Analemma Analysis

This is a question about analemma. You can go through this after finishing positional astronomy, that is, celestial coordinate systems. (Note: The answer might contain mistakes as it wasn't verified by anyone)

Question An obscure stellar object seems to move through the sky. Scientists do not know what it is, and have only the following trajectory available, which traces the object's path throughout the sky, as seen from a fixed location and at a fixed time everyday for a year.

Assume you are a scientist. You have to figure out the stellar object's relative (w.r.t. Earth-like planet) orbital motion in space. Proceed as follows: Let the mass of your planet be M and the mass of the object be 1000M and the mass of your planet be M. Consider a binary star system. Consider the orbits around the centre of mass. Take into account the planet's axial inclination (24 degrees). What can you deduce regarding your planet's orbital shape, considering the given data and above graph? Justify with required calculation, diagram and reasoning. If the trajectory in the graph was lopsided, would your answer to the above question change? If so, why? (Hint: Assume Kepler's laws of orbital motion to be valid)

Analemma : Problem 1.2

Astrophysics Problem Set

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Summary

1. The planet revolves around the stellar object and the orbit is circular because the trajectory (analemma) is symmetric about a vertical line, a horizontal line and is not degenerate. (Figure 3)

2. If the diagram becomes lopsided, it means that the orbit became elliptical. This is because of variable angular velocity of revolution. (Figure 5)

Height of the graph will be twice the axial tilt of the planet.
This trajectory will be degenerate (a horizontal line segment or a point) if axial tilt $= 0°$.

Detailed Reasoning

Clearly the planet and the stellar object seem to have a periodic motion with time period of 1 year. So they must be gravitating bodies , that is, the object is planet's own star - like Sun is to Earth (it can't be any other body like fellow planet/asteroid/moon/comet etc because its highly unlikely to have period of 1 yr of that planet).
Let us call stellar object as S and planet as P.
Normally if we photograph our Sun everyday of the year at same time, say 12 noon, we'd expect to get a point on the plot. But we do not because of two factors:

1. Axial tilt of planet wrt. orbital plane (plane of motion of two bodies).

2. Eccentricity of the planet's orbit, and its orientation in space.

The position of observer obviously doesn't effect the shape of analemma, it only effects the orientation of analemma. So, without loss of generality consider an observer at north pole of P. On the day of spring equinox, at 12:00 GMT (lets say there is some standard time like GMT for the sake of argument) he sees S just behind a small bush (Fig. 2) at horizon.
If the axial tilt of planet was $0°$ then on the next day, day after and every day at 12:00 GMT we would see S behind the bush (because that is how we defined our time system!). However, if the tilt is non-zero and orbit is perfectly circular, consider the equatorial coordinate digram (Fig. 1) showing equator, ecliptic (the path traced by S over year in the sky), α-Right Ascension, δ-Declination, λ-geocentric ecliptic longitude of S and ϵ-axial tilt. Let ω be angular velocity of revolution of P.

