6. Celestial Mechanics

Celestial mechanics, the study of motions of celestial bodies, together with spherical astronomy, was the main branch of astronomy until the end of the 19th century, when astrophysics began to evolve rapidly. The primary task of classical celestial mechanics was to explain and predict the motions of planets and their satellites. Several empirical models, like epicycles and Kepler's laws, were employed to describe these motions. But none of these models explained why the planets moved the way they did. It was only in the 1680's that a simple explanation was found for all these motions – Newton's law of universal gravitation. In this chapter, we will derive some properties of orbital motion. The physics we need for this is simple indeed, just Newton's laws. (For a review, see *Newton's Laws, p. 126)

This chapter is mathematically slightly more involved than the rest of the book. We shall use some vector calculus to derive our results, which, however, can be easily understood with very elementary mathematics. A summary of the basic facts of vector calculus is given in Appendix A.4.

6.1 Equations of Motion

We shall concentrate on the systems of only two bodies. In fact, this is the most complicated case that allows a neat analytical solution. For simplicity, let us call the bodies the Sun and a planet, although they could quite as well be a planet and its moon, or the two components of a binary star.

Let the masses of the two bodies be m_1 and m_2 and the radius vectors in some fixed inertial coordinate frame $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Fig. 6.1). The position of the planet relative to the Sun is denoted by $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. According to Newton's law of gravitation the planet feels a gravitational pull proportional to the masses m_1 and m_2 and inversely proportional to the square of the distance r. Since the force is directed towards the Sun, it can be expressed as

$$\boldsymbol{F} = \frac{Gm_1m_2}{r^2}\frac{-\boldsymbol{r}}{r} = -Gm_1m_2\frac{\boldsymbol{r}}{r^3} , \tag{6.1}$$

where G is the *gravitational constant*. (More about this in Sect. 6.5.)

Newton's second law tells us that the acceleration $\ddot{\boldsymbol{r}}_2$ of the planet is proportional to the applied force:

$$\boldsymbol{F} = m_2\ddot{\boldsymbol{r}}_2 . \tag{6.2}$$

Combining (6.1) and (6.2), we get the *equation of motion* of the planet

$$m_2\ddot{\boldsymbol{r}}_2 = -Gm_1m_2\frac{\boldsymbol{r}}{r^3} . \tag{6.3}$$

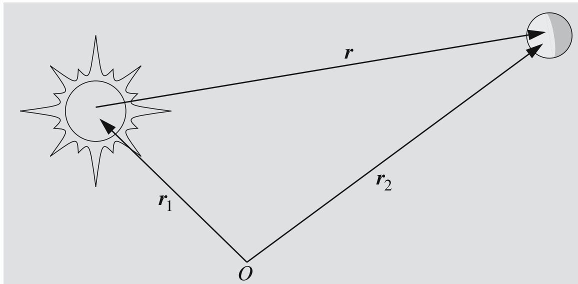

Fig. 6.1. The radius vectors of the Sun and a planet in an arbitrary inertial frame are $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the position of the planet relative to the Sun

Since the Sun feels the same gravitational pull, but in the opposite direction, we can immediately write the equation of motion of the Sun:

$$m_1\ddot{\boldsymbol{r}}_1 = +Gm_1m_2\frac{\boldsymbol{r}}{r^3} . \tag{6.4}$$

We are mainly interested in the relative motion of the planet with respect to the Sun. To find the equation of the relative orbit, we cancel the masses appearing on both sides of (6.3) and (6.4), and subtract (6.4) from (6.3) to get

$$\ddot{\boldsymbol{r}} = -\mu\frac{\boldsymbol{r}}{r^3} , \tag{6.5}$$

where we have denoted

$$\mu = G(m_1 + m_2) . \tag{6.6}$$

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The solution of (6.5) now gives the relative orbit of the planet. The equation involves the radius vector and its second time derivative. In principle, the solution should yield the radius vector as a function of time, $\boldsymbol{r} = \boldsymbol{r}(t)$. Unfortunately things are not this simple in practice; in fact, there is no way to express the radius vector as a function of time in a closed form (i.e. as a finite expression involving familiar elementary functions). Although there are several ways to solve the equation of motion, we must resort to mathematical manipulation in one form or another to figure out the essential properties of the orbit. Next we shall study one possible method.

6.2 Solution of the Equation of Motion

The equation of motion (6.5) is a second-order (i.e. contains second derivatives) vector valued differential equation. Therefore we need six integration constants or *integrals* for the complete solution. The solution is an infinite family of orbits with different sizes, shapes and orientations. A particular solution (e.g. the orbit of Jupiter) is selected by fixing the values of the six integrals. The fate of a planet is unambiguously determined by its position and velocity at any given moment; thus we could take the position and velocity vectors at some moment as our integrals. Although they do not tell us anything about the geometry of the orbit, they can be used as initial values when integrating the orbit numerically with a computer. Another set of integrals, the *orbital elements*, contains geometric quantities describing the orbit in a very clear and concrete way. We shall return to these later. A third possible set involves certain physical quantities, which we shall derive next.

