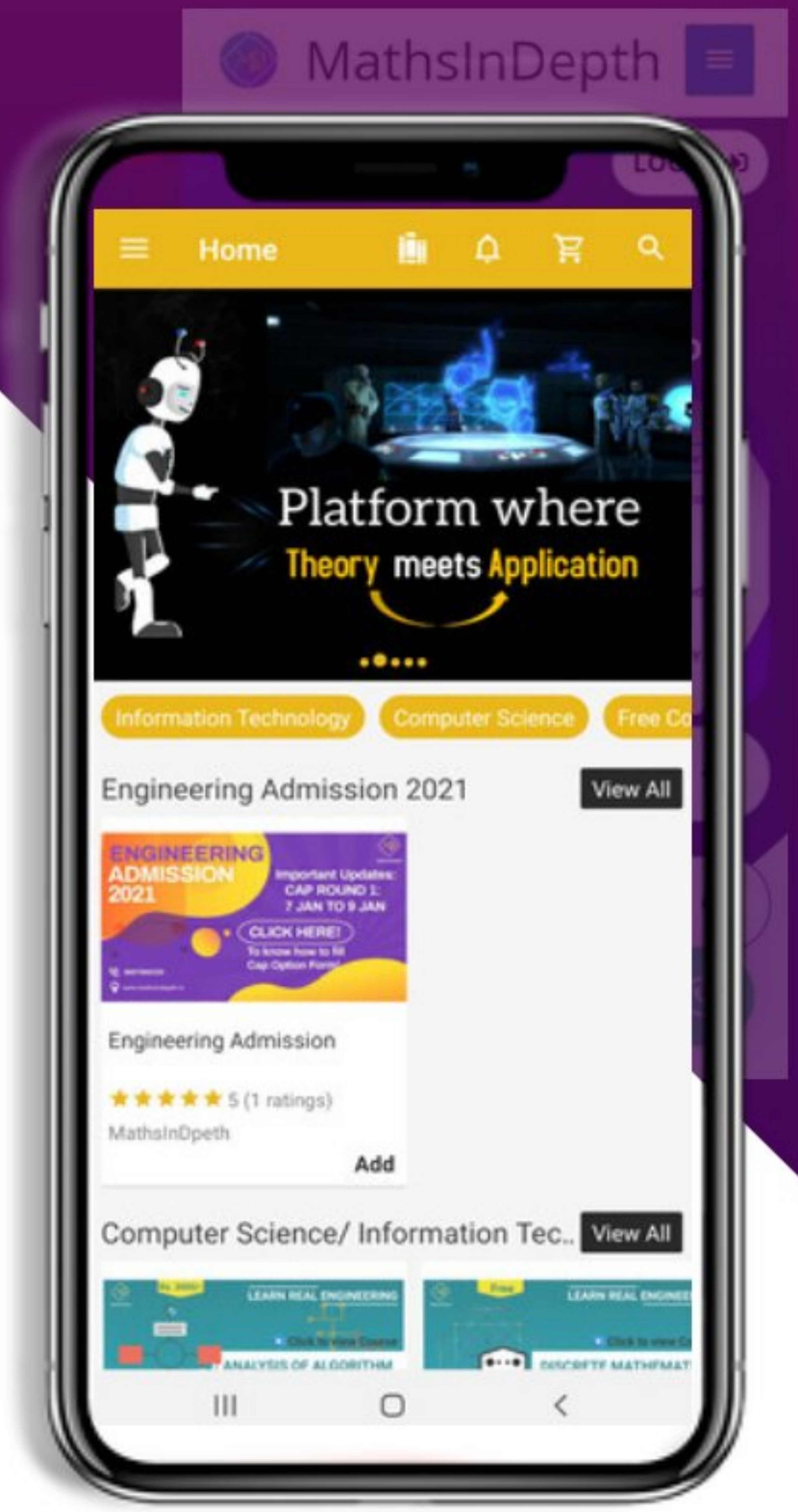
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Chapter 1

Set Theory

- *Sets,*
- *Venn Diagrams, operations on sets*
- *Laws of Set theory : power set and products*
- *Partitions of Sets ; the principle of Inclusion - Exclusion*

Introduction :

Set is a fundamental concept in the theory of Discrete Structures. Any algebraic structure be it a *'group'* or *'graph'* has its *'underlying structure'*. Hence one ought to have a clear understanding of the term "set".

The theory of sets was first introduced by the German Mathematician G. Cantor (1845 – 1918), who defined a set simply as a collection of objects.

In this Chapter we present some basic mathematical ideas about sets.

1.1 Sets :

A **set** is any well defined collection of objects, called the "elements" or "members" of the set.

A specific set can be defined in 2 ways :

1. If there are only a few elements, they can be listed individually, by writing them between braces (curly brackets) and placing commas in between. For examp!e of the set of positive odd members less than 10 can be written as, $\{1, 3, 5, 7, 9\}$

If there is a clear pattern to the elements an ellipsis (three dots) can be used.

For example, the set of positive odd numbers between 0 and 50 can be written as $\{1, 3, 5, \ldots, 49\}$. Some infinite sets can also be written in this way ; for example, the set of all positive odd numbers can be written as $\{1, 3, 5, 7, \ldots\ldots\}$.

A set written in any of these ways is said to be written in enumerated form.

2. The second way of writing down a set is to use a property that defines the elements of the set. Braces are used in this notation also.

For example, the set of odd numbers between 0 and 100 can be written as,

$\{x \mid x \text{ is odd and } 0 < x < 100\}$.

And the definition reads the set of all x such that x is odd and $0 < x < 100$.

The expression following the slash is a predicate containing the variable x.

A set written in the form $\{x \mid P(x)\}$, where P(x) is a predicate is said to be written in predicate form or 'set builder form'.

1.1.1 Notation :

A set is generally denoted by capital letters A, B, C, … X, Y, Z.

Elements of a set are usually denoted by lower–case letters a, b, c, ..., x, y, z.

If x is an element of set A, we express this fact by writing.

$'x \in A'$ ($\in$ means 'belongs to')

If x is not an element of A, we write

$'x \notin A'$

There are various ways of describing a set

1. **Listing method :** In this method, the elements are listed within braces.

Example :

(i) $A = \{a, b, c, d\}$

(ii) $A = \{2, 4, 6, 8, ...\}$

2. **Statement form :** A statement describing the set, especially where the elements share a common characteristic.

Example : (i) The set of all equilateral triangles.

(ii) The set of all Prime Ministers of India.

3. **Set–Builder Notation :** It is not always possible or convenient to describe a set by the Listing method or the Statement form. A more concise or compact way of describing the set is to specify the property shared by all the elements of the set. This property is denoted by P(x), where P is a statement concerning on element x of the set. The set is then simply written as $\{x \mid P(x)\}$ where the braces { } denote the clause "the set of", and the slash or stroke | denotes "such that" (read as "A is the set of all x such that x is greater than 10)

Examples :

(i) $A = \{x \mid x > 10\}$

(ii) $B = \{x \mid x \text{ is real and } x^8 - 5x^4 + 4 = 0\}$

1.1.2 Some Special Sets (Number Sets) :

The following sets occur frequently in our discussion. We give below the standard notations used to denote these sets

N – the set of all natural numbers

Z – the set of all integers

Z^+ – the set of all positive integers

Q – the set of rational numbers

Q^+ – the set of non–negative rational numbers

R – the set of real numbers

R^+ – the set of positive real numbers

C – the set of complex numbers

Note : For example :

Suppose set A contain + ve integer that are less than 4 can be written as

A = {1, 2, 3}

The order in which the elements of a set are listed is not important. Thus {1, 3, 2}, {3, 2, 1}, {3, 1, 2}, {2, 1, 3} and {2, 3, 1} are all representations of the same set.

1.1.3 Definitions :

1.1.3.1 Subset :

If every element of a set A is also an element of a set B, we say A is a **subset** of B or A is contained in B. This is denoted by writing,

$A \subseteq B$

If A is not a subset of B, this is indicated by writing

$A \not\subseteq B$

Examples :

(i) If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 5\}$, $C = \{1, 2, 3, 4, 5\}$ then $B \subseteq A$, $B \subseteq C$ and $C \subseteq A$

(ii) If $A = \{1, 3, 6\}$, $B = \{-1, 1, 2, 3, 4, 6\}$, $C = \{1, 2, 3\}$ then $A \subseteq B$ but $A \not\subseteq C$

It is clear from the definition that **Every set is a subset of itself.**

1.1.3.2 Equal Sets :

If A is a subset of B and B is also a subset of A, then two sets A and B are equal

i.e. $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

i.e. if set A consist of exactly those elements that are contained in set B then we say set A and B are equal.

Example :

(i) If $A = \{1, 2, 3, 4\}$ $B = \{3, 4, 2, 1\}$

then $A = B$

(ii) If $A = \{1, 2, 3\}$

$B = \{x \mid x \text{ is positive integer and } x^2 < 12\}$

then $A = B$

1.1.3.3 Proper Subset :

A set **A** is called **proper subset** of set **B** if,

(i) A is subset of B and

(ii) B is not a subset of A

and it is denoted as $A \subset B$

Example :

Let $A = \{x, y\}$ $B = \{x, y, z\}$

then $A \subseteq B$ but $B \not\subseteq A$

(i.e.) When $A \subseteq B$ but $A \neq B$ we say that A is a proper subset of B. $\therefore$ $A \subset B$

Example : $A = \{1, 3\}$

$B = \{1, 2, 3\}$

$C = \{1, 3, 2\}$

Then A and B are both subsets of C, i.e. $A \subseteq C$ and $B \subseteq C$.

But A is a proper subset of C. Whereas 'B' is not a proper subset of C since $B = C$.

1.1.3.4 Universal Set :

If all sets, considered during a 'specific discussion' are subsets of a given set, then this set is called as the **Universal Set** and is denoted by 'U'.

Hence the universal set is a relative concept dependent on the specific discussion. Therefore, it is also referred to as the 'Universe of discourse'.

1.1.3.5 Null Set or Empty Set :

The set with no elements is called as an **Empty set** or **Null Set.** A null set is denoted by symbol ϕ. Also a null set is a subset of every set.

The null set can be written in enumerated form, like this : { }

(Note that the symbol ϕ is not the same as the Greek letter ϕ (phi)).

Note : Do not make the mistake of writing $\{\phi\}$ for the null set. The set $\{\phi\}$ is not the null set, it is a set with one element, and that element is a null set.

1.1.3.6 Singleton Set :

A set having only one element is called a **singleton set.**

Example :

(i) $A = \{8\}$ (ii) $B = \{\phi\}$

1.1.3.7 Super Set :

If set A is subset of set B, then set B is **super set** of set A

Example : Let $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

If $A \subseteq B$

then B is super set of set A.

1.1.3.8 Finite Set :

A set is said to be **finite** if it has finite number of elements.

Example : $A = \{1, 2, 3\}$

1.1.3.9 Infinite Set :

➤➤➤ [**University Exam – May 2007 !!!**]

A set which is not finite is said to be **infinite set**

Example : (i) Set of natural numbers

(ii) Number of points on a line

1.1.3.10 Disjoint Sets :

➤➤➤ [**University Exam – May 2010, Dec. 2011 !!!**]

Two sets are said to be **disjoint** if they have no elements in common.

Example : $A = \{0, 4, 7, 9\}$ and

$B = \{3, 17, 15\}$ are disjoint.

1.1.3.11 Cardinality of Finite Set :

The cardinality of a finite set is the number of elements in the set. The cardinality of a set A is denoted by $|A|$.

Example : Let $A = \{a, b, c, d\}$

then A is a finite set and $|A| = 4$

1.1.4 Exercise Set - 1 (Solved)

Example 1 :

Consider the set $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$

(i) What are the elements of A ?

(ii) Determine whether each of the following is true or false.

(a) $1 \in A$ (b) $\{6, 7, 8\} \in A$ (c) $\{1, 2, 3\} \subseteq A$

(d) $\{\{4, 5\}\} \subseteq A$ (e) $\phi \in A$ (f) $\phi \subseteq A$

Solution :

A is a class of sets.

(i) Elements of A are :

$\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}$

(ii) (a) False

1 is not an element of A. 1 does not belongs to set A.

(b) True

$\{6, 7, 8\}$ is one of the elements of A.

(c) False

$\{1, 2, 3\}$ is not a subset of set A but is an element of A.

(d) True

$\{\{4, 5\}\}$ is a subset of A

(e) False

The empty set 'ϕ' is not an element of A since it is not listed in A.

(f) True

The empty set is a subset of every set.

Example 2 :

Let A = $\{a, b, \{a, b\}, \{\{a, b\}\}\}$

Identify each of the following statements are true or false. Justify your answers.

(a) $a \in A$
(b) $\{a\} \in A$
(c) $\{a, b\} \in A$
(d) $\{\{a, b\}\} \subseteq A$
(e) $\{a, b\} \subseteq A$
(f) $\{a, \{b\}\} \subseteq A$

Solution :

(a) True, as a is an element of A.
(b) False, as $\{a\}$ is not an element but a subset of A.
(c) True, as $\{a, b\}$ is an element of A, listed third in the set.
(d) True, as a subset containing the single element $\{a, b\}$ of A.
(e) True, as the subset containing the elements $\{a, b\}$ of A.
(f) False, as $\{b\}$ is not an element of A.

Example 3 :

Determine whether each of the following statements is true for arbitrary sets A, B, C. Justify your answers. **(May 2012)**

(a) If $A \in B$ and $B \subseteq C$, then $A \in C$
(b) If $A \in B$ and $B \subseteq C$, then $A \subseteq C$
(c) If $A \subseteq B$ and $B \in C$, then $A \in C$
(d) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

Solution :

(a) True, as A being an element of B, it should also belong to C as B is a subset of C
(b) False, as A is not a subset but an element of B
(c) False, consider $A = \{a\}$, $B = \{a, b\}$, $C = \{\{a, b\}\}$
(d) False, consider the same example as in (c)

Example 4 :

Identify whether each of the following set is infinite or finite

(a) {days in a week}
(b) {ways to order the numbers 1 through 100}
(c) {lines through the origin}
(d) {lines that satisfy the equation $3x = y$}

Solution :

(a) Finite : There are seven days in a week, hence the set is finite.

(b) Finite : Though the number of combinations is very large, there are finite number of possibilities, hence the set is finite.

(c) Infinite : There are infinitely many lines passing through the origin. Hence the set is infinite.

(d) Finite : The equation specifies one single line passing through the origin. Hence the set is finite.

1.1.5 Set Properties :

1. Every set A is a subset of the universal set 'U'. Since, by definition, all the elements of A belong to 'U'. Also the empty set ϕ is a subset of A. For any set A, we have

   $\phi \subseteq A \subseteq U$

2. Every set A is a subset of itself, trivially the elements of A belong to A. For any set A, we have $A \subseteq A$

3. If every element of A belongs to set B, and every element of B belongs to set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

4. If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, that is $A = B$.

   Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$.

   (Since every set is subset of itself. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.)

1.1.6 Computer Representation of Sets :

Some programming languages, such as Pascal, allow sets to be handled as a compound data type, where the elements of the sets belong to one of the data types available in the language, such as integers or characters.

A set is always defined in a program with reference to a universal set U. Here we assume that the elements of U are listed in a definite order. Any set A arising in the program and defined with reference to this universal set U, is a subset of U.

A set A is represented by a string of n bits $b_1, b_2, \ldots, b_n$, where n is the cardinality of U. The bit string $b_1, b_2, \ldots, b_n$ can be regarded as the element $(b_1, b_2, \ldots, b_n)$ of the cartesian product of n sets viz.

$[0, 1] \times [0, 1] \times \ldots \times [0, 1]$

The bits are determined according to the rule

b_i = 1 if the i^{th} element of U is in A

= 0 if the i^{th} element of U is not in A,

where i ranges over the values 1, 2, ... n.

Example :

Let $U = \{1, 2, 3, \ldots, 10\}$

(a) Find the representation of $\{2, 3, 5, 7\}$ as a bit string

(b) Find the set represented by the bit string 1001011011

Solution :

(a) Looking in turn at each element of U, and writing down 1 if the element is in $\{2, 3, 5, 7\}$ and 0 if it is not, we obtain the answer : 0110101000

(b) The answer is obtained by writing down each element of U that corresponds to a 1 in the bit string : $\{1, 4, 6, 7, 9, 10\}$

1.2 Venn Diagrams and Set Operations :

1.2.1 Venn Diagrams :

A **Venn diagram** (named after the British logician John Venn) is a pictorial depiction of a set.

A rectangle represents the universal set. The interior of the rectangle represent the elements in the set. A circle drawn within the rectangle, depicts an arbitrary set. It is not compulsory to show an arbitrary set, always by a circle. An oval shaped or elliptical curve could also be drawn to represent a set. In fact any closed curve of any shape can be used to depict a set.

Venn Diagrams

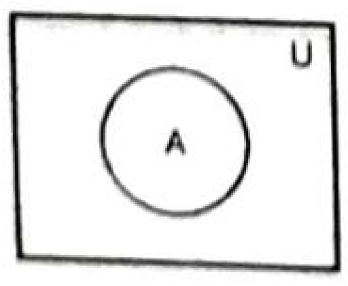
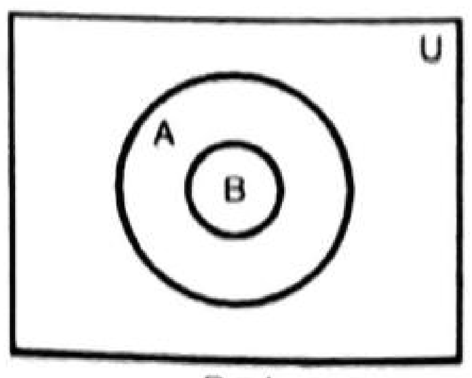


Fig. 1.1



B⊆A

Fig. 1.2

1.2.2 Set Operations :

We shall now define various set operations, that will combine the given sets to yield new sets. These operations are analogous to the algebraic operations of addition, multiplication of numbers.

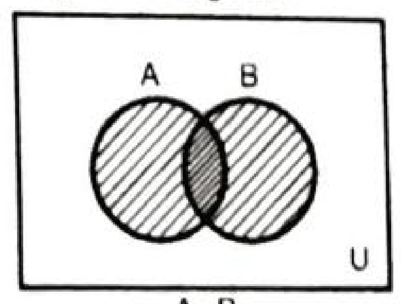
(1) Union :

Let A and B are sets, the union of two sets A and B is the set consisting of all elements which are in A, or in B, or in both sets A and B. It is denoted by **A ∪ B**.

In the set builder notation

$$\boxed{A \cup B = \{x \mid x \in A \text{ or } x \in B\}}$$

The union of A and B is depicted by the shaded region of the Venn diagram in Fig. 1.3.



A∪B

Fig. 1.3

Examples :

(i) If $A = \{2, 4, 6, 8, 10\}$

$B = \{1, 2, 6, 8, 12, 15\}$

then $A \cup B = \{1, 2, 4, 6, 8, 10, 12, 15\}$.

(ii) If $A = \{x \mid x \in Z, \text{ and } x \geq 3\}$

$B = \{x \mid x \in Z, \text{ and } x \geq 8\}$ ≥ 3

then $A \cup B = \{x \mid x \in Z \geq 3\}$. ≥ 8

where Z denotes the set of integers.

(iii) If $A = \{n \mid n \in N, 4 < n < 12\}$ ≥ 3

$B = \{n \mid n \in N, 8 < n < 15\}$

then $A \cup B = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

(iv) If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \cup B = \{\phi, a, \{\phi\}\}$

$= B$

This is because $A \subseteq B$.

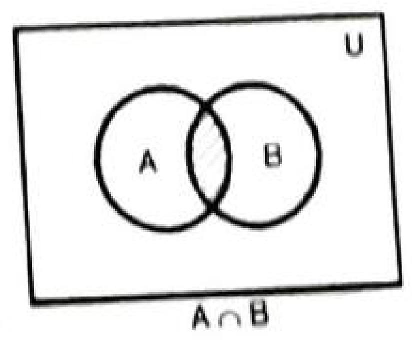
Union of More than two sets :

If $A_1, A_2, A_3, \ldots, A_n$ denote sets, then the union of these sets is denoted by,

$\bigcup_{i=1}^{n} A_i$ is defined as,

$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for atleast one set } A_i\}$

(2) Intersection :



A∩B

Fig. 1.4

Let A and B are sets, the intersection of two sets A and B, denoted by **A ∩ B** is the set consisting of elements which are in A as well as in B (i.e. belongs to both A and B).

In the set-builder notation

$$\boxed{A \cap B = \{x \mid x \in A \text{ and } x \in B\}.}$$

If $A \cap B = \phi$, the sets are said to **disjoint**.

The intersection of A and B is depicted by the shaded region of the Venn diagram in Fig. 1.4.

Examples :

(i) If $A = \{a, b, c, g\}$

$B = \{d, e, f, g\}$

then $A \cap B = \{g\}$.

(ii) If $A = \{a, b, c, e, f\}$ $B = \{b, e, f, r, s\}$ $C = \{a, t, u, v\}$

then $A \cap B = \{b, e, f\}$.

$A \cap C = \{a\}$.

$B \cap C = \{ \}$.

(Two sets that have no common elements such as B and C are called **disjoint set**)

(iii) If $A = \{n \mid n \in N, 4 < n < 12\}$

$B = \{n \mid n \in N, 5 < n < 10\}$

then $A \cap B = \{6, 7, 8, 9\} = B$

(iv) If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \cap B = \{\phi\} = A$

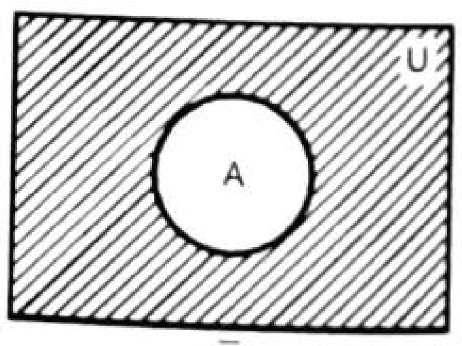
Intersection of More than 2 sets :

If $A_1, A_2, \ldots, A_n$ denote sets, then the intersection of these sets is denoted by,

$\bigcap_{i=1}^{n} A_i$ and is defined as follows :

$$\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for every } i\}$$

(3) The complement of a set :



$\bar{A}$

Fig. 1.5

If U is a universal set containing set A, then U – A is called the **complement** of A and is denoted by $\bar{A}$

$$\boxed{\bar{A} = \{x \mid x \in U \text{ and } x \notin A\}}$$

The complement of A is shown by the shaded region of the Venn diagram shown in Fig. 1.5.

