Black Hole Mass Analysis

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1. Introduction

In the year 1915, Albert Einstein published his formulation of the General Theory of Relativity, the most advanced theory for Gravitation we have to this day in modern physics. The theory relates the curvature of spacetime to the energy and momentum of the matter and radiation present in it. This relation is defined by the Einstein Field Equations, a system of partial differential equations. Later that same year, Karl Schwarzchild, a German Physicist, derived the first exact solution to the Einstein Field Equations for a single spherical non-rotating mass. This solution came to be known as the Schwarzchild solution and the phenomenon is now what physicists refer to as a Black Hole. In 1960, Roy Kerr, a New Zealand Mathematician came up with a more general class of solutions for dense rotating objects, known as Kerr Black Holes.

Black Holes are extremely dense stellar objects formed by the collapse of a very massive star. When a sufficiently massive star goes supernova at the end of its life cycle, its own gravitational effects are so large in magnitude that the escaping gas from the supernova is pulled back towards the centre of the mass. Depending on how strong the gravitational effects are, this matter can be pulled so close together to form a Neutron Star, a densely packed star consisting only of Neutrons or, the more extreme, a Black Hole. The Gravitational effects of a Black Hole are so strong that within a certain radius from its centre (the infinitely dense singularity), even light cannot escape the curvature of its spacetime. The edge of this spherical region is called the Event Horizon.

The condition for a Black Hole to be formed is the collapse of the matter down to the Schwarzchild radius. This radius is given below,

$$R = \frac{2GM}{c^2}$$

Where R is the Schwarzschild Radius, G is Newton's Gravitational Constant, M is the mass of the body under collapse and c is the speed of light. However, a star unable to prevent collapse with Nuclear Fusion, collapses into a Black Hole under its own mass at around 3 Solar Masses.

The first Astrophysical Black Hole discovered was Cygnus A and the first Stellar Mass Black Hole discovered was Cygnus X-1. More recently, in 2019, the first image of a Supermassive Black Hole was taken by EHT Collaboration. The image was that of the Supermassive Black Hole at the centre of the M87 galaxy.

The method used to estimate the mass of a Black Hole is the same method which has been used to estimate the mass of Neutron Stars in a binary star system prior to this. A binary system of a Black Hole and a Star are taken and properties of the system are analysed. Kepler's Third Law states that the 'The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit'. Kepler's Third is accurate under the Newtonian Formulation of Gravitation however this concept can be generalised to the binary system described above without major loss in accuracy. A more detailed derivation with general elliptical orbits (which is omitted here) gives the following relation.

$$\frac{a^3}{P^2} = \frac{G(M + m)}{4\pi^2}$$

Where a is the semi-major axis of the elliptical orbit, P the orbital period, M the mass of the larger body and m the mass of the smaller body. From this law however, it is difficult to deduce the mass of the Black Hole as it is technically very tedious to measure the value of the semi-major axis of the orbit. Instead, the radial velocity v_r is observed as the star in the system is directly moving away from or towards the observer in a straight path. This can be carried out by analysing the collection of spectroscopic observations collected throughout the orbital period for redshift and blue-shift phenomena. Using this information, we can apply the binary mass function, the minimum mass the Black Hole must be, in order to get an estimate for this property.

$$f = \frac{P(v'_r)^3}{2\pi G} = \frac{M^3 sin^3(i)}{(M+m)^2}$$

(1.1)

Where v'_r is the peak radial velocity during the orbit and i is the angle of inclination from observation. Clearly, the angle at which the binary system is observed will affect how the radial component of velocity is calculated, this is discussed further where relevant.

Finally the ratio q of the mass ratio of the Black Hole to the star can be substituted in 1.1 to more easily calculate the estimation of the mass of the Black Hole.

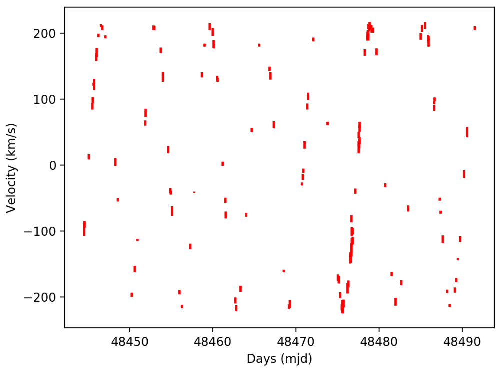
$$q = \frac{m}{M}$$

$$f = \frac{M sin^3(i)}{(1 + q)^2}$$

(1.2)

2. Binary System Orbital Analysis

The data of the binary system gives measurements of the date, the velocity and the error in velocity of the star in the binary system. The velocities with the errors are plotted below.



It can be observed from this data (as well as with our understanding of orbital mechanics) that the velocities are periodic in nature. Hence a sinusoidal function is fit to the data using the function below.

$$v_r = v_0 sin(\omega t + \phi)$$

Where v_0 is the amplitude, ω the angular frequency, t the independent time variable and ϕ the phase. An additional offset was not added to this function as it is assumed any systematic errors are taken into account from the given error in velocity. The data points along with the best fit from Least-Squares are plotted below.

