
# Introduction to Random Processes



## 8.1 Introduction

Chapters 1 to 7 were devoted to the study of probability theory. In those chapters we were concerned with outcomes of random experiments and the random variables used to represent them. This chapter deals with the dynamics of probability theory. The concept of random processes enlarges the random variable concept to include time. Thus, instead of thinking of a random variable $X$ that maps an event $s \in S$, where $S$ is the sample space, to some number $X(s)$, we think of how the random variable maps the event to different numbers at different times. This implies that instead of the number $X(s)$ we deal with $X(t, s)$, where $t \in T$ and $T$ is called the *parameter set* of the process and is usually a set of times.

Random processes are widely encountered in such fields as communications, control, management science, and time series analysis. Examples of random processes include the population growth, the failure of a piece of equipment, the price of a given stock over time, and the number of calls that arrive at a switchboard.

If we fix the sample point $s$, $X(t)$ is some real function of time. For each $s$, we have a function $X(t)$. Thus, $X(t,s)$ can be viewed as a collection of time functions, one for each sample point $s$, as shown in Figure 8.1.

On the other hand, if we fix $t$, we have a function $X(s)$ that depends only on $s$ and thus is a random variable. Thus, a random process becomes a random variable when time is fixed at some particular value. With many values of $t$ we obtain a collection of random variables. Thus, we can define a random process as a family of random variables $\{X(t,s) \mid t \in T, s \in S\}$ defined over a given probability space and indexed by the time parameter $t$.

A random process is also called a *stochastic process*. Consider a communication system example. Assume we have a set of possible messages that can be transmitted over a channel. The set of possible messages then constitutes our sample space. For each message $M$ generated by our source, we transmit an associated waveform $X(t,s)$ over the channel. The channel is not perfect; it selectively adds a noise waveform $N(t,s)$ to the original waveform so that what is seen at the receiver is a random signal $R(t,s)$ that is the sum of the transmitted waveform and the noise waveform. That is,

$$R(t,s) = X(t,s) + N(t,s)$$

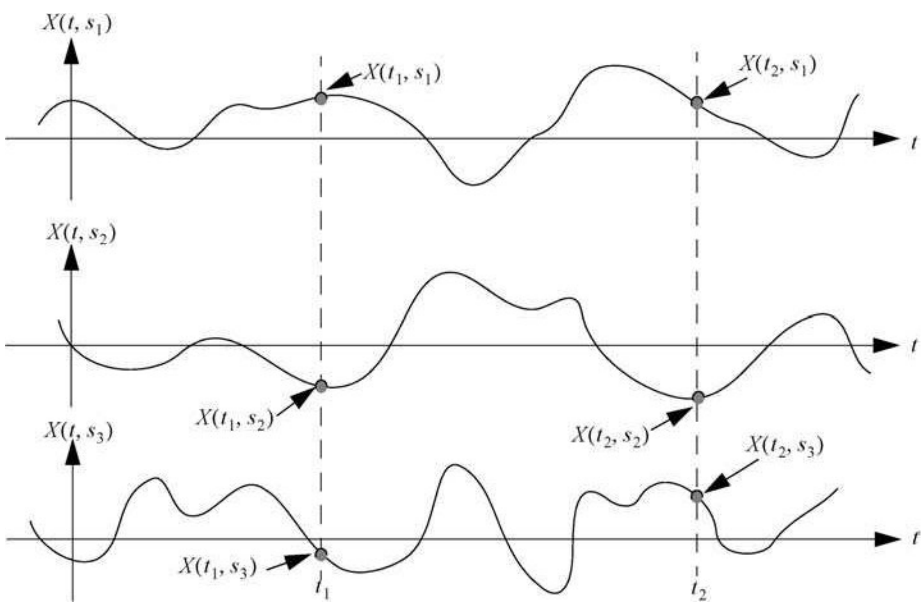


**Figure 8.1** A Sample Random Process

Because the noise waveform is probabilistically selected by the channel, different noise waveforms can be associated not only with the same transmitted waveform but also with different transmitted waveforms. Thus, the plot of $R(t,s)$ will be different for different values of $s$.

## 8.2 Classification of Random Processes

A random process can be classified according to the nature of the time parameter and the values that $X(t,s)$ can take on. As discussed earlier, $T$ is called the parameter set of the random process. If $T$ is an interval of real numbers and hence is continuous, the process is called a *continuous-time* random process. Similarly, if $T$ is a countable set and hence is discrete, the process is called a *discrete-time* random process. A discrete-time random process is also called a *random sequence*, which is denoted by $\{X[n] \mid n = 1, 2, \ldots\}$.

The values that $X(t,s)$ assumes are called the *states* of the random process. The set of all possible values of $X(t,s)$ forms the *state space*, $E$, of the random process. If $E$ is continuous, the process is called a *continuous-state* random process. Similarly, if $E$ is discrete, the process is called a *discrete-state* random process.

## 8.3 Characterizing a Random Process

In the remainder of the discussion we will represent the random process $X(t,s)$ by $X(t)$; that is, we will suppress $s$, the sample space parameter. A random process is completely described or characterized by the joint CDF. Since the value of a random process $X(t)$ at time $t_i$, $X(t_i)$, is a random variable, let

$$\begin{aligned} F_X(x_1,t_1) &= F_X(x_1) = P[X(t_1) \le x_1] \\ F_X(x_2,t_2) &= F_X(x_2) = P[X(t_2) \le x_2] \\ &\cdots \\ F_X(x_n,t_n) &= F_X(x_n) = P[X(t_n) \le x_n] \end{aligned}$$

where $0 < t_1 < t_2 < \cdots < t_n$. Then the joint CDF, which is defined by

$$F_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n) = P[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n] \quad \text{for all } n$$

completely characterizes the random process. If $X(t)$ is a continuous-time random process, then it is specified by a collection of PDFs:

$$f_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_X(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n)$$

Similarly, if $X(t)$ is a discrete-time random process, then it is specified by a collection of PMFs:

$$p_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

### 8.3.1 Mean and Autocorrelation Function of a Random Process

The mean of $X(t)$ is a function of time called the *ensemble average* and is denoted by

$$\mu_X(t) = E[X(t)]$$

The autocorrelation function provides a measure of similarity between two observations of the random process $X(t)$ at different points in time $t$ and $s$. The autocorrelation function of $X(t)$ and $X(s)$ is denoted by $R_{XX}(t, s)$ and defined as follows:

$$\begin{aligned} R_{XX}(t, s) &= E[X(t)X(s)] = E[X(s)X(t)] = R_{XX}(s, t) \\ R_{XX}(t, t) &= E\left[X^2(t)\right] \end{aligned}$$

It is common to define $s = t + \tau$, which gives the autocorrelation function as

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

The parameter $\tau$ is sometimes called the *delay time* (or *lag time*). The autocorrelation function of a deterministic periodic function of period $T$ is given by

$$R_{XX}(t, t+\tau) = \frac{1}{2T}\int_{-T}^{T} f_X(t)f_X(t+\tau)dt$$

Similarly, for an aperiodic function the autocorrelation function is given by

$$R_{XX}(t, t+\tau) = \int_{-\infty}^{\infty} f_X(t)f_X(t+\tau)dt$$

Basically the autocorrelation function defines how much a signal is similar to a time-shifted version of itself. Noise is known to correlate only with an exact replica of itself; that is, when both functions match perfectly. Communication engineers take advantage of this phenomenon in communication systems that use a modulation scheme called *spread spectrum*. In this modulation scheme, finding different noise-like functions is a major task.

A random process $X(t)$ is called a *second order process* if $E[X^2(t)] < \infty$ for each $t \in T$.

### 8.3.2 The Autocovariance Function of a Random Process

The autocovariance function of the random process $X(t)$ is another quantitative measure of the statistical coupling between $X(t)$ and $X(s)$. It is denoted by $C_{XX}(t,s)$ and defined as follows:

$$\begin{aligned} C_{XX}(t,s) &= Cov(X(t),X(s)) = E[\{X(t)-\mu_X(t)\}\{X(s)-\mu_X(s)\}] \\ &= E[X(t)X(s)] - \mu_X(s)E[X(t)] - \mu_X(t)E[X(s)] + \mu_X(t)\mu_X(s) \\ &= R_{XX}(t,s) - \mu_X(t)\mu_X(s) \end{aligned}$$

If $X(t)$ and $X(s)$ are independent, then $R_{XX}(t,s) = \mu_X(t)\mu_X(s)$, and we have $C_{XX}(t,s) = 0$, which means that there is no coupling between $X(t)$ and $X(s)$. This is equivalent to saying that $X(t)$ and $X(s)$ are uncorrelated, but the reverse is not true. That is, $C_{XX}(t,s) = 0$ does not mean that $X(t)$ and $X(s)$ are independent.

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**Example 8.1** A random process is defined by

$$X(t) = K\cos wt \quad t \geq 0$$

where $w$ is a constant and $K$ is uniformly distributed between 0 and 2. Determine the following:

(a) $E[X(t)]$

(b) The autocorrelation function of $X(t)$

(c) The autocovariance function of $X(t)$

**Solution** The expected value and variance of $K$ are given by $E[K] = (2+0)/2 = 1$ and $\sigma_K^2 = (2-0)^2/12 = 1/3$. Thus, $E\left[K^2\right] = \sigma_K^2 + (E[K])^2 = 4/3$.

