N: size of population = 1500

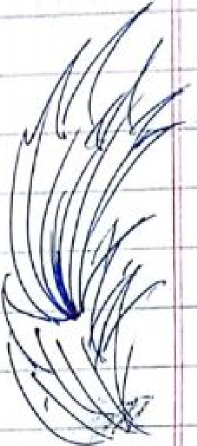
n = size of sample = 36

σ = 48 $\quad$ N_s = no. of samples = 300

μ = 22.4

② The mean height of college students is 155, and S.D = 15. Probability that mean height of 36 students is less than 157 cm is $\underline{0.7881}$ (answer?)

③ A random sample of size 64, is taken from a normal population with mean $\mu = 51.4$ & $\sigma = 68$. What is the probability that the mean of sample will



a) Exceed 52.9
b) Fall b/w 50.5 & 52.3
c) Be less than 50.6

✓ (mark incount)

0.5 ~~34~~ 0.~~1~~392

0.7108

0.1736

$P(\bar{X} > 52.9)$

$= 1 - P(\bar{X} < 52.9)$

$\frac{52.9 - 51.4}{\sqrt{8}}$

$1 - P(Z < 0.176)$ $\frac{1.5}{\cancel{7}\,6.7}$

$1 - (0.5 - 1 - (0.5 + 0.25)$

0.242

25/2/19

saturday
↓
Tuesday

Random Process
(or)
Stochastic Process

A random process is defined as follows:

$X(t, s)$ where t is the time parameter & s is an outcome.

(i.e. Initially we studied random variables $X(s)$ for s as an outcome in the overall sample space)

When t is fixed & s varies, X becomes a random variable.

i.e. $\{X(t)$ for a value of s.

Taking all X(t) for every s in S, we get a random process.

If t, s are both fixed, you get a number.

If s is fixed and t is varied, X(t, s) is a function of time.

Thus, a random process is defined as a collection of random variables

$$\{ X(t) \;/\; t \in (0, T)$$

Assume a sample space S, say

S = { set of all messages }

where s is a message.

Let N(t, s) be the noise factor which is probabilistic, that is added to the signal

X(t, s) is a r.v representing a number which associates a message with time.

∴ Your random process

$$R(t, s) = X(t, s) + N(t, s)$$

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② In a.c current, the voltage is

$$x(t) = \gamma \cos(\omega t + \phi)$$

where γ = amplitude, ϕ = phase

This can also be a random process.

③ Brownian motion is another example, where collision of molecules can be used as a factor.

Classification:

① Continuous time process

② Discrete time process denotable by a
↳ random sequence

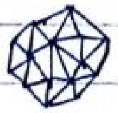
Note:
$x(t,s) \rightarrow$ state

State space = $\left\{ x(t,s) \;\middle|\; \begin{array}{l} t \in (0,T] \\ s \text{ is outcome} \end{array} \right\}$

Characterisation
↳ How to represent a process

Let $\{ x(t) \,/\, t \in (0,T] \}$ be a random process.

Depending on range of t, we can either have disc. or cont. process.



And depending on X, we can define type of distributions.

Any random process can be characterized by joint cumulative distribution function.

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(cumulative)
Distribution function for $X(t)$

$F_X(x) = P(X \le x)$ normally and
$F_X(x,t) = P(X(t) \le x)$ when time is in a varying state

For convenience, t can be assumed discrete here.

Now, let

$$F_X(x, t_2) = P(X(t_2) \le x)$$
$$\vdots$$
$$F_x(x, t_n) = P(X(t_n) \le x)$$

such that $0 < t_1 < t_2 \dots < t_n$
↓
here, $t_1 = t$

Then joint cumulative distribution function

$$\boxed{F(x_1, x_2 \dots x_n, t_1, t_2 \dots t_n) = P[X(t_1) \le x_1, X(t_2) \le x_2 \dots X(t_n) \le x_n]}$$

Get pdf using n^{th} order partial derivative:

If $X(t)$ is a cont. process, then $X(t)$ can be specified by joint pdf

$$f(x_1, x_2, x_3 \dots x_n, t_1, t_2 \dots t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \dots \partial x_n} F(x_1, x_2 \dots x_n, t_1, t_2 \dots t_n)$$

If $X(t)$ is discrete, the process is more tedious.