Examples :

(i) If $U = \{a, b, c, d, e, f\}$ $A = \{b, d, e\}$

then $U - A = \bar{A} = \{a, c, f\}$.

(ii) If $A = \{x \mid x \text{ is a real number and } x \leq 7\}$, then

$\bar{A} = \{x \mid x \text{ is a real number and } x > 7\}$.

(iii) If $U = N = \{1, 2, 3, 4, 5, \ldots\}$

and $E = \{2, 4, 6, \ldots\}$

E is set of even numbers

then $\bar{E} = \{1, 3, 5, \ldots\}$

(4) Difference of sets :

Set Representation

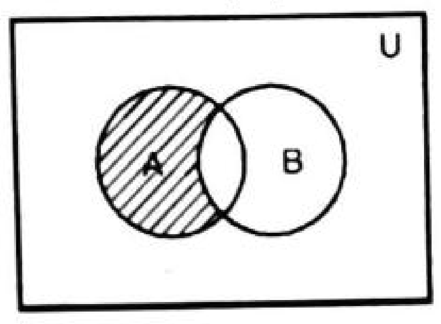
Let A and B be any two sets. The difference **A – B** is the set defined as

$$\boxed{A - B = \{x \mid x \in A \text{ and } x \notin B\}}$$ Set-builder Roaster

is the (Relative) complement of B in A. That means set of all elements that belong to A but not to B.

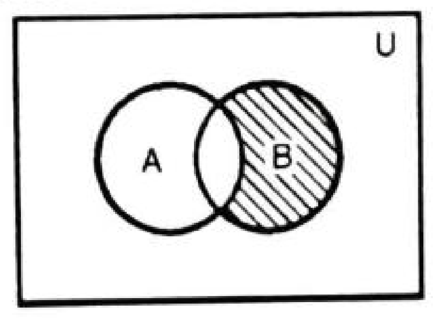
Similarly,

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$ is the complement of A in B.



A – B

Fig. 1.6



B – A

Fig. 1.7

The difference of A and B depicted by the shaded region of the Venn diagram. Refer Fig. 1.6 and Fig. 1.7.

Examples :

(i) If $A = \{a, b, c\}$

$B = \{b, c, d, e\}$

then $A - B = \{a\}$.

$B - A = \{d, e\}$.

(ii) If $A = \{1, 2, 3, \ldots, 10\}$

$B = \{1, 3, 5, \ldots, 9\}$

then $A - B = \{2, 4, 6, 8, 10\}$

$B - A = \phi$.

(iii) If $A = \{a, b, \{a, c\}, \phi\}$

$A - \{a, b\} = \{\{a, c\}, \phi\}$.

(5) Symmetric Difference :

[University Exam – May 2007, May 2010, Dec. 2011 !!!]

Let A and B are two sets, the symmetric difference of two sets A and B, is the set of all elements that belong to A or to B, but not to both A and B, denoted by **A ⊕ B**.

In the set - builder notation,

$$\boxed{A \oplus B = \{x \mid x \in A - B \text{ or } x \in B - A\}.}$$

In other words,

$$\boxed{A \oplus B = (A - B) \cup (B - A).}$$

The symmetric difference of A and B depicted by the shaded region of the Venn diagram is shown in Fig. 1.8.

Examples :

(i) If $A = \{a, b, e, g\}$

$B = \{d, e, f, g\}$

then $A \oplus B = \{a, b, d, f\}$.

(ii) If $A = \{2, 4, 5, 9\}$

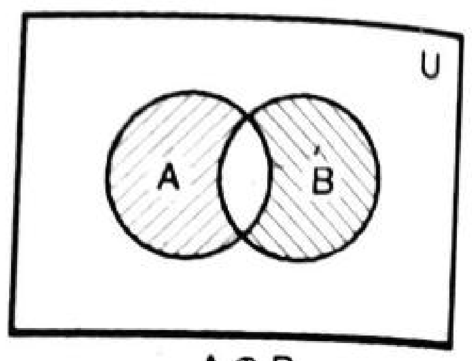
$B = \{x \in Z^+ \mid x^2 \leq 16\}$.

then, $A \oplus B = \{0, 1, 3, 5, 9\}$.

(iii) If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \oplus B = \{a, \{\phi\}\}$.



$A \oplus B$

Fig. 1.8

(6) Cartesian product :

[University Exam – Dec. 2011 !!!]

The Cartesian product of 2 sets A and B is defined by

$$\boxed{A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}}$$

(i.e.) **A × B** is the set of all ordered pairs in which the first element comes from set A and the second element comes from set B.

More generally, the Cartesian product of the 'n' set $A_1, A_2, \ldots, A_n$ is defined by,

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i, i = 1 \text{ to } n\}$$

Example :

If $A = \{a, b\}$

$B = \{1, 2, 3\}$ **(May 1999)**

then $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Sometimes we form a Cartesian product of a set with itself. If A is any set, then

$$A^n = A \times A \times \ldots \times A \text{ (n times)}$$
$$= (x_1, x_2, \ldots, x_n) \mid x_i \in A, i = 1, 2, \ldots, n\}$$

Consider the Cartesian product $[0, 1]^n$. The elements of this set are ordered n tuples in which each elements is either 0 or 1. For example (1, 0, 0, 1, 0, 1, 1, 1) is an element of $[0, 1]^8$.

1.2.3 Theorems :

(1) The Addition Principle :

Let A and B be finite sets which are disjoint.

Then $\boxed{|A \cup B| = |A| + |B|}$.

Proof :

If A or B is the empty set, the proof is trivial. Hence let us assume that $A \neq \phi$, $B \neq \phi$. Since A and B are finite disjoint sets.

Let $A = \{a_1, a_2, \ldots a_m\}$ and

$B = \{b_1, b_2, \ldots b_n\}$, where $a_i \neq b_j$ for $1 \leq i \leq m$,

$1 \leq j \leq n$.

$|A| = m$, $|B| = n$

Then $|A \cup B| = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots b_n\}$,

i.e. $A \cup B$ contains exactly m + n elements.

Hence $|A \cup B| = m + n = |A| + |B|$.

Thus the theorem is proved.

The above theorem can be extended to a finite collection of finite mutually disjoint sets.

Corollary :

Let $A_1, A_2, \ldots, A_n$ be a finite collection of mutually disjoint finite sets.

Then $\boxed{|A_1 \cup A_2 \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|}$.

Proof is left as an exercise.

(2) Let A be a finite set and let B be any set (not necessarily finite).

Then $\boxed{|A - B| = |A| - |A \cap B|}$.

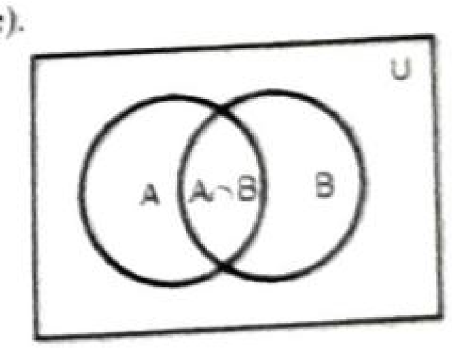


Fig. 1.9

Proof :

Consider the Venn diagram shown in Fig. 1.9.

From the Venn diagram, it is clear that

$A = (A - B) \cup (A \cap B)$ (Disjoint union of two sets)

Hence by the Addition Principle,

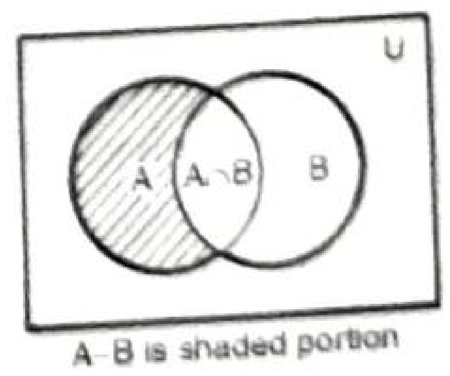
$|A| = |A - B| + |A \cap B|$,

So that $|A - B| = |A| - |A \cap B|$.

(3) Principle of Inclusion – Exclusion :

Let A and B be finite sets.

Then $\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$.



A-B is shaded portion

Fig. 1.10

Proof :

Consider the Venn diagram, shown in Fig. 1.10.

We may express $A \cup B$ as disjoint union of two sets, by writing

$A \cup B = (A - B) \cup B$.

Hence by the Addition Principle

$$|A \cup B| = |A - B| + B$$
$$= |A| - |A \cap B| + |B| \qquad \text{(By the previous theorem)}$$

Hence $|A \cup B| = |A| + |B| - |A \cap B|$.

(4) Mutual Inclusion – Exclusion Principle for three sets :

Let A, B, C be finite sets.

Then $\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|}$

Proof :

Let D denote the union set $B \cup C$.

Then $A \cup B \cup C = A \cup D$

$$|A \cup D| = |A| + |D| - |A \cap D| \qquad \text{(by the previous theorem)} \qquad \ldots(1)$$
$$|D| = |B \cup C|$$
$$= |B| + |C| - |B \cap C| \qquad \ldots(2)$$
$$|A \cap D| = |A \cap (B \cup C)|$$
$$= |(A \cap B) \cup (A \cap C)|$$
$$= |A \cap B| + |A \cap C| - |A \cap B \cap C| \qquad \ldots(3)$$

Substituting Equations (2) and (3) in Equation (1), we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Thus the principle is proved for three sets.

(5) We now have the general theorem for a finite collection of finite sets.

Theorem : Let $|A_1, A_2, \ldots, A_n|$ be a finite collection of sets.

Then

$$\boxed{|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|}$$

Proof :

Proof is by induction (for induction refer second chapter) on n. We have already proved the theorem for n = 2, 3.

Hence, let us assume the theorem for (n – 1) numbers of sets and prove it for n sets.

Regarding $A_1 \cup A_2 \cup \ldots \cup A_n$ as $(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n$, we have

$$|A_1 \cup A_2 \ldots \cup A_n| = |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n|$$
$$= |A_1 \cup A_2 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| \qquad \ldots(1)$$

By induction hypothesis

$$|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = \sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j|$$
$$+ \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \qquad \ldots(2)$$

Now $|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| = |(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)|$

$$= \sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| + \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k \cap A_n| -$$
$$\ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n| \qquad \ldots(3)$$

Substituting Equations (2) and (3) in Equation (1) we obtain the equation

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j < n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$
$$- \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

1.2.4 Exercise Set 2 - (Solved)

Example 1 :

How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1 and 18 elements in A_2 and

(i) $A_1 \cap A_2 = \phi$ (ii) $|A_1 \cap A_2| = 1$

(iii) $|A_1 \cap A_2| = 6$ (iv) $A_1 \subseteq A_2$

Solution :

Given : $|A_1| = 12$ and $|A_2| = 18$.

(i) $A_1 \cap A_2 = \phi$

$\therefore |A_1 \cap A_2| = 0$

By principle of Inclusion -Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 12 + 18 - 0$$
$$= 30$$
$$\boxed{|A_1 \cup A_2| = 30}$$

(ii) Given : $|A_1 \cap A_2| = 1$

By principle of Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 12 + 18 - 1$$
$$= 29$$
$$\boxed{|A_1 \cup A_2| = 29}$$

(iii) Given : $|A_1 \cap A_2| = 6$

By principle of Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 12 + 18 - 6 = 24$$
$$\boxed{|A_1 \cup A_2| = 24}$$

(iv) Given : $A_1 \subseteq A_2$

$\therefore |A_1 \cap A_2| = 12$

By principle of Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 12 + 18 - 12 = 18$$
$$\boxed{|A_1 \cup A_2| = 18}$$

Example 2 :

A company manufactures cranes. These are three basic type of crane, labelled A, B and C. Each crane is assembled from a sub–assembly set {a, b, c, d, e, f} as follows

A is assembled from {a, b, c, d}

B is assembled from {a, c, f}

C is assembled from {b, d, e}

In turn, the sub–assemblies are manufactured from basic components {p, q, r, s, t, u, v, w, x, y, z} as follows :

a is manufactured from {p, q, r, s}

b is manufactured from {q, r, t, v}

c is manufactured from {p, r, s, t}

d is manufactured from {p, w, y}

e is manufactured from {u, z}

f is manufactured from {p, r, u, v, y, z}

(a) Give the makeup of the following sub–assemblies

(i) $a \cup b$ (ii) $a \cup c \cup f$ (iii) $d \cup e$

(b) Given that A is made in Bombay and B and C are made in Calcutta, what components need to be available on both sites ?

Solution :

(a) Using the definition of union of two sets, we write :

(i) $a \cup b = \{p, q, r, s, t, v\}$

(ii) $a \cup c \cup f = \{p, q, r, s, t, u, v, y, z\}$

(iii) $d \cup e = \{p, u, w, y, z\}$

(b) Note that A is made from sub–assemblies {a, b, c, d}, where as B and C require {a, b, c, d, e, f} in all of them.

Now, inspection of those components required to make all six sub–assemblies reveal that the sub–assemblies a, b, c and d do not require components u and z.

Hence only components u and z need not be made available in both sites.

Hence the components that constitute $a \cup b \cup c \cup d$ have to be made available on both sites.

Example 3 :

Define Cartesian product of two sets.

If $A = \{ x \mid x \text{ is real and } -2 \le x \le 3 \}$

and $B = \{ y \mid y \text{ is real and } 1 \le y \le 5 \}$

Sketch the set A × B in the Cartesian plan.

(Dec. 98)

Solution :

Let A and B be two non-empty sets. Then we define Cartesian product of A and B as a set of collection of ordered pairs.

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

Sketch is as shown in Fig. 1.11.

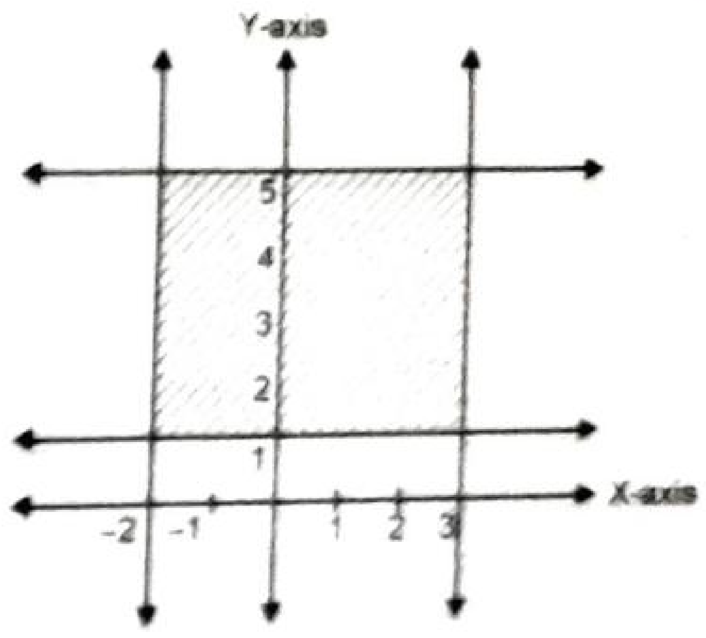


Fig. 1.11

Example 4 :

Given A = {1, 2}, B = {x, y, z}, C = {3, 4} then find A × B × C

Solution : Note that A × B × C consists of all ordered triplets (a, b, c) where $a \in A, b \in B, c \in C$.

We can draw a tree diagram by which we can obtain the elements of A × B × C.

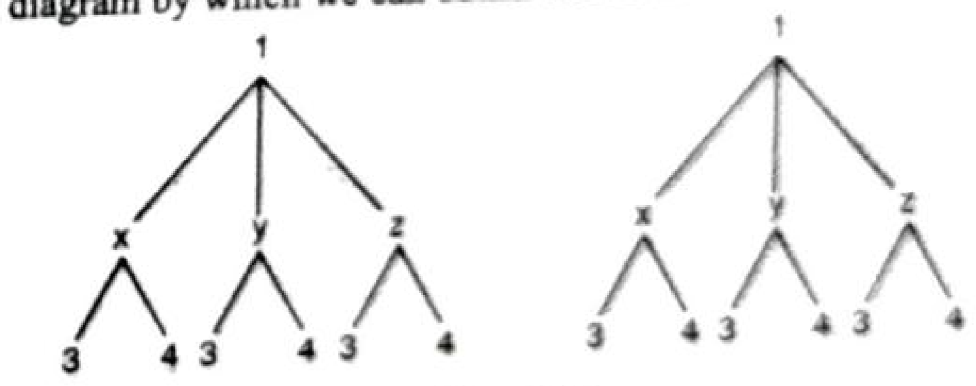


Fig. 1.12

$\therefore$ A × B × C = {(1, x, 3), (1, x, 4), (1, y, 3), (1, y, 4), (1, z, 3), (1, z, 4), (2, x, 3), (2, x, 4), (2, y, 3), (2, y, 4), (2, z, 3), (2, z, 4)}

Example 5 :

A computer company must hire 25 programmers to handle system programming jobs and 40 programmers for application programming. Of those hired 10 will be expected to perform jobs of both types. How many programmers must be hired?

Solution : Let, A be the set of system programmers hired.

B be the set of applications programmers hired.

We want to find $|A \cup B|$

We have, $|A| = 25$,

$|B| = 40$

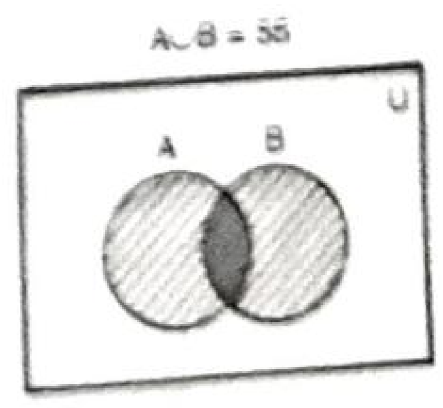


Fig. 1.13

Using the Principle of Inclusion and Exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 40 - 10 = 55$$

Thus 55 programmers must be hired.

Example 6 :

A sample of 80 people have revealed that 24 like cinema and 62 like television programmes. Find the number of people who like both cinema and television programmes.

Solution :

Let A = Set of people who like cinema

B = Set of people who like television programmes

Then, we have

$|A| = 24,$

$|B| = 62.$

$|A \cup B| = 80$

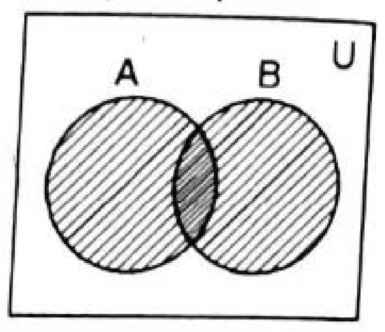


Fig. 1.14

Using principle of Inclusion and Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

We obtain

$$80 = 24 + 62 - |A \cap B|$$

Therefore,

$$|A \cap B| = 6$$

Thus 6 people like both cinema and television programmes.

Example 7 :

In a group of 120 students studying computer course, 84 can program in 'pascal' and 66 can program in 'C'. If 45 can program in both 'pascal' and 'C', how many of the students cannot program in either of these languages ?

Solution :

Let U = {Students in computer course}

P = {'Pascal' programming students}

C = {'C' programing students}

We want to find $|\overline{P \cup C}|$.

We have, $|P| = 84,$

$|C| = 66,$

$|P \cap C| = 45$

$|P \cap C| = 45$

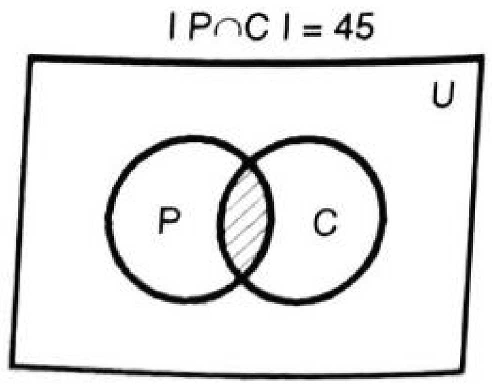


Fig. 1.15

Using the Principle of Inclusion and Exclusion

$$|P \cup C| = |P| + |C| - |P \cap C| = 84 + 66 - 45$$
$$= 105$$

$|P \cup C| = 105$

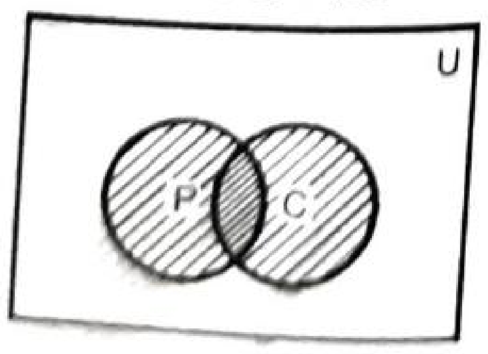


Fig. 1.16

$$\therefore \quad |\overline{P \cup C}| = |U| - |P \cup C|$$
$$= 120 - 105$$
$$= 15$$

There are 15 students who cannot program in either of these languages.

Example 8 :

In a survey of 260 college students, the following data were obtained

64 had taken a Mathematics course.

94 had taken a computer science course.

58 had taken a business course.

28 had taken both mathematics and business course.

26 had taken both a mathematics and computer science course.

22 had taken both a computer science and Business course.

14 had taken all three types of courses.

(i) How many students were surveyed who had taken none of the three types of courses ?

(ii) Of the students surveyed, how many had taken only a computer science course ?

(Dec. 2003, 2007)

Solution :

Let A be the set of students taken Mathematics course.

B be the set of students taken Computer Science course.

C be the set of students taken Business course.

We have $|A| = 64, |B| = 94, |C| = 58,$

$|A \cap C| = 28,$ taken mathematics and business course

$|A \cap B| = 26,$ taken mathematics and computer science course

$|B \cap C| = 22,$ taken computer science and business course

$|A \cap B \cap C| = 14,$ taken mathematics, computer science and business course.