(a) The mean of $X(t)$ is given by $E[X(t)] = E[K\cos wt] = E[K]\cos wt = \cos wt$

(b) The autocorrelation function of $X(t)$ is given by

$$\begin{aligned} R_{XX}(t,s) &= E[X(t)X(s)] = E\left[K^2\cos(wt)\cos(ws)\right] = E\left[K^2\right]\cos(wt)\cos(ws) \\ &= \frac{4}{3}\cos(wt)\cos(ws) \end{aligned}$$

(c) The autocovariance function of $X(t)$ is given by

$$\begin{aligned} C_{XX}(t,s) &= R_{XX}(t,s) - E[X(t)]E[X(s)] \\ &= \frac{4}{3}\cos(wt)\cos(ws) - \cos(wt)\cos(ws) \\ &= \frac{1}{3}\cos(wt)\cos(ws) \end{aligned}$$

▲

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## 8.4 Crosscorrelation and Crosscovariance Functions

Let $X(t)$ and $Y(t)$ be two random processes defined on the same probability space and with means $\mu_X(t)$ and $\mu_Y(t)$, respectively. The crosscorrelation function $R_{XY}(t,s)$ of the two random processes is defined by

$$R_{XY}(t,s) = E[X(t)Y(s)] = R_{YX}(s,t)$$

for all $t$ and $s$. The crosscorrelation function essentially measures how similar two different processes (or *signals*) are when one of them is shifted in time relative to the other. If $R_{XY}(t,s) = 0$ for all $t$ and $s$, we say that $X(t)$ and $Y(t)$ are *orthogonal processes*. If the two processes are statistically independent, the crosscorrelation function becomes

$$R_{XY}(t,s) = E[X(t)]E[Y(s)] = \mu_X(t)\mu_Y(s)$$

The crosscovariance functions of $X(t)$ and $Y(t)$, denoted by $C_{XY}(t,s)$, is defined by

$$\begin{aligned}
C_{XY}(t,s) &= E[\{X(t) - \mu_X(t)\}\{Y(s) - \mu_Y(s)\}] \\
&= E[X(t)Y(s)] - E[X(t)\mu_Y(s)] - E[\mu_X(t)Y(s)] + E[\mu_X(t)\mu_Y(s)] \\
&= E[X(t)Y(s)] - E[X(t)]\mu_Y(s) - \mu_X(t)E[Y(s) + \mu_X(t)\mu_Y(s) \\
&= R_{XY}(t,s) - \mu_X(t)\mu_Y(s)
\end{aligned}$$

The random processes $X(t)$ and $Y(t)$ are said to be uncorrelated if $C_{XY}(t,s) = 0$ for all $t$ and $s$. That is, $X(t)$ and $Y(t)$ are said to be uncorrelated if for all $t$ and $s$, we have that

$$R_{XY}(t,s) = \mu_X(t)\mu_Y(s)$$

In many situations the random process $Y(t)$ is the sum of the random process $X(t)$ and a statistically independent noise process $N(t)$.

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**Example 8.2** A random process $Y(t)$ consists of the sum of the random process $X(t)$ and a statistically independent noise process $N(t)$. Find the crosscorrelation function of $X(t)$ and $Y(t)$.

**Solution** By definition the crosscorrelation function is given by

$$\begin{aligned}
R_{XY}(t,s) &= E[X(t)Y(s)] = E[X(t)\{X(s) + N(s)\}] \\
&= E[X(t)Y(s)] + E[X(t)N(s)] \\
&= R_{XX}(t,s) + E[X(t)]E[N(s)] \\
&= R_{XX}(t,s) + \mu_X(t)\mu_N(s)
\end{aligned}$$

where the fourth equality follows from the fact that $X(t)$ and $N(s)$ are independent. Using the results obtained earlier, the crosscovariance function of $X(t)$ and $Y(t)$ is given by

$$\begin{aligned}
C_{XY}(t,s) &= E[\{X(t)-\mu_X(t)\}\{Y(s)-\mu_Y(s)\}] = R_{XY}(t,s)-\mu_X(t)\mu_Y(s) \\
&= R_{XX}(t,s)+\mu_X(t)\mu_N(s)-\mu_X(t)\{\mu_X(s)+\mu_N(s)\} \\
&= R_{XX}(t,s)-\mu_X(t)\mu_X(s) \\
&= C_{XX}(t,s)
\end{aligned}$$

▲

Thus, the crosscovariance function is identical to the autocovariance function.

### 8.4.1 Review of Some Trigonometric Identities

Some of the problems that we encounter in this chapter deal with trigonometric functions. As a result we summarize some of the relevant trigonometric identities. These identities are derived from expansions of $\sin(A \pm B)$ and $\cos(A \pm B)$. Specifically, we know that

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B
\end{aligned}$$

Adding the two equations, we obtain the following identity:

$$\sin A \cos B = \frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$$

Similarly,

$$\begin{aligned}
\cos(A-B) &= \cos A \cos B + \sin A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B
\end{aligned}$$

Adding the two equations, we obtain the following identity:

$$\cos A \cos B = \frac{1}{2}\{\cos(A-B)+\cos(A+B)\}$$

Similarly, subtracting the second equation from the first, we obtain the following identity:

$$\sin A \sin B = \frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$$

**Example 8.3** Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(wt + \Theta)$$
$$Y(t) = B\sin(wt + \Theta)$$

where $A, B$, and $w$ are constants and $\Theta$ is a random variable that is uniformly distributed between 0 and $2\pi$. Find the crosscorrelation function of $X(t)$ and $Y(t)$.

**Solution** The crosscorrelation function of $X(t)$ and $Y(t)$ is given by

$$\begin{aligned}
R_{XY}(t,s) &= E[X(t)Y(s)] = E[A\cos(wt+\Theta)B\sin(ws+\Theta)] \\
&= E[AB\cos(wt+\Theta)\sin(ws+\Theta)] \\
&= ABE[\cos(wt+\Theta)\sin(ws+\Theta)] \\
&= ABE\left[\frac{1}{2}\{\sin(wt+ws+2\Theta) - \sin(wt-ws)\}\right] \\
&= \frac{AB}{2}\{E[\sin(wt+ws+2\Theta)] - \sin w(t-s)\}
\end{aligned}$$

Now, since $f_\Theta(\theta) = 1/2\pi$ for $0 \le \Theta \le 2\pi$, we have that

$$\begin{aligned}
E[\sin(wt+ws+2\theta)] &= \int_{-\infty}^{\infty} \sin(wt+ws+2\theta) f_\Theta(\theta)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sin(wt+ws+2\theta)\,d\theta \\
&= \frac{1}{2\pi}\left[-\frac{\cos(wt+ws+2\theta)}{2}\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\{\cos(wt+ws) - \cos(wt+ws+4\pi)\} \\
&= \frac{1}{4\pi}\{\cos(wt+ws) - \cos(wt+ws)\} = 0
\end{aligned}$$

Thus,

$$R_{XY}(t,s) = \frac{AB}{2}[0 - \sin w(t-s)] = -\frac{AB}{2}\sin w(t-s)$$

If we define $s = t + \tau$, then

$$\begin{aligned}
R_{XY}(t,s) = R_{XY}(t, t+\tau) &= -\frac{AB}{2}\sin w(-\tau) \\
&= \frac{AB}{2}\sin(w\tau)
\end{aligned}$$

▲

## 8.5 Stationary Random Processes

There are several ways to define a stationary random process. At a high level, it is a process whose statistical properties do not vary with time. In this book we consider only two types of stationary processes. These are the *strict-sense stationary* processes and the *wide-sense stationary* processes.

### 8.5.1 Strict-Sense Stationary Processes

A random process is defined to be a strict-sense stationary process if its CDF is invariant to a shift in the time origin. This means that the process $X(t)$ with the CDF $F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$ is a strict-sense stationary process if its CDF is identical to that of $X(t+\varepsilon)$ for any arbitrary $\varepsilon$. Thus, we have that being a strict-sense stationary process implies that for any arbitrary $\varepsilon$,

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_X(x_1, x_2, \ldots, x_n; t_1+\varepsilon, t_2+\varepsilon, \ldots, t_n+\varepsilon) \quad \text{for all } n$$

When the CDF is differentiable, the equivalent condition for strict-sense stationarity is that the PDF is invariant to a shift in the time origin; that is,

$$f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = f_X(x_1, x_2, \ldots, x_n; t_1+\varepsilon, t_2+\varepsilon, \ldots, t_n+\varepsilon) \quad \text{for all } n$$

If $X(t)$ is a strict-sense stationary process, then the CDF $F_{X_1X_2}(x_1, x_2; t_1, t_1+\tau)$ does not depend on $t$ but it may depend on $\tau$. Thus, if $t_2 = t_1 + \tau$, then $F_{X_1X_2}(x_1, x_2; t_1, t_2)$ may depend on $t_2 - t_1$, but not on $t_1$ and $t_2$ individually. This means that if $X(t)$ is a strict-sense stationary process, then the autocorrelation and autocovariance functions do not depend on $t$. Thus, we have that for all $\tau \in T$:

$$\begin{aligned} \mu_X(t) &= \mu_X(0) \\ R_{XX}(t, t+\tau) &= R_{XX}(0, \tau) \\ C_{XX}(t, t+\tau) &= C_{XX}(0, \tau) \end{aligned}$$

If the condition $\mu_X(t) = \mu_X(0)$ holds for all $t$, the mean is constant and denoted by $\mu_X$. Similarly, if the equation $R_{XX}(t, t+\tau)$ does not depend on $t$ but is a function of $\tau$, we write $R_{XX}(0, \tau) = R_{XX}(\tau)$. Finally, whenever the condition $C_{XX}(t, t+\tau) = C_{XX}(0, \tau)$ holds for all $t$, we write $C_{XX}(0, \tau) = C_{XX}(\tau)$.

### 8.5.2 Wide-Sense Stationary Processes

Many practical problems that we encounter require that we deal with only the mean and autocorrelation function of a random process. Solutions to these problems are simplified if these quantities do not depend on absolute time. Random

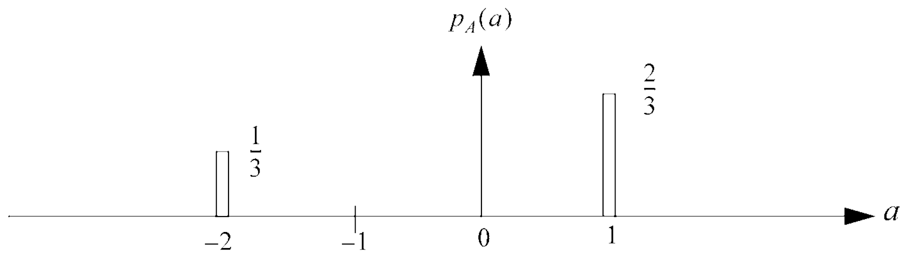


**Figure 8.2** PMF of $A$ and $B$

processes in which the mean and autocorrelation function do not depend on absolute time are called wide-sense stationary (WSS) processes. Thus, for a wide-sense stationary process $X(t)$,

$$E[X(t)] = \mu_X \quad \text{(constant)}$$
$$R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

Note that a strict-sense stationary process is also a wide-sense stationary process. However, in general the converse is not true; that is, a WSS process is not necessarily stationary in the strict sense.