We begin by showing that the angular momentum remains constant. The angular momentum of the planet in the heliocentric frame is

$$\boldsymbol{L} = m_2 \boldsymbol{r} \times \dot{\boldsymbol{r}} \, . \tag{6.7}$$

Celestial mechanicians usually prefer to use the angular momentum divided by the planet's mass

$$\boldsymbol{k} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \, . \tag{6.8}$$

Let us find the time derivative of this:

$$\dot{\boldsymbol{k}} = \boldsymbol{r} \times \ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} \, .$$

The latter term vanishes as a vector product of two parallel vectors. The former term contains $\ddot{\boldsymbol{r}}$, which is given by the equation of motion:

$$\dot{\boldsymbol{k}} = \boldsymbol{r} \times (-\mu \boldsymbol{r}/r^3) = -(\mu/r^3)\boldsymbol{r} \times \boldsymbol{r} = 0 \, .$$

Thus $\boldsymbol{k}$ is a constant vector independent of time (as is $\boldsymbol{L}$, of course).

Since the angular momentum vector is always perpendicular to the motion (this follows from (6.8)), the motion is at all times restricted to the invariable plane perpendicular to $\boldsymbol{k}$ (Fig. 6.2).

To find another constant vector, we compute the vector product $\boldsymbol{k} \times \ddot{\boldsymbol{r}}$:

$$\begin{aligned} \boldsymbol{k} \times \ddot{\boldsymbol{r}} &= (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \times (-\mu \boldsymbol{r}/r^3) \\ &= -\frac{\mu}{r^3}[(\boldsymbol{r} \cdot \boldsymbol{r})\dot{\boldsymbol{r}} - (\boldsymbol{r} \cdot \dot{\boldsymbol{r}})\boldsymbol{r}] \, . \end{aligned}$$

The time derivative of the distance r is equal to the projection of $\dot{\boldsymbol{r}}$ in the direction of $\boldsymbol{r}$ (Fig. 6.3); thus, using the properties of the scalar product, we get $\dot{r} = \boldsymbol{r} \cdot \dot{\boldsymbol{r}}/r$, which gives

$$\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = r\dot{r} \, . \tag{6.9}$$

Hence,

$$\boldsymbol{k} \times \ddot{\boldsymbol{r}} = -\mu(\dot{\boldsymbol{r}}/r - \boldsymbol{r}\dot{r}/r^2) = \frac{\mathrm{d}}{\mathrm{d}t}(-\mu \boldsymbol{r}/r) \, .$$

The vector product can also be expressed as

$$\boldsymbol{k} \times \ddot{\boldsymbol{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{k} \times \dot{\boldsymbol{r}}) \, ,$$

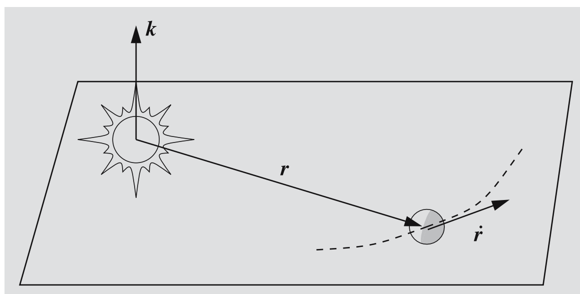

Fig. 6.2. The angular momentum vector $\boldsymbol{k}$ is perpendicular to the radius and velocity vectors of the planet. Since $\boldsymbol{k}$ is a constant vector, the motion of the planet is restricted to the plane perpendicular to $\boldsymbol{k}$

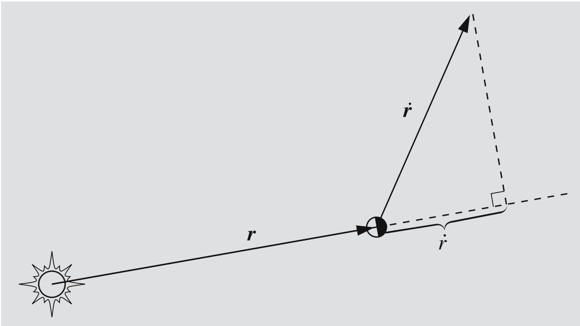

Fig. 6.3. The radial velocity $\dot{r}$ is the projection of the velocity vector $\dot{\boldsymbol{r}}$ in the direction of the radius vector $\boldsymbol{r}$

since $\boldsymbol{k}$ is a constant vector. Combining this with the previous equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{k}\times\dot{\boldsymbol{r}}+\mu\boldsymbol{r}/r)=0$$

and

$$\boldsymbol{k}\times\dot{\boldsymbol{r}}+\mu\boldsymbol{r}/r=\text{const}=-\mu\boldsymbol{e}\,. \tag{6.10}$$

Since $\boldsymbol{k}$ is perpendicular to the orbital plane, $\boldsymbol{k}\times\dot{\boldsymbol{r}}$ must lie in that plane. Thus, $\boldsymbol{e}$ is a linear combination of two vectors in the orbital plane; so $\boldsymbol{e}$ itself must be in the orbital plane (Fig. 6.4). Later we shall see that it points to the direction where the planet is closest to the Sun in its orbit. This point is called the *perihelion*.

One more constant is found by computing $\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}}$:

$$\begin{aligned}\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}}&=-\mu\dot{\boldsymbol{r}}\cdot\boldsymbol{r}/r^3=-\mu r\dot{r}/r^3\\&=-\mu\dot{r}/r^2=\frac{\mathrm{d}}{\mathrm{d}t}(\mu/r)\,.\end{aligned}$$

Since we also have

$$\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}\right)\,,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}-\frac{\mu}{r}\right)=0$$

or

$$\frac{1}{2}v^2-\mu/r=\text{const}=h\,. \tag{6.11}$$

Here v is the speed of the planet relative to the Sun. The constant h is called the *energy integral*; the total energy of the planet is m_2h. We must not forget that energy and angular momentum depend on the coordinate frame used. Here we have used a heliocentric frame, which in fact is in accelerated motion.