(i) Using Principle of Inclusion and Exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$
$$= 154$$

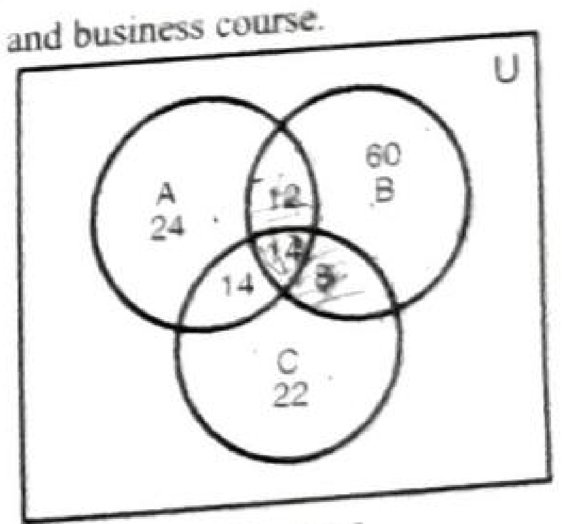


Fig. 1.17

$\therefore$ Students who have not taken either of subjects,

$$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$$
$$= 260 - 154 = 106$$

Thus 106 students had taken none of the three types of course.

(ii) $\therefore$ Students who have taken only computer science course

$$= |B| - |B \cap A| - |B \cap C| + |A \cap B \cap C| = 94 - 26 - 22 + 14 = 60$$

60 students had taken only computer science course.

Example 9 :

In a survey of 60 people, it was found that 25 read Newsweek magazine, 26 read Time and 26 read Fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune, 8 read no magazine at all.

(a) Find the number of people who read all three magazines.

(b) Fill in the correct number of people in each of eight regions of Venn diagram of Fig. 1.18. Here N, T and F denote the set of people who read Newsweek, Time and Fortune respectively.

(c) Determine the number of people who read exactly one magazine. Refer Fig. 1.18.

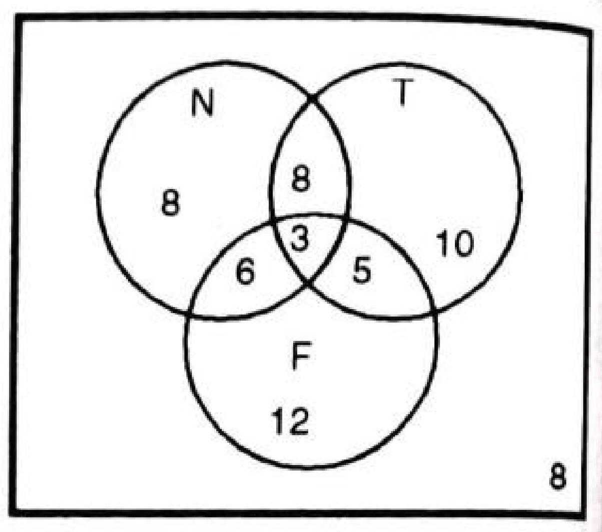


Fig. 1.18

(May 98, May 2002, May 2006, May 2007, May 2012, May 201

Solution :

(a) By the principle of inclusion and exclusion we have,

$$|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |T \cap F| + |N \cap T \cap F| \quad ...(1$$

We have,

$$|N \cup T \cup F| = 60 - 8 = 52 \quad \because \text{ 8 read no magazine at all}$$

$$|N| = 25, \quad |T| = 26, \quad |F| \ 26,$$

and $|N \cap T| = 11, \ |N \cap F| = 9, \ |T \cap F| = 8.$

Hence from Equation (1),

$$52 = 25 + 26 + 26 - 11 - 9 - 8 + |N \cap T \cap F|$$

$$|N \cap T \cap F| = 3$$

Hence, 3 people read all three magazines.

(b) To draw Venn Diagram,

3 read all three magazines

$\therefore |N \cap T| - |N \cap T \cap F| = 11 - 3 = 8$ read Newsweek and Time but not all three magazines.

$|N \cap F| - |N \cap T \cap F| = 9 - 3 = 6$ read Newsweek and Fortune but not all three magazines.

$|T \cap F| - |N \cap T \cap F| = 8 - 3 = 5$ read Time and Fortune but not all three magazines.

$|N| - |N \cap T| - |N \cap F| + |N \cap T \cap F|$

$= 25 - 11 - 9 + 3 = 8$ read only Newsweek

$|T| - |T \cap F| - |N \cap T| + |N \cap T \cap F|$

$= 26 - 8 - 11 + 3 = 10$ read only Time

$|F| - |T \cap F| - |N \cap F| + |N \cap T \cap F|$

$= 26 - 8 - 9 + 3 = 12$ read only Fortune

(c) People who read only one magazine is $8 + 10 + 12 = 30$

Example 10 :

Suppose that 100 of the 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose 65 study French,

20 study French and German,

45 study German,

25 study French and Russian,

42 study Russian,

15 study German and Russian,

(a) Find the number of students who study all three languages.

(b) Fill in the correct number of students in each of the eight regions of Venn diagram.

Here F, G and R denote the sets of students studying French, German and Russian respectively.

(c) Determine the number k of students who study

(i) exactly one language (ii) exactly two languages

Solution :

(a) Since 100 of the students study at least one of languages.

$$\therefore \ |F \cup G \cup R| = 100$$

Again, from given data

$$|F| = 65 \qquad |G| = 45 \qquad |R| = 42$$

$$|F \cap G| = 20 \qquad |F \cap R| = 25 \qquad |G \cap R| = 15$$

Using formula of inclusion and exclusion, we have

$$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$$

$$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$$

$$|F \cap G \cap R| = 8$$

Eight students study all three languages.

(b) Using the above result, we draw the Venn Diagram

8 study all three languages

$\therefore \ |F \cap G| - |F \cap G \cap R| = 20 - 8 = 12$ study French and German but not Russian.

$\therefore \ |F \cap R| - |F \cap G \cap R| = 25 - 8 = 17$ study French and Russian but not German.

$\therefore \ |G \cap R| - |F \cap G \cap R| = 15 - 8 = 7$ study German and Russian but not French

Only French $= |F| - |F \cap G| - |F \cap R| + |F \cap G \cap R|$

$$= 65 - 20 - 25 + 8 = 28$$

$\therefore$ 28 student study only French language

Only German $= |G| - |G \cap R| - |F \cap G| + |F \cap G \cap R|$

$= 45 - 15 - 20 + 8$

$= 18$

$\therefore$ 18 students study only German language.

Only Russian $= |R| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$

$= 42 - 25 - 15 + 8$

$= 10$

$\therefore$ 10 students study only Russian language.

$\therefore |U| - |F \cup G \cup R| = 120 - 100$

$= 20$ do not study any of the languages.

(c) From Venn diagram

(i) $k = 28 + 18 + 10 = 56$

Thus 56 mathematics students study exactly one language.

(ii) $k = 12 + 17 + 7 = 36$

Thus 36 mathematics students study exactly two languages.

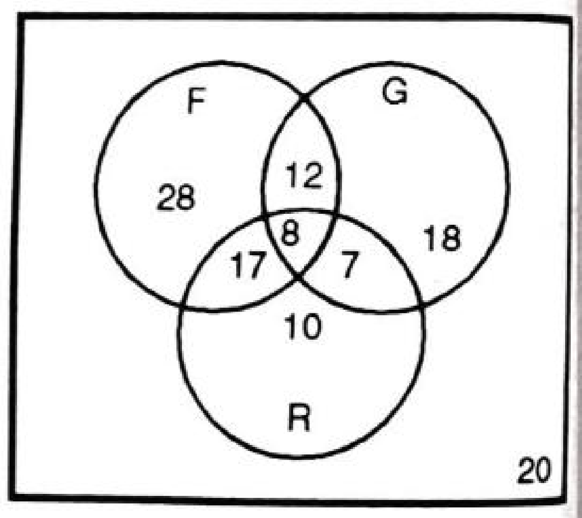


Fig. 1.19

Example 11 :

One hundred students were asked whether they had taken course in any of three areas, sociology, anthropology and history. The results were

45 had taken sociology,

38 had taken anthropology,

21 had taken history,

18 had taken sociology and anthropology,

9 had taken sociology and history,

4 had taken history and anthropology and 23 had taken no courses in any of the three areas.

(a) Draw a Venn diagram that will show the results of the survey

(b) Determine the number k of students who had taken classes in exactly (i) one of the areas (ii) two of the areas.

Solution :

Let S, A and H denote the sets of students who have taken courses in sociology, anthropology and history respectively

(a) We have $|S| = 45$, $|A| = 38$, $|H| = 21$,

$|S \cap A| = 18$, $|S \cap H| = 9$, $|H \cap A| = 4$ and,

$|S \cup A \cup H| = 100 - 23 = 77$.

Hence by principle of inclusion and exclusion, we have

$|S \cup A \cup H| = |S| + |A| + |H| - |S \cap A| - |S \cap H| - |A \cap H| + |S \cap H \cap A|$

$77 = 45 + 38 + 21 - 18 - 4 - 9 + |S \cap H \cap A|$

$|S \cap H \cap A| = 4$

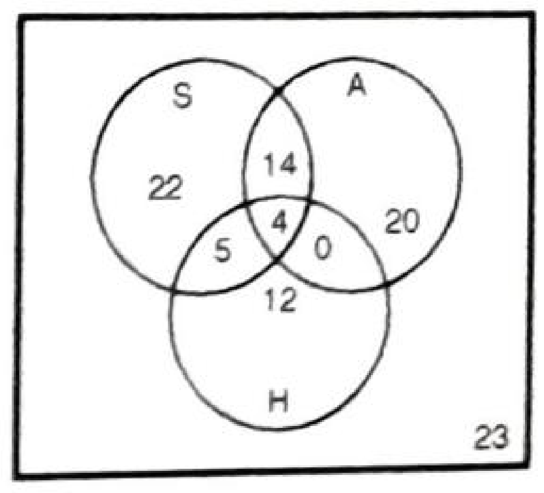


Fig. 1.20

To draw Venn diagram we write

4 belong to all the three sets

$|S \cap A| - |S \cap H \cap A| = 18 - 4 = 14$ belong to S and A but not H

$|S \cap H| - |S \cap H \cap A| = 9 - 4 = 5$ belong to S and H but not A

$|H \cap A| - |S \cap H \cap A| = 4 - 4 = 0$ belong to A and H but not S

Number of students who had taken only sociology.

$= |S| - |S \cap A| - |S \cap H| + |S \cap H \cap A|$

$= 45 - 18 - 9 + 4$

$= 22$

Number of students who had taken only anthropology

$= |A| - |H \cap A| - |S \cap A| + |S \cap H \cap A|$

$= 38 - 4 - 18 + 4$

$= 20$

Number of students who had taken only history

$= |H| - |S \cap H| - |H \cap A| + |S \cap H \cap A|$

$= 21 - 9 - 4 + 4$

$= 12$

and 23 belong to none of the three sets

(b) Using Venn diagram

(i) $k = 22 + 20 + 12 = 54$

$\therefore$ 54 students had taken classes in exactly one of the area.

(ii) $k = 14 + 5 + 0 = 19$

$\therefore$ 19 students had taken classes in exactly two of the areas.

Example 12 :

A survey of a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options air-conditioning (A), radio (R) and power windows (W), were already installed. **(Dec. 2011)**

The survey found :

15 had air-conditioning,

12 had radio,

11 had power windows,,

5 had air-conditioning and power windows,

9 had air-conditioning and radio,

4 had radio and power windows,

3 had all three options,

2 had no options.

Find the number of cars that had :

(a) Only power windows.

(b) Only air-conditioning.

(c) Only radio

(d) Radio and power windows but not air-conditioning.

(e) Air-conditioning and radio but not power windows.

(f) Only one of the options.

Solution :

We have,

$|A| = 15,\ |R| = 12,\ |W| = 11,\ |A \cap W| = 5,$
$|A \cap R| = 9,$

$|R \cap W| = 4$, and $\therefore\ |A \cap R \cap W| = 3.$

3 had all three options.

We draw Venn diagram as shown in Fig. 1.21.

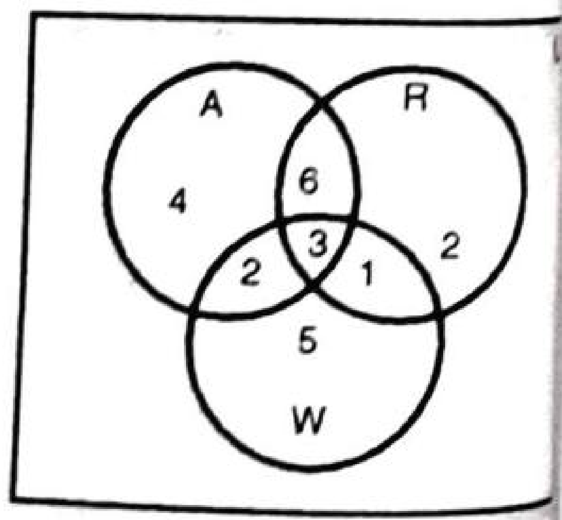


Fig. 1.21

Using the Venn diagram, we have

(a) Number of cars that had only power windows

$$= |W| - |R \cap W| - |A \cap W| + |A \cap R \cap W|$$
$$= 11 - 4 - 5 + 3 = 5$$

(b) Number of cars that had only air-conditioning

$$= |A| - |A \cap W| - |A \cap R| + |A \cap R \cap W|$$
$$= 15 - 5 - 9 + 3 = 4$$

(c) Number of cars that had only radio

$$= |R| - |A \cap R| - |R \cap W| - |A \cap R \cap W|$$
$$= 12 - 9 - 4 + 3$$
$$= 2$$

(d) Number of cars that had Radio (R) and power windows (W) but not air conditioning (A)

$$= |R \cap W| - |A \cap R \cap W|$$
$$= 4 - 3$$
$$= 1$$

(e) Number of cars that had air-conditioning (A) and radio (R) but not power windows (W)

$$= |A \cap R| - |A \cap R \cap W|$$
$$= 9 - 3 = 6$$

(f) $4 + 2 + 5 = 11$

$\therefore$ 11 cars had only one of the options.

Example 13

An investigator interviewed 100 students to determine their preferences for three drinks–Milk (M), Coffee (C) and Tea (T). He reported the following 10 students had all the three drinks 20 had 'M' and 'C', 30 had 'C' and 'T', 25 had 'M' and 'T', 12 had 'M' only, 5 had 'C' only and 8 had 'T' only.

(i) How many did not take any of the three drinks ?

(ii) How many take milk but not coffee ?

(iii) How many take tea and coffee but not milk ? **(May 97)**

Solution :

Consider the Venn diagram

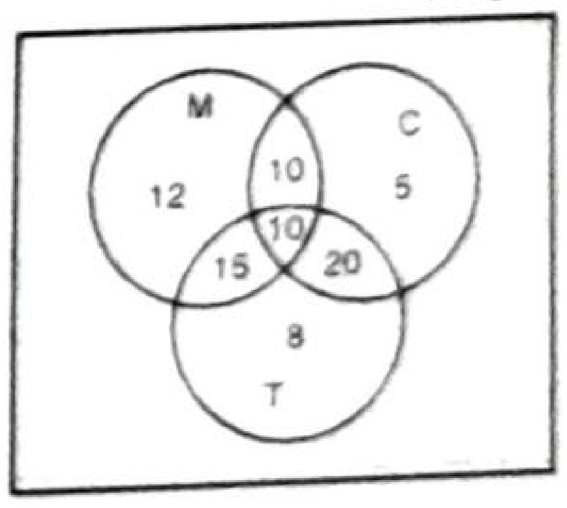


Fig. 1.22

(i) Taking the cardinalities of the disjoints sets into account.

$$|\bar{M} \cap \bar{C} \cap \bar{T}| = 100 - |M \cup C \cup T|$$
$$= 100 - (12 + 10 + 10 + 15 + 20 + 8 + 5)$$
$$= 100 - 80 = 20$$

Hence 20 students did not take any drink.

(ii) The set of students taking milk but not coffee is

$$= 12 + 15 = 27$$

(iii) The set of students taking tea and coffee, but not milk is

$$= |T \cap C| - |T \cap C \cap M|$$
$$= 30 - 10 = 20$$

Example 14

In a survey, it is reported that of 1000 programmers, 650 habitually flowchart their programs, 788 are skilled COBOL programmers, 675 are men, 278 of the women are skilled COBOL programmers, 440 programmers both habitually flowchart and are skilled in COBOL, 210 women habitually flowchart and 166 women are both skilled in COBOL and habitually flowchart. Would you accept these data as being accurately reported? Justify your answer **(Dec. 92)**

Solution :

Let F denote the set programs (both men and women) who habitually flowchart their programs. Let C denote the set of all skilled COBOL programmers. Let M and W denote the set of men women programmers respectively.

$$|M| = 675 \qquad \therefore \qquad |W| = 1000 - 675 = 325$$

$$|F| = 650 \qquad |C| = 788$$

$$|W \cap C| = 278 \qquad |W \cap F| = 210$$

$$|F \cap C| = 440$$

$$|W \cap F \cap C| = 166$$

$$|M \cap C| = |C| - |W \cap C|$$
$$= 788 - 278$$
$$= 510$$

$$|M \cap F| = |F| - |W \cap F|$$
$$= 650 - 210$$
$$= 440$$

$$|M \cap F \cap C| = |F \cap C| - |W \cap F \cap C|$$
$$= 440 - 166$$
$$= 274$$

The set of $M \cap (F \cap C)$ is the set of male programmers who habitually flowchart their program are, skilled COBOL programmers.

$$|M \cap (F \cup C)| = |M \cap F| + |M \cap C| - |M \cap F \cap C|$$
$$= 510 + 440 - 274$$
$$= 676$$

Hence there should be atleast 676 men programmers. But this contradicts the given data that the are in all 675 men programmers. Hence the data is inaccurately reported.

Example 15 :

75 children went to an amusement park, where they can ride on the merry–go–round, roller coaste and the ferris wheel. It is known that 20 of them have taken all three rides. and 55 of them have take atleast 2. Each ride costs 5 rupees and the total collection of the park was 700 rupees. Determine th number of children who did not try any of the rides.

Solution :

The total number of rides $= \frac{700}{5} = 140$

The number of children who have taken exactly 2 rides $= 55 - 20 = 35$

The number of children who have taken only one ride.

$$= 140 - 2 \times 35 - 3 \times 20$$
$$= 140 - 70 - 60$$
$$= 10$$

Hence the number of children who have not taken any rides

$$= 75 - (35 + 20 + 10)$$
$$= 75 - 65$$
$$= 10$$

Example 16 :

When carrying out a survey on the popularity of three different brands X, Y and Z of washing powder, 100 housewives were interviewed and the results were shown as follows:

30 used brand X only

22 used brand Y only

18 used brand Z only

8 used brands X and Y

9 used brands X and Z

7 used brands Z and Y

14 used none of the brands

(a) How many housewives used brands X, Y and Z?

(b) How many housewives used brands X and Z but not brand Y?

Solution :

(a) Let the number of housewives using brands X, Y and Z be 'm'.

Let also the number of housewives using brands X, Y and Z as being elements of the sets X, Y and Z respectively.

Now we can draw Venn–diagram :

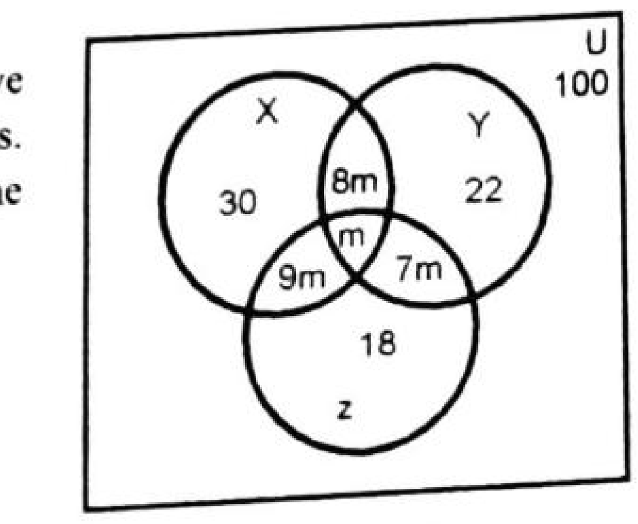


Fig. 1.23

Since 14 housewives used none of the three brands, we have that $100 - 14 = 86$ housewives used one or more of the brands. So number of elements of $|X \cup Y \cup Z| = 86$. Hence, from the Venn diagram.

$$30 + (8 - m) + m + (9 - m) + 22 + (7 - m) + 18 = 86$$

$$\therefore \qquad 94 - 2m = 86$$

$$m = 4$$

$\therefore$ 4 housewives used all three brands X, Y, and Z

(b) The number of housewives using brands X and Z and not Y is the number of elements in $(x \cap z) \cap \bar{Y}$, which is the region indicated as having $9 - m$ elements in the Venn diagram.

Thus the required answer is

$$9 - m = 9 - 4 = 5 \text{ housewives}$$

Example 17 :

Find how many integers between 1 and 60 are not divisible by 2 nor by 3 and nor by 5 ?

(Dec. 2006, 2008)

Solution :

Let A_1, A_2 and A_3 be the set of integers between 1 and 60 divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A_1| = \left|\frac{60}{2}\right| = 30$$

$$|A_2| = \left|\frac{60}{3}\right| = 20$$

$$|A_3| = \left|\frac{60}{5}\right| = 12$$

and $$|A_1 \cap A_2| = \left|\frac{60}{2 \times 3}\right| = 10$$

$$|A_1 \cap A_3| = \left|\frac{60}{2 \times 5}\right| = 6$$

$$|A_2 \cap A_3| = \left|\frac{60}{3 \times 5}\right| = 4$$

and $$|A_1 \cap A_2 \cap A_3| = \left|\frac{60}{2 \times 3 \times 5}\right| = 2$$

Number of integers between 1 and 60 which are divisible by 2, 3 or 5 are

$$\begin{aligned} &= |A_1 \cup A_2 \cup A_3| \\ &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\ &= 30 + 20 + 12 - 10 - 6 - 4 + 2 \\ &= 44 \end{aligned}$$

Hence the number of integers between 1 and 60 are not divisible by 2, 3 or 5 $= 60 - 44 = 16$

Example 18 :

Determine number of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

(Dec. 2005)

Solution : Let A_1 be the set of integers between 1 and 100 divisible by 2.