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**Example 8.4** A random process $X(t)$ is defined by

$$X(t) = A\cos t + B\sin t \qquad -\infty < t < \infty$$

where $A$ and $B$ are independent random variables each of which has a value $-2$ with probability 1/3 and a value 1 with probability 2/3. Show that $X(t)$ is a wide-sense stationary process.

**Solution** The PMF of $A$ and $B$ is shown in Figure 8.2.

$$E[A] = E[B] = \frac{1}{3}(-2) + \frac{2}{3}(1) = 0$$
$$E\left[A^2\right] = E\left[B^2\right] = \frac{1}{3}(-2)^2 + \frac{2}{3}(1)^2 = 2$$

Since $A$ and $B$ are independent, $E[AB] = E[A]E[B] = 0$. Thus,

$$\begin{aligned} R_{XX}(t,s) &= E[X(t)X(s)] = E[\{A\cos(t) + B\sin(t)\}\{A\cos(s) + B\sin(s)\}] \\ &= \left[A^2\cos(t)\cos(s) + AB\cos(t)\sin(s) + AB\sin(t)\cos(s) \right. \\ &\quad \left. + B^2\sin(t)\sin(s)\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[A^2\right]\cos(t)\cos(s) + E[AB]\{\cos(t)\sin(s) + \sin(t)\cos(s)\} \\
&\quad + E\left[B^2\right]\sin(t)\sin(s) \\
&= 2\{\cos(t)\cos(s) + \sin(t)\sin(s)\} \\
&= 2\cos(t-s)
\end{aligned}
$$

Since the mean is constant and the autocorrelation function is a function of the difference between the two times, we conclude that the random process $X(t)$ is wide-sense stationary. ▲

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**Example 8.5** Assume that $X(t)$ is a random process defined as follows:

$$X(t) = A\cos(2\pi t + \Phi)$$

where $A$ is a zero-mean normal random variable with variance $\sigma_A^2 = 2$ and $\Phi$ is a uniformly distributed random variable over the interval $-\pi \leq \phi \leq \pi$. $A$ and $\Phi$ are statistically independent. Let the random variable $Y$ be defined as follows:

$$Y = \int_0^1 X(t)dt$$

Determine

1. the mean $E[Y]$ of $Y$.
2. the variance of $Y$.

**Solution** The mean of $X(t)$ is given by

$$E[X(t)] = E[A\cos(2\pi t + \Phi)] = E[A]E[\cos(2\pi t + \Phi)] = 0$$

Similarly the variance of $X(t)$ is given by

$$
\begin{aligned}
\sigma_{X(t)}^2 &= E\left[\{X(t) - E[X(t)]\}^2\right] = E\left[X^2(t)\right] \\
&= E\left[(A\cos(2\pi t + \Phi))^2\right] = E\left[A^2\right]E\left[\{\cos(2\pi t + \Phi)\}^2\right] \\
&= 2E\left[\frac{1+\cos(4\pi t + 2\Phi)}{2}\right] = 2\times\frac{1}{2}\left\{1 + \int_{-\pi}^{\pi}\cos(4\pi t + 2\phi)f_\Phi(\phi)d\phi\right\} \\
&= 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(4\pi t + 2\phi)d\phi = 1
\end{aligned}
$$

(1) The mean of $Y$ is given by

$$E[Y] = E\left[\int_0^1 X(t)dt\right] = \int_0^1 E[X(t)]dt = 0$$

(2) The variance of $Y$ is given by

$$\begin{aligned}
\sigma_Y^2 &= E\left[\{Y - E[Y]\}^2\right] = E\left[Y^2\right] \\
&= E\left[\left(\int_0^1 X(t)dt\right)^2\right] = E\left[\left\{\int_0^1 A\cos(2\pi t + \Phi)dt\right\}^2\right] \\
&= E\left[\left\{\frac{A\sin(2\pi t + \Phi)}{2\pi}\bigg|_0^1\right\}^2\right] \\
&= \frac{1}{4\pi^2}E\left[\{A\sin(2\pi + \Phi) - A\sin(\Phi)\}^2\right] = \frac{1}{4\pi^2}E\left[\{A\sin(\Phi) - A\sin(\Phi)\}^2\right] = 0
\end{aligned}$$

▲

Note that

$$Y = \int_0^1 X(t)dt = \int_0^1 A\cos(2\pi t + \Phi)dt = \frac{A[\sin(2\pi + \Phi) - \sin(\Phi)]}{2\pi} = 0$$

which is why we got the results for the mean and variance of $Y$.

#### 8.5.2.1 Properties of Autocorrelation Functions for WSS Processes

As defined earlier, the autocorrelation function of a wide-sense stationary random process $X(t)$ is defined as

$$R_{XX}(t, t + \tau) = R_{XX}(\tau)$$

The properties of autocorrelation functions of wide-sense stationary processes include the following:

1. $|R_{XX}(\tau)| \le R_{XX}(0)$, which means that $R_{XX}(\tau)$ is bounded by its value at the origin (or the largest value of $R_{XX}(\tau)$ occurs at $\tau = 0$)
2. $R_{XX}(\tau) = R_{XX}(-\tau)$, which means that $R_{XX}(\tau)$ is an even function
3. $R_{XX}(0) = E\left[X^2(t)\right]$, which means that the largest value of the autocorrelation function, $R_{XX}(0)$ (according to property 1 above), is equal to the second moment of the random process. $E\left[X^2(t)\right]$ is usually referred to as the *mean-square value*.

4. If $X(t)$ has no periodic components and is ergodic (where the concept of "ergodic processes" is discussed later), and $E[X(t)] = \mu_X \neq 0$, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \mu_X^2$.
5. If $X(t)$ has a direct-current (dc) component or mean value, then $R_{XX}(\tau)$ will have a constant component. Thus, if $X(t) = K + N(t)$, where $K$ is a constant, then $R_{XX}(\tau) = K^2 + R_{NN}(\tau)$.
6. If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
7. $R_{XX}(\tau)$ cannot have an arbitrary shape; this means that any arbitrary function cannot be an autocorrelation function.

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**Example 8.6** Compute the variance of the random process $X(t)$ whose autocorrelation function is given by

$$R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2}$$

**Solution** By property 4, the square of the mean is given by: $\mu_X^2 = \lim_{|\tau|\to\infty} R_{XX}(\tau) = 25$. Thus, $\mu_X = \sqrt{25} = \pm 5$. Note that the property yields only the magnitude of the mean but not its sign. Also, from property 3,

$$E\left[X^2(t)\right] = R_{XX}(0) = 25 + 4 = 29$$

Thus, the variance is given by

$$\sigma_X^2 = E\left[X^2(t)\right] - \mu_X^2 = 29 - 25 = 4$$

▲

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**Example 8.7** A random process has the autocorrelation function

$$R_{XX}(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1}$$

Find the mean-square value, the mean value and the variance of the process.

**Solution** We first decompose the function to obtain its dc component as follows:

$$R_{XX}(\tau) = \frac{4\tau^2 + 6}{\tau^2 + 1} = \frac{4(\tau^2 + 1) + 2}{\tau^2 + 1} = 4 + \frac{2}{\tau^2 + 1}$$

Thus,

$$E\left[X^2(t)\right] = R_{XX}(0) = 6$$

$$E[X(t)] = \pm\sqrt{\lim_{|\tau|\to\infty} R_{XX}(\tau)} = \pm\sqrt{4} = \pm 2$$

$$\sigma_X^2 = E\left[X^2(t)\right] - (E[X(t)])^2 = 6 - 4 = 2$$

▲

#### 8.5.2.2 Autocorrelation Matrices for WSS Processes

Consider a WSS random signal $X(t)$ that is sampled at periodic time instants and assume that we accumulate $N$ such samples. If the sampling times are $t_1, t_2, \ldots, t_N$, the vector representing the different samples of $X(t)$ is given by

$$\mathbf{X} = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \cdots \\ X(t_N) \end{bmatrix}$$

Since each sample is a random variable, we can define an $N \times N$ autocorrelation matrix that gives the autocorrelation function for every pair of random variables in the vector $\mathbf{X}$. Also, if the interval between two consecutive samples is $\Delta t$, we have that

$$\begin{aligned} t_2 &= t_1 + \Delta t \\ t_3 &= t_2 + \Delta t = t_1 + 2\Delta t \\ &\cdots \\ t_N &= t_{N-1} + \Delta t = t_1 + (N-1)\Delta t \end{aligned}$$