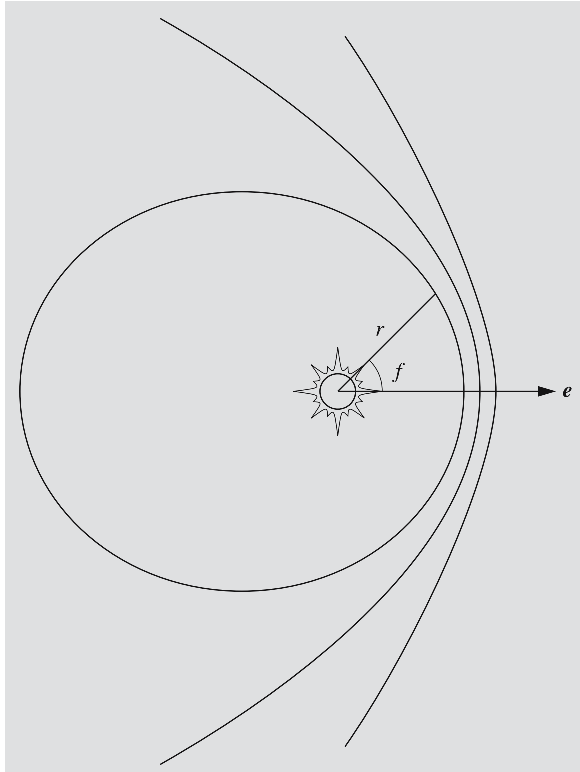

Fig. 6.4. The orbit of an object in the gravitational field of another object is a conic section: ellipse, parabola or hyperbola. Vector $\boldsymbol{e}$ points to the direction of the pericentre, where the orbiting object is closest to central body. If the central body is the Sun, this direction is called the perihelion; if some other star, periastron; if the Earth, perigee, etc. The true anomaly f is measured from the pericentre

So far, we have found two constant vectors and one constant scalar. It looks as though we already have seven integrals, i.e. one too many. But not all of these constants are independent; specifically, the following two relations hold:

$$\boldsymbol{k}\cdot\boldsymbol{e}=0\,, \tag{6.12}$$

$$\mu^2(e^2-1)=2hk^2\,, \tag{6.13}$$

where e and k are the lengths of $\boldsymbol{e}$ and $\boldsymbol{k}$. The first equation is obvious from the definitions of $\boldsymbol{e}$ and $\boldsymbol{k}$. To prove (6.13), we square both sides of (6.10) to get

$$\mu^2 e^2 = (\boldsymbol{k} \times \dot{\boldsymbol{r}}) \cdot (\boldsymbol{k} \times \dot{\boldsymbol{r}}) + \mu^2 \frac{\boldsymbol{r} \cdot \boldsymbol{r}}{r^2} + 2(\boldsymbol{k} \times \dot{\boldsymbol{r}}) \cdot \frac{\mu \boldsymbol{r}}{r} .$$

Since $\boldsymbol{k}$ is perpendicular to $\dot{\boldsymbol{r}}$, the length of $\boldsymbol{k} \times \dot{\boldsymbol{r}}$ is $|\boldsymbol{k}||\dot{\boldsymbol{r}}| = kv$ and $(\boldsymbol{k} \times \dot{\boldsymbol{r}}) \cdot (\boldsymbol{k} \times \dot{\boldsymbol{r}}) = k^2 v^2$. Thus, we have

$$\mu^2 e^2 = k^2 v^2 + \mu^2 + \frac{2\mu}{r} (\boldsymbol{k} \times \dot{\boldsymbol{r}} \cdot \boldsymbol{r}) .$$

The last term contains a scalar triple product, where we can exchange the dot and cross to get $\boldsymbol{k} \cdot \dot{\boldsymbol{r}} \times \boldsymbol{r}$. Next we reverse the order of the two last factors. Because the vector product is anticommutative, we have to change the sign of the product:

$$\begin{aligned} \mu^2 (e^2 - 1) &= k^2 v^2 - \frac{2\mu}{r} (\boldsymbol{k} \cdot \boldsymbol{r} \times \dot{\boldsymbol{r}}) = k^2 v^2 - \frac{2\mu}{r} k^2 \\ &= 2k^2 \left(\frac{1}{2} v^2 - \frac{\mu}{r} \right) = 2k^2 h . \end{aligned}$$

This completes the proof of (6.13).

The relations (6.12) and (6.13) reduce the number of independent integrals by two, so we still need one more. The constants we have describe the size, shape and orientation of the orbit completely, but we do not yet know where the planet is! To fix its position in the orbit, we have to determine where the planet is at some given instant of time $t = t_0$, or alternatively, at what time it is in some given direction. We use the latter method by specifying the time of perihelion passage, the *time of perihelion* τ.