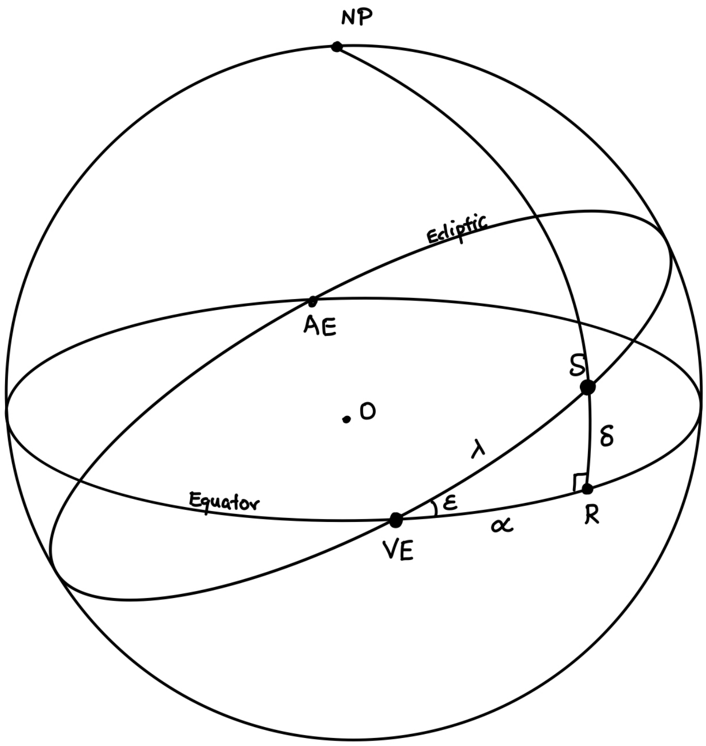

Figure 1: Equatorial Coordinate System. VE represents the Spring Equinox (Vernal Equinox) and AE is the Autumnal Equinox. S is the position of stellar object at time t. R is the point of intersection of great circle drawn through north pole (NP) and S with the equator.

We can say $\lambda = \omega t$, for a circular orbit ($t = 0$ represents spring equinox). Consider the spherical triangle VE-S-R, using the spherical trigonometric formulae, we get, $\delta = \sin^{-1}(\sin\epsilon\sin\lambda) = \sin^{-1}(\sin\epsilon\sin(\omega t))$ and $\alpha = \cos^{-1}\left(\frac{\cos\lambda}{\cos\delta}\right) = \cos^{-1}\left(\frac{\cos(\omega t)}{\cos(\sin^{-1}(\sin\epsilon\sin(\omega t)))}\right)$. The plot given in the problem is a-Altitude vs A-Azimuth at same time everyday. For situation described above,

$$a = \delta \quad \text{and} \quad A = \alpha - \omega t$$

The above equations satisfy if we take $\epsilon = 0$, i.e., we get $a = 0$ and $A = 0$ as expected.
We can conclude that the two lobes will be equal and analemma has symmetry about x and y axes as the orbit is circular (and vice versa).
Here a-altitude component of graph arises because declination (δ) is not constant. And A-azimuth component arises because right acension (α) sometimes changes slower and sometimes changes faster than ωt.

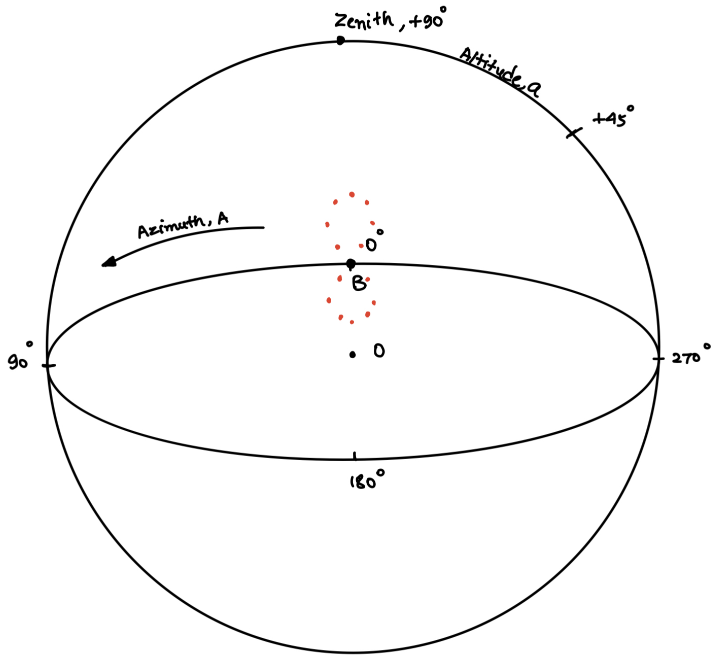

Figure 2: Alt-azimuth system of an observer at north pole of P. B is the ‘bush’

Having elliptical orbit complicates the problem λ will no longer be simply ωt, it will be a complicated function of time. The analemma will also depend on the angle the line of eqinoxes makes with the major axis of the ellipse - ϕ.
The analemma will lose symmetry about y-axis. The two lobes will be of unequal size except when $\phi = 0°$. This happens because λ and hence α and δ will change at a faster/slower rate depending on position. Here A- azimuth need not be zero at solstices and equinoxes.