Thus, the autocorrelation matrix becomes

$$\begin{aligned} \mathbf{R_{XX}} = E\left[\mathbf{XX}^T\right] &= E\left\{\begin{bmatrix} X(t_1)X(t_1) & X(t_1)X(t_2) & \cdots & X(t_1)X(t_N) \\ X(t_2)X(t_1) & X(t_2)X(t_2) & \cdots & X(t_2)X(t_N) \\ \cdots & \cdots & \cdots & \cdots \\ X(t_N)X(t_1) & X(t_N)X(t_2) & \cdots & X(t_N)X(t_N) \end{bmatrix}\right\} \\ &= \begin{bmatrix} R_{XX}(t_1,t_1) & R_{XX}(t_1,t_2) & \cdots & R_{XX}(t_1,t_N) \\ R_{XX}(t_2,t_1) & R_{XX}(t_2,t_2) & \cdots & R_{XX}(t_2,t_N) \\ \cdots & \cdots & \cdots & \cdots \\ R_{XX}(t_N,t_1) & R_{XX}(t_N,t_2) & \cdots & R_{XX}(t_N,t_N) \end{bmatrix} \\ &= \begin{bmatrix} R_{XX}(0) & R_{XX}(\Delta t) & \cdots & R_{XX}([N-1]\Delta t) \\ R_{XX}(\Delta t) & R_{XX}(0) & \cdots & R_{XX}([N-2]\Delta t) \\ \cdots & \cdots & \cdots & \cdots \\ R_{XX}([N-1]\Delta t) & R_{XX}([N-2]\Delta t) & \cdots & R_{XX}(0) \end{bmatrix} \end{aligned}$$

where $\mathbf{X}^T$ is the transpose of $\mathbf{X}$, and we have taken advantage of the fact that $R_{XX}(-\tau) = R_{XX}(\tau)$. Thus, for a wide-sense stationary process, $\mathbf{R_{XX}}$ is a symmetric matrix. In a similar manner we can obtain the autocovariance matrix $\mathbf{C_{XX}}$, which is defined as follows:

$$\mathbf{C_{XX}} = E\left[(\mathbf{X} - \bar{\mathbf{X}})(\mathbf{X}^T - \bar{\mathbf{X}}^T)\right] = \mathbf{R_{XX}} - \bar{\mathbf{X}}\bar{\mathbf{X}}^T$$

---

**Example 8.8** Determine the missing elements denoted by $xx$ in the following autocorrelation matrix of a WSS random process $Y(t)$:

$$\mathbf{R_{YY}} = \begin{bmatrix} 2 & 1.3 & 0.4 & xx \\ xx & 2 & 1.2 & 0.8 \\ 0.4 & 1.2 & xx & 1.1 \\ 0.9 & xx & xx & 2 \end{bmatrix}$$

**Solution** Since the autocorrelation matrix of a WSS process is a symmetric matrix, the real matrix is as follows:

$$\mathbf{R_{YY}} = \begin{bmatrix} 2 & 1.3 & 0.4 & 0.9 \\ 1.3 & 2 & 1.2 & 0.8 \\ 0.4 & 1.2 & 2 & 1.1 \\ 0.9 & 0.8 & 1.1 & 2 \end{bmatrix}$$

▲

---

### 8.5.2.3 Properties of Crosscorrelation Functions for WSS Processes

As defined earlier, the crosscorrelation function $R_{XY}(t,s)$ of the two random processes $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(t,s) = E[X(t)Y(s)]$$

If we set $s = t + \tau$, we may write

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

We say that $X(t)$ and $Y(t)$ are jointly wide-sense stationary if $R_{XY}(t, t+\tau)$ is independent of the absolute time. That is, $X(t)$ and $Y(t)$ are jointly wide-sense stationary random processes if

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

Generally the crosscorrelation function is not an even function, as is true for the autocorrelation function. Also, it does not necessarily have a maximum value at the origin as is true for the autocorrelation function. Some of the properties of $R_{XY}(\tau)$ include the following:

1. $R_{XY}(\tau) = R_{YX}(-\tau)$
2. $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)}$
3. $|R_{XY}(\tau)| \le [R_{XX}(0) + R_{YY}(0)]/2$

## 8.6 Ergodic Random Processes

One desirable property of a random process is the ability to estimate its parameters from measurement data. Consider a random process $X(t)$ whose observed samples are $x(t)$. The time average of the function $x(t)$ is defined by

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

The statistical average of the random process $X(t)$ is the expected value $E[X(t)]$ of the process. The expected value is also called the *ensemble average*. An ergodic random process is a stationary process in which every member of the ensemble exhibits the same statistical behavior as the ensemble. This implies that it is possible to determine the statistical behavior of the ensemble by examining only one typical sample function. Thus, for an ergodic random process, the mean values and moments can be determined by time averages as well as by ensemble averages (or expected values), which are equal. That is,

$$E\left[X^n\right] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^n(t)dt$$

A random process $X(t)$ is defined to be *mean-ergodic* (or *ergodic in the mean*) if $E[X(t)] = \bar{x}$.

---

**Example 8.9** A random process has sample functions of the form

$$X(t) = A\cos(wt + \Theta)$$

where $w$ is constant, $A$ is a random variable that has a magnitude of $+1$ and $-1$ with equal probability, and $\Theta$ is a random variable that is uniformly distributed between 0 and $2\pi$. Assume that the random variables $A$ and $\Theta$ are independent.

(a) Is $X(t)$ a wide-sense stationary process?

(b) Is $X(t)$ a mean-ergodic process?

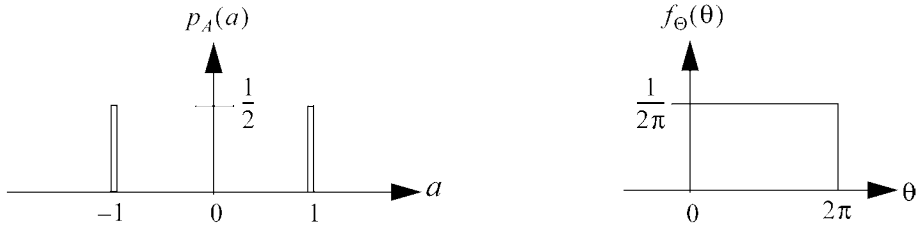


**Figure 8.3** PMF of $A$ and PDF of $\Theta$ for Example 8.9

**Solution** The PMF of $A$ and the PDF of $\Theta$ are shown in Figure 8.3.

$$
\begin{aligned}
E[A] &= 0 \\
\sigma_A^2 &= \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1 = E\left[A^2\right] \\
E[\Theta] &= \pi \\
\sigma_\Theta^2 &= \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}
\end{aligned}
$$

(a) Since $A$ and $\Theta$ are independent, $E[X(t)] = E[A]E[\cos(wt+\Theta)] = 0$, which is a constant. Also, the autocorrelation function of $X(t)$ is given by

$$
\begin{aligned}
R_{XX}(t, t+\tau) &= E[A\cos(wt+\Theta)A\cos(wt+w\tau+\Theta)] \\
&= E\left[A^2\right]E[\cos(wt+\Theta)\cos(wt+w\tau+\Theta)] \\
&= \frac{1}{2}E[\cos(-w\tau)+\cos(2wt+w\tau+2\Theta)] \\
&= \frac{1}{2}E[\cos(-w\tau)] + \frac{1}{2}E[\cos(2wt+w\tau+2\Theta)] \\
&= \frac{1}{2}\cos(w\tau) + \frac{1}{2}E[\cos(2wt+w\tau+2\Theta)] \\
&= \frac{1}{2}\cos(w\tau) + \frac{1}{2}\int_0^{2\pi}\frac{\cos(2wt+w\tau+2\theta)}{2\pi}d\theta \\
&= \frac{1}{2}\cos(w\tau) + \frac{1}{8\pi}[\sin(2wt+w\tau+2\theta)]_0^{2\pi} \\
&= \frac{1}{2}\cos(w\tau) + \frac{1}{8\pi}\{\sin(2wt+w\tau+4\pi) - \sin(2wt+w\tau)\} \\
&= \frac{1}{2}\cos(w\tau)
\end{aligned}
$$

Since the mean is constant and the autocorrelation function depends only on the difference between the two times and not on $t$, we conclude that the process is wide-sense stationary.

(b)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{2\pi}A\cos(wt+\Theta)dt = \lim_{T\to\infty}\frac{A}{2Tw}[\sin(wt+\Theta)]_0^{2\pi}$$
$$= \lim_{T\to\infty}\frac{A}{2Tw}[\sin(2\pi w+\Theta)-\sin\Theta] = 0$$

Thus, $X(t)$ is mean-ergodic.

▲

## 8.7 Power Spectral Density

So far we have been able to characterize a random process by its mean, autocorrelation function, and covariance function. All these functions deal with the time domain; we have not said anything about the spectral (or frequency domain) properties of the process. For a deterministic signal $y(t)$, it is well known that its spectral properties are contained in its Fourier transform $Y(w)$, which is given by

$$Y(w) = \int_{-\infty}^{\infty} y(t)e^{-jwt}dt$$

Conversely, given $Y(w)$ we can recover $y(t)$ by means of the inverse Fourier transform:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(w)e^{jwt}dw$$

Thus, $Y(w)$ provides a complete description of $y(t)$ and vice versa. Unfortunately the same argument cannot be applied to a random process $X(t)$ because the Fourier transform may not exist for most sample functions of the process. One of the conditions for the function $y(t)$ to be Fourier transformable is that it must be absolutely integrable, which means that

$$\int_{-\infty}^{\infty} |y(t)|dt < \infty$$

Recall that for wide-sense stationary processes the autocorrelation function, $R_{XX}(\tau)$, is bounded: $|R_{XX}(\tau)| \leq R_{XX}(0) = E[X^2(t)]$. Thus, instead of working directly with the random process $X(t)$, we work with its autocorrelation function, which is bounded and hence absolutely integrable.

For a wide-sense stationary process, the Fourier transform of the autocorrelation function is called the *power spectral density*, $S_{XX}(w)$, of the random process. Thus,

$$S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-jw\tau}d\tau$$

We can recover $R_{XX}(\tau)$ via the inverse Fourier transform operation as follows:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)e^{jw\tau}dw$$

The statement that the autocorrelation function of a random process and the power spectral density of the process constitute a Fourier transform pair is called the *Wiener–Khintchine theorem*. Note that the mean-square value of the random process, $E[X^2(t)]$, which is also called the *average power*, is given by

$$E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)dw$$

Thus, the properties of the power spectral density include the following:

1. $S_{XX}(w) \geq 0$, which means that $S_{XX}(w)$ is a nonnegative function
2. $S_{XX}(-w) = S_{XX}(w)$, which means that $S_{XX}(w)$ is an even function
3. The power spectral density is a real function if $X(t)$ is real because we have that

$$\begin{aligned} S_{XX}(w) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-jw\tau}d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\{\cos(w\tau) - j\sin(w\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau - j\int_{-\infty}^{\infty} R_{XX}(\tau)\sin(w\tau)d\tau \end{aligned}$$

   We know from the second property above that $S_{XX}(w)$ is an even function. Since $R_{XX}(\tau)\cos(w\tau)$ is an even function of $\tau$ and $R_{XX}(\tau)\sin(w\tau)$ is an odd function of $\tau$, the imaginary part in the above equation vanishes. Thus, we have that

$$S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau$$

   where the second equality follows from the fact that $S_{XX}(w)$ is an even function.