6.3 Equation of the Orbit and Kepler's First Law

In order to find the geometric shape of the orbit, we now derive the equation of the orbit. Since $\boldsymbol{e}$ is a constant vector lying in the orbital plane, we choose it as the reference direction. We denote the angle between the radius vector $\boldsymbol{r}$ and $\boldsymbol{e}$ by f. The angle f is called the *true anomaly*. (There is nothing false or anomalous in this and other anomalies we shall meet later. Angles measured from the perihelion point are called anomalies to distinguish them from longitudes measured from some other reference point, usually the vernal equinox.) Using the properties of the scalar product we get

$$\boldsymbol{r} \cdot \boldsymbol{e} = re \cos f .$$

But the product $\boldsymbol{r} \cdot \boldsymbol{e}$ can also be evaluated using the definition of $\boldsymbol{e}$:

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{e} &= -\frac{1}{\mu} (\boldsymbol{r} \cdot \boldsymbol{k} \times \dot{\boldsymbol{r}} + \mu \boldsymbol{r} \cdot \boldsymbol{r}/r) \\ &= -\frac{1}{\mu} (\boldsymbol{k} \cdot \dot{\boldsymbol{r}} \times \boldsymbol{r} + \mu r) = -\frac{1}{\mu} (-k^2 + \mu r) \\ &= \frac{k^2}{\mu} - r . \end{aligned}$$

Equating the two expressions of $\boldsymbol{r} \cdot \boldsymbol{e}$ we get

$$r = \frac{k^2/\mu}{1 + e \cos f} . \tag{6.14}$$

This is the general equation of a *conic section* in polar coordinates (Fig. 6.4; see Appendix A.2 for a brief summary of conic sections). The magnitude of $\boldsymbol{e}$ gives the *eccentricity* of the conic:

$e = 0$	circle ,
$0 < e < 1$	ellipse ,
$e = 1$	parabola ,
$e > 1$	hyperbola .

Inspecting (6.14), we find that r attains its minimum when $f = 0$, i.e. in the direction of the vector $\boldsymbol{e}$. Thus, $\boldsymbol{e}$ indeed points to the direction of the perihelion.

Starting with Newton's laws, we have thus managed to prove Kepler's first law:

> The orbit of a planet is an ellipse, one focus of which is in the Sun.

Without any extra effort, we have shown that also other conic sections, the parabola and hyperbola, are possible orbits.

6.4 Orbital Elements

We have derived a set of integrals convenient for studying the dynamics of orbital motion. We now turn to another collection of constants more appropriate for describing the geometry of the orbit. The following six quantities are called the *orbital elements* (Fig. 6.5):

– semimajor axis a,
– eccentricity e,
– inclination i (or ι),

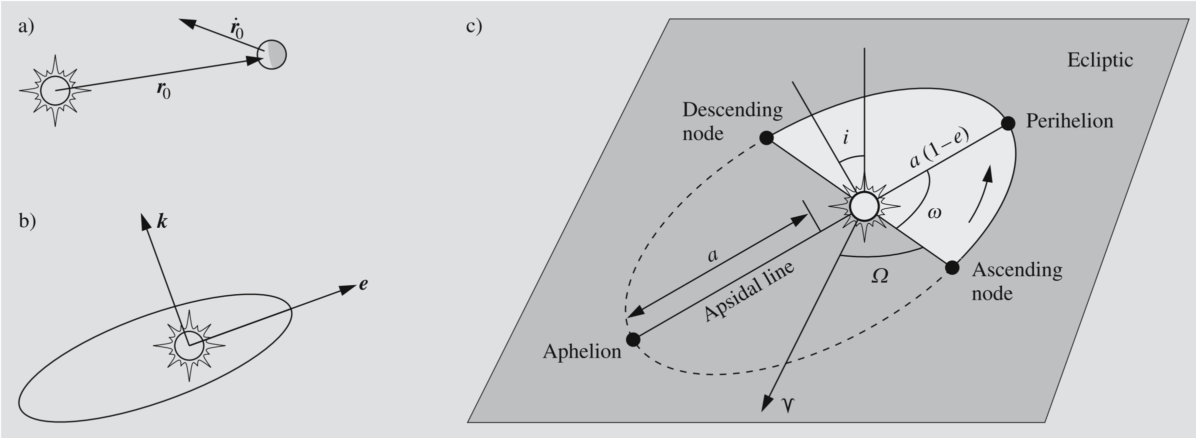

Fig. 6.5a–c. Six integration constants are needed to describe a planet's orbit. These constants can be chosen in various ways. (**a**) If the orbit is to be computed numerically, the simplest choice is to use the initial values of the radius and velocity vectors. (**b**) Another possibility is to use the angular momentum $\boldsymbol{k}$, the direction of the perihelion $\boldsymbol{e}$ (the length of which gives the eccentricity), and the perihelion time τ. (**c**) The third method best describes the geometry of the orbit. The constants are the longitude of the ascending node Ω, the argument of perihelion ω, the inclination i, the semimajor axis a, the eccentricity e and the time of perihelion τ

– longitude of the ascending node Ω,
– argument of the perihelion ω,
– time of the perihelion τ.

The eccentricity is obtained readily as the length of the vector $\boldsymbol{e}$. From the equation of the orbit (6.14), we see that the *parameter* (or semilatus rectum) of the orbit is $p = k^2/\mu$. But the parameter of a conic section is always $a|1-e^2|$, which gives the semimajor axis, if e and k are known:

$$a = \frac{k^2/\mu}{|1-e^2|} \,. \tag{6.15}$$

By applying (6.13), we get an important relation between the size of the orbit and the energy integral h:

$$a = \begin{cases} -\mu/2h \,, & \text{if the orbit is an ellipse} \,, \\ \mu/2h \,, & \text{if the orbit is a hyperbola} \,. \end{cases} \tag{6.16}$$

For a bound system (elliptical orbit), the total energy and the energy integral are negative. For a hyperbolic orbit h is positive; the kinetic energy is so high that the particle can escape the system (or more correctly, recede without any limit). The parabola, with $h = 0$, is a limiting case between elliptical and hyperbolic orbits. In reality parabolic orbits do not exist, since hardly any object can have an energy integral exactly zero. However, if the eccentricity is very close to one (as with many comets), the orbit is usually considered parabolic to simplify calculations.