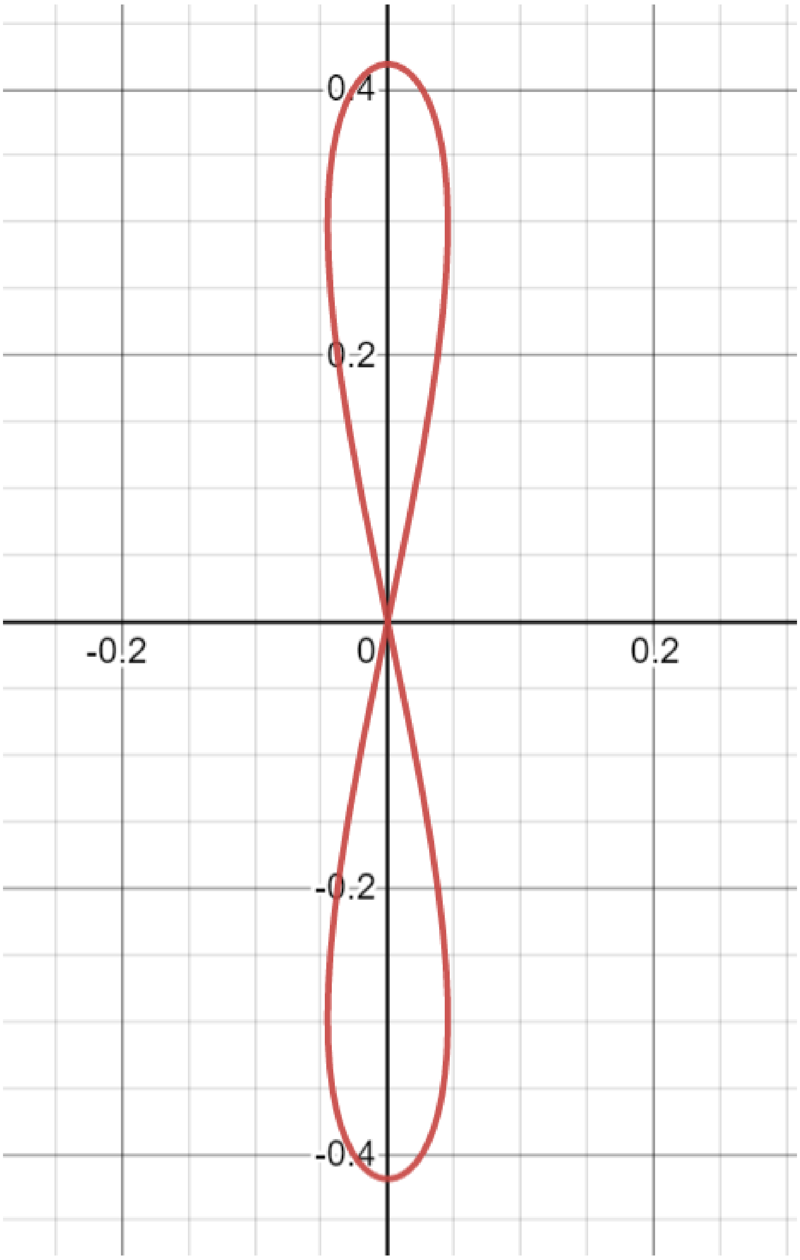

Figure 3: Analemma for circular orbit with axial tilt 24°. y-axis: a in radians, x-axis: A in radians. The top point will be summer solstice, bottom - winter solstice and origin- both the equinoxes.

Analemma of Earth (Elliptical Orbit)

(This is shown as an example that elliptical orbit makes the analemma lopsided and asymmetric)
Consider the polar equation of ellipse of eccentricity e, semi-major axis a.