Example:

① In a fair coin experiment, we define the process x(t) as follows:

$$x(t) = \begin{matrix} \sin \pi t & \text{if heads show} \\ 2t & \text{if tails show} \end{matrix}$$

Find:

i) $E(x(t))$

ii) Find $F(X,t)$ for $t = 0.25$, $t = 0.5$ & $t = 1$

Random process:

$$\{x(t) \,/\, t = 0.25, 0.5 \,\&\, 1\}$$

$$= \{x(0.25), x(0.5), x(1)\}$$

Now, the experiment here is tossing a fair coin.

$$x(0.25) = \begin{cases} \overset{\nearrow 1/\sqrt{2}}{\sin \pi/4} & \text{if head} \\ \underset{\hookrightarrow 1/2}{0.5} & \text{if tail} \end{cases}$$

$$x(0.5) = \begin{matrix} \sin \overset{\nearrow 1}{\pi/2} & \text{if head} \\ 1 & \text{if tail} \end{matrix}$$

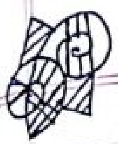
$$x(1) = \begin{matrix} \sin \overset{\nearrow 0}{\pi} & \text{if head} \\ 2 & \text{if tail} \end{matrix}$$

$$E(x(t)) = \sum_{x} x \cdot f(x,t)$$

$$= \underset{\substack{\downarrow \\ \text{prob of} \\ \text{getting head}}}{\sin \pi t \times \frac{1}{2}} + \underset{\substack{\downarrow \\ \text{prob of} \\ \text{getting tail}}}{2t \times \frac{1}{2}} = \frac{1}{2} \sin \pi t + t$$

? Check

Date 26/2/19
Page



Pmf : Now, $X(t)$ is a random var on sample space $\{H, T\} \to \mathbb{R}$

=> Pmf is defined as follows.

x	$\sin \pi t$	$2t$
$P[X(t)=x]$	$1/2$	$1/2$

P(getting head) P(getting tail)

Hence we have defined $E(X(t)) = \sum x P(X(t)=x)$

$$= \frac{1}{2} \times \sin \pi t + \frac{1}{2} \times 2t$$

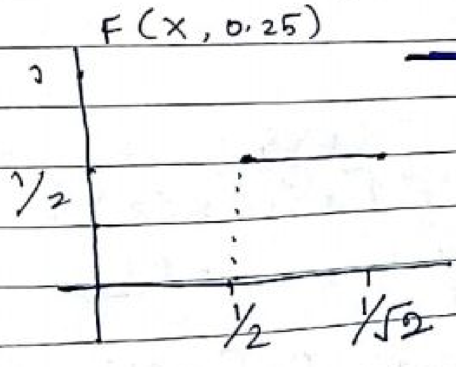
$$= \frac{\sin \pi t}{2} + t$$

ii) $F(X, t)$: For $t = 0.25$

$$F(x, t=0.25) = P(X(0.25) \leq x)$$

From our previous value of $X(0.25)$,

$X(0.25, H) = \frac{1}{\sqrt{2}}$ $X(0.25, T) = \frac{1}{2}$

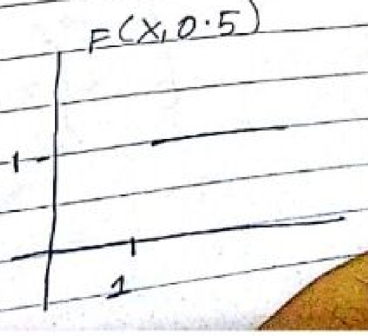


Pmf :

x	~~[illegible]~~ $1/2$	~~$1/2$~~ $1/\sqrt{2}$
$P(X(0.25)=x)$	$1/2$	$1/2$
$F(X, 0.25)$	$1/2$	1

Similarly,

	x	1	1
~~t~~ $=0.5$	$P(X(0.25))$	$1/2$	$1/2$
	$F(X,t)$	$1/2$	1



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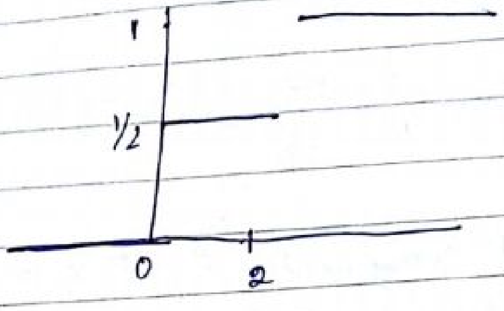
		heads	
$X(1)$	x :	0	2
	$p(x)$:	$1/2$	$1/2$
	$F(x)$:	$1/2$	1



- Let $X(t),\ t \in (0,T)$ be a random process.

