Kolmogorov Smirnov One Sample Test

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Contents

Goodness of Fit

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- The goodness of fit test is used to test if sample data fits a distribution from a certain population (i.e. a population with a normal distribution or one with a Weibull distribution).
- Goodness of fit checks whether there is any significant differences between an observed frequency distribution and a given theoretical (expected) frequency distribution.

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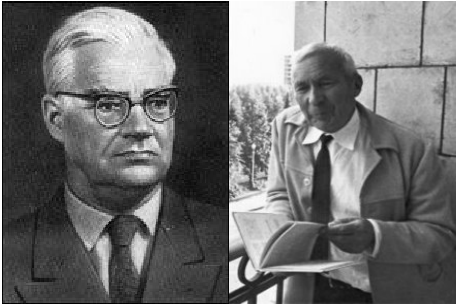
  - Designed for null hypothesis concerning a continuous distribution
    Kolmogorov Smirnov test and Lilliefor's test.

Kolmogorov Smirnov One sample test

- It is named after Andrey Kolmogorov and Nikolai Smirnoff.

Figure: Nikolai Smirnov and Andrey Kolmogorov



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- Before jumping into the test , let us take a look at some points.

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- The empirical distribution function (denoted by $S_n(x)$) of the sample is defined as the proportion of sample observations that are less than or equal of x for all real numbers x, that is,

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- Or in terms of order statistics

$$S_n(x) = \begin{cases} 0 & \text{if } x \leqslant X_{(1)} \\ \frac{i}{n} & X_{(i)} \leqslant x \leqslant X_{(i+1)} \;\; i = 1, 2, ..., n-1 \\ 1 & x \geqslant X_{(n)} \end{cases}$$

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- In case of tied observations,the edf is still a step function but it jumps only at the ordered sample values $X_{(j)}$ and the height of the jump is equal to k/n ,where k is the number of values tied at $X_{(j)}$.
- In terms of indicator variables we can express it as $S_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$

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- which implies for any fixed value x, $S_n(x)$ is a consistent estimator of $F_X(x)$

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$$P\{\lim_{n\to\infty} \sup_{-\infty<x<\infty} [|S_n(x) - F(x)|] = 0\} = 1$$

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- This D_n statistic , called **K-S one sample statistic.**

Test Statistic

- $$D_n^+ = \sup_x (S_n(x) - F_0(x))$$
  $$D_n^- = \sup_x (F_0(x) - S_n(x))$$

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- D_n is particularly useful in non-parametric statistical inference because the probability distribution of D_n does not depend on $F_0(x)$ as long as F_0 is continuous.
- Therefore, D_n is a distribution free statistic.

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- Let us define the inverse of F_0 by $F_0^{-1}(y) = min\{x : F_0(x) \geqslant y\}$.
- Then by making change of variables $y = F_0(x)$ or $x = F_0^{-1}(y)$ we can write,

$$P(D_n \leqslant d)$$

$$= P(\sup_x |S_n(x) - F_0(x)| \leqslant d)$$

$$= P(\sup_{0<y<1} |S_n(F_0^{-1}(y)) - y| \leqslant d)$$

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- Now using the definition of the empirical cdf S_n we can write,

$$S_n(F_0^{-1}(y)) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leqslant F_0^{-1}(y)) = \frac{1}{n}\sum_{i=1}^{n} I(F_0(X_i) \leqslant y)$$

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- Therefore,

$$P(\sup_{0<y<1} |S_n(F_0^{-1}(y)) - y| \leqslant d)$$

$$= P(\sup_{0<y<1} |\frac{1}{n}\sum_{i=1}^{n} I(F_0(X_i) \leqslant y) - y| \leqslant d)$$

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- is clearly independent of F_0

Results

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- This is stated in the following two results:

Result-I

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- $P\{D_n < \frac{1}{2n} + \gamma\}=$

$$\begin{cases} 0 & for\ \gamma < 0 \\ \int_{\frac{1}{2n}-\gamma}^{\frac{1}{2n}+\gamma} \int_{\frac{1}{3n}-\gamma}^{\frac{1}{3n}+\gamma} \dots \int_{\frac{2n-1}{2n}-\gamma}^{\frac{2n-1}{2n}+\gamma} f(u_1, u_2, \dots, u_n) du_n \dots d_{u_1} & for\ 0 < \gamma < \frac{2n-1}{2n} \\ 1 & for\ \gamma > \frac{2n-1}{2n} \end{cases}$$

Result-I

- For $D_n = \sup_x |S_n(x) - F_0(x)|$, where $F_0(.)$ is any specific continuous cdf, we have under H_0
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- where

$$f(u_1, u_2, \ldots, u_n) = \begin{cases} n! & for\ 0 < u_1 < u_2 < \ldots < u_n < 1 \\ 0 & o.w \end{cases}$$