Let A_2 be the set of integers between 1 and 100 divisible by 3.

Let A_3 be the set of integers between 1 and 100 divisible by 5.

Then $$|A_1| = \left|\frac{100}{2}\right| = 50$$

$$|A_2| = \left|\frac{100}{3}\right| = 33$$

$$|A_3| = \left|\frac{100}{5}\right| = 20$$

$$|A_1 \cap A_2| = \left|\frac{100}{2 \times 3}\right| = 16$$

$$|A_1 \cap A_3| = \left|\frac{100}{2 \times 5}\right| = 10$$

$$|A_1 \cap A_3| = \left|\frac{100}{3 \times 5}\right| = 6$$

and $$|A_1 \cap A_2 \cap A_3| = \left|\frac{100}{2 \times 3 \times 5}\right| = 3$$

Number of integers between 1 and 100 which are divisible by 2 or 3 or 5 i.e.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\quad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\ &= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74 \end{aligned}$$

Hence, number of integer between 1 and 100 are not divisible by 2, 3, or 5 i.e.

$$\overline{|A_1 \cup A_2 \cup A_3|} = 100 - 74 = 26$$

Example 19 :

Among the integers 1 and 300, how many of them are divisible by 3, 5 or 7 and are not divisible by 3, nor by 5, nor by 7? How many of them are divisible by '3' but not by '5' nor by '7'? **(May 2009)**

Solution : Let A_1 denote the set of integers between 1 and 300 divisible by '3'. Similarly A_2 and A_3 be the sets of integers divisible by 5 and 7 respectively.

Then $$|A_1| = \left|\frac{300}{3}\right| = 100$$

$$|A_2| = \left|\frac{300}{5}\right| = 60$$

$$|A_3| = \left|\frac{300}{7}\right| = 42$$

and $|A_1 \cap A_2|$ = Number of integers divisible by 3 and 5

$$= \left|\frac{300}{3 \times 5}\right| = 20$$

$$|A_1 \cap A_3| = \left|\frac{300}{3 \times 7}\right| = 14$$

$$|A_2 \cap A_3| = \left|\frac{300}{5 \times 7}\right| = 8$$

and $$|A_1 \cap A_2 \cap A_3| = \left|\frac{300}{3 \times 5 \times 7}\right| = 2$$

Hence, using the principle of inclusion and exclusion, we have number of integers which divisible by 3 or 5 or 7.

$$= |A_1 \cup A_2 \cup A_3|$$

$$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$$

Hence, there are 162 numbers between 1 and 300 divisible by 3, 5 or 7.

∴ Number of integers which are not divisible by 3, nor by 5, and nor by 7 $= 300 - 162 = 1$

Again number of integers between 1 and 300 which are divisible by 3 but not by nor by 7 $= |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 100 - 20 - 14 + 2 = 6$

Example 20 :

How many integers between 1 and 2000 are divisible by 2, 3, 5 or 7 ? **(May**

Solution : Let A be the set of numbers between 1 to 2000 divisible by 2.

B be the set of numbers divisible by 3.

C be the set of numbers divisible by 5, and

D be the set of numbers divisible by 7

∴ $|A| = \left|\frac{2000}{2}\right| = 1000.$ $|B| = \left|\frac{2000}{3}\right| = 666.$

$|C| = \left|\frac{2000}{5}\right| = 400.$ $|D| = \left|\frac{2000}{7}\right| = 285.$

∴ $|A \cap B| = \left|\frac{2000}{2 \times 3}\right| = 333.$ $|A \cap C| = \left|\frac{2000}{2 \times 5}\right| = 200.$

$|A \cap D| = \left|\frac{2000}{2 \times 7}\right| = 142.$ $|B \cap C| = \left|\frac{2000}{3 \times 5}\right| = 133.$

$|B \cap D| = \left|\frac{2000}{3 \times 7}\right| = 95.$ $|C \cap D| = \left|\frac{2000}{5 \times 7}\right| = 57.$

Also, $$|A \cap B \cap C| = \left|\frac{2000}{2 \times 3 \times 5}\right| = 66.$$

$$|A \cap B \cap D| = \left|\frac{2000}{2 \times 3 \times 7}\right| = 47.$$

$$|B \cap C \cap D| = \left|\frac{2000}{3 \times 5 \times 7}\right| = 19.$$

$$|A \cap C \cap D| = \left|\frac{2000}{2 \times 5 \times 7}\right| = 28.$$

and $$|A \cap B \cap C \cap D| = \left|\frac{2000}{2 \times 3 \times 5 \times 7}\right| = 9.$$

By the principle of inclusion and exclusion

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (1000 + 666 + 400 + 285) - (333 + 200 + 142 + 133 + 95 + 57) + (66 + 47 + 19 + 28) - 9 = 1542$$

Example 21 :

Determine the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7. **(Dec. 97, May 2003)**

Solution : Let the sets A_1, A_2, A_3, A_4 denote the set of integers between 1 and 250 that are divisible respectively by 2, 3, 5 and 7.

Then $|A_1|$ = number of elements in A_1

= largest number of elements in A_1 which are divisible by 2

$$= \left|\frac{250}{2}\right| = 125.$$

Similarly, $$|A_2| = \left|\frac{250}{3}\right| = 83.$$

$$|A_3| = \left|\frac{250}{5}\right| = 50.$$

$$|A_4| = \left|\frac{250}{7}\right| = 35.$$

Similarly, $|A_1 \cap A_2|$ = largest number of elements that are divisible by 2 and 3.

$$= \left|\frac{250}{2 \times 3}\right| = 41.$$

and, $$|A_1 \cap A_3| = \left|\frac{250}{2 \times 5}\right| = 25.$$

and $$|A_2 \cap A_3| = \left|\frac{250}{3 \times 5}\right| = 16.$$

$$|A_1 \cap A_4| = \left\lfloor \frac{250}{2 \times 7} \right\rfloor = 17.$$

$$|A_3 \cap A_4| = \left\lfloor \frac{250}{5 \times 7} \right\rfloor = 7.$$

$$|A_2 \cap A_4| = \left\lfloor \frac{250}{3 \times 7} \right\rfloor = 11.$$

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{2 \times 3 \times 5} \right\rfloor = 8.$$

$$|A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{250}{2 \times 3 \times 7} \right\rfloor = 5.$$

$$|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2 \times 5 \times 7} \right\rfloor = 3.$$

$$|A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{3 \times 5 \times 7} \right\rfloor = 2.$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{2 \times 3 \times 5 \times 7} \right\rfloor = 1.$$

Now we use the formula,

$$\begin{aligned}
|A_1 \cap A_2 \cap A_3 \cap A_4| &= \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| \\
&\quad - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
&\quad + |A_1 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 125 + 83 + 50 + 35 - 41 - 25 - 11 - 16 - 17 - 7 + 8 + 5 + 3 + 2 - 1 \\
&= 193
\end{aligned}$$

Example 22 :

Among the integers 1 to 1000 :

(i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

(ii) How many are not divisible by 5 and 7 but divisible by 3 ? **(Dec. 2010)**

Solution :

Let A, B, C denote respectively the set of integers from 1 to 1000 divisible by 3 by 5 and by 7.

Then $\bar{A} \cap \bar{B} \cap \bar{C}$ denote the set of integers not divisible by 3, nor by 5, nor by 7.

By De Morgan's Law $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{(A \cup B \cup C)}$

Hence, $|\overline{A \cup B \cup C}| = 1000 - |A \cup B \cup C|$

$$|A| = \left\lfloor \frac{1000}{3} \right\rfloor = 333$$

$$|B| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$|C| = \left\lfloor \frac{1000}{7} \right\rfloor = 142.$$

$$|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

$$|B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor = 28$$

$$|A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor = 47$$

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{105} \right\rfloor = 9$$

By principle of inclusion and exclusion,

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| \\
&\quad + |A \cap B \cap C|. \\
&= 333 + 200 + 142 - 66 - 28 - 47 + 9. = 543.
\end{aligned}$$

Hence, $|\overline{A \cup B \cup C}| = 1000 - 543 = 457.$

(ii) Consider the Venn diagram shown in Fig. 1.24

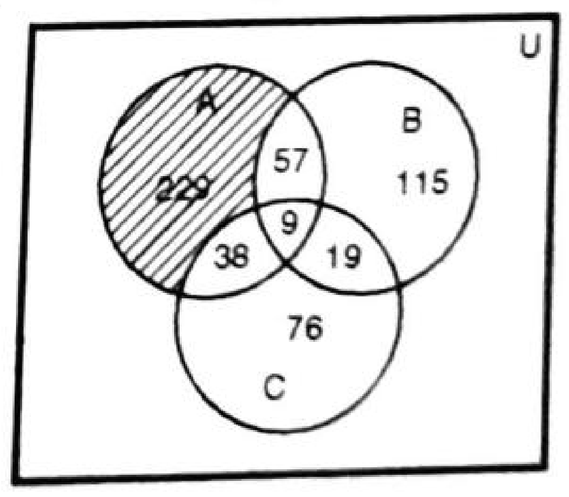


Fig. 1.24

The set of integers not divisible by 5 and 7 but divisible by 3 is the set $A \cap \bar{B} \cap \bar{C}$.

$A \cap \bar{B} \cap \bar{C} = A \cap (\overline{B \cup C}) = A - (B \cup C)$, the shaded portion shown in Venn diagram.

It is clear from the diagram that,

$$|A - (B \cup C)| = |A| - |(A \cap B) \cup (A \cap C)|$$

Now, $|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$

$$= 66 + 47 - 9 = 104$$

$\therefore \quad |A - (B \cup C)| = 333 - 104 = 229$

Hence, 229 integers from 1 to 1000 are not divisible by 5 and 7 but divisible by 3.

Example 23 :

A software company writes a new package which integrates a word processing program with a spread sheet program and they wish it to run on a 64 k machine. The word processor requires 40 k for program and data and the spread sheet requires 32 k for the same. If 16 k must be reserved for the code integrator, what is the minimum amount of overlapping space that will be necessary ?

Solution :

Let A denote the memory space reserved for word processor

and B denote the memory space reserved for spread sheet.

Given : $|A| = 40$

$|B| = 32$

Available memory is $64 - 16 = 48$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$\because \quad |A \cup B| \le 48$

$\therefore |A| + |B| - |A \cap B| \le 48$

i.e. $\quad |A \cap B| \ge |A| + |B| - 48$

$$= 40 + 32 - 48 = 24$$

Hence the minimum amount of overlapping space that will be necessary is 24 k.

Example 24 :

A survey of 500 television watchers produced the following information :

285 watch football games

195 watch hockey games

115 watch basketball games

45 watch football and basketball games

70 watch football and hockey games

50 watch hockey and basketball games

50 do not watch any of the 3 kinds of games.

(i) How many people in the survey watch all 3 kinds of games ?

(ii) How many people watch exactly one of the sports ?

(May 2009, 2011)

Solution :

Let F be the set of people who watch football

Let H be the set of people who watch hockey.

Let B be the set of people who watch basketball.

Given : $|F| = 285$

$|H| = 195$

$|B| = 115$

$|F \cap B| = 45$

$|F \cap H| = 70$

$|H \cap B| = 50$

50 people do not watch any of the 3 kinds of games.

(i) People in survey who watch all 3 kinds of games i.e. football or hockey or basketball,

$$|F \cup H \cup B| = 500 - 50$$
$$= 450$$

450 people watch atleast one of the 3 games.

Using the formula of inclusion and exclusion, we have,

$$|F \cup H \cup B| = |F| + |H| + |B| - |F \cap H| - |F \cap B| - |H \cap B| + |F \cap H \cap B|$$

$\therefore \quad 450 = 285 + 195 + 115 - 45 - 70 - 50 + |F \cap H \cap B|$

$\therefore \quad |F \cap H \cap B| = 20$

20 people in survey watch all 3 kinds of games.

(ii) People who watch only football.

$$= |F| - |F \cap H| - |F \cap B| + |F \cap H \cap B|$$
$$= 285 - 70 - 45 + 20$$
$$= 190$$

People who watch only hockey

$$= |H| - |H \cap F| - |H \cap B| + |F \cap H \cap B|$$
$$= 195 - 70 - 50 + 20 = 95$$

People who watch only basketball.

$$= |B| - |F \cap B| - |B \cap H| + |F \cap H \cap B|$$
$$= 115 - 45 - 50 + 20 = 40$$

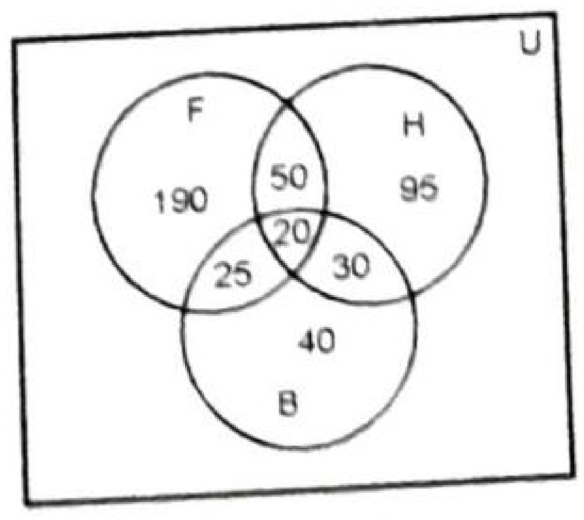


Fig. 1.25

Number of people watching exactly one kind of game

$$= 190 + 95 + 40 = 325$$

$\therefore \quad$ 325 people watch exactly one kind of game.

Example 25 :

Among 130 students, 60 study Mathematics, 51 study Physics and 30 study both mathemati
physics. Out of 54 students studying Chemistry 26 study Mathematics, 21 study Physics and 12
both mathematics and Physics. All the students studying neither mathematics nor Physics are stu
biology. Find :

(i) How many are studying biology ?

(ii) How many are not studying Chemistry and studying Mathematics but not Physics ?

(iii) How many students are studying neither mathematics nor Physics nor Chemistry?

Solution :

(i) $|M \cup P| = |M| + |P| - |M \cap P|.$

$= 60 + 51 - 30$

$= 81.$

$\therefore$ Number of students studying neither Mathematics nor Physics

$= 130 - |M \cup P|$

$= 130 - 81 = 49$

Hence the number of students studying Biology is 49.

(ii) The set of students studying Mathematics but neither Chemistry nor Physics is,

$|M - [M \cap (C \cup P)]| = |M| - |M \cap C| - |M \cap P| + |M \cap C \cap P|.$

$= 60 - 26 - 30 + 12 = 16.$

(iii) Set of students studying neither mathematics nor Physics nor Chemistry is $|\overline{M \cup P \cup C}|$

$\therefore \quad |\overline{M \cup P \cup C}| = 130 - |M \cup P \cup C|.$

$= 130 - |M| - |P| - |C| + |M \cap P| + |M \cap C| + |P \cap C|$
$- |M \cap P \cap C|.$

$= 130 - 60 - 51 - 54 + 30 + 26 + 21 - 12 = 30.$

Example 26 :

It was found that in first year of Computer Science of 80 students 50 know Cobol, 55 know 'C',
know Pascal. It was also known that 37 know 'C' and Cobol, 28 know 'C' and Pascal, 25 know Pas
and Cobol. 7 students however know none of the languages. Find:

(i) How many know all the 3 languages ?

(ii) How many know exactly two languages ?

(iii) How many know exactly 1 language ?

Solution :

Let B, C and P denote the set of students who know Cobol, 'C' and Pascal respectively.

Then $|B \cup C \cup P| = 80 - 7$

$= 73$

is the number of students who know at least one of the languages.

(i) $|B \cup C \cup P| = |B| + |C| + |P| - |B \cap C| - |B \cap P| - |C \cap P| + |B \cap C \cap P|$

Hence the number of students who know all the three languages is

$|B \cap C \cap P| = 73 - 50 - 55 - 46 + 37 + 28 + 25 = 12$

(ii) The number of students who know Cobol and 'C' but not Pascal is

$|B \cap C \cap \bar{P}| = |B \cap C| - |B \cap C \cap P|$

$= 37 - 12 = 25$

Similarly, the number of students who know Cobol and Pascal but not 'C' is,

$|B \cap P \cap \bar{C}| = |B \cap P| - |B \cap C \cap P|$

$= 25 - 12 = 13.$

and the number of students who known Pascal and 'C' but not Cobol is

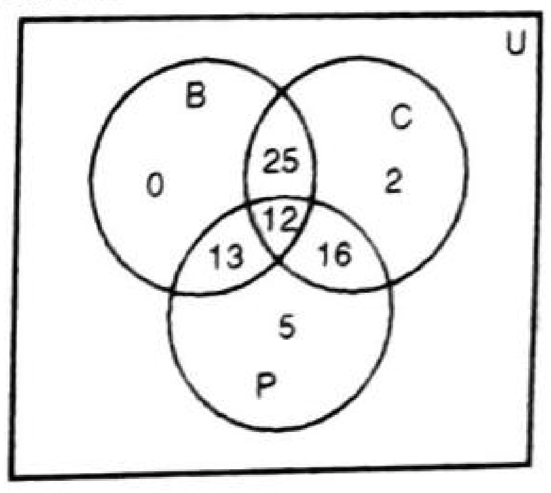


Fig. 1.26

$|\bar{B} \cap P \cap C| = |P \cap C| - |B \cap C \cap P|$

$= 28 - 12 = 16.$

Hence, number of students who know exactly two languages is,

$25 + 13 + 16 = 54.$

(iii) The number of students who know only Cobol (i.e. neither 'C' nor Pascal) is

$|B| - |B \cap C| - |B \cap P| + |B \cap P \cap C|$

$= 50 - 37 - 25 + 12 = 0.$

Similarly the number who know only 'C' is

$|C| - |C \cap P| - |C \cap B| + |B \cap P \cap C|$

$= 55 - 28 - 37 + 12 = 2.$

The number of students knowing only Pascal is,

$|P| - |P \cap C| - |P \cap B| + |B \cap P \cap C|$

$= 46 - 28 - 25 + 12 = 5.$

Hence the number of students who know exactly one language is

$0 + 2 + 5 = 7.$

Example 27 :

It is known that at the university, 60 percent of the professors play tennis, 50 percent of them play bridge, 70% jog, 20% play tennis and bridge, 30% play tennis and jog, 40% play bridge and jog. If someone claimed that 20% of the professors jog and play bridge and tennis, would you believe this claim ? Why ? **(Dec. 2000)**

Solution :

Let T be the set of professors playing tennis

B be the set of professors playing bridge

J be the set of professors who jog.

Since, the data is given in percent, we assume that the total number of professors be 100 x. Hence from the given data,

$$|T| = 60x$$
$$|B| = 50x$$
$$|J| = 70x$$
$$|T \cap B| = 20x$$
$$|T \cap J| = 30x$$
$$|B \cap J| = 40x$$
$$|B \cap J \cap T| = 20x$$

Using the formula of inclusion and exclusion, we have,

$$|B \cup J \cup T| = |B| + |J| + |T| - |B \cap J| - |J \cap T| - |B \cap T| + |B \cap J \cap T|$$
$$= 50x + 70x + 60x - 40x - 30x - 20x + 20x$$
$$= 110x$$

$\therefore$ $|B \cup J \cup T|$ is 110% which absurd.

$\therefore$ I will not believe the claim made by someone.

Example 28 :

Out of 250 candidates who failed in an examination, it was revealed that 128 failed in mathematics, 87 in physics and 134 in aggregate. 31 failed in mathematics and in Physics, 54 failed in the aggregate and in mathematics. 30 failed in the aggregate and in physics. Find how many candidates failed.

(i) in all the three subjects.

(ii) in mathematics but not in physics.

(iii) in the aggregate but not in mathematics.

(iv) in physics but not in aggregate or in mathematics. **(Dec. 2012)**

Solution :

Let $|M| = 128$, $|P| = 87$, $|A| = 134$

$|M \cap P| = 31$, $|A \cap M| = 54$, $|A \cap P| = 30$

(i) By the principle of inclusion and exclusion we have,

$$|M \cup P \cup A| = |M| + |P| + |A| - |M \cap P| - |A \cap M| - |A \cap P| + |M \cap P \cap A|$$
$$250 = 128 + 87 + 134 - 31 - 54 - 30 + |M \cap P \cap A|$$
$$250 = 234 + |M \cap P \cap A|$$
$$250 - 234 = |M \cap P \cap A|$$
$$\therefore |M \cap P \cap A| = 16$$

16 candidates are failed in all three subjects.

(ii) Candidates failed in Mathematics but not in Physics

$$= |M| - |M \cap P|$$
$$= 128 - 31 = 97$$

(iii) Candidates Failed in aggregate but not in Mathematics

$$= |A| - |A \cap M|$$
$$= 134 - 54 = 80$$

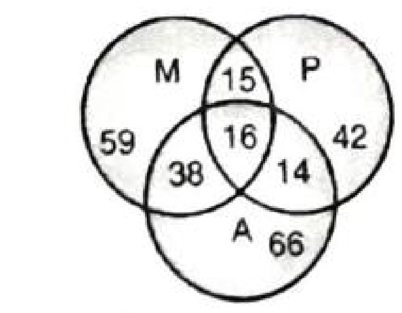


Fig. 1.26 (a)

(iv) Candidates Failed in Physics but not in aggregate or in Mathematics

$$|P| - |M \cap P| - |A \cap P| + |M \cap P \cap A| = 87 - 31 - 30 + 16$$
$$= 42$$

Example 29 :

A college record gives the following information : 119 students enrolled in Introductory computer Science of these 96 took Data structures, 53 took Foundations, 39 took Assembly language, 31 took both foundations and Assembly language, 32 took both data structures and assembly language, 38 took data structures and foundations and 22 took all the three courses. Is the information correct ? Why ? **(Dec. 93)**

Solution :

Let D be the set of students who took data structure.

F be the set of students who took data Foundations.