4. The average power of $X(t)$ is given by $E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi} \times \int_{-\infty}^{\infty} S_{XX}(w)dw$, as stated earlier.

**Table 8.1** Some Common Fourier Transform Pairs

| $x(\tau)$ | $X(w)$ |
|---|---|
| $e^{-a\|\tau\|}, a > 0$ | $\dfrac{2a}{a^2 + w^2}$ |
| $e^{-a\tau}, a > 0, \tau \geq 0$ | $\dfrac{1}{a + jw}$ |
| $e^{b\tau}, b > 0, \tau < 0$ | $\dfrac{1}{b - jw}$ |
| $\tau e^{-a\tau}, a > 0, \tau \geq 0$ | $\dfrac{1}{(a + jw)^2}$ |
| $1$ | $2\pi\delta(w)$ |
| $\delta(\tau)$ | $1$ |
| $e^{jw_0\tau}$ | $2\pi\delta(w - w_0)$ |
| $\begin{cases} 1 & -T/2 < \tau < T/2 \\ 0 & \text{otherwise} \end{cases}$ | $T\dfrac{\sin(wT/2)}{(wT/2)}$ |
| $\begin{cases} 1 - \|\tau\|/T & \|\tau\| < T \\ 0 & \text{otherwise} \end{cases}$ | $T\left[\dfrac{\sin(wT/2)}{(wT/2)}\right]^2$ |
| $\cos(w_0\tau)$ | $\pi\delta(w - w_0) + \pi\delta(w + w_0)$ |
| $\sin(w_0\tau)$ | $-j\pi[\delta(w - w_0) - \delta(w + w_0)]$ |
| $e^{-a\|\tau\|}\cos(w_0\tau)$ | $\dfrac{a}{a^2 + (w - w_0)^2} + \dfrac{a}{a^2 + (w + w_0)^2}$ |

5. $S^*_{XX}(w) = S_{XX}(w)$, where $S^*_{XX}(w)$ is the complex conjugate of $S_{XX}(w)$. This means that $S_{XX}(w)$ cannot be a complex function; it must be a real function.
6. If $\int_{-\infty}^{\infty} |R_{XX}(\tau)|d\tau < \infty$, then $S_{XX}(w)$ is a continuous function of $w$.

Table 8.1 shows some of the common Fourier transform pairs used in random processes analysis.

Note that because of the fact that the power spectral density must be an even, nonnegative, real function, some of the entries for $x(\tau)$ cannot be autocorrelation functions of wide-sense stationary processes. In particular, the functions $e^{-a\tau}$, $\tau e^{-a\tau}$, and $\sin(w_0\tau)$ cannot be autocorrelation functions of wide-sense stationary processes because their Fourier transforms are complex functions.

For two random processes $X(t)$ and $Y(t)$ that are jointly wide-sense stationary, the Fourier transform of their crosscorrelation function $R_{XY}(\tau)$ is called the *cross-power spectral density*, $S_{XY}(w)$, of the two random processes. Thus,

$$S_{XY}(w) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-jw\tau}d\tau$$

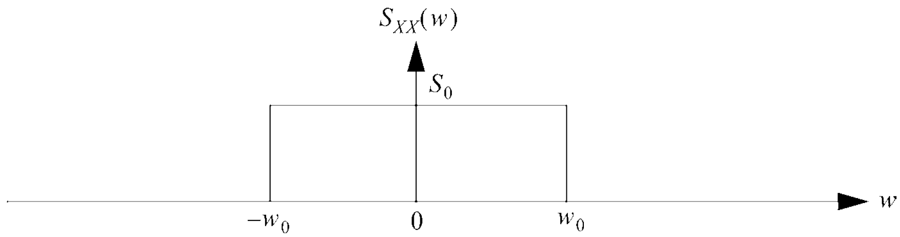


**Figure 8.4** Plot of $S_{XX}(w)$ for Example 8.10

The cross-power spectral density is generally a complex function even when both $X(t)$ and $Y(t)$ are real. Thus, since $R_{YX}(\tau) = R_{XY}(-\tau)$, we have that

$$S_{YX}(w) = S_{XY}(-w) = S_{XY}^{*}(w)$$

where $S_{XY}^{*}(w)$ is the complex conjugate of $S_{XY}(w)$.

---

**Example 8.10** Determine the autocorrelation function of the random process with the power spectral density given by

$$S_{XX}(w) = \begin{cases} S_0 & |w| < w_0 \\ 0 & \text{otherwise} \end{cases}$$

**Solution** $S_{XX}(w)$ is plotted in Figure 8.4.

$$\begin{aligned} R_{XX}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)e^{jw\tau}dw \\ &= \frac{1}{2\pi}\int_{-w_0}^{w_0} S_0 e^{jw\tau}dw = \frac{S_0}{2j\pi\tau}\left[e^{jw\tau}\right]_{-w_0}^{w_0} \\ &= \frac{S_0}{2j\pi\tau}\left[e^{jw_0\tau} - e^{-jw_0\tau}\right] = \frac{S_0}{\pi\tau}\left(\frac{e^{jw_0\tau} - e^{-jw_0\tau}}{2j}\right) \\ &= \frac{S_0}{\pi\tau}\sin(w_0\tau) \end{aligned}$$

▲

---

**Example 8.11** A stationary random process $X(t)$ has the power spectral density

$$S_{XX}(w) = \frac{24}{w^2 + 16}$$

Find the mean-square value of the process.

**Solution** **Method 1 (Brute-Force Method)**: The mean-square value is given by

$$E\left[X^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)dw = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{24}{w^2+16}dw$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{24}{16[1+(w/4)^2]}dw$$

Let $w/4 = \tan\theta$. Then

$$dw = 4\sec^2(\theta)d\theta$$
$$1+(w/4)^2 = 1+\tan(\theta)^2 = \sec^2(\theta)$$

Also, when $w = -\infty$, $\theta = -\pi/2$; and when $w = \infty$, $\theta = \pi/2$. Thus, we obtain

$$E\left[X^2(t)\right] = \frac{24}{32\pi}\int_{-\pi/2}^{\pi/2}\frac{4\sec^2(\theta)d\theta}{\sec^2(\theta)} = \frac{3}{\pi}\int_{-\pi/2}^{\pi/2}d\theta$$
$$= \frac{3}{\pi}[\theta]_{-\pi/2}^{\pi/2} = \frac{3}{\pi}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\} = \frac{3}{\pi}\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\}$$
$$= 3$$

**Solution** **Method 2 (Smart-Move Method)**: From Table 8.1 we observe that

$$e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2+w^2}$$

That is, $e^{-a|\tau|}$ and $2a/(a^2+w^2)$ are Fourier transform pairs. Thus, if we can identify the parameter $a$ in the given problem, we can readily obtain the autocorrelation function. Rearranging the power spectral density, we obtain

$$S_{XX}(w) = \frac{24}{w^2+16} = \frac{24}{w^2+4^2} = 3\left\{\frac{2(4)}{w^2+4^2}\right\} \equiv 3\left\{\frac{2a}{w^2+a^2}\right\}$$

This means that $a = 4$ and the autocorrelation function is

$$R_{XX}(\tau) = 3e^{-4|\tau|}$$

Therefore, the mean-square value of the process is

$$E\left[X^2(t)\right] = R_{XX}(0) = 3$$

▲

### 8.7.1 White Noise

White noise is the term used to define a random function whose power spectral density is constant for all frequencies. Thus, if $N(t)$ denotes white noise,

$$S_{NN}(w) = N_0/2$$

where $N_0$ is a real positive constant. The inverse Fourier transform of $S_{NN}(w)$ gives the autocorrelation function of $N(t)$, $R_{NN}(\tau)$, as follows:

$$R_{NN}(\tau) = (N_0/2)\delta(\tau)$$

where $\delta(\tau)$ is the impulse function. The two functions are shown in Figure 8.5.

---

**Example 8.12** Let $Y(t) = X(t) + N(t)$ be a wide-sense stationary process where $X(t)$ is the actual signal and $N(t)$ is a zero-mean noise process with variance $\sigma_N^2$ and independent of $X(t)$. Find the power spectral density of $Y(t)$.

**Solution** Since $X(t)$ and $N(t)$ are independent random processes, the autocorrelation function of $Y(t)$ is given by

$$\begin{aligned}
R_{YY}(\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t)+N(t)\}\{X(t+\tau)+N(t+\tau)\}] \\
&= E[X(t)X(t+\tau) + X(t)N(t+\tau) + N(t)X(t+\tau) + N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)] + E[X(t)]E[N(t+\tau)] + E[N(t)]E[X(t+\tau)] \\
&\quad + E[N(t)N(t+\tau)] \\
&= R_{XX}(\tau) + R_{NN}(\tau) = R_{XX}(\tau) + \sigma_N^2\delta(\tau)
\end{aligned}$$

Thus, the power spectral density of $Y(t)$ is given by

$$S_{YY}(w) = S_{XX}(w) + \sigma_N^2$$

▲

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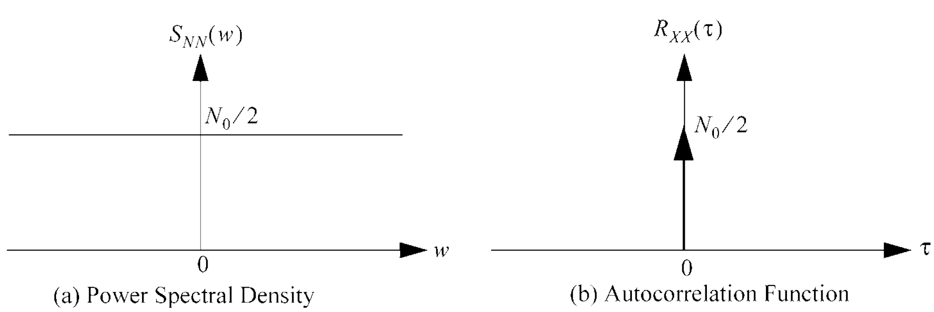


**Figure 8.5** Power Spectral Density and Autocorrelation Function of White Noise

## 8.8 Discrete-Time Random Processes

All the discussion thus far assumes that we are dealing with continuous-time random processes. In this section we extend the discussion to discrete-time random processes $\{X[n], n = 0, 1, \ldots\}$, which are also called random sequences. A discrete-time random process can be obtained by sampling a continuous-time random process. Thus, if the sampling interval is $T_s$, then for such a case we have that

$$X[n] = X(nT_s) \quad n = 0, \pm 1, \pm 2, \ldots$$

We provide a summary of the key results as follows.