The orientation of the orbit is determined by the directions of the two vectors $\boldsymbol{k}$ (perpendicular to the orbital plane) and $\boldsymbol{e}$ (pointing towards the perihelion). The three angles i, Ω and ω contain the same information.

The inclination i gives the obliquity of the orbital plane relative to some fixed reference plane. For bodies in the solar system, the reference plane is usually the ecliptic. For objects moving in the usual fashion, i.e. counterclockwise, the inclination is in the interval $[0°, 90°]$; for retrograde orbits (clockwise motion), the inclination is in the range $(90°, 180°]$. For example, the inclination of Halley's comet is 162°, which means that the motion is retrograde and the angle between its orbital plane and the ecliptic is $180° - 162° = 18°$.

The longitude of the ascending node, Ω, indicates where the object crosses the ecliptic from south to north. It is measured counterclockwise from the vernal equinox. The orbital elements i and Ω together determine the orientation of the orbital plane, and they correspond to the direction of $\boldsymbol{k}$, i.e. the ratios of its components.

The argument of the perihelion ω gives the direction of the perihelion, measured from the ascending node in the direction of motion. The same information is contained in the direction of $\boldsymbol{e}$. Very often another angle, the *longitude of the perihelion* ϖ (pronounced as pi), is used instead of ω. It is defined as

$$\varpi = \Omega + \omega\,. \tag{6.17}$$

This is a rather peculiar angle, as it is measured partly along the ecliptic, partly along the orbital plane. However, it is often more practical than the argument of perihelion, since it is well defined even when the inclination is close to zero in which case the direction of the ascending node becomes indeterminate.

We have assumed up to this point that each planet forms a separate two-body system with the Sun. In reality planets interfere with each other by disturbing each other's orbits. Still their motions do not deviate very far from the shape of conic sections, and we can use orbital elements to describe the orbits. But the elements are no longer constant; they vary slowly with time. Moreover, their geometric interpretation is no longer quite as obvious as before. Such elements are *osculating elements* that would describe the orbit if all perturbations were to suddenly disappear. They can be used to find the positions and velocities of the planets exactly as if the elements were constants. The only difference is that we have to use different elements for each moment of time.

Table C.12 (at the end of the book) gives the mean orbital elements for the nine planets for the epoch J2000.0 as well as their first time derivatives. In addition to these secular variations the orbital elements suffer from periodic disturbations, which are not included in the table. Thus only approximate positions can be calculated with these elements. Instead of the time of perihelion the table gives the *mean longitude*

$$L = M + \omega + \Omega\,, \tag{6.18}$$

which gives directly the mean anomaly M (which will be defined in Sect. 6.7).

6.5 Kepler's Second and Third Law

The radius vector of a planet in polar coordinates is simply

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r\,, \tag{6.19}$$

where $\hat{\boldsymbol{e}}_r$ is a unit vector parallel with $\boldsymbol{r}$ (Fig. 6.6). If the planet moves with angular velocity $\dot{f}$, the direction of this unit vector also changes at the same rate:

$$\dot{\hat{\boldsymbol{e}}}_r = \dot{f}\hat{\boldsymbol{e}}_f\,, \tag{6.20}$$

where $\hat{\boldsymbol{e}}_f$ is a unit vector perpendicular to $\hat{\boldsymbol{e}}_r$. The velocity of the planet is found by taking the time derivative of (6.19):

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\hat{\boldsymbol{e}}}_r = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{f}\hat{\boldsymbol{e}}_f\,. \tag{6.21}$$

The angular momentum $\boldsymbol{k}$ can now be evaluated using (6.19) and (6.21):

$$\boldsymbol{k} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = r^2\dot{f}\hat{\boldsymbol{e}}_z\,, \tag{6.22}$$

where $\hat{\boldsymbol{e}}_z$ is a unit vector perpendicular to the orbital plane. The magnitude of $\boldsymbol{k}$ is

$$k = r^2\dot{f}\,. \tag{6.23}$$

The *surface velocity* of a planet means the area swept by the radius vector per unit of time. This is obviously the time derivative of some area, so let us call it $\dot{A}$. In terms of the distance r and true anomaly f, the surface velocity is

$$\dot{A} = \frac{1}{2}r^2\dot{f}\,. \tag{6.24}$$

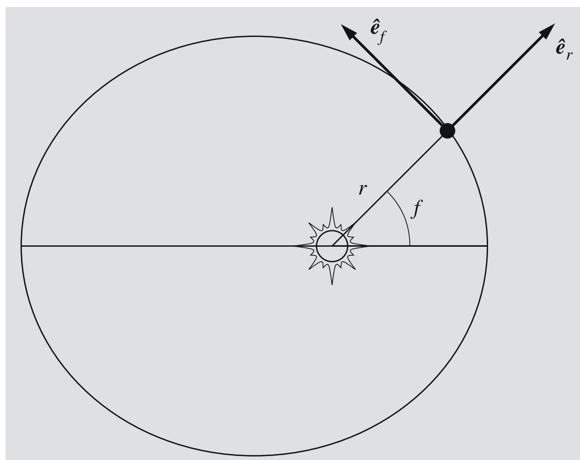

Fig. 6.6. Unit vectors $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_f$ of the polar coordinate frame. The directions of these change while the planet moves along its orbit

By comparing this with the length of $\boldsymbol{k}$ (6.23), we find that

$$\dot{A} = \frac{1}{2}k \,. \tag{6.25}$$

Since k is constant, so is the surface velocity. Hence we have Kepler's second law:

> The radius vector of a planet sweeps equal areas in equal amounts of time.