A be the set of students who took data Assembly Language.

Hence from given data,

$|D| = 96$, $|F| = 53$, $|A| = 39$

$|F \cap A| = 31$, $|D \cap A| = 32$, $|D \cap F| = 38$

$|F \cap D \cap A| = 22$

Using the formula of **principle of inclusion and exclusion** we have,

$$\therefore |F \cup D \cup A| = |F| + |D| + |A| - |F \cap A| - |D \cap A| - |D \cap F| + |F \cap D \cap A|$$
$$= 53 + 96 + 39 - 31 - 32 - 38 + 22$$
$$= 109 \text{ which is less than } 119.$$

Since there were 119 students enrolled for the course, assuming that all these students had taken atleast one course, the given information is not correct.

Example 30 :

Suppose that 109 of 150 computer science students at one of the Mumbai college take at least one of the following computer language : VB, VC++ and Java.

Suppose 45 study VB, 61 study VC++, 53 study Java. 18 study VB and VC++, 53 study VC++ and Java, and 23 study VB and Java.

(i) How many students study all 3 languages ?

(ii) How many students study only VC++?

(iii) How many students do not study any of the language ? **(May 20**

Solution :

Let V be the set of students who study VB

Let C be the set of students who study VC++

Let J be the set of students who study Java.

By using the formula of principle of inclusion and exclusion

(i) $|V \cup C \cup J| = |V| + |C| + |J| - |V \cup C| - |V \cup J| - |C \cup J| + |V \cap C \cap J|$

$$109 = 45 + 61 + 53 - 18 - 23 - 53 + |V \cap C \cap J|$$

$$|V \cap C \cap J| = 109 - 65$$

$$= 44$$

4 students study all 3 languages.

(ii) $|C| - |V \cup C| - |C \cup J| + |V \cap C \cap J|$

$$= 61 - 18 - 53 + 44 = 34$$

34 students study only VC++.

(iii) $|\overline{V \cup C \cup J}| = |U| - |V \cup C \cup J|$

$$= 150 - 109 = 41$$

41 students do not study any of the language.

Example 31 :

30 cars were assembled in a factory. The options available were a radio, an air conditioner, white - wall tires. It is known that 15 of the cars have radios, 8 of them have air conditioners, and 6 them have white-wall tires. Moreover, 3 of them have all 3 options. How many cars do not have options at all.

Solution :

Let A_1 be the sets of cars with a radio.

Let A_2 be the sets of cars with an air-conditioner.

Let A_3 be the sets of cars with wall tires.

Given : $|A_1| = 15$

$|A_2| = 8$

$|A_3| = 6$

$|A_1 \cap A_2 \cap A_3| = 3$

By using formula of principle of inclusive and exclusion, we have,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

$$= 15 + 8 + 6 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + 3$$

$$= 32 - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|.$$

Since, $|A_1 \cap A_2| \geq |A_1 \cap A_2 \cap A_3|$.

$|A_1 \cap A_3| \geq |A_1 \cap A_2 \cap A_3|$.

$|A_2 \cap A_3| \geq |A_1 \cap A_2 \cap A_3|$.

Hence we consider that,

Since, $|A_1 \cap A_2| = 3$.

$|A_1 \cap A_3| = 3$.

$|A_2 \cap A_3| = 3$.

$$|A_1 \cup A_2 \cup A_3| \leq 32 - 3 - 3 - 3.$$

$$\leq 23.$$

That is, there are at most 23 cars that have one or more options. There are at least 7 cars that do not have any options.

Example 32 :

Find the number of positive integers not exceeding 100, that are either odd or the square of an integer.

Solution :

Let A be the set of odd integers between 1 and 100.

B be the set of integers between 1 and 100 that are squares of an integer.

$$A = \{1, 3, 5, 7, 9, \ldots\ldots\ldots, 99\}$$

$$B = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 50 + 10 - 5$$

$$= 55$$

Example 33 :

Let A denote the set of staff who has mastery in 'C" language. B denotes the set of staff having mastery in C++. C denote the set of staff having mastery in COBOL, D denote the set of staff members having mastery in JAVA. E denote the set of staff staying in staff quarters. F denote the set of staff who went to play games on computer. Express the following statements in set theoretic notation.

(i) All staff staying in staff quarter has mastery neither in C nor in C++ went to play games on computer.

(ii) The staff who went to play games are only those who has mastery in COBOL.

(iii) No staff who has mastery in C language went to play games.

(iv) Those and only those staff who has mastery in JAVA and COBOL went for a cricket match.

(v) All staff went to play games.

Solution :

(i) $E = F - (A \cup B)$.

(ii) $C \subseteq F$.

(iii) $A \cap F$.

(iv) $F = C \cup D$.

(v) $A \cup B \cup C \cup D \cup E \subseteq F$.

Example 34 :

Let A denote the set of students who study data structures, B denote the set of students who study discrete structures. C denote the set of students who study assembly language programming. D denote the set of students studying Theory of Computer Science. Let E denote the set of students who are staying in Hostel and F denote the set of students who went to watch a cricket match last Monday. Express the following statements in set theoretic notation.

(i) All hostelites who study neither data structure nor discrete structure went to watch cricket match last Monday.

(ii) The students who went to see cricket match are only those who study assembly language programming or data structure.

(iii) No student who is studying data structures went to see cricket match.

(iv) Those and only those students who are studying theory of computer science and discrete structure went to see for a cricket match.

(v) All went to see cricket match.

Solution : **(May 9…)**

(i) $E \cap \bar{A} \cap \bar{B} \subseteq F$ or $(E - A) - B \subseteq F$

(ii) $F = C \cup A$

(iii) $A \cap F = \phi$

(iv) $F = D \cap B$

(v) If the universal set is $A \cup B \cup C \cup D$, then $\bar{E} \subseteq F$, otherwise $(A \cup B \cup C \cup D) - E \subseteq F$.

1.3 Laws of Set Theory :

1.3.1 Laws :

The set operations obey the same rules as those of numbers, such as associativity, commutativity and distributivity. However, the cancellation rule which is true for numbers, is not true for sets in general. In addition, there are rules such as Idempotent laws, Absorption laws, De Morgans laws, which are true only for sets

Theorem : The operations defined on sets satisfy the following properties, for any sets A, B, C.

1. Commutativity :

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

2. Associativity :

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributivity :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. Idempotent laws :

(i) $A \cup A = A$

(ii) $A \cap A = A$

5. De Morgans laws :

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

6. Absorption laws :

(i) $A \cup (A \cap B) = A$

(ii) $A \cap (A \cup B) = A$

7. Properties of the complement :

(i) $\bar{A} = U - A$ (ii) $A \cup \bar{A} = U$ (iii) $A \cap \bar{A} = \phi$

(iv) $\bar{\phi} = U$ (v) $\bar{U} = \{\ \}$

$\phi \neq \{\phi\}$

8. Double complement :

$\bar{\bar{A}} = A$

9. Properties of a universal set :

(i) $A \cup U = U$

(ii) $A \cap U = A$

$\phi = \{\ \}$

10. Properties of the Empty set :

(i) $A \cup \phi = A$ or $A \cup \{\ \} = A$

(ii) $A \cap \phi = \phi$ or $A \cap \{\ \} = \{\ \}$

11. Properties of Difference :

(i) $A - A = \phi$ (ii) $A - \bar{A} = A$

(iii) $\bar{A} - A = \bar{A}$ (iv) $A - \phi = A$

(v) $A - B = A \cap \bar{B}$

(vi) $A - B = B - A$ if and only if $A = B$

(vii) $A - B = A$ if and if $A \cap B = \phi$

(viii) $A - B = \phi$ if and only if $A \subseteq B$

12. Properties of symmetric difference :

(i) $A \oplus A = \phi$ (ii) $A \oplus \phi = A$ (iii) $A \oplus U = \bar{A}$

(iv) $A \oplus \bar{A} = U$ (v) $A \oplus B = A \cup B - A \cap B$

Proof :

We shall prove properties 1, 3, 5, 8. The remaining are easy exercises for the reader.

1. Commutativity :

(i) $A \cup B = B \cup A$

Let $x \in (A \cup B)$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in (B \cup A)$

$\therefore (A \cup B) \subseteq (B \cup A)$ ...(1)

Conversly, let us assume $x \in (B \cup A)$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in (A \cup B)$

$\therefore (B \cup A) \subseteq (A \cup B)$ ...(2)

From Equations (1) or (2), we can deduce

$A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

Let $x \in (A \cap B)$

then $x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in (B \cap A)$

Hence $(A \cap B) \subseteq (B \cap A)$ ...(3)

conversely, let us assume $x \in (B \cap A)$

then $x \in B$ and $x \in A$

$\Rightarrow x \in (A \cap B)$

Hence $(B \cap A) \subseteq (A \cap B)$

From Equations (3) and (4) we can deduce, ...(4)

$A \cap B = B \cap A$

3. Distributive laws:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$

Let $x \in A \cup (B \cap C)$

Then $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A$ or ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Hence one can prove

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can be proved on similar lines.

5. De Morgans laws :

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cup B} = \{x \mid x \notin A \cup B\}$

$= \{x \mid x \notin A \text{ and } x \notin B\}$

$= \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\}$

$= \bar{A} \cap \bar{B}$

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$ can be proved in the some way, as above.

6. Double complement :

$\bar{\bar{A}} = A$

$\bar{\bar{A}} = \{x \mid x \notin \bar{A}\}$

$= \{x \mid x \in A\}$

$= A$

The above properties can also be demonstrated by drawing suitable Venn diagrams, as shown below

For commutative laws :

A∪B

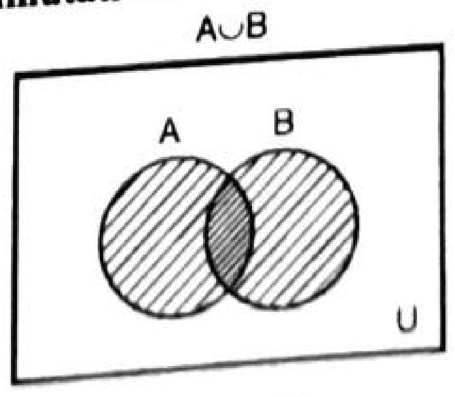


Fig. 1.27

B∪A

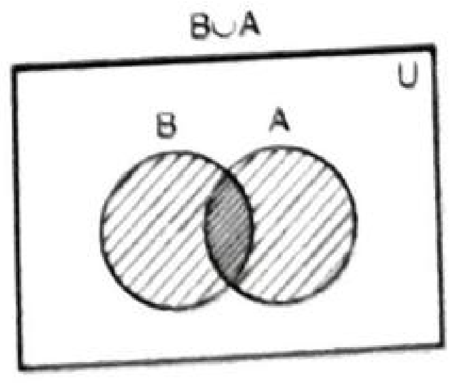


Fig. 1.28

Fig. 1.27 and Fig. 1.28 are same, hence, $A \cup B = B \cup A$

For distributive laws :

B∩C

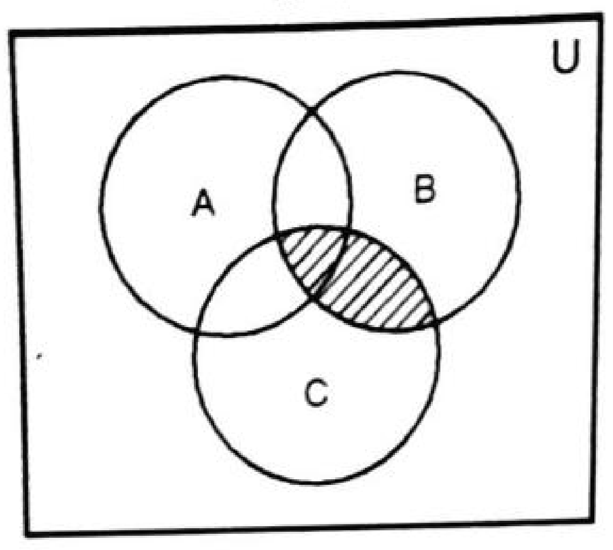


Fig. 1.29

A∪(B∩C)

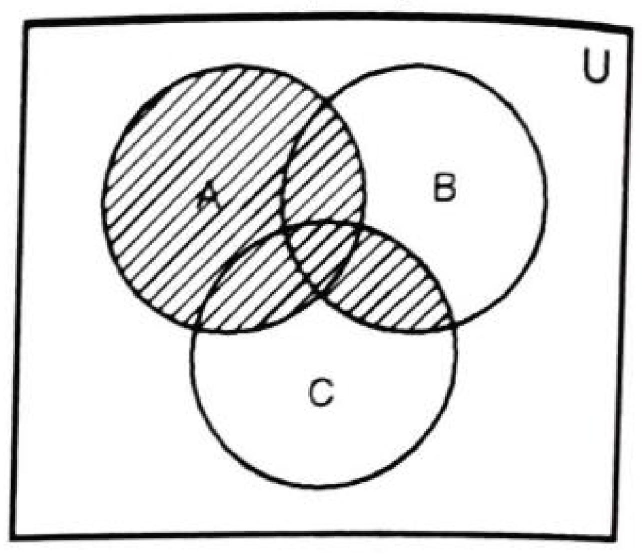


Fig. 1.30

A∪B

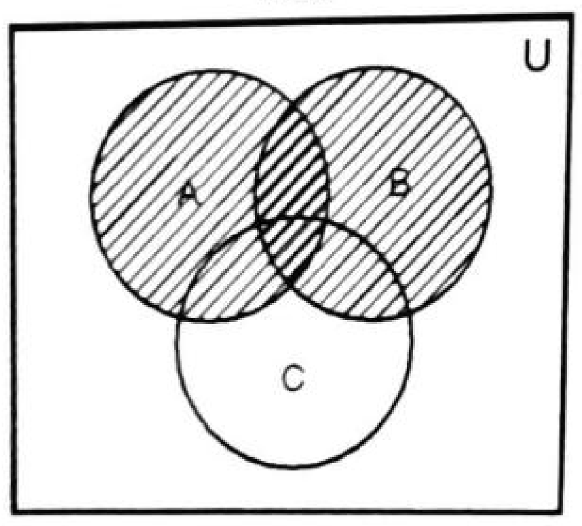


Fig. 1.31

A∪C

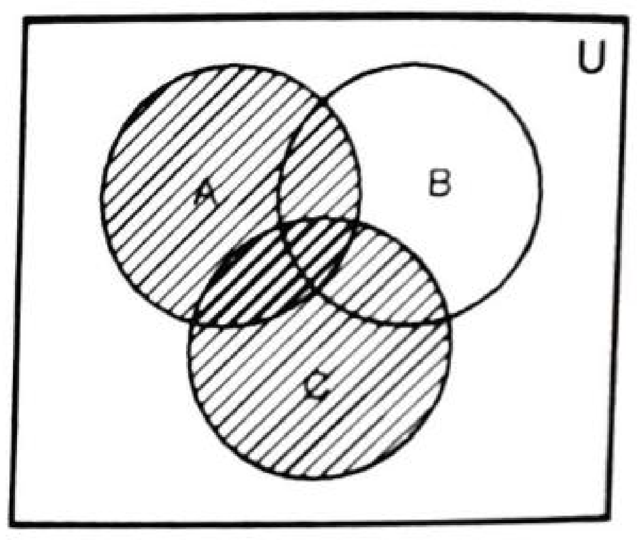


Fig. 1.32

(A∪B)∩(A∪C)

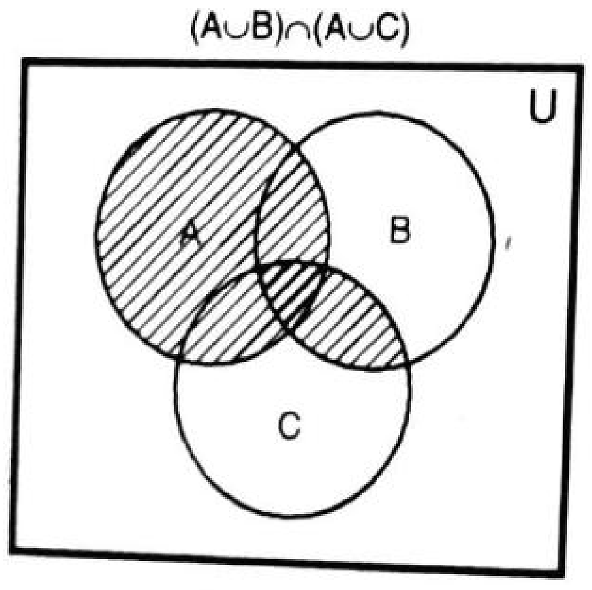


Fig. 1.33

Fig. 1.30 and Fig. 1.33 are same

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

For De Morgans laws :

$\overline{A \cup B}$

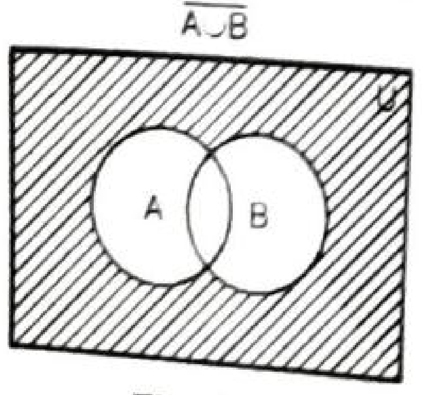


Fig. 1.34

$\bar{A}$

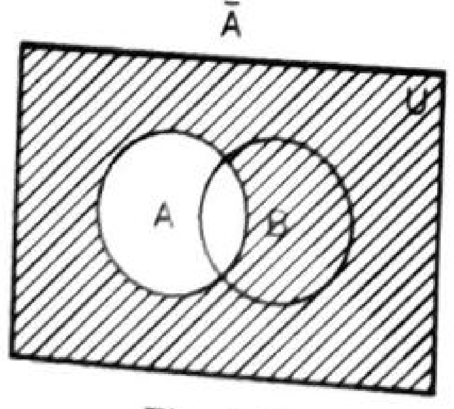


Fig. 1.35

$\bar{B}$

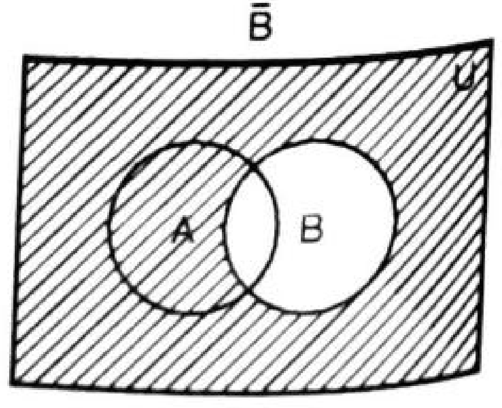


Fig. 1.36

$\bar{A} \cap \bar{B}$

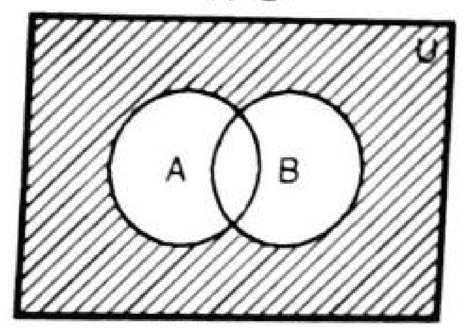


Fig. 1.37

Fig. 1.34 and Fig. 1.35 are same

Hence $\overline{A \cup B} = \bar{A} \cap \bar{B}$

1.3.2 Principle of Duality :

The principle of duality states that any established result involving sets and complements and operations of union and intersection gives a corresponding dual result by replacing U by ϕ and $\cup$ by $\cap$, and vice versa.

Example :

Consider $A \cup \bar{A} = U$ applying the principle of duality.

We get, $A \cap \bar{A} = \phi$

1.3.3 Exercise Set 3 - (Solved) :

Example 1 :

Let A, B and C are subset of U (universal set) prove that

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ **(Dec. 2003, Dec. 2006)**

Solution : Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow \quad x \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad x \in A$ and $y \in B$ or $y \in C$

$\Rightarrow \quad (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow \quad (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Rightarrow \quad (x, y) \in (A \times B) \cup (A \times C)$

$\therefore \quad A \times (B \cup C) = (A \times B) \cup (A \times C)$

Example 2 :

Show that (using laws of logic)

(a) $A \cup (A^C \cap B) = A \cup B$.

(b) $A \cap (A^C \cup B) = A \cap B$.

Solution :

Note : A^C means complement of A. This is another notation.

(a)

$$\text{L.H.S} = A \cup (A^C \cap B)$$

$$= (A \cup A^C) \cap (A \cup B) \quad \text{...Distributive law}$$

$$= U \cap (A \cup B) \quad \text{...Complement laws}$$

$$= A \cup B$$

Hence $A \cup (A^C \cap B) = A \cup B$

(b) $A \cap (A^C \cup B) = A \cap B$

$$\text{L.H.S} = A \cap (A^C \cup B)$$

$$= (A \cap A^C) \cup (A \cap B) \quad \text{...Distributive law}$$

$$= \phi \cup (A \cap B) \quad \text{...Complement law}$$

$$= A \cap B \quad \text{...Property of Empty set}$$

Hence $A \cap (A^C \cup B) = A \cap B$

Example 3 :

Let A, B, C be subsets of the universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer. **(Dec. 9**

Solution :

Yes, $B = C$

We can express B as

$$B = B \cap U$$

$$= B \cap (A \cup \bar{A})$$

$$= (B \cap A) \cup (B \cap \bar{A}) \quad \text{...Distributive law}$$

$$= (A \cap B) \cup (\bar{A} \cap B) \quad \text{...Commutative law}$$

$$= (A \cap C) \cup (\bar{A} \cap C) \quad \text{...Given condition}$$

$$= (A \cup \bar{A}) \cap C \quad \text{...Distributive law}$$

$$= U \cap C$$

$$= C$$

Let us justify the answer by considering an example

Let $U = \{1, 2, 3, \ldots, 10\}$

$A = \{1, 3, 4, 5, 7\}$

Let $B = \{3, 5, 7, 8, 9\}$

$C = \{2, 3, 5, 6, 7, 8\}$

Now, $A \cap B = \{3, 5, 7\}$

and $A \cap C = \{3, 5, 7\}$

As $A \cap B = A \cap C$, condition 1 is satisfied.

