Papoulis Question 5.14

Suryaansh Jain

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Outline

Question

x and y are independent Gamma random variables with common parameters α and β. Find the p.d.f. of
(a) $x + y$.
(b) x/y.
(c) $x/(x + y)$.

Genaral

$$f(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-x/\beta}, & x \geq 0 \\ 0, & otherwise \end{cases} \tag{1}$$

$$f(y) = \begin{cases} \frac{y^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-y/\beta}, & y \geq 0 \\ 0, & otherwise \end{cases} \tag{2}$$

Genaral

$$\phi_x(\omega) = \frac{1}{(1 - j\omega\beta)^{\alpha}} \tag{3}$$

$$\phi_y(\omega) = \frac{1}{(1 - j\omega\beta)^{\alpha}} \tag{4}$$

Part (a)

As x and y are independent $\phi_{x+y}(\omega) = \phi_x(\omega)\phi_y(\omega)$.

$$\phi_{x+y}(\omega) = \frac{1}{(1 - j\omega\beta)^{2\alpha}} \tag{5}$$

$$\implies \phi_{x+y}(\omega) \sim \text{Gamma}(2\alpha, \beta) \tag{6}$$

Part (b)

Let $z = x/y$ then we know,

$$f_z(z) = \int_0^\infty y \; f_{xy}(yz, y)\, dy \tag{7}$$

$$\implies f_z(z) = \int_0^\infty y \; \frac{(y^2 z)^{\alpha-1}}{(\Gamma(\alpha)\beta^\alpha)^2} e^{-(1+z)y/\beta}\, dy \tag{8}$$

$$= \frac{z^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha)^2} \int_0^\infty y^{(2\alpha-1)} e^{-(1+z)y/\beta}\, dy \tag{9}$$

$$= \frac{z^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha)^2} \frac{\beta^{2\alpha}}{(1+z)^{2\alpha}} \int_0^\infty u^{2\alpha-1} e^{-u}\, du \tag{10}$$

$$= \frac{\Gamma(2\alpha)\; z^{\alpha-1}}{(\Gamma(\alpha))^2 (1+z)^{2\alpha}}, \qquad z > 0 \tag{11}$$

Part (c)

Let,

$$w = \frac{x}{x+y} = \frac{z}{z+1} \tag{12}$$

$$F_w(w) = P\left(\frac{z}{1+z} \leq w\right) = P\left(\frac{w}{1-w} \geq z\right) = F_z\left(\frac{w}{1-w}\right) \tag{13}$$

Differentiation the above eqn. we get,

$$f_w(w) = \frac{1}{(1-w)^2} f_z\left(\frac{w}{1-w}\right) \tag{14}$$

$$= \frac{\Gamma(2\alpha}{(\Gamma(\alpha))^2} w^{\alpha-1}(1-w)^{\alpha-1} \tag{15}$$

$$\sim \text{Beta}(\alpha, \alpha) \tag{16}$$