Mean of $X(t)$

$$\mu_X(t) = E(X(t))$$

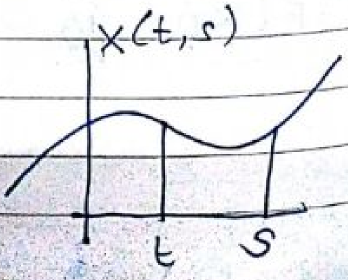
$$= \begin{cases} \int_{-\infty}^{\infty} x f(x,t)\, dx & x(t) \text{ is cont.} \\ \sum_{-\infty}^{\infty} x\, P[X(t)=x] & x(t) \text{ is discrete} \end{cases}$$

Autocorrelation

$$R_{XX}(\overset{t_1}{t}, s) = E[X(\overset{t_1}{t})\, X(\overset{t_2}{s})]$$

↳ 2 diff. time variables

$$= \begin{cases} \sum\sum x_1 x_2\, P[X(t)=x_1,\ X(s)=x_2] & \text{for discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f(x_1, x_2, t, s)\, dx_1 dx_2 \end{cases}$$



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Average power:

$$R(t,t) = E[X^2(t)]$$

↳ expectation of diagonal values

Auto covariance:

$$C_{XX}(t_1,t_2) = \operatorname{cov}(X(t_1), X(t_2))$$

$$= R_{XX}(t_1,t_2) - E(X(t_1))\,E(X(t_2))$$

↳ $\mu_X(t_1)$

↳ $\mu_X(t_2)$

$$= R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

$$E(XY) - E(X)\cdot E(Y)$$

Example:

Suppose $X(t)$ is a process with $\mu_X(t) = 3$

$R_{XX}(t_1,t_2) = 9 + 4e^{-0.2|t_1-t_2|}$

Find mean, var & covariance of r.v

$Z = X(5)$ and $W = X(8)$

$$\text{Var} = E[X^2(t)] - (E(X(t)))^2$$

$$= \cancel{9} + 4e^{-0.2|t_1-t_2|} - \cancel{9}$$

$$= 4e^{-0.2|t_1-t_2|} = 4 \quad (t_1, t_2 = 5)$$

Similarly, Var(W) $= 4$

$E(X) = E(X(5)) = 3$

$E(W) = E(X(8)) = 3$

For Covariance (Z,W) $= R_{XX}(5,8) - E(X(5))\,E(X(8))$

$$= ~~9+4e^{0}~~ -$$

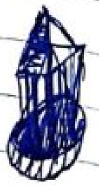
$$\cancel{9} + 4e^{-0.2|3|} - \cancel{9}$$

$$= 4e^{-0.6} = 2.195$$

Gypsy Caravan

Determine the autocorrelation of
$X(t) = r\cos(\omega t + \phi)$

r, ϕ are independent r. vs
ϕ is uniform in $(-\pi, \pi)$



$R(t_1, t_2) = E[X(t_1) * X(t_2)]$

$= E[r\cos(\omega t_1 + \phi)\, r\cos(\omega t_2 + \phi)]$

$= [illegible]\; E[r^2 \cdot \frac{1}{2}(\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1+t_2) + 2\phi))]$

$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$

$= \frac{1}{2} E[r^2]\, E[\cos\omega(t_1 - t_2) + \cos\omega(t_1+t_2) + 2\phi]$

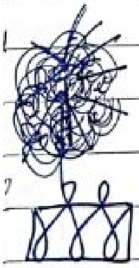
since r, ϕ are independent

$= \frac{1}{2} E[r^2] \times (\underbrace{\cos\omega(t_1 - t_2)}_{\text{constant value}} + 0)$

$0 \downarrow E(\underbrace{\cos\omega(t_1+t_2) + 2\phi}_{0})$

~~since~~

$= \frac{1}{2} E[r^2] \times \cos\omega(t_1 - t_2)$



$\downarrow$

$\underbrace{\int_{-\pi}^{\pi} \cos\omega(t_1+t_2) + 2\phi \times \frac{1}{2}\pi}$

for uniform distribution, pdf $= 1/b-a$
so we find expectation as $\int_{a}^{b} x \cdot f(x)\,dx$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\omega(t_1+t_2) + 2\phi)\, d\phi$

$= 0$

$\downarrow$

$\int \cos(a + x)\,dx$

substitution.

