^{\pi} (A\cos\theta + 1) P_n(\cos\theta) \sin\theta\, d\theta$$

$$= \frac{1}{2A} \cdot \left(2 - \frac{2}{3}\right) = \frac{2}{3A} ,$$