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- where,

$$L(d) = 1 - 2\Sigma_{i=1}^{\propto}(-1)^{i-1}e^{-2i^2d^2}$$

Result-II

- $P_{H_0}\{D_n^+ < c\} =$
$$\begin{cases} 0 & \textit{for } c \leq 0 \\ \int_{1-c}^{1} \int_{\frac{n-1}{n}-c}^{u_n} \dots \int_{\frac{2}{n}-c}^{u_3} \int_{\frac{1}{n}-c}^{u_2} f(u_1, u_2, \dots u_n) du_1 \dots . du_n & \textit{for } 0 < c < 1 \\ 1 & \textit{for } c \geq 1 \end{cases}$$

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- $$D_n^+ = \sup_x (S_n(x) - F_0(x))$$
$$= \max_{0 \leq 1 \leq n} [\sup_{X_{(i)} \leq x \leq X_{(i+1)}} (\frac{i}{n} - F_0(x))]$$
$$= \max_{0 \leq 1 \leq n} (\frac{i}{n} - U_{(i)})$$
$$= \max\{\max_{1 \leq i \leq n} (\frac{i}{n} - U_{(i)})$$
$$= \max\{\max_{1 \leq i \leq n} (\frac{i}{n} - U_{(i)}), 0\}$$

Result-II

- For all $0 < c < 1$, we have,

$$P(D_n^+ < c)$$

$$= P[\max_{1 \le i \le n} (\frac{i}{n} - X_{(i)}) < c]$$

$$= P(\frac{i}{n} - X_{(i)} < c \text{ for all } i = 1, 2, ..., n)$$

$$= P(X_{(i)} > \frac{i}{n} - c \text{ for all } i = 1, 2, ..., n)$$

$$= \int_{1-c}^{\infty} \int_{(n-1)/n-c}^{\infty} \cdots \int_{2/n-c}^{\infty} \int_{1/n-c}^{\infty} f(x_1, x_{2,}, ..., x_n) dx_1 dx_2 ... dx_n$$

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- which is equivalent to the stated integral. **(Proved)**

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Proof.

- We have $D_n^+ < d/\sqrt{n}$ if and only if $4nD_n^{+2} < 4d^2$ or $V < 4d^2$

$\square$

Note

- Therefore

$$\lim_{n\to\infty} P(V < 4d^2)$$

$$= \lim_{n\to\infty} P(D_n^+ < d/\sqrt{n})$$

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- Therefore
$$\lim_{n\to\infty} P(V < 4d^2)$$
$$= \lim_{n\to\infty} P(D_n^+ < d/\sqrt{n})$$
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$$= 1 - e^{-4d^2/2}$$
- So, $\lim_{n\to\infty} P(V < c) = 1 - e^{-c/2} \quad$ for all $c > 0$

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- So, $\lim_{n \to \infty} P(V < c) = 1 - e^{-c/2}$ for all $c > 0$
- The right-hand side is the cdf of a chi square distribution with 2 degrees of freedom. **(Proved)**

Applications of the K-S One Sample Statistics

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Applications of the K-S One Sample Statistics

- The differences between $S_n(x)$ & $F_0(x)$should be small for all x except for sampling variation , if the H_0 is true.
- For $H_1 : F_X(x) \neq F_0(x)$ for some x, large absolute values of these deviations tend to discredit the H_0.
- Therefore , the K-S goodness of fit test with significance level α is to reject H_0 when $D_n > D_{n,\alpha}$.

Applications of the K-S One Sample Statistics

- The following expression is considerably easier for algebraic calculations & applies when ties are present:

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$$= max[|S_n(x) - F_0(x)|, |S_n(x - \epsilon) - F_0(x)|]$$

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- where ϵ denotes any small +ve number.

One Sided Test

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- the approximate rejection region is $D_n^+ > D_{n,\alpha}^+$
- Most test of goodness of fit are two-sided.
- The tail probabilities for the one sided statistic are approximately one-half of the corresponding tail probabilities for the two sided statistic.

Confidence Band

- Now

$$P\{D_n > D_{n,\alpha}\} = \alpha$$

$$\iff P\{D_n < D_{n,\alpha}\} = 1 - \alpha$$

$$\iff P\{\sup_x |S_n(x) - F_0(x)| < D_{n,\alpha}\} = 1 - \alpha$$

$$\iff P\{S_n(x) - D_{n,\alpha} < F_0(x) < S_n(x) + D_{n,\alpha}\} = 1 - \alpha$$

Confidence Band

- Thus we define,

$$L_n(x) = max(S_n(x) - D_{n,\alpha}, 0)$$

$$U_n(x) = min(S_n(x) + D_{n,\alpha}, 1)$$

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- as lower & upper confidence bands with associated confidence coefficient $(1 - \alpha)$

Determination of sample size

- The statistic $D_n$enables us to determine the minimum sample size occupied to guarantee with a certain probability $1 - \alpha$, that the error in the estimate never exceeds a fixed value .