$\frac{\cos a \cos x}{2}$

/

Q17

Find autocorrelation of the process $x(t) = K\cos\omega t$ $\quad t \geq 0$, K is uniformly distributed b/w 0 & 2

$R_{XX}(t, s) = E(X(t)X(S))$

$= E(K\cos\omega t \cdot K\cos\omega s)$

$= E[K^2 \cos\omega t \cos\omega s]$

$= E[K^2] \cdot (\cos\omega t \cos\omega s)$

$$K(x) = \begin{cases} \frac{1}{2-0} = \frac{1}{2} & \text{when } 0 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$E(K^2) = ?$ $\qquad$ $Var(K) = E(K^2) - \mu(k)^2$

$\Rightarrow E(K^2)$

$\mu = \frac{2+0}{2} = 1$ $\qquad$ $Var(K) = \frac{(b-a)^2}{12}$

$= \frac{2^2}{12} = \frac{4}{12} = 1/3$

$\Rightarrow E(K^2) = Var(K) + \mu_K^2$

$= \frac{1}{3} + 1 = \frac{4}{3}$

$R_{XX} = \frac{4}{3} \cos\omega t \cos\omega s$

CROSS CORRELATION & CROSS COVARIANCE

Let X, Y be 2 random processes

$R_{XY}(t, s) = E[X(t)Y(S)]$

$C_{XY}(t, S) = Cov(X(t), Y(S))$

$= R_{XY}(t, s) - \mu_X(t)\mu_Y(s)$

The above 2 denote relationship b/w 2 signals.

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Find the cross-correlation of $X(t) = A\cos(\omega t + \phi)$
$y(t) = B\sin(\omega t + \phi)$

where ω, A, B are ~~independent~~ constants
ϕ is uniformly distributed b/w 0 & 2π

~~f(x)(φ)~~ pdf for $\phi = \frac{1}{2\pi} \quad 0 \le \phi \le 2\pi$

0 otherwise

$\mu: \frac{2\pi + 0}{2} = \pi \qquad \text{Var} = \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}$

$R_{xy}(t, s) = E(x(t)(y(s)]$
$= E(A\cos(\omega t + \phi)\, B\sin(\omega s + \phi))$

cosa sinb

$= \frac{AB}{2} \cdot E(\cancel{\cos} \sin(\omega t + \phi + \omega s + \phi) \cancel{\varepsilon} - \sin(\omega t + \cancel{\phi} - \omega s - \cancel{\phi})$

=

$= \frac{AB}{2} \; E(\underbrace{\sin(\omega (t+s) + 2\phi)}_{0} - \sin(\omega(t-s)))$

$\hookrightarrow \sin(\cancel{2\phi}\,\pi + \cdots)$

$= \frac{AB}{2} \; E(-\sin(\omega(t-s)))$

$= -\frac{AB}{2} \; E(\sin\omega(t-s))$

If $R_{xy}^{(t,s)} = 0$, the 2 processes, $x(t)$ & $y(t)$ are orthogonal
$\forall\, t, s$

Suppose $X(t)$ is a normal process with
mean $q(t) = 3$ & $C(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$
find $P(X(5) \le 2)$
$P(|X(\cancel{8})\cancel{7} - X(5)| \le 1)$

Auto covariance $C_{xx}(t_1,t_2) = E(x(t_1)x(t_2)) - \mu_x(t_1)\mu_x(t_2)$

when $t_1 = t_2 = t$,
$C_{xx}(t,t) = \sigma^2_{x(t)}$

$\sigma^2 = 4e^{-0.2(0)}$
$= 4$
$\sigma = 2$

i) $P(x(5) \le 2) = P\left(\frac{x(5)-3}{2} \le \frac{2-3}{2}\right)$

$= P\left(z(5) \le \frac{-1}{2}\right)$

$= 0.5 - P(0 \le z \le 1/2)$
$= 0.5 - 0.1916$
$= 0.3084$

$$\begin{array}{r} \overset{499}{0.5000} - \\ 0.1916 \\ \hline 0.3084 \end{array}$$

ii) $P(|x(8) - x(5)| \le 1)$

$= P(-1 \le |x(8) - x(5)| \le 1)$

$E(x(8) - x(5)) = E(x(8)) - E(x(5))$
$= 3 - 3 = 0$

$E(x(8) - x(5))^2$
$= 0$

$\text{Var}(x(8) - x(5)) = E((x(8) - x(5))^2) - 0$

$\text{Var}(8) - E(x(8))^2$

$= E(x(8))^2 + E(x(5))^2 - 2E(x(8)x(5))$

$-2E(x(8))E(x(5))$

$= 4^{-9} + 4^{-9} - 2(4e^{-0.2(3)} - 2 \times 9)$

$= 8 + 9 - (4e^{-0.6} - 18)$

$= 17 - (4e^{-0.6} - 18) = -1 - 4e^{+0.6}$

Since $E(x^2(t)) = R_{xx}(t,t) = C_x(t,t) + \mu_x(t)\mu_x(t)$
$= C(t,t) + (\mu_x(t))^2$

we get

$\text{Var}(x(8) - x(5)) = C(8,8) + \mu_x(8)^2 + C(5,5) + \mu_x(5)^2 - 2R(8,5)$

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$\frac{13\times}{2}$

$= 4+9+4+9 - 2(E(x(8)x(5))$

$= 26 - 2(4e^{-0.2(3)} + 9)$

$= 26 - 2(4e^{-0.2(3)}) \mp 18 \quad = 3.6$

$\sigma = 1.89$

Substituting,

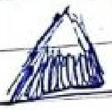
$P\left(\frac{-1-0}{1.89} \leq z \leq \frac{1-}{1.89}\right)$

