Learners space Quantum Computing Week-0 assignment

Suryansh Patidar- 22B1036

Contents

1 Q1

Before moving on to part(a), we should understand that the eigenvalues of *Kronecker* product of two matrices are also the product of the eigenvalues of those two individual matrices. To prove this, we do

$$A|v\rangle = \lambda_1|v\rangle$$
$$B|w\rangle = \lambda_2|w\rangle$$

Now take *Kronecker* product of both equations;

$$(A|v\rangle) \otimes (B|w\rangle) = \lambda_1\lambda_2(|v\rangle \otimes |w\rangle)$$

RHS can be written as;

$$(A|v\rangle) \otimes (B|w\rangle) = (A \otimes B)(|v\rangle \otimes |w\rangle) = \lambda_1\lambda_2(|v\rangle \otimes |w\rangle)$$

So we get that eigenvalues of $A \otimes B$ is equal to product of eigenvalues of A and B

1.1 Part(a)

$$\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$$

Looking at the above matrix, we can see that it is *Kronecker product* of

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We can easily calculate eigenvalues of above two matrices by $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 5-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} = 0$$
$$(5-\lambda)(2-\lambda) - 12 = 0$$
$$\lambda^2 - 7\lambda - 2 = 0$$
$$\lambda = \frac{7 \pm \sqrt{57}}{2}$$

Same for the second matrix;

$$det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$

So the eigenvalues of the required matrix are;

$$Eigenvalues = \frac{7+\sqrt{57}}{2}, \frac{7-\sqrt{57}}{2}, \frac{-7-\sqrt{57}}{2}, \frac{-7+\sqrt{57}}{2} \tag{1}$$

1.2 Part(b)

$$\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}$$

This is also *Kronecker* product of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

So the result would be the same as part(a);
So,

$$Eigenvalues = \frac{7+\sqrt{57}}{2}, \frac{7-\sqrt{57}}{2}, \frac{-7-\sqrt{57}}{2}, \frac{-7+\sqrt{57}}{2} \tag{2}$$

1.3 Part(c)

$$\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix}$$

This is also *Kronecker* product of

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

So the eigenvalues will be;

$$Eigenvalues = (\frac{7+\sqrt{57}}{2})^2, (\frac{7-\sqrt{57}}{2})^2, (\frac{7+\sqrt{57}}{2}) \times (\frac{7-\sqrt{57}}{2}), (\frac{7-\sqrt{57}}{2}) \times (\frac{7+\sqrt{57}}{2}) \tag{3}$$

So finally the eigenvalues are;

$$Eigenvalues = \frac{106+14\sqrt{57}}{4}, \frac{106+14\sqrt{57}}{4}, -2, -2 \tag{4}$$

2 Q2

As the eigenvalues are ± 1,
So we can write,

$$\mathcal{O} = P_1 - P_{-1}$$

Also, the summation of projectors is equal to I

$$\mathcal{I} = P_1 + P_{-1}$$

By Solving the above two equations we can write,

$$P_1 = \frac{\mathcal{I} + \mathcal{O}}{2}$$
$$P_{-1} = \frac{\mathcal{I} - \mathcal{O}}{2}$$

So, this gives us the final result,

$$P_{\pm 1} = \frac{\mathcal{I} \pm \mathcal{O}}{2}$$

3 Q3

Let's say an operator A is unitary. I.e.,

$$A^*A = AA^* = I$$

So if we see and try to calculate $\langle Ax|Ay\rangle$, We get,

$$\langle Ax|Ay\rangle = (Ax)^*(Ay) = x * A * Ay = x * y = \langle x|y\rangle$$

Now this also holds is $x = y$.

$$\langle Ax|Ax\rangle = \langle x|x\rangle$$

And we also know that these represents the square norm,

$$|Ax|^2 = |x|^2$$

i.e we can also say that

$$|Ax| = |x|$$

We get that it is norm preserving