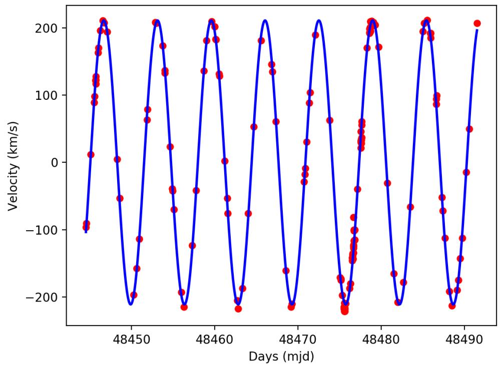


Figure 2.2

The optimal parameters arrived at by gradient descent are:

$$v_0 = 211.13666966757324 \pm 0.6243736167934268$$
$$\omega = 0.9721166616372593 \pm 0.00013744385122586243$$
$$\phi = 1389.9663693501705 \pm 6.661589491918894$$

The respective units are km/s, rad/mjd and rad. It is possible that these optimal parameters were arrived at by finding a local minimum in the parameter space for optimization but since the fit seems highly accurate qualitatively, accept these values. Also, due to this local minima phenomenon, we get a phase value outside the principal interval but we ignore this as it does not affect our calculations further. The goodness of fit is measured using the Chi-squared metric and the Reduced Chi-squared metric. The number of data points available to us is 153 and the degrees of freedom in the model is 3.

$$\chi^2 = 599.4778066058172$$

$$\chi_v{}^2 = 3.9965187107054483$$

Here, the observed $\chi_v{}^2$ values are higher than we would ideally want ($\chi_v{}^2 \approx 1$ is ideal). This is possibly due to numerous outliers at the peaks and troughs of the fit function which aren't fully modelled by the function given. We still accept the given optimal parameter as the $\chi_v{}^2$ values are still relatively close to what we would want.

From this, we can calculate the maximum radial velocity and orbital time period. The maximum radial velocity is simply the maximum value which v_r can take. This occurs when the $sin(\omega t + \phi)$ function takes a value of 1 so the peak velocity is simply,

$$v'_r = v_0$$

$$v'_r = 211.13666966757324 \pm 0.6243736167934268$$

Similarly, we derive the orbital time period from ω.

$$\omega = 2\pi f$$

$$f = \frac{1}{P}$$

$$P = \frac{2\pi}{\omega}$$

The uncertainty in this value can be calculated as follows (calculated in mjd units),

$$\sigma_P{}^2 = (\frac{\partial P}{\partial \omega})^2 \sigma_\omega{}^2$$

$$P = 6.463406662115341 \pm 0.0009138363107405669$$

3. Binary Mass Function and Mass Estimation

Now that we have calculated the peak velocity and orbital period, we can use Eq 1.1, to find the minimum possible mass of the Black Hole using the Binary Mass Function. The Universal Gravitational Constant is taken to be $6.673 * 10^{-11}$ and the uncertainty in this value is neglected as it has a high degree of accuracy which will not significantly affect the resulting uncertainty in the Binary Mass Function.

$$f = \frac{P(v'_r)^3}{2\pi G} = \frac{M^3 sin^3(i)}{(M+m)^2}$$

$$\sigma_f{}^2 = (\frac{\partial f}{\partial P})^2 \sigma_P{}^2 + (\frac{\partial f}{\partial v'_r})^2 \sigma_{v'_r}{}^2$$

The results are:

$$f = (1.2536180361115522 \pm 0.01112301462683895) * 10^{31}$$

This is the minimum mass of the Black Hole in kg.

After several years of gathering addition data on this binary system, it was observed that the angle of inclination and the q- ratio were determined to be,

$$i = 56^\circ \pm 4^\circ$$

$$q = \frac{m}{M} = 0.067 \pm 0.005$$

Using this newly acquired data, we can make use of Eq 1.2 to find the estimated mass of the Black Hole.

$$M = \frac{(1 + q)^2 f}{sin^3(i)}$$

$$\sigma_M{}^2 = (\frac{\partial M}{\partial q})^2 \sigma_q{}^2 + (\frac{\partial M}{\partial f})^2 \sigma_f{}^2 + (\frac{\partial M}{\partial i})\sigma_i{}^2$$

The result is the following values for the mass of the Black Hole in kg,

$$M = (2.504790630581239 \pm 0.3541977317686197) * 10^{31}$$

4. Conclusion

Analysing the values obtained for the Binary Mass Function and the mass of the Black Hole, we see that both of these values are well above the mass of formation of Black Holes. The general mass of formation of a Black Hole is around 3 Solar Masses which is approximately $6 *$

$10^{30}\ kg$. Even with maximum negative error, the Binary Mass Function is consistent with having the mass of a Black Hole.

It can be concluded therefore, that the more massive object in the analysed binary system is in fact very likely to be a Black Hole as it exhibits the mass characteristics of one and its mass is approximately 12.5 Solar Masses. The Orbital Period of orbiting star is estimated to be 6.46 mjd with a peak radial velocity of 211.13 km/s. Using the Schwarzchild radius equation given in the first section, the radius of the Black Hole is estimated to be

$$R = 37194.707242433826 \pm 5259.633591016271\ km.$$

5. References

[1] Narayan, R., & Mcclintock, J.E. (2013). Observational Evidence for Black Holes, https://arxiv.org/abs/1312.6698.