$\downarrow$

$-0.5291 \qquad 0.5291$

Should get $\simeq 0.4$

$= 2\times$

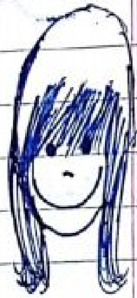


FILL NOTES LATER

6/2/19 Properties of cross-correlation function.

WKT $R_{xy}(t,s) = E[x(t), y(s)]$

If $s = t+\tau$, then $R_{xy}(t, t+\tau) = E[x(t), y(t+\tau)]$



① $R_{xy}(\tau) = R_{yx}(-\tau)$

i.e. ~~auto~~ cross correlation function is not even.

(even function implies that $f(x) = f(-x)$)

$R_{xy}(t, \tau+t) = E[x(t)\, y(t+\tau)]$

$= R_{xy}(\tau)$ where $\tau = s-t$

② $|R_{xy}(\tau)| \le \frac{1}{2}\{R_{xx}(0)+R_{yy}(0)\}$

③ $R_{xy}(\tau) \le \sqrt{R_{xx}(0) \cancel{*} R_{yy}(0)}$ (↳ Arithmetic mean.)

↳ Geom·mean

<u>Proof:</u>

By Cauchy-Schwartz Inequality,

$$\{E(XY)\}^2 \le E[X^2]\,E[Y^2]$$

Let $x(t)$, $y(t+\tau)$ be two processes.

then

Proof for ③

$$\{E[x(t)\,y(t+\tau)]\}^2 \le E[x^2(t)]\,E[y^2(t+\tau)]$$

$$\Rightarrow [R_{xy}(\tau)]^2 \le R_{xx}(0)\,R_{yy}(0)$$

$$\Rightarrow R_{xy}(\tau) \le \sqrt{R_{xx}(0)\,R_{yy}(0)}$$

$\frac{a+b}{2}$ $\sqrt{ab}$

Now, w.r.t A.M ≥ G.M

$$\Rightarrow R_{xy}(t) \le \sqrt{R_{xx}(0)\,R_{yy}(0)} \le \frac{1}{2}(R_{xx}(0)+R_{yy}(0))$$

Direct proof for ②:

$$E[x(t+\tau) - \overset{y}{\cancel{x}}(t)]^2$$

$\to E[x^2(t+\tau)]$ $\to E[y^2(t)]$

$$= R_{xx}(0) + R_{yy}(0) - 2R_{xy}(\tau)$$

$$\Rightarrow R_{xy}(t) = \frac{1}{2}\left(R_{xx}(0)+R_{yy}(0) - E[x(t+\tau)-y(t)]^2\right)$$

$$\le \frac{1}{2}(R_{xx}(0)+R_{yy}(0))$$

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Similarly, we can prove for

$$E[x(t+\tau) + y(t)]^2$$

$$= R_{xx}(0) + R_{yy}(0) + 2(R_{xy}(\tau))$$

where $\tau = t - s$

$$R_{xy}(\tau) = -\frac{1}{2}\left(R_{xx}(0) + R_{yy}(0) - E(\cancel{xy}(t+\tau) + y(t)\right)$$

$$\therefore R_{xy}(\tau) \geq -\frac{1}{2}(R_{xx}(0) + R_{yy}(0))$$

- Let $x(t) = 3\cos(\omega t + \theta)$
  $y(t) = 2\cos(\omega t + \theta - \pi/2)$

where θ is a r.v uniformly distributed in $(0, 2\pi)$

Verify that $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)\, R_{yy}(0)}$

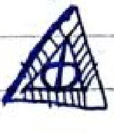
LHS:

$$R_{xy}(\tau) = E[x(t+\tau)\,y(t)]$$

$$= E[3\cos[\omega(t+\tau) + \theta] \times 2\cos[\omega t + \theta - \pi/2]]$$

$$= \frac{6}{2}\left[\cos(\omega(t+\tau) + \theta + \omega t + \theta - \pi/2)\right]$$

$$= 3\left[\cos(2\omega t + \omega\tau + 2\theta - \pi/2) - \cos(\underset{\omega\tau}{\omega\tau} - \pi/2)\right]$$



$$= 3\left[\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta - \pi/2) \cdot \frac{1}{2\pi}\, d\theta\right]$$