Now, $\bar{A} = \{2, 6, 8, 9\}$

$\therefore$ $\bar{A} \cap B = \{8, 9\}$

and $\bar{A} \cap C = \{2, 6, 8\}$

Here $\bar{A} \cap B \neq \bar{A} \cap C$. Hence second condition is not satisfied. We modify set C.

Now, in order that $A \cap B = A \cap C$.

Hence C must contain the element 3, 5, 7. For the second condition to be satisfied, C must contain the elements 8 and 9.

$\therefore$ $C = \{3, 5, 7, 8, 9\}$

$\therefore$ $B = C$

Example 4 :

Prove the following (use law of set theory) **(May 2006, 6 Marks)**

$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))] = B \cap (A \cup C)$

Solution : $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))] = B \cap (A \cup C)$

$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$ ...Distributive Law

$(A \cap B) \cup [B \cap (C \cap U)]$ ...Complement law

$(A \cap B) \cup [B \cap C]$ ...Property of universal set

$(B \cap A) \cup [B \cap C]$ ...Commutative Law

$B \cap (A \cup C)$

Example 5 : **(May 2005)**

$\overline{\overline{(A \cup B)} \cap C} \cup \bar{B}$

Solution :

$$\overline{\overline{(A \cup B)} \cap C \cup \bar{B}} = \overline{\overline{(A \cup B)} \cap C \cup \bar{B}} \quad \text{De- Morgan's Law}$$

$$= \overline{(\bar{A} \cap \bar{B}) \cap C} \cap (\bar{\bar{B}}) \quad \text{De- Morgan's Law}$$

$$= \left[\overline{(\bar{A} \cap \bar{B})} \cup \bar{C}\right] \cap B \quad \text{De- Morgan's Law}$$

$$= \overline{\bar{A} \cap \bar{B} \cap C} \cap (\bar{\bar{B}}) = \left[(\overline{\bar{A} \cap \bar{B}} \cup \bar{C})\right] \cap B$$

$$= B \cup (\bar{C} \cap B) \quad \text{Absorption law}$$

$$= B \quad \text{Absorption law}$$

Example 6 :

Let the universal set be $U = \{1, 2, 3, \ldots, 10\}$

Let $A = \{2, 4, 7, 9\}$, $B = \{1, 4, 6, 7, 10\}$ and $C = \{3, 5, 7, 9\}$.

Find : (i) $A \cup B$ (ii) $A \cap B$ (iii) $B \cap \bar{C}$ (iv) $(A \cap \bar{B}) \cup C$ (v) $\overline{(B \cup C)} \cap C$

Solution :

(i) $A \cup B = \{1, 2, 4, 6, 7, 9, 10\}$

(ii) $A \cap B = \{4, 7\}$

(iii) $\bar{C} = \{1, 2, 4, 6, 8, 10\}$

$\therefore$ $B \cap \bar{C} = \{1, 4, 6, 10\}$

(iv) $\bar{B} = \{2, 3, 5, 8, 9\}$

$\therefore$ $A \cap \bar{B} = \{2, 9\}$ $(A \cap \bar{B}) \cup C = \{2, 3, 5, 7, 9\}$

(v) $B \cup C = \{1, 3, 4, 5, 6, 7, 9, 10\}$

$\overline{B \cup C} = \{2, 8\}$

$\overline{(B \cup C)} \cap C = \phi$

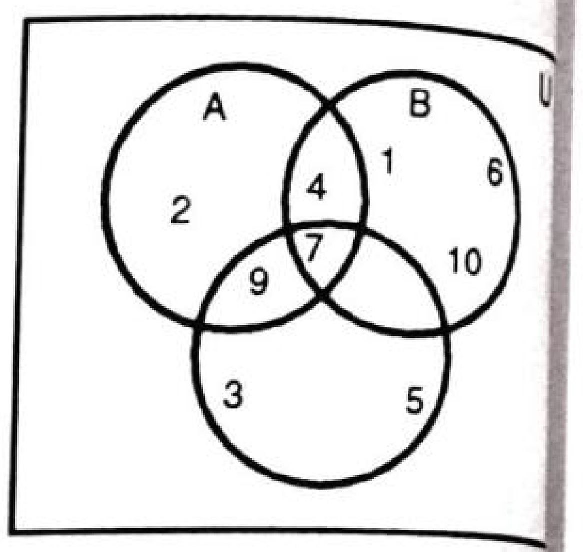


Fig. 1.38

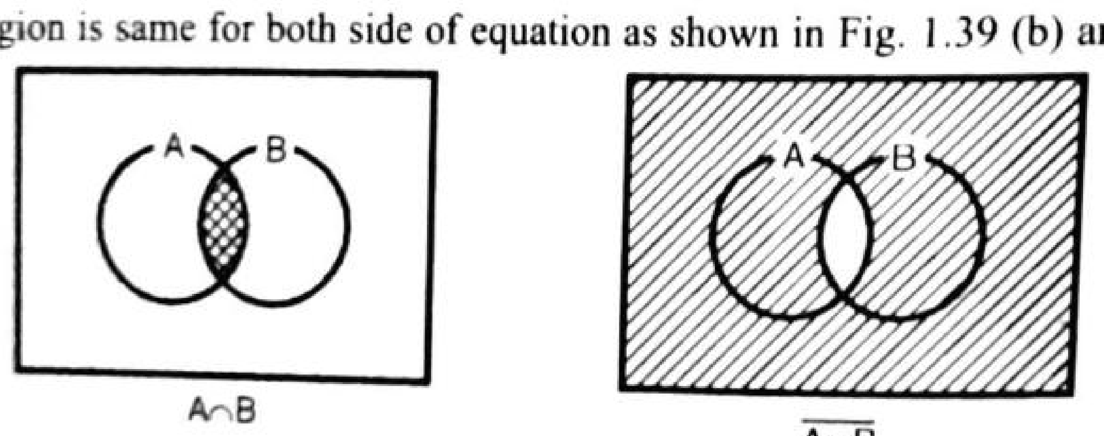
Example 7 :

Use Venn diagram to illustrate De Morgan's law for sets, viz.

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Solution :

The shaded region is same for both side of equation as shown in Fig. 1.39 (b) and Fig. 1.40 (c).



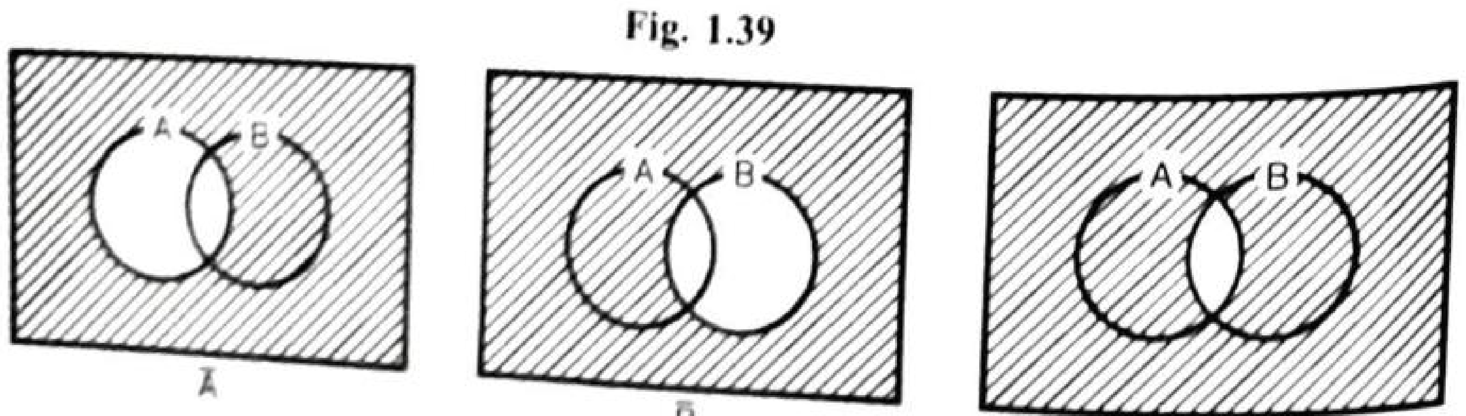
$A \cap B$

(a)

$\overline{A \cap B}$

(b)

Fig. 1.39



$\bar{A}$

(a)

$\bar{B}$

(b)

$\bar{A} \cap \bar{B}$

(c)

Fig. 1.40

Example 8 :

In the Venn diagram below shade

(i) $A \cup (B \cup C)$ (ii) $(A \cap B) \cup (A \cap C)$

Solution :

(i) $A \cup (B \cup C)$

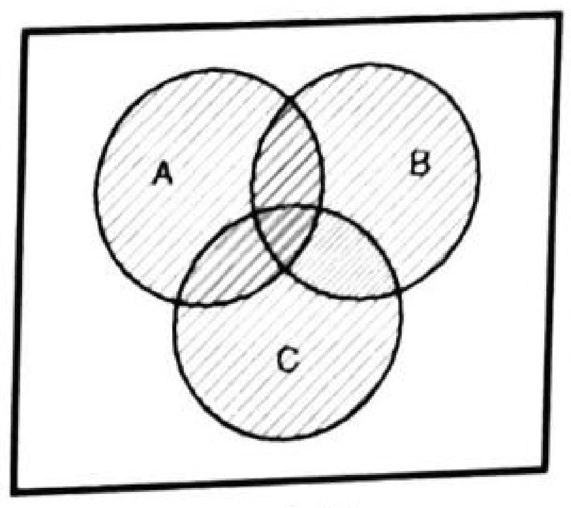


Fig. 1.41

(ii) $(A \cap B) \cup (A \cap C)$

(A∩B)

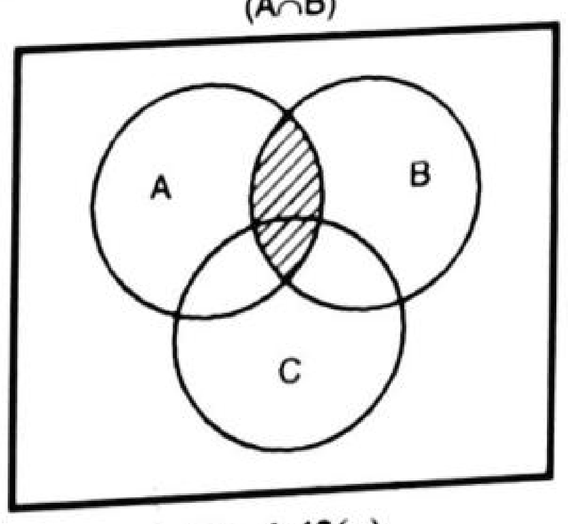


Fig. 1.42(a)

(A∩C)

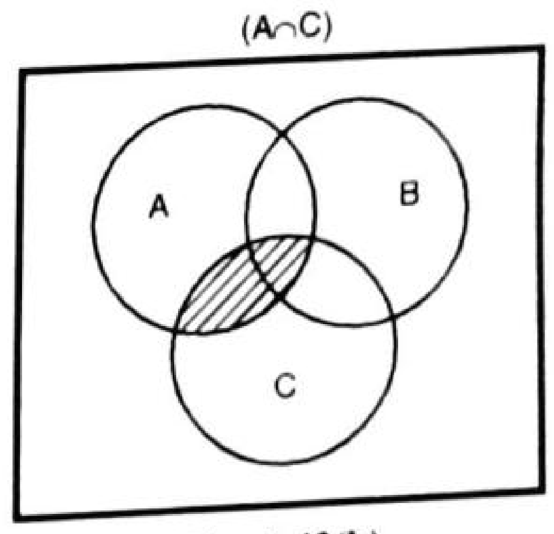


Fig. 1.42(b)

(A∩B)∪(A∩C)

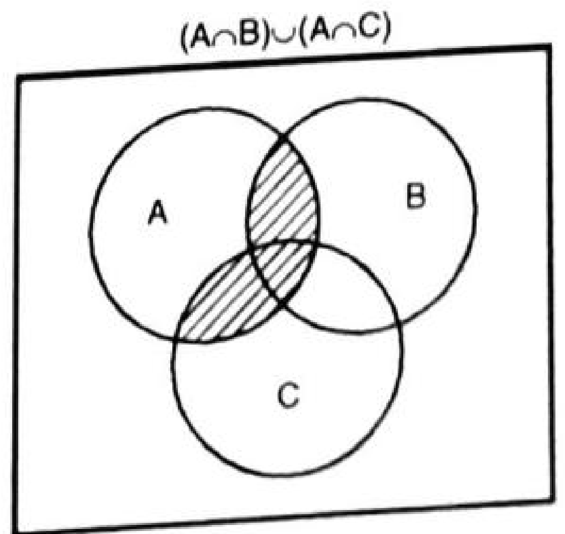


Fig. 1.42(c)

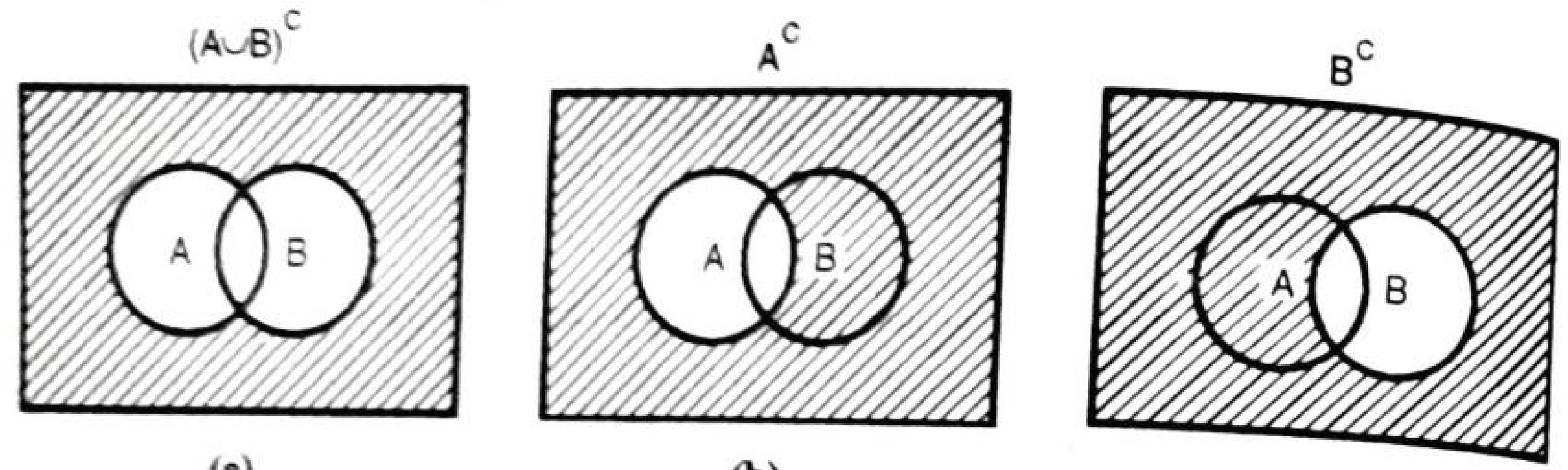
Example 9 :

Show by means of Venn diagram that $(A \cup B)^C = A^C \cap B^C$

Solution : A^C means complement of A

$(A \cup B)^C$ is shaded part.

We shade A^C and B^C separately.



(a) (b) (c)

Fig. 1.43

With this we shade $A^C \cap B^C$

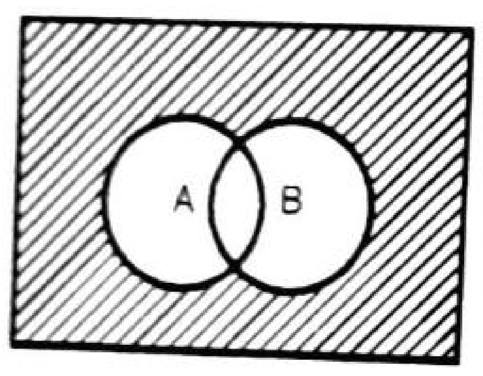


Fig. 1.43 (d)

Hence, Fig. 1.43 (a) and Fig. 1.43 (d) are same

$\therefore \quad (A \cup B)^C = A^C \cap B^C$

Example 10 :

Establish the following by Venn diagram.

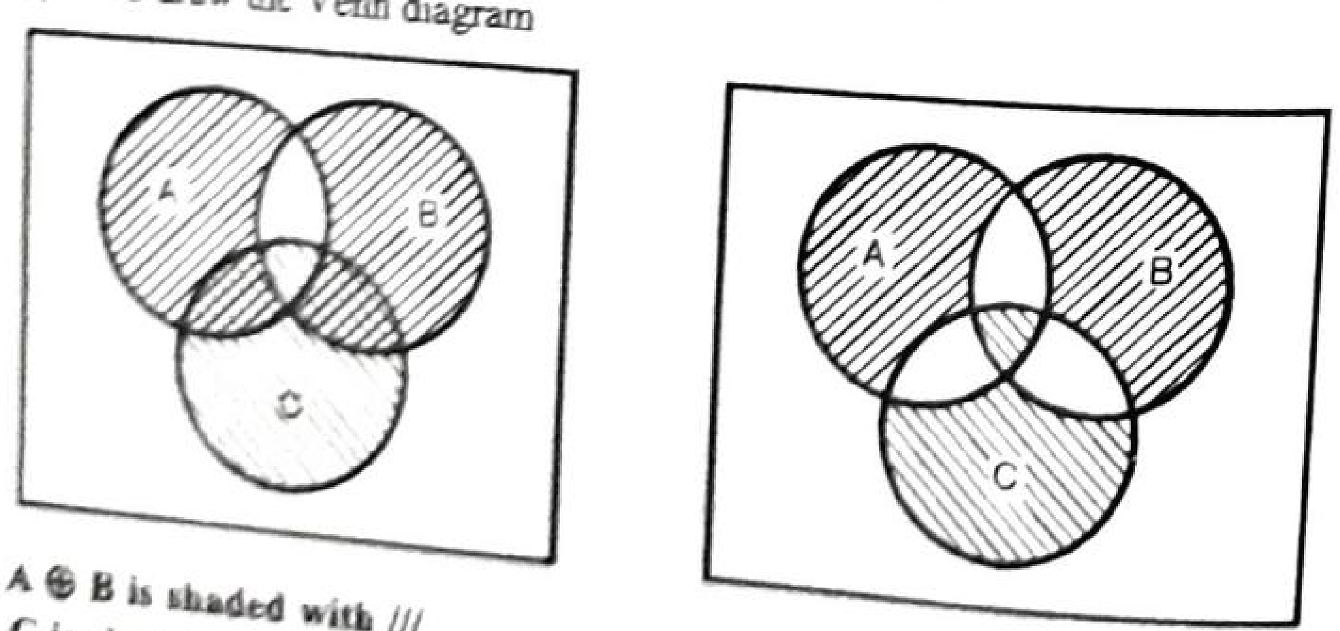
(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (Associative law)

(ii) $A \oplus B = B \oplus A$ (Commutative law)

(iii) If $A \oplus B = A \oplus C$ then $B = C$ (Cancellation law)

(iv) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ (Distributive law)

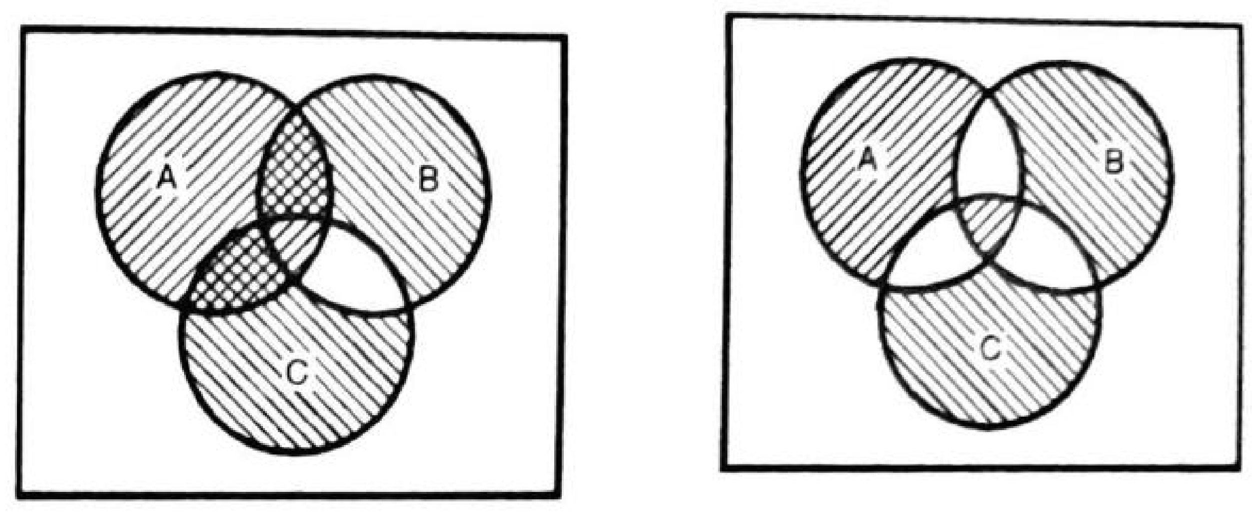
Solution : (i) We draw the Venn diagram



(a) $A \oplus B$ is shaded with ///
**C is shaded with **

(b) $(A \oplus B) \oplus C$ is shaded

Fig. 1.44



(c) (d)

(c) A is shaded with ///
**$B \oplus C$ is shaded with **

(d) $A \oplus (B \oplus C)$ is shaded

Fig. 1.44

Fig. 1.44 (b) and (d) are same. Hence $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(ii) $A \oplus B = \{x/x \in A \cup B$ but $x \notin A \cap B\}$