### 8.8.1 Mean, Autocorrelation Function, and Autocovariance Function

The mean of $X[n]$ is given by

$$\mu_X[n] = E[X[n]]$$

The autocorrelation function is given by

$$R_{XX}[n, n+m] = E[X[n]X^*[n+m]]$$

where $X^*[n]$ is the complex conjugate of $X[n]$. The random process is wide-sense stationary if $\mu_X[n] = \mu$, a constant, and $R_{XX}[n, n+m] = R_{XX}[m]$.

Finally, the autocovariance function of $X[n]$, $C_{XX}[n_1, n_2]$, which is one measure of the coupling between $X[n_1]$ and $X[n_2]$, is defined by

$$\begin{aligned} C_{XX}[n_1, n_2] &= E[\{X[n_1] - \mu_X[n_1]\}\{X[n_2] - \mu_X[n_2]\}] \\ &= E[X[n_1]X[n_2]] - \mu_X[n_1]\mu_X[n_2] \\ &= R_{XX}[n_1, n_2] - \mu_X[n_1]\mu_X[n_2] \end{aligned}$$

If $X[n_1]$ and $X[n_2]$ are independent, then $R_{XX}[n_1, n_2] = \mu_X[n_1]\mu_X[n_2]$, and we have $C_{XX}[n_1, n_2] = 0$, which means that $X[n_1]$ and $X[n_2]$ are uncorrelated.

A discrete-time random process is called a white noise if the random variables $X[n_k]$ are uncorrelated. If the white noise is a Gaussian wide-sense stationary process, then $X[n]$ consists of a sequence of independent and identically distributed random variables with variance $\sigma^2$ and the autocorrelation function is given by

$$R_{XX}[m] = \sigma^2 \delta[m]$$

$$\delta[m] = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

### 8.8.2 Power Spectral Density

The power spectral density of $X[n]$ is given by the following discrete-time Fourier transform of its autocorrelation function:

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m}$$

Note that $e^{-j\Omega n}$ is periodic with period $2\pi$. That is, $e^{-j(\Omega+2\pi)n} = e^{-j\Omega n}e^{-j2\pi n} = e^{-j\Omega n}$ because $e^{-j2\pi n} = 1$. Thus, $S_{XX}(\Omega)$ is periodic with period $2\pi$, and it is sufficient to define $S_{XX}(\Omega)$ only in the range $(-\pi, \pi)$. This means that the autocorrelation function is given by

$$R_{XX}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)e^{j\Omega m}d\Omega$$

The properties of $S_{XX}(\Omega)$ include the following:

1. $S_{XX}(\Omega + 2\pi) = S_{XX}(\Omega)$, which means that $S_{XX}(\Omega)$ is periodic with period $2\pi$ as stated earlier.
2. $S_{XX}(-\Omega) = S_{XX}(\Omega)$, which means that $S_{XX}(\Omega)$ is an even function.
3. $S_{XX}(\Omega)$ is real, which means that $S_{XX}(\Omega) \geq 0$.
4. $E[X^2[n]] = R_{XX}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)d\Omega$, which is the average power of the process.

---

**Example 8.13** Assume that $X[n]$ is a real process, which means that $R_{XX}[-m] = R_{XX}[m]$. Find the power spectral density $S_{XX}(\Omega)$.

**Solution** The power spectral density is the discrete-time Fourier transform of the autocorrelation function and is given by

$$\begin{aligned} S_{XX}(\Omega) &= \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m} = \sum_{m=-\infty}^{-1} R_{XX}[m]e^{-j\Omega m} + \sum_{m=0}^{\infty} R_{XX}[m]e^{-j\Omega m} \\ &= \sum_{k=1}^{\infty} R_{XX}[-k]e^{j\Omega k} + \sum_{m=0}^{\infty} R_{XX}[m]e^{-j\Omega m} \\ &= \sum_{k=1}^{\infty} R_{XX}[-k]e^{j\Omega k} + \sum_{m=1}^{\infty} R_{XX}[m]e^{-j\Omega m} + R_{XX}[0] \end{aligned}$$

$$= R_{XX}[0] + 2\sum_{m=1}^{\infty} R_{XX}[m]\frac{\{e^{j\Omega m} + e^{-j\Omega m}\}}{2}$$

$$= R_{XX}[0] + 2\sum_{m=1}^{\infty} R_{XX}[m]\cos(m\Omega)$$

▲

**Example 8.14** Find the power spectral density of a random sequence $X[n]$ whose autocorrelation function is given by $R_{XX}[m] = a^{|m|}$.

**Solution** The power spectral density is given by

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m} = \sum_{m=-\infty}^{\infty} a^{|m|}e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{-1} a^{-m}e^{-j\Omega m} + \sum_{m=0}^{\infty} a^{m}e^{-j\Omega m}$$

$$= \sum_{k=1}^{\infty} a^{k}e^{j\Omega k} + \sum_{m=0}^{\infty} a^{m}e^{-j\Omega m} = \sum_{k=1}^{\infty}\{ae^{j\Omega}\}^{k} + \sum_{m=0}^{\infty}\{ae^{-j\Omega}\}^{m}$$

$$= \frac{1}{1 - ae^{j\Omega}} - 1 + \frac{1}{1 - ae^{-j\Omega}}$$

$$= \frac{1 - a^2}{1 + a^2 - 2\cos(\Omega)}$$

▲

### 8.8.3 Sampling of Continuous-Time Processes

As discussed earlier, one of the methods that can be used to generate discrete-time processes is by sampling a continuous-time process. Thus, if $X(t)$ is a continuous-time process that is sampled at constant intervals of $T_s$ time units (that is, $T_s$ is the sampling period), then the samples constitute the discrete-time process defined by

$$X[n] = X(nT_s) \quad n = 0, \pm 1, \pm 2, \ldots$$

If $\mu_X(t)$ and $R_{XX}(t_1, t_2)$ are the mean and autocorrelation function, respectively, of $X(t)$, the mean and autocorrelation function of $X[n]$ are given by

$$\mu_X[n] = \mu_X(nT_s)$$

$$R_{XX}[n_1, n_2] = R_{XX}(n_1 T_s, n_2 T_s)$$

It can be shown that if $X(t)$ is a wide-sense stationary process, then $X[n]$ is also a wide-sense stationary process with mean $\mu_X[n] = \mu_X$ and autocorrelation function $R_{XX}[m] = R_{XX}(mT_s)$. If $X(t)$ is a wide-sense stationary process, then the power spectral density of $X[n]$ is given by

$$\begin{aligned} S_{XX}(\Omega) &= \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m} = \sum_{m=-\infty}^{\infty} R_{X_cX_c}(mT_s)e^{-j\Omega m} \\ &= \frac{1}{T_s}\sum_{m=-\infty}^{\infty} S_{X_cX_c}\left(\frac{\Omega - 2\pi m}{T_s}\right) \end{aligned}$$

where $S_{X_cX_c}(w)$ and $R_{X_cX_c}(\tau)$ are the power spectral density and autocorrelation function, respectively, of $X(t)$.

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**Example 8.15** A wide-sense stationary continuous-time process $X_c(t)$ has the autocorrelation function given by

$$R_{X_cX_c}(\tau) = e^{-4|\tau|}$$

If $X_c(t)$ is sampled with a sampling period of 20 seconds to produce the discrete-time process $X[n]$, find the power spectral density of $X[n]$.

**Solution** The discrete-time process $X[n] = X_c(20n)$. From Table 8.1 we see that the power spectral density of the continuous-time process is given by

$$S_{X_cX_c}(w) = \frac{2 \times 4}{4^2 + w^2} = \frac{8}{16 + w^2}$$

Thus, the power spectral density of the discrete-time process $X[n]$ is given by

$$S_{XX}(\Omega) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} S_{X_cX_c}\left(\frac{\Omega - 2\pi m}{T_s}\right) = \frac{1}{20}\sum_{m=-\infty}^{\infty} \frac{8}{16 + \left[\frac{\Omega - 2\pi m}{20}\right]^2}$$

▲

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## 8.9 Chapter Summary

This chapter presented an introduction to random (or stochastic) processes. It provided different classifications of random processes including discrete-state random processes, continuous-state random processes, discrete-time random processes, and continuous-time random processes. It also discussed two types of stationarity for random processes. A random process whose CDF is invariant to a shift in the time origin is defined to be a strict-sense stationary random process. In many practical situations, this stringent condition is not required. For these situations two conditions are required: the mean value of the process must be a constant, and the autocorrelation function must be dependent only on the difference between the two observation times and not on the absolute time. Such processes are said to be stationary in the wide sense.

The power spectral density of a continuous-time random process is defined as the Fourier transform of its autocorrelation function. Thus, the autocorrelation function and the power spectral density are Fourier transform pairs, which means that given one of them the other can be obtained by an inverse transformation. Similarly, the power spectral density of a discrete-time random process is defined as the discrete-time Fourier transform of its autocorrelation function.