Since the Sun–planet distance varies, the orbital velocity must also vary (Fig. 6.7). From Kepler's second law it follows that a planet must move fastest when it is closest to the Sun (near perihelion). Motion is slowest when the planet is farthest from the Sun at *aphelion*.

We can write (6.25) in the form

$$\mathrm{d}A = \frac{1}{2}k\,\mathrm{d}t \,, \tag{6.26}$$

and integrate over one complete period:

$$\int\limits_{\text{orbital ellipse}} \mathrm{d}A = \frac{1}{2}k \int\limits_0^P \mathrm{d}t \,, \tag{6.27}$$

where P is the orbital period. Since the area of the ellipse is

$$\pi ab = \pi a^2\sqrt{1-e^2} \,, \tag{6.28}$$

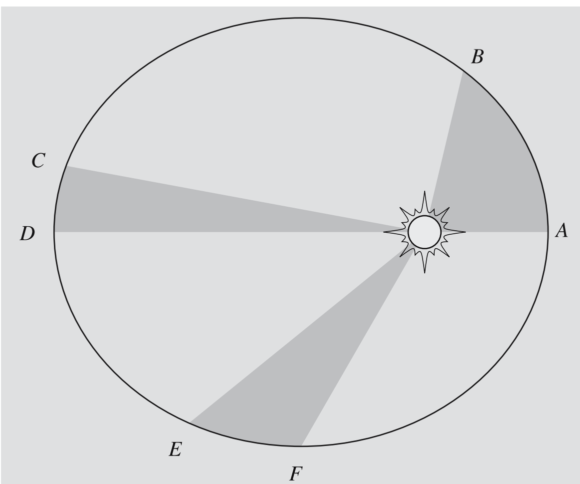

Fig. 6.7. The areas of the shaded sectors of the ellipse are equal. According to Kepler's second law, it takes equal times to travel distances AB, CD and EF

where a and b are the semimajor and semiminor axes and e the eccentricity, we get

$$\pi a^2\sqrt{1-e^2} = \frac{1}{2}kP \,. \tag{6.29}$$

To find the length of $\boldsymbol{k}$, we substitute the energy integral h as a function of semimajor axis (6.16) into (6.13) to get

$$k = \sqrt{G(m_1+m_2)a\,(1-e^2)} \,. \tag{6.30}$$

When this is substituted into (6.29) we have

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}\,a^3 \,. \tag{6.31}$$

This is the exact form of Kepler's third law as derived from Newton's laws. The original version was

> The ratio of the cubes of the semimajor axes of the orbits of two planets is equal to the ratio of the squares of their orbital periods.

In this form the law is not exactly valid, even for planets of the solar system, since their own masses influence their periods. The errors due to ignoring this effect are very small, however.

Kepler's third law becomes remarkably simple if we express distances in astronomical units (AU), times in sidereal years (the abbreviation is unfortunately a, not to be confused with the semimajor axis, denoted by a somewhat similar symbol a) and masses in solar masses ($M_\odot$). Then $G = 4\pi^2$ and

$$a^3 = (m_1+m_2)P^2 \,. \tag{6.32}$$

The masses of objects orbiting around the Sun can safely be ignored (except for the largest planets), and we have the original law $P^2 = a^3$. This is very useful for determining distances of various objects whose periods have been observed. For absolute distances we have to measure at least one distance in metres to find the length of one AU. Earlier, triangulation was used to measure the parallax of the Sun or a minor planet, such as Eros, that comes very close to the Earth. Nowadays, radiotelescopes are used as radar to very accurately measure, for example, the distance to Venus. Since changes in the value of one AU also change all other distances, the International Astronomical Union decided in 1968 to adopt the value 1 AU $= 1.496000 \times 10^{11}$ m. The semimajor axis of Earth's orbit is then slightly over one AU.

But constants tend to change. And so, after 1984, the astronomical unit has a new value,

$$1\ \mathrm{AU} = 1.49597870 \times 10^{11}\ \mathrm{m}\,.$$

Another important application of Kepler's third law is the determination of masses. By observing the period of a natural or artificial satellite, the mass of the central body can be obtained immediately. The same method is used to determine masses of binary stars (more about this subject in Chap. 9).

Although the values of the AU and year are accurately known in SI-units, the gravitational constant is known only approximately. Astronomical observations give the product $G(m_1 + m_2)$, but there is no way to distinguish between the contributions of the gravitational constant and those of the masses. The gravitational constant must be measured in the laboratory; this is very difficult because of the weakness of gravitation. Therefore, if a precision higher than 2–3 significant digits is required, the SI-units cannot be used. Instead we have to use the solar mass as a unit of mass (or, for example, the Earth's mass after $Gm_\oplus$ has been determined from observations of satellite orbits).