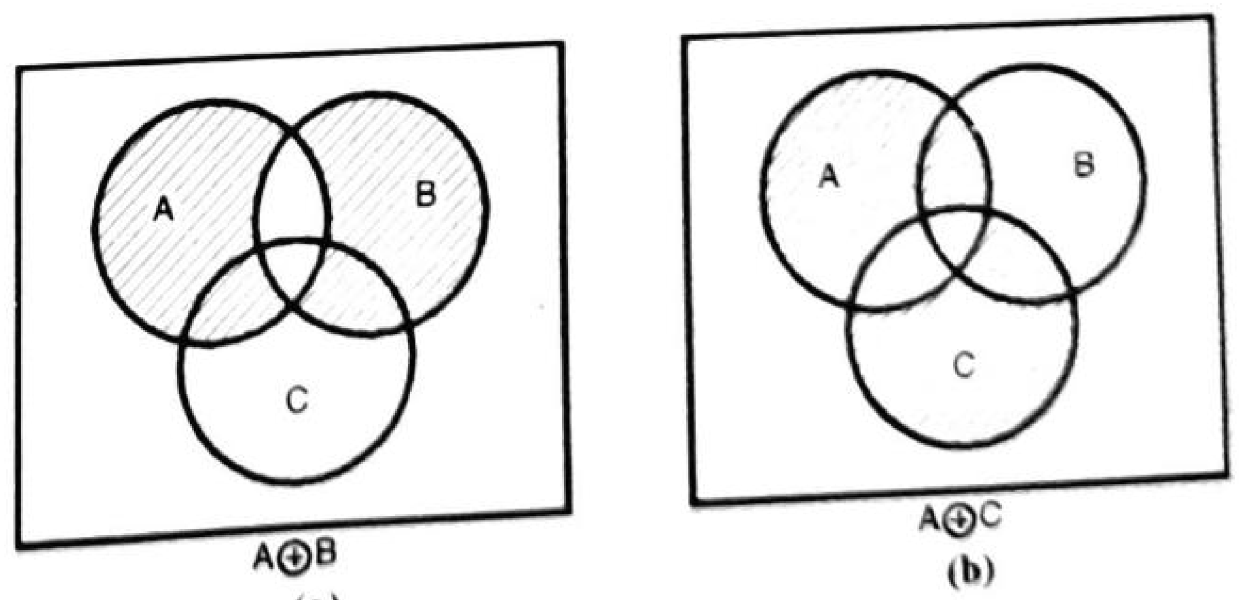
$= \{x/x \in B \cup A$ but $x \notin B \cap A\} = B \oplus A$

(a) (b)

Fig. 1.45

Fig. 1.45 (a) and (b) are same. Hence $A \oplus B = B \oplus A$

(iii)



(a) (b)

Fig. 1.46

$\therefore \quad B = C$

(iv)

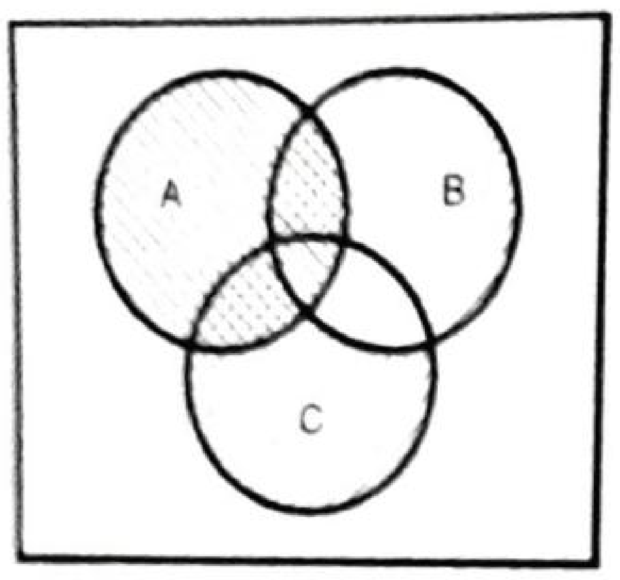


Fig. 1.47 (a) B ⊕ C is shaded with ///
A is shaded with \\\

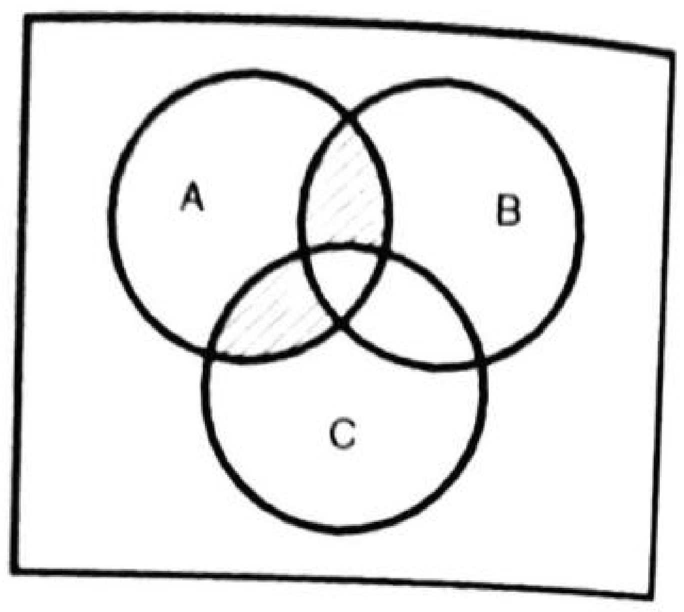
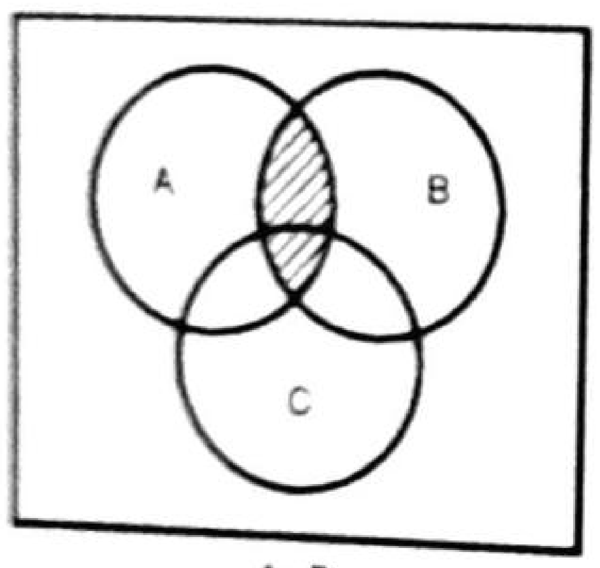
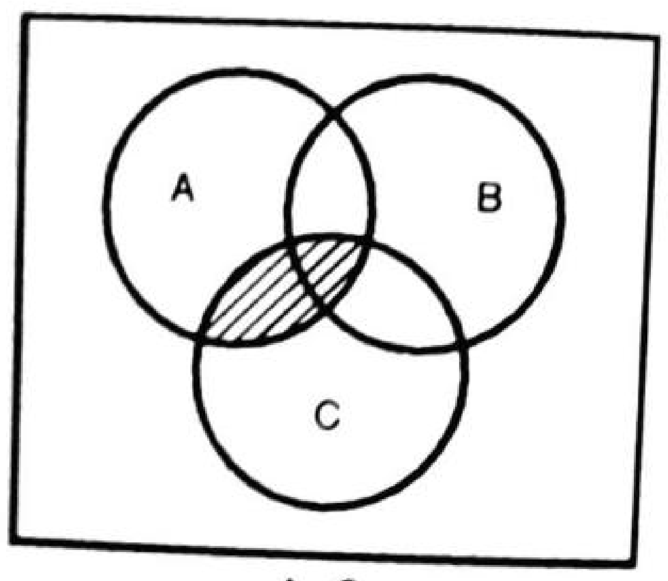


Fig. 1.47 (b) A ∩ (B ⊕ C) is shaded



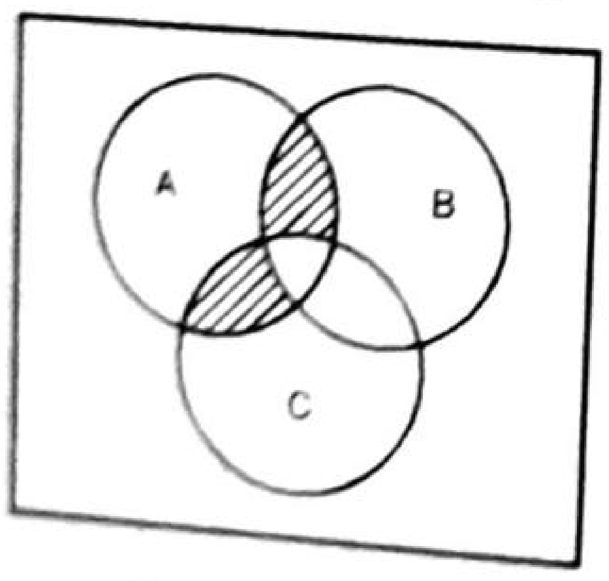
A∩B

Fig. 1.47 (c)



A∩C

Fig. 1.47 (d)



$(A \cap B) \oplus (A \cap C)$

Fig. 1.47 (e)

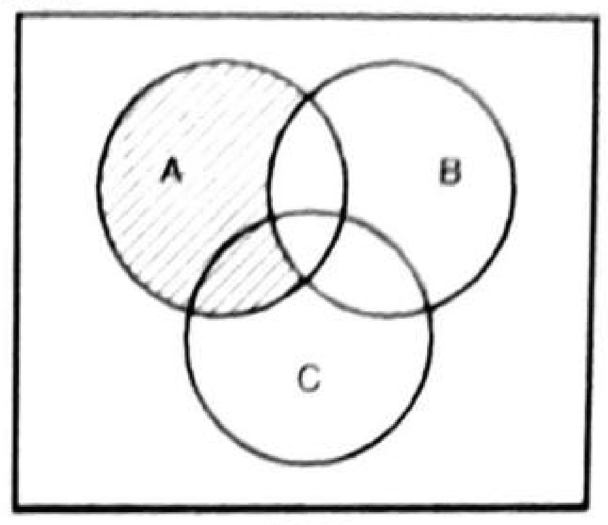
Fig. 1.47 (b) and (e) are same

∴ $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

Example 11 :

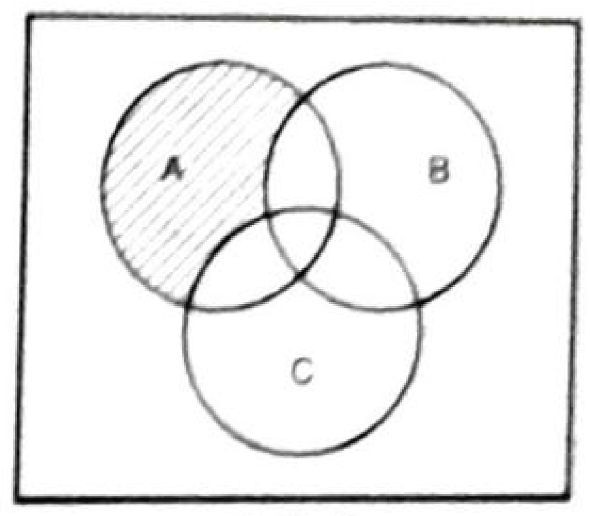
Show that $(A - B) - C = A - (B \cup C)$ using Venn diagram. **(May 9**

Solution :



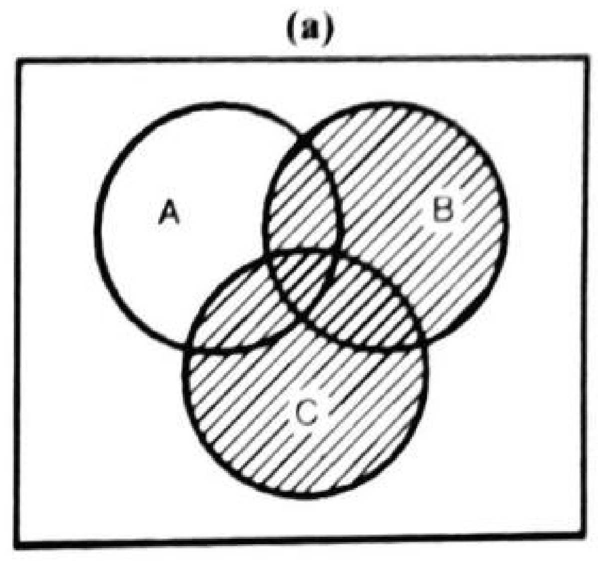
(A–B)

(a)



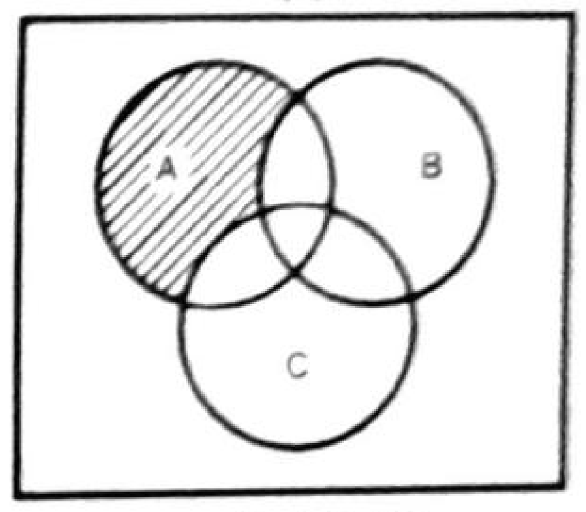
(A–B)–C

(b)



$(B \cup C)$

(c)



$A - (B \cup C)$

(d)

Fig. 1.48

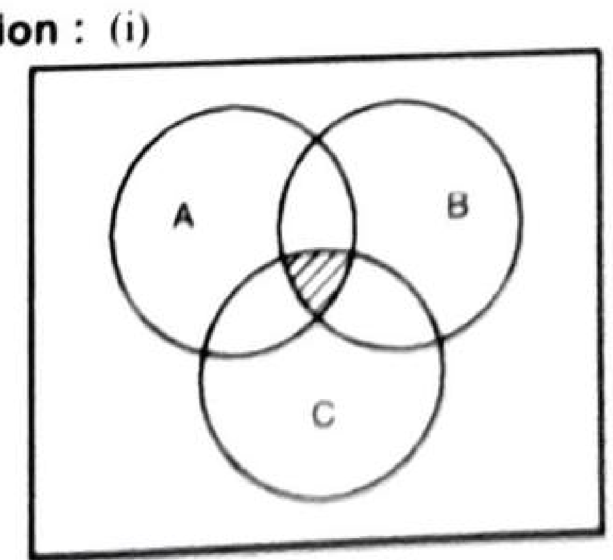
Fig. 1.48 (b) and Fig. 1.48 (d) are same

∴ $(A - B) - C = A - (B \cup C)$

Example 12 :

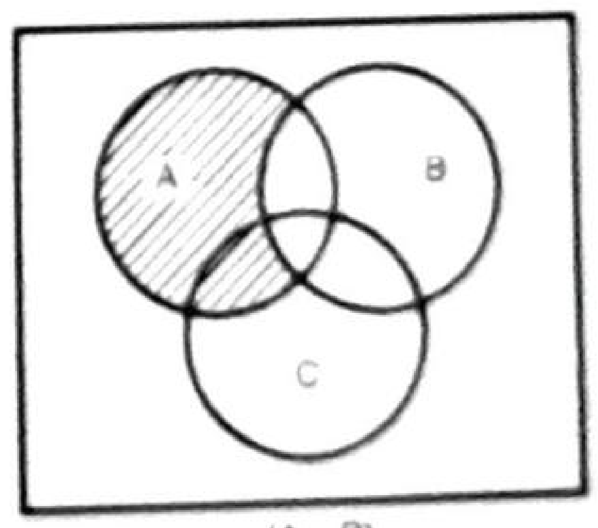
Using Venn diagram, prove or disprove $A \cap B \cap C = [(A - B) \cup (A - C)]$

Solution : (i)



$(A \cap B \cap C)$

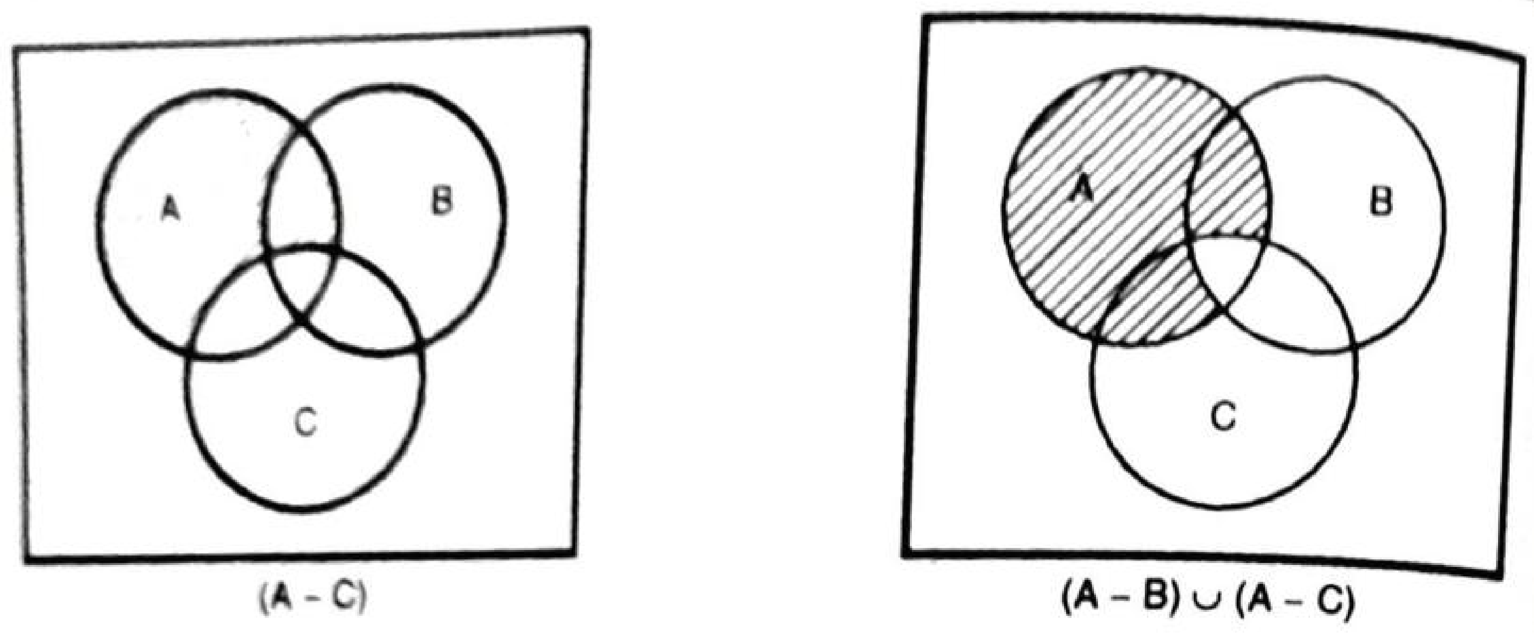
(a)



(A – B)

(b)

Fig. 1.49



(A - C)

(c)

(A – B) ∪ (A – C)

(d)

Fig. 1.49

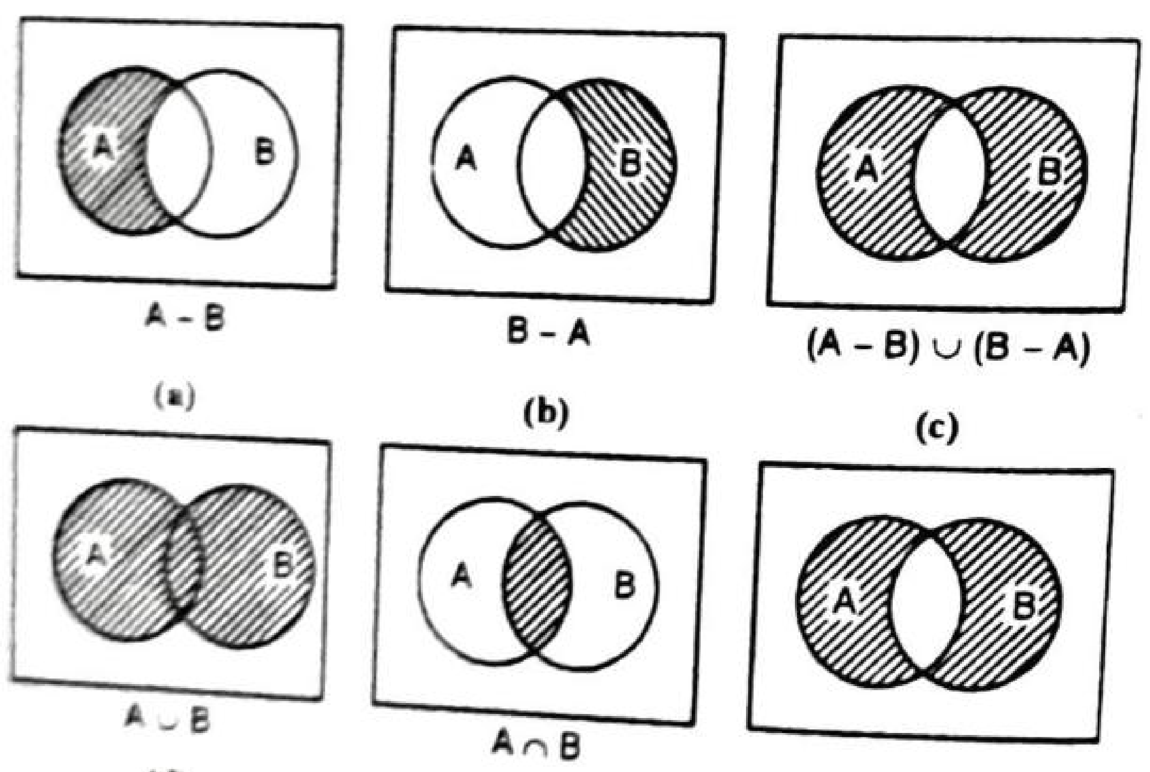
Fig. 1.49 (a) and Fig. 1.49 (d) are not same

$A \cap B \cap C \neq (A - B) \cup (A - C)$

Example 13 :

Prove that following $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ **(Dec. 20**

Solution :



A – B (a)

B – A (b)

(A – B) ∪ (B – A) (c)

A ∪ B (d)

A ∩ B (e)

(f)

Fig. 1.49 (A)

Example 14 :

Consider the sets :

$A = \{x : x^2 - 4x + 3 = 0\}$

$B = \{x : x^2 - 3x + 2 = 0\}$

$C = \{x : x \in N, x < 3\}$

$D = \{x : x \in N, x \text{ is odd}, x < 5\}$

$E = \{1, 2\}$

$F = \{1, 2, 1\}$

$G = \{3, 1\}$

which of the given sets are equal.