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$-3\cos(\omega t - \pi/2)$

$$= 3\left(\frac{\sin(2\omega t + \omega\tau + 2\theta - \pi/2)}{2}\right)_0^{2\pi} - 3\cos(\omega t - \pi/2)$$

$$= 0 - 3\cos(\omega t - \pi/2) = -3\sin(\omega\tau)$$

$$|R_{xy}(\tau)| = |3\sin\omega t|$$
$$\leq 3\,|\sin\omega t|$$
$$\leq 3$$

RHS:

$$R_{xx}(\overset{\tau}{0}) = E(x(t)\,x(t+\tau))$$
$$= E(3\cos(\omega t + \theta)\cdot 3\cos(\omega t + \omega\tau + \theta))$$

$$= 9\,E[\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)]$$
$$= \frac{9}{2}\,E[\cos(\omega t + \theta + \omega t + \omega\tau + \theta) - \underset{0}{\cos}(\cancel{\omega t} + \cancel{\theta} - \cancel{\omega t} - \omega\tau - \cancel{\theta})$$

$$= \frac{9}{2}\left(\overbrace{E[\cos(2\omega t + \omega\tau + 2\theta)]}^{0} + E[\cos(-\omega\tau)]\right)$$

$$= \frac{9}{2}\cos(-\omega\tau)$$

At 0, $= 9/2$

$$R_{yy}(\tau) = E(y(t)\,y(t+\tau))$$
$$= E(2\cos(\omega t + \theta - \pi/2) \times 2\cos(\omega t + \omega\tau + \theta - \pi/2))$$

$$\overset{(2\omega t + 2\theta - \pi + \omega\tau) = 0}{}$$
$$= \frac{4}{2}\,E[\cos(\omega t + \theta - \pi/2 + \omega t + \omega\tau + \theta - \pi/2) + \cos(\cancel{\omega t} + \cancel{\theta} - \cancel{\pi/2} - \cancel{\omega t} - \omega\tau - \cancel{\theta} + \cancel{\pi/2})$$

$$= 2\,E(+\cos(-\omega\tau)) = 2\cos(\omega\tau)$$ At 0, $= 2$

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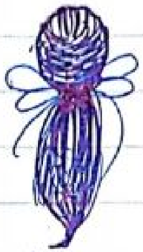
RHS = $\sqrt{R_{xx}(0)\, R_{yy}(0)}$

= $\sqrt{9/2 \times 2}$ = $\sqrt{9}$ = 3

≥ LHS

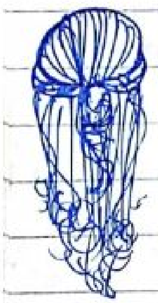
Hence verified



POISSON PROCESS

11/3/19 ① Counting process

If $X(t)$ represents the number of events that occur in the time interval $[0,t)$ then $X(t)$ is a counting process.



The Poisson process is a counting process, provided $X(t)$ follows Poisson distribution.

Properties of ①

① $X(t) \geq 0$

② $X(0) = 0$

③ $X(t)$ is an integer value.

④ When $s<t$, => $X(s) \leq X(t)$

⑤ $X(t) - X(s)$: No. of events in $[s,t]$

② Independent increment process

Assume a time interval, such that

$t_0 < t_1 < t_2 \ldots < t_n$

for a discrete, finite set of t_i.

Now, we calculate $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$ ....

$X(t_n) - X(t_{n-1})$

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If all the variables thus calculated are independent variables, we say that $X(t)$ is an independent increment process.

i.e. $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$.... etc. are mutually independent.

③ Stationary increments ~~process~~ :

For any finite number of discrete points from the time domain, say $t_0 < t_1 < t_2 .. < t_n$ the increments

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1) \dots X(t_n) - X(t_{n-1}$$

should be identically distributed.

(or)

$E(X(t)) = mt$ where m is $E(X(1))$.