6.6 Systems of Several Bodies

This far we have discussed systems consisting of only two bodies. In fact it is the most complex system for which a complete solution is known. The equations of motion are easily generalized, though. As in (6.5) we get the equation of motion for the body k, $k = 1, \ldots, n$:

$$\ddot{\boldsymbol{r}}_k = \sum_{i=1,\, i \neq k}^{i=n} G m_i \frac{\boldsymbol{r}_i - \boldsymbol{r}_k}{|\boldsymbol{r}_i - \boldsymbol{r}_k|^3}\,, \tag{6.33}$$

where m_i is the mass of the ith body and $\boldsymbol{r}_i$ its radius vector. On the right hand side of the equation we now have the total gravitational force due to all other objects, instead of the force of just one body. If there are more than two bodies, these equations cannot be solved analytically in a closed form. The only integrals that can be easily derived in the general case are the total energy, total momentum, and total angular momentum.

If the radius and velocity vectors of all bodies are known for a certain instant of time, the positions at some other time can easily be calculated numerically from the equations of motion. For example, the planetary positions needed for astronomical yearbooks are computed by integrating the equations numerically.

Another method can be applied if the gravity of one body dominates like in the solar system. Planetary orbits can then be calculated as in a two-body system, and the effects of other planets taken into account as small perturbations. For these perturbations several series expansions have been derived.

The *restricted three-body problem* is an extensively studied special case. It consists of two massive bodies or *primaries*, moving on circular orbits around each other, and a third, massless body, moving in the same plane with the primaries. This small object does in no way disturb the motion of the primaries. Thus the orbits of the massive bodies are as simple as possible, and their positions are easily computed for all times. The problem is to find the orbit of the third body. It turns out that there is no finite expression for this orbit.

The Finnish astronomer *Karl Frithiof Sundman* (1873–1949) managed to show that a solution exists and derive a series expansion for the orbit. The series converges so slowly that it has no practical use, but as a mathematical result it was remarkable, since many mathematicians had for a long time tried to attack the problem without success.

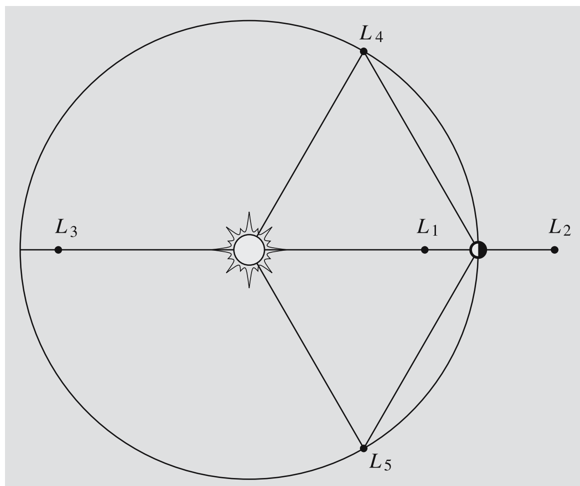

Fig. 6.8. The Lagrangian points of the restricted three-body problem. The points L_1, L_2 and L_3 are on the same line with the primaries, but the numbering may vary. The points L_4 and L_5 form equilateral triangles with the primaries

The three-body problem has some interesting special solutions. It can be shown that in certain points the third body can remain at rest with respect to the primaries. There are five such points, known as the *Lagrangian points* $L_1, \ldots, L_5$ (Fig. 6.8). Three of them are on the straight line determined by the primaries. These points are unstable: if a body in any of these points is disturbed, it will escape. The two other points, on the other hand, are stable. These points together with the primaries form equilateral triangles. For example, some *asteroids* have been found around the Lagrangian points L_4 and L_5 of Jupiter and Mars. The first of them were named after heroes of the Trojan war, and so they are called *Trojan asteroids*. They move around the Lagrangian points and can actually travel quite far from them, but they cannot escape. Fig. 7.56 shows two distinct condensations around the Lagrangian points of Jupiter.

6.7 Orbit Determination

Celestial mechanics has two very practical tasks: to determine orbital elements from observations and to predict positions of celestial bodies with known elements. Planetary orbits are already known very accurately, but new comets and minor planets are found frequently, requiring orbit determination.

The first practical methods for orbit determination were developed by *Johann Karl Friedrich Gauss* (1777–1855) at the beginning of the 19th century. By that time the first minor planets had been discovered, and thanks to Gauss's orbit determinations, they could be found and observed at any time.

At least three observations are needed for computing the orbital elements. The directions are usually measured from pictures taken a few nights apart. Using these directions, it is possible to find the corresponding absolute positions (the rectangular components of the radius vector). To be able to do this, we need some additional constraints on the orbit; we must assume that the object moves along a conic section lying in a plane that passes through the Sun. When the three radius vectors are known, the ellipse (or some other conic section) going through these three points can be determined. In practice, more observations are used. The elements determined are more accurate if there are more observations and if they cover the orbit more completely.

Although the calculations for orbit determination are not too involved mathematically, they are relatively long and laborious. Several methods can be found in textbooks of celestial mechanics.

6.8 Position in the Orbit

Although we already know everything about the geometry of the orbit, we still cannot find the planet at a given time, since we do not know the radius vector $\boldsymbol{r}$ as a function of time. The variable in the equation of the orbit is an angle, the true anomaly f, measured from the perihelion. From Kepler's second law it follows that f cannot increase at a constant rate with time. Therefore we need some preparations before we can find the radius vector at a given instant.