## 8.10 Problems

### Section 8.3: Mean, Autocorrelation Function, and Autocovariance Function

8.1 Calculate the autocorrelation function of the rectangular pulse shown in Figure 8.6; that is,

$$X(t) = A \quad 0 \leq t \leq T$$

where $A$ and $T$ are constants.

8.2 Calculate the autocorrelation function of the periodic function $X(t) = A\sin(wt + \phi)$, where the period $T = 2\pi/w$, and $A$, $\phi$, and $w$ are constants.

8.3 The random process $X(t)$ is given by

$$X(t) = Y\cos(2\pi t) \quad t \geq 0$$

where $Y$ is a random variable that is uniformly distributed between 0 and 2. Find the expected value and autocorrelation function of $X(t)$.

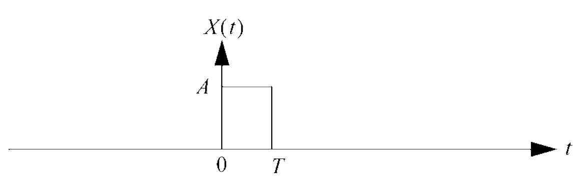


**Figure 8.6** Figure for Problem 8.1

8.4 The sample function $X(t)$ of a stationary random process $Y(t)$ is given by

$$X(t) = Y(t)\sin(wt + \Theta)$$

where $w$ is a constant, $Y(t)$ and $\Theta$ are statistically independent, and $\Theta$ is uniformly distributed between 0 and $2\pi$. Find the autocorrelation function of $X(t)$ in terms of $R_{YY}(\tau)$.

8.5 The sample function $X(t)$ of a stationary random process $Y(t)$ is given by

$$X(t) = Y(t)\sin(wt + \Theta)$$

where $w$ is a constant, $Y(t)$ and $\Theta$ are statistically independent, and $\Theta$ is uniformly distributed between 0 and $2\pi$. Find the autocovariance function of $X(t)$.

8.6 The random process $X(t)$ is given by

$$X(t) = A\cos(wt) + B\sin(wt)$$

where $w$ is a constant, and $A$ and $B$ are independent standard normal random variables (i.e., zero mean and variance of 1). Find the autocovariance function of $X(t)$.

8.7 Assume that $Y$ is a random variable that is uniformly distributed between 0 and 2. If we define the random process $X(t) = Y\cos(2\pi t)$, $t \geq 0$, find the autocovariance function of $X(t)$.

8.8 A random process $X(t)$ is given by

$$X(t) = A\cos(t) + (B+1)\sin(t) \qquad -\infty < t < \infty$$

where $A$ and $B$ are independent random variables with $E[A] = E[B] = 0$ and $E[A^2] = E[B^2] = 1$. Find the autocovariance function of $X(t)$.

8.9 Determine the missing elements of the following autocovariance matrix of a zero-mean wide-sense stationary random process $X(t)$, where the missing elements are denoted by $xx$.

$$\mathbf{C_{XX}} = \begin{bmatrix} 1 & xx & 0.4 & xx \\ 0.8 & xx & 0.6 & 0.4 \\ xx & 0.6 & 1 & 0.6 \\ 0.2 & xx & xx & 1 \end{bmatrix}$$

8.10 The random process $X(t)$ is defined as follows:

$$X(t) = A + e^{-B|t|}$$

where $A$ and $B$ are independent random variables. $A$ is uniformly distributed over the range $-1 \leq a \leq 1$, and $B$ is uniformly distributed over the range $0 \leq b \leq 2$. Find the following:

a. the mean of $X(t)$

b. the autocorrelation function of $X(t)$

8.11 The random process $X(t)$ has the autocorrelation function $R_{XX}(\tau) = e^{-2|\tau|}$. The random process $Y(t)$ is defined as follows:

$$Y(t) = \int_0^t X^2(u)du$$

Find $E[Y(t)]$.

**Section 8.4: Crosscorrelation and Crosscovariance Functions**

8.12 Two random processes $X(t)$ and $Y(t)$ are both zero-mean and wide-sense stationary processes. If we define the random process $Z(t) = X(t) + Y(t)$, determine the autocorrelation function of $Z(t)$ under the following conditions:

a. $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

b. $X(t)$ and $Y(t)$ are orthogonal.

8.13 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(wt) + B\sin(wt)$$
$$Y(t) = B\cos(wt) - A\sin(wt)$$

where $w$ is a constant, and $A$ and $B$ zero-mean and uncorrelated random variables with variances $\sigma_A^2 = \sigma_B^2 = \sigma^2$. Find the crosscorrelation function $R_{XY}(t, t+\tau)$.

8.14 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(wt + \Theta)$$
$$Y(t) = B\sin(wt + \Theta)$$

where $w, A$, and $B$ are constants, and $\Theta$ is a random variable that is uniformly distributed between 0 and $2\pi$.

a. Find the autocorrelation function $R_{XX}(t, t+\tau)$, and show that $X(t)$ is a wide-sense stationary process.

b. Find the autocorrelation function $R_{YY}(t, t+\tau)$, and show that $Y(t)$ is a wide-sense stationary process.

c. Find the crosscorrelation function $R_{XY}(t, t+\tau)$, and show that $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

**Section 8.5: Wide-Sense Stationary Processes**

8.15 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(w_1 t + \Theta)$$
$$Y(t) = B\sin(w_2 t + \Phi)$$

where $w_1$, $w_2$, $A$, and $B$ are constants, and $\Theta$ and $\Phi$ are statistically independent random variables, each of which is uniformly distributed between 0 and $2\pi$.

a. Find the crosscorrelation function $R_{XY}(t, t+\tau)$, and show that $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

b. If $\Theta = \Phi$, show that $X(t)$ and $Y(t)$ are not jointly wide-sense stationary.

c. If $\Theta = \Phi$, under what condition are $X(t)$ and $Y(t)$ jointly wide-sense stationary?

8.16 Explain why the following matrices can or cannot be valid autocorrelation matrices of a zero-mean wide-sense stationary random process $X(t)$.

a.

$$G = \begin{bmatrix} 1 & 1.2 & 0.4 & 1 \\ 1.2 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 1.3 \\ 1 & 0.9 & 1.3 & 1 \end{bmatrix}$$

b.

$$H = \begin{bmatrix} 2 & 1.2 & 0.4 & 1 \\ 1.2 & 2 & 0.6 & 0.9 \\ 0.4 & 0.6 & 2 & 1.3 \\ 1 & 0.9 & 1.3 & 2 \end{bmatrix}$$

c.

$$K = \begin{bmatrix} 1 & 0.7 & 0.4 & 0.8 \\ 0.5 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 0.3 \\ 0.1 & 0.9 & 0.3 & 1 \end{bmatrix}$$

8.17 Two jointly stationary random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = 2\cos(5t + \phi)$$
$$Y(t) = 10\sin(5t + \phi)$$

where $\phi$ is a random variable that is uniformly distributed between 0 and $2\pi$. Find the crosscorrelation functions $R_{XY}(\tau)$ and $R_{YX}(\tau)$.

8.18 State why each of the functions, $F(\tau)$, $G(\tau)$, and $H(\tau)$, shown in Figure 8.7, can or cannot be a valid autocorrelation function of a wide-sense stationary process.

8.19 A random process $Y(t)$ is given by

$$Y(t) = A\cos(wt + \phi)$$

where $A$, $w$, and $\phi$ are independent random variables. Assume that $A$ has a mean of 3 and a variance of 9, $\phi$ is uniformly distributed between $-\pi$ and $\pi$, and $w$ is uniformly distributed between $-6$ and 6. Determine if the process is stationary in the wide sense.

8.20 A random process $X(t)$ is given by

$$X(t) = A\cos(t) + (B+1)\sin(t) \qquad -\infty < t < \infty$$

where $A$ and $B$ are independent random variables with $E[A] = E[B] = 0$ and $E[A^2] = E[B^2] = 1$. Is $X(t)$ wide-sense stationary?

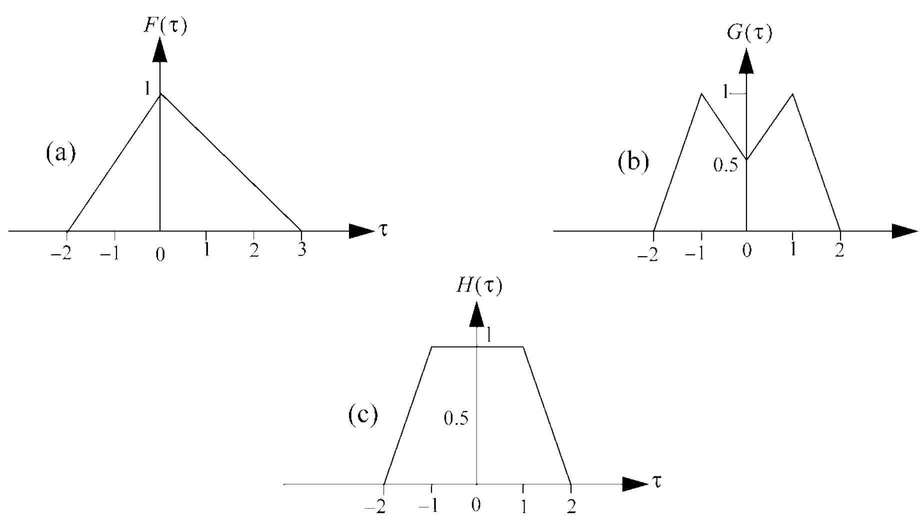


**Figure 8.7** Figure for Problem 8.18

8.21 A random process has the autocorrelation function

$$R_{XX}(\tau) = \frac{16\tau^2 + 28}{\tau^2 + 1}$$

Find the mean-square value, the mean value, and the variance of the process.