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- $'n'$ can be read directly from table as that sample size corresponding to $D_{n,\alpha} = c$.
- If no $n \leq 40$ will meet the specified accuracy, the asymptotic distribution of D_n ($\lim\limits_{n \to \propto} P\{D_n \leqslant \frac{d}{\sqrt{n}}\} = L(d)$ where $L(d) = 1 - 2\Sigma_{i=1}^{\propto}(-1)^{i-1}e^{-2i^2d^2}$ can be used by solving $c = d/\sqrt{n}$ for n, where $d/\sqrt{n}$ is given in the last row of the table.

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- This entry is 0.247 which corresponds to $n = 36$.

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- Non-restriction of sample size.
- The $D-$ value is an easy to compute and the graph can be understood easily.

Drawbacks

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- It works best only when the CDF's deviate the most near the center of the distribution.
- The situation in which normality tests are needed –small sample sizes–is also a situation when they perform poorly.

Practical Method

Steps in K-S test:

1. sort the data from smallest to largest.

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4. If D is greater than critical D , then it can be concluded that the distribution are indeed different , otherwise there is not enough evidence to prove the difference between the two data-set.
5. A P-value can also be calculated from the D value and the sample size of the two data sets.

Example

- Here is data for 100 observations.

Figure: Given Data

Suppose you are given the following 100 observations.

-0.16	-0.68	-0.32	-0.85	0.89	-2.28	0.63	0.41	0.15	0.74
1.30	-0.13	0.80	-0.75	0.28	-1.00	0.14	-1.38	-0.04	-0.25
-0.17	1.29	0.47	-1.23	0.21	-0.04	0.07	-0.08	0.32	-0.17
0.13	-1.94	0.78	0.19	-0.12	-0.19	0.76	-1.48	-0.01	0.20
-1.97	-0.37	3.08	-0.40	0.80	0.01	1.32	-0.47	2.29	-0.26
-1.52	-0.06	-1.02	1.06	0.60	1.15	1.92	-0.06	-0.19	0.67
0.29	0.58	0.02	2.18	-0.04	-0.13	-0.79	-1.28	-1.41	-0.23
0.65	-0.26	-0.17	-1.53	-1.69	-1.60	0.09	-1.11	0.30	0.71
-0.88	-0.03	0.56	-3.68	2.40	0.62	0.52	-1.25	0.85	-0.09
-0.23	-1.16	0.22	-1.68	0.50	-0.35	-0.35	-0.33	-0.24	0.25

Do they come from N(0,1)?

Example

- 1st we order the data.

Figure: Ordered Data

-3.68	-2.28	-1.97	-1.94	-1.69	-1.68	-1.60	-1.53	-1.52	-1.48
-1.41	-1.38	-1.28	-1.25	-1.23	-1.16	-1.11	-1.02	-1.00	-0.88
-0.85	-0.79	-0.75	-0.68	-0.47	-0.40	-0.37	-0.35	-0.35	-0.33
-0.32	-0.26	-0.26	-0.25	-0.24	-0.23	-0.23	-0.19	-0.19	-0.17
-0.17	-0.17	-0.16	-0.13	-0.13	-0.12	-0.09	-0.08	-0.06	-0.06
-0.04	-0.04	-0.04	-0.03	-0.01	0.01	0.02	0.07	0.09	0.13
0.14	0.15	0.19	0.20	0.21	0.22	0.25	0.28	0.29	0.30
0.32	0.41	0.47	0.50	0.52	0.56	0.58	0.60	0.62	0.63
0.65	0.67	0.71	0.74	0.76	0.78	0.80	0.80	0.85	0.89
1.06	1.15	1.29	1.30	1.32	1.92	2.18	2.29	2.40	3.08

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- then we compute the empirical distribution.

Example

- Here

$$S_{100}(-3.68) = \frac{1}{100}$$

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$$\vdots$$

$$S_{100}(3.08) = 1$$

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$$S_{100}(3.08) = 1$$

- If our data is ordered, x_1 being the least and x_n being the largest,

$$S_n(x_i) = \frac{i}{100}$$

Example

Figure

0.000	0.011	0.024	0.026	0.045	0.047	0.055	0.064	0.064	0.070
0.080	0.084	0.101	0.107	0.110	0.123	0.133	0.154	0.158	0.189
0.198	0.215	0.226	0.249	0.321	0.343	0.356	0.362	0.363	0.369
0.375	0.396	0.399	0.400	0.407	0.409	0.410	0.423	0.425	0.432
0.432	0.434	0.437	0.447	0.449	0.453	0.464	0.468	0.476	0.477
0.484	0.484	0.485	0.490	0.496	0.505	0.508	0.526	0.535	0.553
0.557	0.560	0.577	0.577	0.582	0.588	0.597	0.610	0.614	0.617
0.627	0.658	0.680	0.692	0.698	0.711	0.720	0.727	0.732	0.735
0.743	0.748	0.761	0.771	0.777	0.783	0.788	0.789	0.803	0.812
0.854	0.874	0.902	0.903	0.907	0.973	0.985	0.989	0.992	0.999