The radius vector can be expressed as

$$\boldsymbol{r} = a(\cos E - e)\hat{\boldsymbol{i}} + b \sin E \hat{\boldsymbol{j}} , \tag{6.34}$$

where $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors parallel with the major and minor axes, respectively. The angle E is the *eccentric anomaly*; its slightly eccentric definition is shown in Fig. 6.9. Many formulas of elliptical motion become very simple if either time or true anomaly is replaced

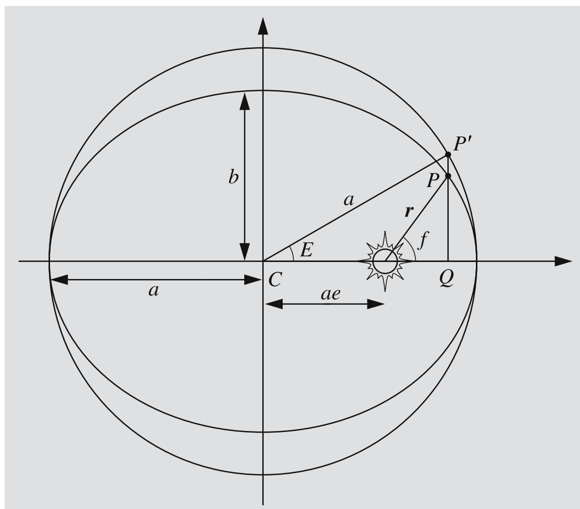

Fig. 6.9. Definition of the eccentric anomaly E. The planet is at P, and $\boldsymbol{r}$ is its radius vector

by the eccentric anomaly. As an example, we take the square of (6.34) to find the distance from the Sun:

$$\begin{aligned} r^2 &= \boldsymbol{r} \cdot \boldsymbol{r} \\ &= a^2(\cos E - e)^2 + b^2 \sin^2 E \\ &= a^2[(\cos E - e)^2 + (1-e^2)(1-\cos^2 E)] \\ &= a^2[1 - 2e\cos E + e^2 \cos^2 E]\,, \end{aligned}$$

whence

$$r = a\,(1 - e\cos E)\,. \tag{6.35}$$

Our next problem is to find how to calculate E for a given moment of time. According to Kepler's second law, the surface velocity is constant. Thus the area of the shaded sector in Fig. 6.10 is

$$A = \pi ab \frac{t-\tau}{P}\,, \tag{6.36}$$

where $t-\tau$ is the time elapsed since the perihelion, and P is the orbital period. But the area of a part of an ellipse is obtained by reducing the area of the corresponding part of the circumscribed circle by the axial ratio b/a. (As the mathematicians say, an ellipse is an affine transformation of a circle.) Hence the area of SPX is

$$\begin{aligned} A &= \frac{b}{a}(\text{area of } SP'X) \\ &= \frac{b}{a}(\text{area of the sector } CP'X \\ &\quad - \text{area of the triangle } CP'S) \\ &= \frac{b}{a}\left(\frac{1}{2}a \cdot aE - \frac{1}{2}ae \cdot a\sin E\right) \\ &= \frac{1}{2}ab(E - e\sin E)\,. \end{aligned}$$

By equating these two expressions for the area A, we get the famous *Kepler's equation*,

$$E - e\sin e = M\,, \tag{6.37}$$

where

$$M = \frac{2\pi}{P}(t-\tau) \tag{6.38}$$

is the *mean anomaly* of the planet at time t. The mean anomaly increases at a constant rate with time. It indicates where the planet would be if it moved in a circular orbit of radius a. For circular orbits all three anomalies f, E, and M are always equal.

If we know the period and the time elapsed after the perihelion, we can use (6.38) to find the mean anomaly. Next we must solve for the eccentric anomaly from Kepler's equation (6.37). Finally the radius vector is given by (6.35). Since the components of $\boldsymbol{r}$ expressed in terms of the true anomaly are $r\cos f$ and $r\sin f$, we find

$$\begin{aligned} \cos f &= \frac{a(\cos E - e)}{r} = \frac{\cos E - e}{1 - e\cos E}\,, \\ \sin f &= \frac{b\sin E}{r} = \sqrt{1-e^2}\,\frac{\sin E}{1 - e\cos E}\,. \end{aligned} \tag{6.39}$$

These determine the true anomaly, should it be of interest.

Now we know the position in the orbital plane. This must usually be transformed to some other previously selected reference frame. For example, we may want to know the ecliptic longitude and latitude, which can later be used to find the right ascension and declination. These transformations belong to the realm of spherical astronomy and are briefly discussed in Examples 6.5–6.7.

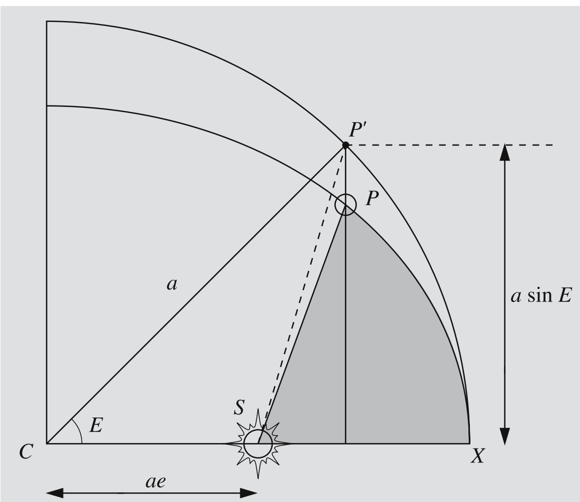

Fig. 6.10. The area of the shaded sector equals b/a times the area $SP'X$. $S =$ the Sun, $P =$ the planet, $X =$ the perihelion