Based on the above statements, we can define Poisson process to be:

" A counting process $X(t)$ in which no. of events in any interval of length t, has a Poisson distribution with mean λt. "

Thus,

$$P[X(s+t) - X(s)] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$= n$

↳ no. of occurrences in time interval of length t = n

$n = 0, 1, 2 \dots$

λ = arrival rate (usually given; or you need to calculate it from set of data)

Identity

☆☆☆ Def: 2

☆ Poisson process $X(t)$ is a counting process with stationary and independent increments such that for a rate $\lambda > 0$ the following conditions hold:

- $P(X(t+\Delta t) - X(t) = 1) = \lambda \Delta t + o(\Delta t)$
- $P(X(t+\Delta t) - X(t) \geq 2] = o(\Delta t)$
- $P(X(t+\Delta t) - X(t) = 0) = 1 - \lambda \Delta t + o(\Delta t)$

P(getting 1 occurrence in time interval Δt $= \lambda$~~Δt~~$+ o(\Delta t)$)

no occurrence

$1-(\lambda\Delta t + \lambda 2 o \Delta t + 4$

To derive 1st def (Pmf) from 2nd def:

Consider

$$P[X(t+\Delta t) = n] = P[X(t) = n]\, P[X(\Delta t) = 0] + P[X(t) = n-1]\, P[X(\Delta t) = 1] + \underbrace{P[X(t) = n-2]\, P(X(\Delta t) = 2]}_{\text{not included}} \ldots$$

But, by the second property stated above, if no. of occurrences ≥ 2 then it approximates to 0, so no more combinations are calculated after the first 2 terms.

$$\therefore \text{LHS} \approx P[X(t)=n](1 - \lambda \Delta t) + P[X(t)=n-1](\lambda \Delta t)$$

from property ① & ③ above

Note: ($o(\Delta t)$ is an error of a very small kind, hence negligible in these calculations)

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$$\frac{P[X(t+\Delta t)=n]-P[X(t)=n]}{\Delta t}$$

$$= -\lambda P[X(t)=n] + \lambda P[X(t)=n-1]$$

As $\Delta t \to 0$, we apply formula $f'(x) = \frac{f(x+h)-f(x)}{h}$

LHS is $\frac{d}{dt} P[X(t)=n]$

RHS remains the same (constants remain as they were during derivation)

$$\therefore \frac{d}{dt} P[X(t)=n] = -\lambda P[X(t)=n] + \lambda P[X(t)=n-1]$$

Rewriting,

$$\frac{d}{dt} P[X(t)=n] + \lambda P[X(t)=n] = \lambda P[X(t)=n-1]$$

This is a first order differential equation.

Now, for $\frac{dy}{dx} + \underbrace{P}_{\text{function of } x} y = Q$

$$y = e^{-\int P dx} \cdot \int Q e^{\int P dx} dx$$

Using this method of solving, inductively for $n = 0, 1, 2, 3 \ldots\ldots$

subject to $P[X(0)=n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

we get the solution.

$P[X(t)=n]$ = The pmf of the no. of events or arrivals in an interval of length t becomes

$$P_X[n,t] = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2 \ldots.$$

(or) $P[X(t)=n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ $n = 0, 1, 2 \ldots$ which is def. 1

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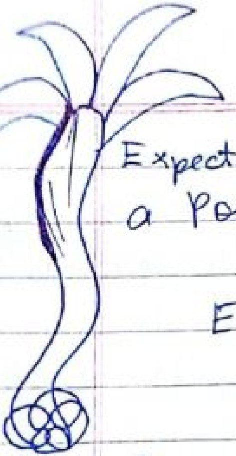


Expectation & variance will be equal for a Poisson process:

$$E(X(t)) = Var[X(t)]$$
$$= \lambda t$$

Sum of 2 Poisson processes is also a Poisson process.

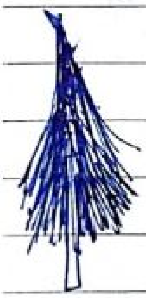
For proof (expanded) refer Stochastic process by Medhi.

13/3/18 Properties of Poisson Process:

① Sum of 2 Poisson Processes is a Poisson Process.

If $X_1(t), X_2(t)$ are Poisson Processes, then $X_1(t) + X_2(t)$ is again a Poisson Process, but $X_1(t) - X_2(t) = (X_1 - X_2)(t)$ need not be a Poisson Process.

$$E(X_1(t) - X_2(t)) = \lambda_1 t - \lambda_2 t$$



Variance $(X_1(t) - X_2(t))$

$$= \text{Variance}(X_1(t)) + \text{Variance}(X_2(t))$$
$$= \lambda_1 t + \lambda_2 t$$

Therefore, since mean ≠ var,

$E(X_1 - X_2) \neq V(X_1 - X_2) \Rightarrow$ $X_1 - X_2$ not necessarily Poisson process

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② Let $X(t)$ be a Poisson process with paramelers λ_{y}, t