- for each observation x_i compute $F_{exp}(x_i) = P(Z \leq x_i)$

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- for each observation x_i compute $F_{exp}(x_i) = P(Z \leq x_i)$
- In this case the expected distribution function is standard normal so use the normal table.

Example

- Then compute the absolute difference between the entries in the two tables.

0.010	0.009	0.006	0.014	0.004	0.014	0.015	0.017	0.026	0.031
0.031	0.036	0.030	0.034	0.041	0.037	0.037	0.026	0.031	0.011
0.012	0.005	0.003	0.008	0.069	0.085	0.086	0.083	0.073	0.071
0.064	0.077	0.067	0.061	0.055	0.049	0.039	0.045	0.035	0.033
0.023	0.013	0.006	0.008	0.002	0.008	0.006	0.012	0.014	0.024
0.026	0.036	0.046	0.052	0.054	0.056	0.062	0.052	0.054	0.048
0.054	0.060	0.055	0.061	0.067	0.073	0.071	0.070	0.076	0.082
0.084	0.061	0.049	0.049	0.052	0.048	0.051	0.054	0.058	0.064
0.068	0.071	0.069	0.070	0.074	0.078	0.082	0.092	0.088	0.087
0.055	0.045	0.029	0.037	0.043	0.013	0.015	0.009	0.002	0.001

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- The Kolmogorov-Smirnov statistic $D_n = 0.092$ is the maximum shown here in blue.

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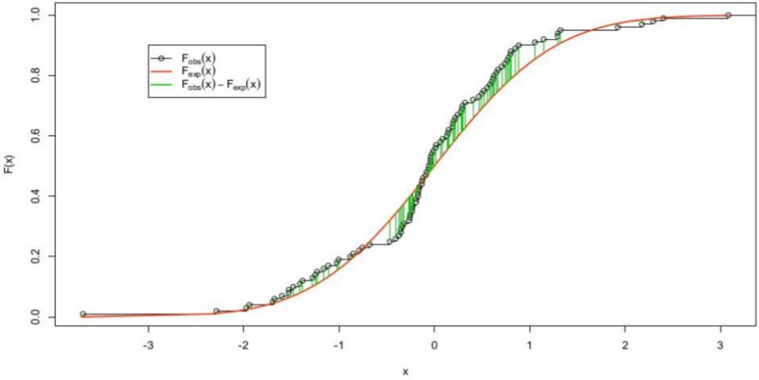
- At the 95% level the critical value is approximately given by,

$$D_{crit,0.05} = \frac{1.36}{\sqrt{n}}$$

- Here we have a sample size of n = 100 so $D_{crit} = 0.136$.
- Since $0.092 < 0.136$ do not reject the null hypothesis.

Example

- We have calculated the maximum absolute distance between the expected and observed distribution functions, in green in the plot below.



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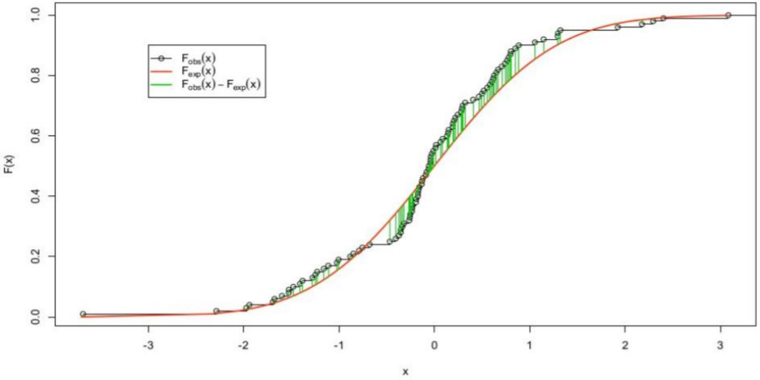
- Here $F_{obs}(x) = S_n(x)$



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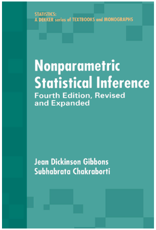
- Nonparametric Statistical Inference (Fourth Edition),by Jean Dickinson Gibbons and Subhabrata Chakraborti



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