1. **Black hole shadow**

The Event Horizon Telescope (EHT) has captured the images of supermassive black holes at the centre of the M87 galaxy and the Milky Way. The images show an almost circular dark central region surrounded by illuminated ring or disc around it. For this problem we consider a spherically symmetric non-rotating black hole (BH) - the Schwarzschild black hole, with an event horizon at r_s. The trajectory of light is described by the following two equations:

$$\dot{r}^2 + r^2\dot{\phi}^2\left(1 - \frac{r_s}{r}\right) = \mathcal{E}$$
$$r^2\dot{\phi} = L$$

Here, $r \in (0, \infty)$, is the radial co-ordinate and $\phi \in (0, 2\pi)$ is the angular co-ordinate. M is the mass of the BH. L is the angular momentum per unit mass of the photon and $\mathcal{E}$ can be regarded as the energy of the photon.

(a) Write down the expression for the effective potential (V_{eff}) for the light trajectory and sketch it as a function of r.

(b) Determine the radius, (r_{ph}) of the circular orbit of the light around the BH.

(c) From the sketch of V_{eff}, show that any ingoing light trajectory, if it crosses the $r = r_{ph}$ sphere, then it will fall into the black hole. Also, determine the value of the $\mathcal{E}$ for this critical circular trajectory in terms of L and M.

(d) Determine the shape equation $\left(\frac{dr}{d\phi}\right)$ for the light trajectory.

(e) Due to the strong gravity of the BH, any light ray escaping from the circular light trajectory will follow a curved path to reach an observer outside. If the observer is far away from the BH, $r_0 \to \infty$, find the projected radius r_{shad} that the observer will see the photon ring at.

(f) The supermassive BH at the centre of our galaxy has a mass of around 4 million solar masses and is at a distance of 27,000 light years away from earth. Determine the minimum angular resolution required for the EHT telescope to see the BH. If the EHT operates in radio wavelength of 1.3 mm, then how much is the baseline needed to reach the required minimum angular resolution.