Papoulis Question 6.60

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Outline

Question

x and y are independent exponential random variables with common parameter λ. Find
(a) $E[\min(x, y)]$
(b) $E[\max(2x, y)]$.

Genaral

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases} \tag{1}$$

$$f(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & otherwise \end{cases} \tag{2}$$

Part (a)

Let $w = \min(x, y)$ Then from inclusion-exclusion,

$$F_w(w) = F_x(w) + F_y(w) - F_x(w).F_y(w) \tag{3}$$

Differentiating,

$$f_w(w) = f_x(w) + f_y(w) - f_x(w).F_y(w) - F_x(w).f_y(w) \tag{4}$$

Plugging the values of $f_x(w)$ and $f_y(w)$,

$$f_w(w) = 2\lambda e^{-2\lambda w} \tag{5}$$

As this an exponential variable its expected value is,

$$E(\min(x, y)) = \frac{1}{2\lambda} \tag{6}$$

Part (b)

Let $z = \max(2x, y)$ Then,

$$\begin{aligned} F_z(z) &= P(\max(2x, y) \leq z) && (7) \\ &= P(2x \leq z, 2x \geq y) + P(y \leq z, y \geq 2x) && (8) \\ &= P(2x \leq z, y \leq z) && (9) \\ &= F_{xy}(z/2, z) && (10) \end{aligned}$$

As x ans y are independent $F_{xy}(z/2, z) = F_x(z/2).F_y(z)$

$$F_z(z) = F_x(z/2).F_y(z) \tag{11}$$

Part (b) Contd.

Differentiating,

$$f_z(z) = F_x(z/2).f_y(z) + \frac{1}{2} f_x(z/2).F_y(z) \tag{12}$$

$$= \left(1 - e^{-\lambda z/2}\right).e^{-\lambda z} + \frac{1}{2}\left(1 - e^{\lambda z}\right).e^{-\lambda z/2} \tag{13}$$

$$= \lambda e^{-\lambda z} - \frac{3}{2}\lambda e^{-\frac{3}{2}\lambda z} + \frac{1}{2}\lambda e^{-\frac{1}{2}\lambda z} \tag{14}$$

$$\implies E(z) = \frac{1}{\lambda} - \frac{2}{3\lambda} + \frac{2}{\lambda} = \frac{7}{3\lambda} \tag{15}$$