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Where θ is the angle of current position of Earth from perihelion.

$$\frac{1}{2}r^2\omega = \frac{\pi a^2\sqrt{1-e^2}}{T}$$

(Kepler's 2$^{\text{nd}}$ law)

$$\int_{\theta_0}^{\theta} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{2\pi t}{T(1-e^2)^{3/2}}$$

Since, eccentricity e of Earth is small, using expansions $[(1+x)^n \approx 1+nx$ when $x \ll 1]$ on LHS and then integrating, we would get function of the form:

$$c(\theta - \theta_0 - 2e(\sin\theta - \sin\theta_0)) = t$$

Substituting values, we get $c \approx 58.1$ days. Using $e = 0.0167$ and $\theta_0 \approx 75°$ (from Internet). θ_0 is the angular position of spring equinox and t is measured from spring equinox day. Now, taking some values of t we can use obtained values of θ to find λ.

$$\lambda = \theta - \theta_0$$

Obtaining the values of a and A and plotting in Python we get Fig. 5. In Fig. 5 the azimuth maybe shifted by $1° - 2°$ compared to diagrams in Internet because I took spring equinox as $(0, 0)$. I think that the shape of this picture matches pretty well with the observed analemma as the maximum width of lower lobe and upper lobe matched fairly well with observed values from the Internet.

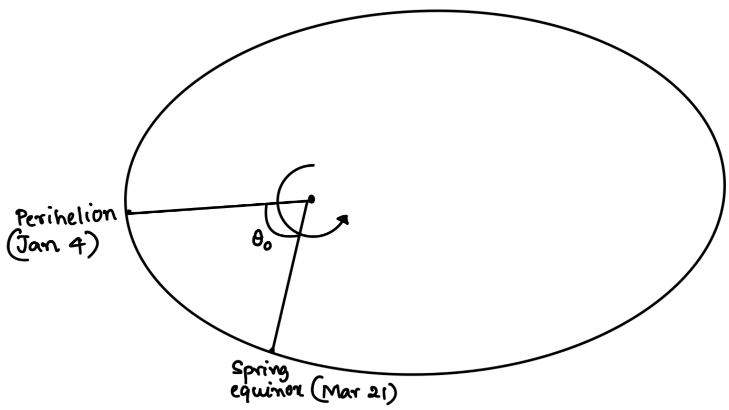

Figure 4: Orbit of the Earth.

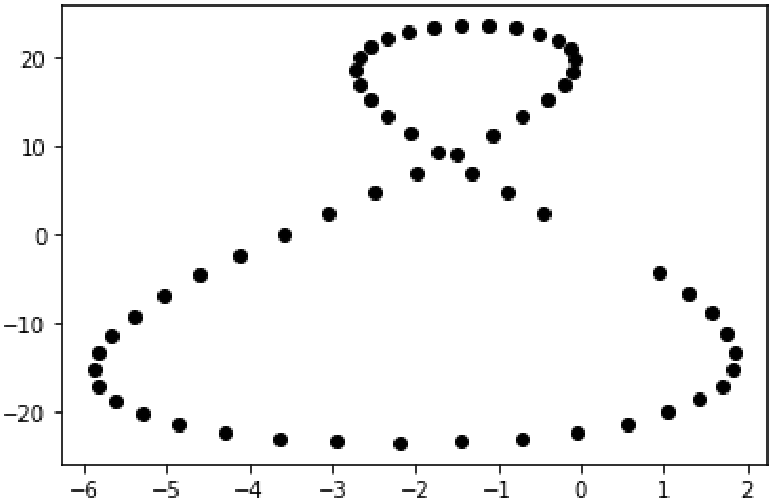

Figure 5: Calculated Analemma of Earth (using Python). x-axis : Azimuth A and y-axis: Altitude a, both in degrees. $a = 0^\circ$, $A = 0^\circ$ represents position of Sun on spring equinox.