Now,

mean $E[x(t)] = \lambda t$

$E[x^2(t)] = \lambda t + (\lambda t)^2$

$Var[x(t)] = \lambda t$

Autocorrelation $R(t_1, t_2) = E[x(t_1) x(t_2)]$

Add & sulctract $x(t_1) x(t_1)$ inside

$$E[x(t_1) x(t_2) - x(t_1) x(t_1) + \underbrace{x(t_1) x(t_1)}]$$

$$= E[x(t_1)[x(t_2) - x(t_1)]] + E[x^2(t_1)],$$

Note: In Poisson Process, disjoint / independent time intervals.

$$\therefore E[x(t_1)] E[x(t_2) - x(t_1)] + E[x^2(t_1)] -$$

$$\therefore \left.\begin{array}{ll} \lambda t_1 + \lambda^2 t_1 t_2 & \text{if } t_1 < t_2 \\ \lambda t_2 + \lambda^2 t_1 t_2 & \text{if } t_2 < t_1 \end{array}\right\} = R(t_1, t_2)$$

$$\boxed{R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)}$$

Since $0 < t_1 < t_2$
& [illegible]
me ⁘

③ $C_{xx}(t_1, t_2) = \begin{array}{ll} \lambda t_1 & t_1 < t_2 \\ \lambda t_2 & t_2 < t_1 \end{array}$

$\therefore C_{xx}(t_1, t_2) = \lambda \min(t_1, t_2)$

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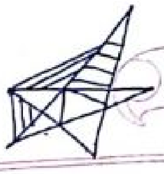


Problem:

1. At a service counter, customers arrive acc. to Poisson process, with mean rate of 3/min. Find the prob. that during a time interval of 2 min,
   - i) exactly 4 customers arrive
   - ii) > 4 customers arrive

$\lambda = 3$

$$P(X(t)=n) = \frac{e^{-\lambda t}\cdot(\lambda t)^{n}}{n!} \quad n=0,1,2\ldots$$

$$\text{Here,} = \frac{e^{-3t}(3t)^{n}}{n!}$$

i) $$P(X(2)=4) = \frac{e^{-3\times2}(3\times2)^{4}}{4!}$$

$$= \frac{e^{-6}\cdot 6^{4}}{4!} = 0.1338$$

ii) $$P(X(2)\geq5)$$

$$= 1 - P(X(2)\leq4)$$

$$= 1 - \sum_{n=0}^{4} \frac{e^{-3}\cdot(6)^{n}}{n!}$$

$$= 1 - 0.285$$

$$= 0.7149$$

0 9 9
~~1~~.~~0~~~~0~~0
0.285
0.715

2) If X(t) is a Poisson Process such that $E[X(9)]=6$ then

a) Find mean, var of $X(8)$

b) $P\{X(4)\leq5 / X(2)=3\}$

c) $P\{X(4)\leq5 / X(2)\leq3\}$

In Poisson process, if vars have non-overlapping intervals, they are independent

a) Given, $\lambda \times 9 = \cancel{8}6$ $\quad \lambda = \frac{2}{3}$

$\therefore$ mean of $x(8) = \frac{2}{3} \times 8 = \frac{16}{3}$ = variance

b) Probability $= \frac{P[x(4) \le 5,\ x(2) = 3]}{P[x(2) = 3]}$ $\quad$ subtract $x(2)=3$ on both sides

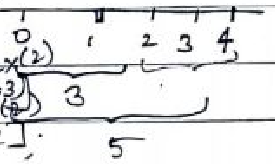
$$= \frac{\cancel{P[x(2) = 3]} \times P[x(4) - x(2) \le 2]}{\cancel{P(x(2) = 3)}}$$

$= P[x(4) - x(2) \le 2]$

$$= \sum_{n=0}^{2} \frac{e^{-2/3 \times (4-2)} \times \left(\frac{2}{3} \times (4-2)\right)^n}{n!} = 0.8494$$

c) Probability: $\frac{P[x(4) \le 5,\ x(2) \le 3]}{P[x(2) \le 3]}$

$$= \frac{P[x(2)=0,\ x(4) \overset{-x(2)}{=} 5] + P[x(2)=1,\ x(4) \overset{-x(2)}{=} 4) + P[x(2)=2,\ x(4) \overset{-x(2)}{=} 3] + P[x(2)=3,\ x(4) \overset{-x(2)}{=} 2]}{P[x(2) \le 3]}$$



$$= \frac{\sum_{k=0}^{3} P[x(2) = k] \times P[x(4) - x(2)] = 5-k}{P[x(2) \le 3]}$$

$= 0.9671$

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