8.22 A wide-sense stationary random process $X(t)$ has a mean-square value (or average power) $E[X^2(t)] = 11$. Give reasons why the functions given below can or cannot be its autocorrelation function.

a. $R_{XX}(\tau) = \dfrac{11\sin(2\tau)}{1+\tau^2}$

b. $R_{XX}(\tau) = \dfrac{11\tau}{1+3\tau^2+4\tau^4}$

c. $R_{XX}(\tau) = \dfrac{\tau^2+44}{\tau^2+4}$

d. $R_{XX}(\tau) = \dfrac{11\cos(\tau)}{1+3\tau^2+4\tau^4}$

e. $R_{XX}(\tau) = \dfrac{11\tau^2}{1+3\tau^2+4\tau^4}$

8.23 An ergodic random process $X(t)$ has the autocorrelation function

$$R_{XX}(\tau) = 36 + \frac{4}{1+\tau^2}$$

Determine the mean value, mean-square value, and variance of $X(t)$.

8.24 Assume that $X(t)$ is the sum of a deterministic quantity $Q$ and a wide-sense stationary noise process $N(t)$. Determine the following:

a. the mean of $X(t)$

b. the autocorrelation function of $X(t)$

c. the autocovariance function of $X(t)$

8.25 Two statistically independent and zero-mean random processes $X(t)$ and $Y(t)$ have the following autocorrelation functions, respectively:

$$R_{XX}(\tau) = e^{-|\tau|}$$

$$R_{YY}(\tau) = \cos(2\pi\tau)$$

Determine the following:

a. the autocorrelation function of the process $U(t) = X(t) + Y(t)$

b. the autocorrelation function of the process $V(t) = X(t) - Y(t)$

c. the crosscorrelation function of $U(t)$ and $V(t)$

**Section 8.6: Ergodic Random Processes**

8.26 A random process $Y(t)$ is given by

$$Y(t) = A\cos(wt + \phi)$$

where $w$ is a constant, and $A$ and $\phi$ are independent random variables. The random variable $A$ has a mean of 3 and a variance of 9, and $\phi$ is uniformly distributed between $-\pi$ and $\pi$. Determine if the process is a mean-ergodic process.

8.27 A random process $X(t)$ is given by

$$X(t) = A$$

where $A$ is a random variable with a finite mean of $\mu_A$ and finite variance $\sigma_A^2$. Determine if $X(t)$ is a mean-ergodic process.

**Section 8.7: Power Spectral Density**

8.28 Assume that $V(t)$ and $W(t)$ are both zero-mean wide-sense stationary random processes and let the random process $M(t)$ be defined as follows:

$$M(t) = V(t) + W(t)$$

a. If $V(t)$ and $W(t)$ are jointly wide-sense stationary, determine the following in terms of those of $V(t)$ and $W(t)$:

1. the autocorrelation function of $M(t)$
2. the power spectral density of $M(t)$

b. If $V(t)$ and $W(t)$ are orthogonal, determine the following in terms of those of $V(t)$ and $W(t)$:

1 the autocorrelation function of $M(t)$

2. the power spectral density of $M(t)$

8.29 A stationary random process $X(t)$ has an autocorrelation function given by

$$R_{XX}(\tau) = 2e^{-|\tau|} + 4e^{-4|\tau|}$$

Find the power spectral density of the process.

8.30 A random process $X(t)$ has a power spectral density given by

$$S_{XX}(w) = \begin{cases} 4 - \dfrac{w^2}{9} & |w| \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Determine (a) the average power and (b) the autocorrelation function of the process.

8.31 A random process $Y(t)$ has the power spectral density

$$S_{YY}(w) = \frac{9}{w^2 + 64}$$

Find (a) the average power in the process and (b) the autocorrelation function.

8.32 A random process $Z(t)$ has the autocorrelation function given by

$$R_{ZZ}(\tau) = \begin{cases} 1 + \dfrac{\tau}{\tau_0} & -\tau_0 \le \tau \le 0 \\ 1 - \dfrac{\tau}{\tau_0} & 0 \le \tau \le \tau_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\tau_0$ is a constant. Calculate the power spectral density of the process.

8.33 Give reasons why the functions given below can or cannot be the power spectral density of a wide-sense stationary random process.

a. $S_{XX}(w) = \dfrac{\sin(w)}{w}$

b. $S_{XX}(w) = \dfrac{\cos(w)}{w}$

c. $S_{XX}(w) = \dfrac{8}{w^2 + 16}$

d. $S_{XX}(w) = \dfrac{5w^2}{1 + 3w^2 + 4w^4}$

e. $S_{XX}(w) = \dfrac{5w}{1 + 3w^2 + 4w^4}$

8.34 A bandlimited white noise has the power spectral density defined by

$$S_{XX}(w) = \begin{cases} 0.01 & 400\pi \le |w| \le 500\pi \\ 0 & \text{otherwise} \end{cases}$$

Find the mean-square value of the process.

8.35 A wide-sense stationary noise process $N(t)$ has an autocorrelation function

$$R_{NN}(\tau) = Ae^{-4|\tau|}$$

where $A$ is a constant. Determine the power spectral density.

8.36 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(w_0 t) + B\sin(w_0 t)$$
$$Y(t) = B\cos(w_0 t) - A\sin(w_0 t)$$

where $w_0$ is a constant, and $A$ and $B$ are zero-mean and uncorrelated random variables with variances $\sigma_A^2 = \sigma_B^2 = \sigma^2$. Find the cross-power spectral density of $X(t)$ and $Y(t)$, $S_{XY}(w)$. (Note that $S_{XY}(w)$ is the Fourier transform of the crosscorrelation function $R_{XY}(\tau)$.)

8.37 Two random processes $X(t)$ and $Y(t)$ are both zero-mean and wide-sense stationary processes. If we define the random process $Z(t) = X(t) + Y(t)$, determine the power spectral density of $Z(t)$ under the following conditions:

a. $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

b. $X(t)$ and $Y(t)$ are orthogonal.

8.38 Two jointly stationary random processes $X(t)$ and $Y(t)$ have the crosscorrelation function given by:

$$R_{XY}(\tau) = 2e^{-2\tau} \quad \tau \geq 0$$

Determine the following:

a. the cross-power spectral density $S_{XY}(w)$

b. the cross-power spectral density $S_{YX}(w)$

8.39 Two jointly stationary random processes $X(t)$ and $Y(t)$ have the cross-power spectral density given by

$$S_{XY}(w) = \frac{1}{-w^2 + j4w + 4}$$

Find the corresponding crosscorrelation function.

8.40 Two zero-mean independent wide-sense stationary random processes $X(t)$ and $Y(t)$ have the following power spectral densities

$$S_{XX}(w) = \frac{4}{w^2 + 4}$$
$$S_{YY}(w) = \frac{4}{w^2 + 4}$$

respectively. A new random process $W(t)$ is defined as follows: $W(t) = X(t) + Y(t)$. Determine the following:

a. the power spectral density of $W(t)$

b. the cross-power spectral density $S_{XW}(w)$

c. the cross-power spectral density $S_{YW}(w)$

8.41 Two zero-mean independent wide-sense stationary random processes $X(t)$ and $Y(t)$ have the following power spectral densities:

$$S_{XX}(w) = \frac{4}{w^2 + 4}$$

$$S_{YY}(w) = \frac{w^2}{w^2 + 4}$$

respectively. Two new random processes $V(t)$ and $W(t)$ are defined as follows:

$$V(t) = X(t) + Y(t)$$

$$W(t) = X(t) - Y(t)$$

respectively. Determine the cross-power spectral density $S_{VW}(w)$.

8.42 A zero-mean wide-sense stationary random process $X(t)$, $-\infty < t < \infty$, has the following power spectral density:

$$S_{XX}(w) = \frac{2}{1 + w^2} \qquad -\infty < w < \infty$$

The random process $Y(t)$ is defined by

$$Y(t) = \sum_{k=0}^{2} X(t + k)$$

a. Find the mean of $Y(t)$.

b. Find the variance of $Y(t)$.

8.43 Consider two individual wide-sense stationary processes $X(t)$ and $Y(t)$. Consider the random process $Z(t) = X(t) + Y(t)$.

a. Show that the autocorrelation function of $Z(t)$ is given by

$$R_{ZZ}(t, t + \tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(t, t + \tau) + R_{YX}(t, t + \tau).$$

b. If $X(t)$ and $Y(t)$ are jointly wide-sense stationary, show that the autocorrelation function of $Z(t)$ is given by

$$R_{ZZ}(t, t+\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau).$$

c. If $X(t)$ and $Y(t)$ are jointly wide-sense stationary, find the power spectral density of $Z(t)$.

d. If $X(t)$ and $Y(t)$ are uncorrelated, find the power spectral density of $Z(t)$.

e. If $X(t)$ and $Y(t)$ are orthogonal, find the power spectral density of $Z(t)$.

**Section 8.8: Discrete-time Random Processes**

8.44 Find the power spectral density of a random sequence $X[n]$ whose autocorrelation function is given by $R_{XX}[m] = a^m$, $m = 0, 1, 2, \ldots$, where $|a| < 1$.

8.45 A wide-sense stationary continuous-time process $X(t)$ has the autocorrelation function given by

$$R_{X_cX_c}(\tau) = e^{-2|\tau|}\cos(w_0\tau)$$

If $X(t)$ is sampled with a sampling period of 10 seconds to produce the discrete-time process $X[n]$, find the power spectral density of $X[n]$. [Hint: Use Table 8.1 to find $S_{X_cX_c}(w)$.]

8.46 Periodic samples of the autocorrelation function of white noise $N(t)$ with period $T$ are defined by

$$R_{NN}(kT) = \begin{cases} \sigma_N^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Find the power spectral density of the discrete-time random process.

8.47 The autocorrelation function $R_{XX}[k]$ of a discrete-time process $X[n]$ is given by

$$R_{XX}[k] = \begin{cases} \sigma_X^2 & k = 0 \\ \dfrac{4\sigma_X^2}{k^2\pi^2} & k = \text{odd} \\ 0 & k = \text{even} \end{cases}$$

Find the power spectral density $S_{XX}(\Omega)$ of the process.