Solution :

$$\begin{aligned}
A &= \{x : x^2 - 4x + 3 = 0\} \\
&= \{x : (x - 3)(x - 1) = 0\} \\
&= \{x : x = 3, x = 1\} \\
&= \{3, 1\} \\
B &= \{x : x^2 - 3x + 2 = 0\} \\
&= \{x : (x - 2)(x - 1) = 0\} \\
&= \{x : x = 2, x = 1\} \\
&= \{1, 2\} \\
C &= \{x : x \in N, x < 3\} \\
&= \{1, 2\} \\
D &= \{x : x \in N, x \text{ is odd } x < 5\} \\
&= \{1, 3\}
\end{aligned}$$

$\therefore \quad A = D = G$

Set A, D and G are equal

$B = C = E = F$

Sets B, C, F and E are equal.

Example 15 :

Consider the following assumptions

A : Poets are happy people

B : Every doctor is wealthy

C : No one who is happy is also wealthy

Determine the validity of each of the following conclusions.

(i) No poet is wealthy (ii) Doctors are happy people

(iii) No one can be both a poet and a doctor.

Solution :

We observe that the set of poets is contained in the set of happy people and the set of doctors is contained in the set of wealthy people. From C, the set of happy people and the set of wealthy people are disjoint sets. Hence we draw Venn diagram.

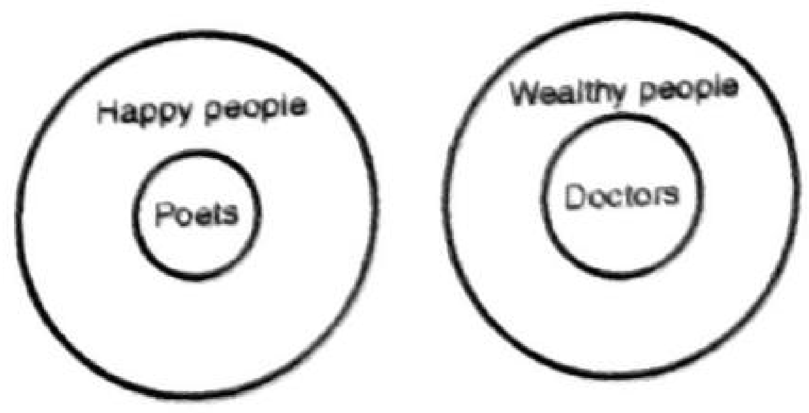


Fig. 1.50

By Venn diagram,

(i) 'No poet is wealthy' is valid.

(ii) 'Doctors are happy people' is invalid.

(iii) 'No one can be both a poet and a doctor' is valid.

Example 16 :

Show that (using Laws of Logic, Venn diagram and truth table).

(i) $A \cup (A^c \cap B) = A \cup B$

(ii) $A \cap (A^c \cup B) = A \cap B$

Solution :

(i) $A \cup (A^c \cap B) = A \cup B$

(a) Using Laws of Logic,

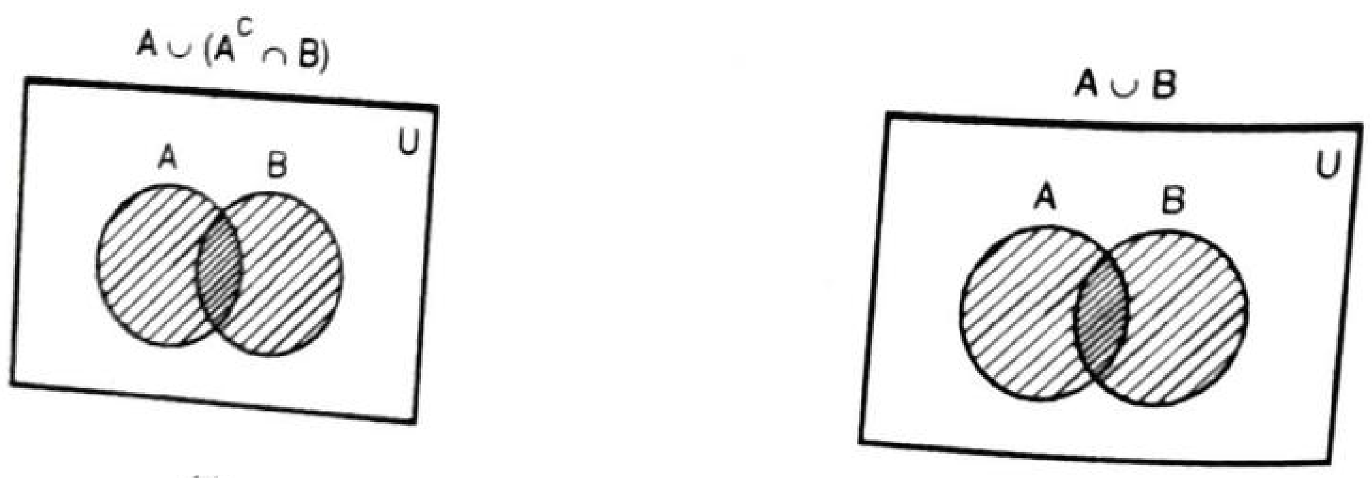
$$\begin{aligned} \text{L.H.S.} &= A \cup (A^c \cap B) \\ &= (A \cup A^c) \cap (A \cup B) && \text{...Distributive Law} \\ &= U \cap (A \cup B) && \text{...Complement Law} \\ &= A \cup B \end{aligned}$$

$\therefore$ L.H.S. = R.H.S.

(b) Using Venn diagrams shown in 1.51 (a) (b) (c) (d)

(a) (b)

Fig. 1.51



(c) (d)

Fig. 1.51

Fig. 1.51 (c) and 1.51 (d) are same.

$\therefore A \cup (A^c \cap B) = A \cup B$

(c) By using truth table

A	B	A^c	$A^c \cap B$	$A \cup (A^c \cap B)$	$A \cup B$
0	0	1	0	0	0
0	1	1	1	1	1
1	0	0	0	1	1
1	1	0	0	1	1

Last 2 columns are same.

Hence, L.H.S. = R.H.S.

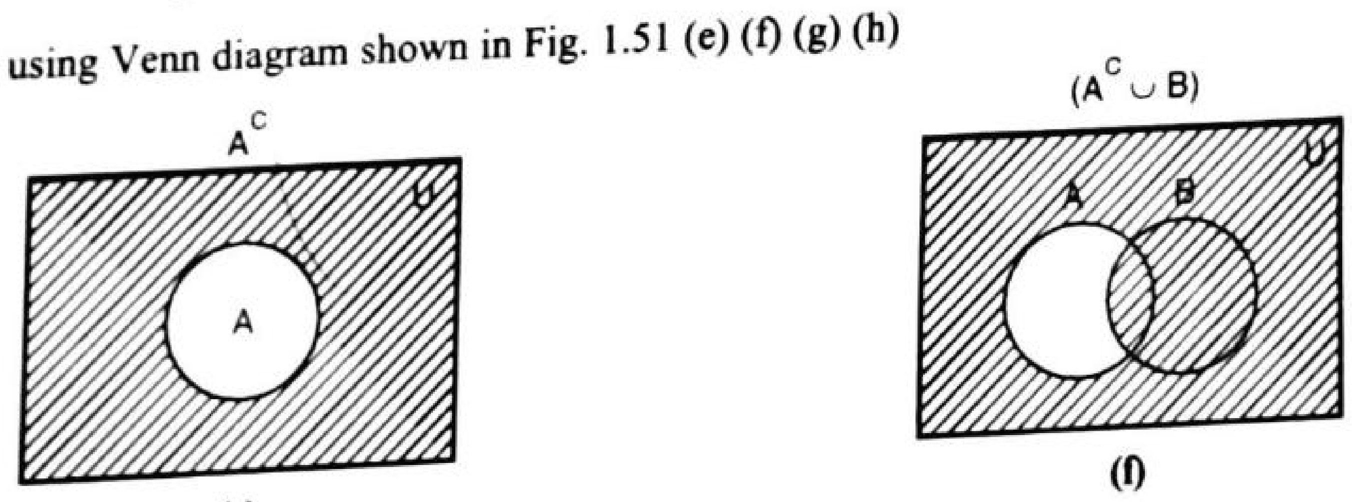
(ii) $A \cap (A^c \cup B) = A \cap B$

(a) By Using Laws of Logic

$$\begin{aligned} \text{L.H.S.} &= A \cap (A^c \cup B) \\ &= (A \cap A^c) \cup (A \cap B) && \text{...Distributive law} \\ &= \phi \cup (A \cap B) && \text{...Complement Law} \\ &= A \cap B \end{aligned}$$

L.H.S. = R.H.S.

(b) By using Venn diagram shown in Fig. 1.51 (e) (f) (g) (h)



(e) (f)

Fig. 1.51

(g) (h)

Fig. 1.51

Fig. 1.51 (g) and (h) are same.

$\therefore A \cap (A^c \cup B) = A \cap B$

(c) By using truth table

A	B	A^c	$A^c \cup B$	$A \cap (A^c \cup B)$	$A \cap B$
0	0	1	1	0	0
0	1	1	1	0	0
1	0	0	0	0	0
1	1	0	1	1	1

Last 2 column are same.

Hence L.H.S. = R.H.S.

Example 17 :

Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **(Dec. 2008, May 20**

Solution :

Let $(x, y) \in A \times (B \cap C)$

$\therefore$ $x \in A$, and $y \in (B \cap C)$

i.e. $x \in A$ and $y \in B$ and $y \in C$

$\therefore$ $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\therefore$ $(x, y) \in (A \times B) \cap (A \times C)$

$\therefore$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...(1)

Again let $(x, y) \in (A \times B) \cap (A \times C)$

$\therefore$ $(x, y) \in A \times B$ and $(x, y) \in (A \times C)$

and since $(x, y) \in (A \times B)$

$\therefore$ $x \in A$, $y \in B$ and

since $(x, y) \in (A \times C)$

$\therefore$ $x \in A$ and $y \in C$

$\therefore$ $x \in A$ and $y \in B \cap C$

$\therefore$ $(x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(2)

From Equations (1) and (2)

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Example 18 :

Check whether the following equations true equal

$A - (B - C) = (A - B) \cup (A \cap B \cap C)$

Solution :

By definition

$$A - (B - C) = \{x \mid x \in A \text{ and } x \notin (B - C)\}$$
$$= \{x \mid x \in A \text{ and } x \notin B \text{ or } x \in C\}$$

$$= \{x \mid x \in A \text{ and } x \notin B\} \text{ or } \{x \mid x \in A \text{ and } x \in C\}$$
$$= \{x \mid x \in A \text{ and } x \notin B\} \text{ or } \{x \mid x \in A \text{ and } x \in B \cap C\}$$
$$= \{x \mid x \in (A - B)\} \text{ or } \{x \mid x \in (A \cap B \cap)\}$$
$$= \{x \mid x \in (A - B) \cup (A \cap B \cap C)\}$$
$$A - (B - C) = (A - B) \cup (A \cap B \cap C)$$

Example 19 :

If A, B, C be any sets, then show that,

(i) $(A - B) - C = A - (B \cup C)$.

(ii) $(A - B) - C = (A - C) - B$.

Solution : By definition

(i) $(A - B) - C = \{x \mid x \in (A - B) \text{ and } x \notin C\}$

$= \{x \mid (x \in A \text{ and } x \notin B) \text{ and } x \notin C\}$

$= \{x \mid x \in A \text{ and } x \notin B \cup C\}$

$= \{x \mid x \in A - (B \cup C)\}$

$\therefore$ $(A - B) - C = A - (B \cup C)$

(ii) $(A - B) - C = \{x \mid x \in (A - B) \text{ and } x \notin C\}$

$= \{x \mid (x \in A \text{ and } x \notin B) \text{ and } x \notin C\}$

$= \{x \mid (x \in A \text{ and } x \notin C) \text{ and } x \notin B\}$

$= \{x \mid x \in (A - C) - B\}$

$= (A - C) - B$

Example 20 :

Prove that,

$$((A \cup B) \cap \bar{A}) \cup \overline{(B \cap A)} = \overline{A \cap B}$$

Solution :

We have,

$((A \cup B) \cap \bar{A}) \cup \overline{(B \cap A)}$

$= [(A \cap \bar{A}) \cup (B \cap \bar{A})] \cup \overline{(B \cap A)}$ ($\because$ Distributive law)

($\because$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$)

$= [\phi \cup (B \cap \bar{A})] \cup (\bar{B} \cup \bar{A})$ ($\because$ $A \cap \bar{A} = \phi$)

$= (B \cap \bar{A}) \cup (\bar{B} \cup \bar{A})$

$= [B \cup (\bar{B} \cup \bar{A})] \cap [\bar{A} \cup (\bar{B} \cup \bar{A})]$

$= U \cap [\bar{A} \cup \bar{B}]$ $[\because B \cup \bar{B} = \text{universal set}]$

$= (\bar{A} \cup \bar{B})$

$= \overline{(A \cap B)}$

$\therefore \quad ((A \cup B) \cap \bar{A}) \cup \overline{(B \cap A)} = \overline{A \cap B}$.

Example 21 :

For all sets A, X and Y show that

$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$

Solution :

Let $(a, x) \in A \times (X \cap Y)$

By the definition of the Cartesian product, this means that

$a \in A$ and $x \in X \cap Y$

Since $x \in X \cap Y$

$\therefore$ $x \in X$ and $x \in Y$

$\therefore$ $(a, x) \in A \times X$ and $(a, x) \in A \times Y$

$\therefore$ $(a, x) \in (A \times X) \cap (A \times Y)$

$\therefore$ $A \times (X \cap Y) \subseteq (A \times X) \cap (A \times Y)$

Again, Let $(a, x) \in (A \times X) \cap (A \times Y)$

$\therefore$ $(a, x) \in (A \times X)$ and $(a, x) \in A \times Y$

$\therefore$ $a \in A, x \in X$ and $x \in Y$

$\therefore$ $a \in A$ and $x \in X \cap Y$

$(a, x) \in A \times (X \cap Y)$

$\therefore$ $(A \times X) \cap (A \times Y) \subseteq A \times (X \cap Y)$

From Equations (1) and (2) we have,

$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$

Example 22 : (May 9...

(i) Given that $A \cup B = A \cup C$, is it necessary that $B = C$?

(ii) Given that $A \cap B = A \cap C$, is it necessary that $B = C$?

Solution :

(i) Let $A = \{1, 2, 3\}, B = \{1\}, C = \{3\}$

$A \cup B = \{1, 2, 3\} = A \cup C$

But $B \neq C$

(ii) Let $A = \{1, 2\}$

$B = \{2, 3, 4, 5\}$

$C = \{2, 6, 7\}$

then $A \cap B = \{2\} = A \cap C$

But $B \neq C$

Example 23 :

Verify the statement : $\overline{(A \cap B)} \cup \overline{\bar{A} \cap \bar{B} \cap C} = U$

Stating clearly the law used in each step.

Solution :

We have L.H.S. $= \overline{(A \cap B)} \cup \overline{(\bar{A} \cap \bar{B} \cap C)} \cup A$

$= (\bar{B} \cup \bar{A}) \cup (\bar{\bar{A}} \cup \bar{\bar{B}} \cup \bar{C}) \cup A$ using De morgan law

$= (\bar{B} \cup \bar{A}) \cup (A \cup B \cup \bar{C}) \cup A$ $(\because \bar{\bar{A}} = A)$

$= \bar{A} \cup (A \cup A) \cup (\bar{B} \cup B) \cup \bar{C}$ Using associativity

$= (\bar{A} \cup A) \cup (\bar{B} \cup B) \cup \bar{C}$

$= (U) \cup (U) \cup \bar{C}$

$= (U \cup U) \cup \bar{C}$

$= U \cup \bar{C}$ Idempotent law

$= U$ Definition of union

$=$ R.H.S

1.4 Partitions of Sets :

➤➤➤ [University Exam – Dec. 96, May 2000, 2001, Dec. 2002, 2011!!!]

Let A be a set. A ***partition*** of A is any set of non-empty subset $A_1, A_2, \ldots\ldots$ of A such that

(i) $A_1 \cup A_2 \cup \ldots\ldots\ldots\ldots = A$ and

(ii) the subset A_i are mutually disjoint, that is

$A_i \cap A_j = \phi$, for $i \neq j$.

1.4.1 Exercise Set 4 - (Sovled) :

Example 1 :

Let $A = \{ a, b, c \}$

Then { {a}, {b, c}} is a partition of A

Let $A_1 = \{ a \}$

$A_2 = \{ b, c \}$

(i) $A_1 \cup A_2 = \{ a \} \cup \{ b, c\}$

$= \{ a, b, c\}$

$= A$

$\therefore \quad A_1 \cup A_2 = A$

(ii) $A_1 \cap A_2 = \{ a \} \cap \{ b, c\}$

$= \phi$

$A_1 \cap A_2 = \phi$

Hence it is satisfying for both codition, Since { {a } , { b, c} } is a partition of A.

Example 2 :

Let S = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }. Determine whether or not each of the following is a partition of S.

(i) { { 1, 3, 5} { 2, 6} , { 4, 8, 9} }

(ii) { {1, 3, 5} , { 2, 4, 6, 8} , {5, 7, 9} }

(iii) { {1, 3, 5} , { 2, 4, 6, 8} , {7, 9} }

(iv) { { S } }

Solution :

(i) { { 1, 3, 5} { 2, 6} , { 4, 8, 9} }

Let $S_1 = \{ 1, 3, 5\}$

$S_2 = \{ 2, 6 \}$

$S_3 = \{ 4, 8, 9 \}$

(a) $S_1 \cup S_2 \cup S_3 = \{ 1, 3, 5 \} \cup \{ 2, 6\} \cup \{ 4, 8, 9 \}$

$= \{ 1, 2, 3, 4, 5, 6, 8, 9 \}$

$S_1 \cup S_2 \cup S_3 \neq S$

(b) $S_1 \cap S_2 \cap S_3 = \{ 1, 3, 5\} \cap \{ 2, 6 \} \cap \{ 4, 8, 9\} = \phi$

$S_1 \cap S_2 \cap S_3 = \phi$

{ {1, 3, 5} , {2, 6} , { 4, 8, 9} } is not partition of S. Since 7 ∈ S does not belong to any cell.

(ii) { {1, 3, 5} , { 2, 4, 6, 8} , {5, 7, 9} }

Let $S_1 = \{ 1, 3, 5 \}$

$S_2 = \{ 2, 4, 6, 8 \}$

$S_3 = \{ 5, 7, 9 \}$

(a) $S_1 \cup S_2 \cup S_3 = \{ 1, 3, 5 \} \cup \{ 2, 4, 6, 8 \} \cup \{ 5, 7, 9 \}$

$= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$

$S_1 \cup S_2 \cup S_3 = S$

(b) $S_1 \cap S_2 \cap S_3 = \{ 1, 3, 5 \} \cap \{ 2, 4, 6, 8 \} \cap \{ 5, 7, 9 \}$

$= \{ 5 \}$

$S_1 \cap S_2 \cap S_3 \neq \phi$

{ {1, 3, 5} , { 2, 4, 6, 8} , {5, 7, 9} } is not partition of S. Since { 1, 3, 5} and { 5, 7, 9} are not disjoint.

(iii) { {1, 3, 5} , { 2, 4, 6, 8} , {7, 9} }

Let $S_1 = \{ 1, 3, 5 \}$

$S_2 = \{ 2, 4, 6, 8 \}$

$S_3 = \{ 7, 9 \}$

(a) $S_1 \cup S_2 \cup S_3 = \{ 1, 3, 5 \} \cup \{ 2, 4, 6, 8 \} \cup \{ 7, 9 \}$

$= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$

$S_1 \cup S_2 \cup S_3 = S$

(b) $S_1 \cap S_2 \cap S_3 = \{ 1, 3, 5 \} \cap \{ 2, 4, 6, 8 \} \cap \{ 7, 9 \}$

$= \phi$

$S_1 \cap S_2 \cap S_3 = \phi$

{ {1, 3, 5} , { 2, 4, 6, 8} , {7, 9} } is partition of S. (Satisfying for both condition).

(iv) { { S } } $\quad S = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$

{ {1, 2, 3, 4, 5, 6, 7 8, 9 } }

{ { S } } is partition of S. Since there is only one cell i. e. { 1,2, 3, 4,......, 9 }. (set S itself)

Example 3 :

Let S = { red, blue, green, yellow }. Determine which of the following is a partition of S.

(i) { {red}, { blue, green } }

(ii) { { red, blue, green, yellow} }

(iii) { ϕ , { red, blue} , { green, yellow} }

(iv) { { blue }, { red, yellow, green} }

Solution :

(i) { {red}, { blue, green }}

Let $S_1 = \{ red \}$

$S_2 = \{ blue, green \}$

(a) $S_1 \cup S_2 = \{red\} \cup \{blue, green\}$

$= \{ red, blue, green \}$

$S_1 \cup S_2 \neq S$

(b) $S_1 \cap S_2 = \{ red \} \cap \{ blue, green \}$

$= \phi$

$S_1 \cap S_2 = \phi$

{ {red} , { blue, green} } is not partition of S. Since yellow ∈ S, does not belong to any cell.

(ii) { {red, blue, green, yellow } }

Let $S_1 = \{ red, blue, green, yellow \}$

∴ $S_1 = S$

∴ $\{ \{S\} \}$

∴ { {S} } is partition of S. Since there is only one cell

i.e. { red, blue, green, yellow } (set S itself).

(iii) { φ, { red, blue} ,{ green, yellow } }

φ (empty) set cannot belong to a partition. Hence φ, { red, blue}, {green, yellow}} is not partition of S.

(iv) { {blue} , {red, yellow, green } }

Let $S_1 = \{ blue \}$

$S_2 = \{ red, yellow, green \}$

(a) $S_1 \cup S_2 = \{ blue \} \cup \{ red, yellow, green \}$

$= \{ blue, red, yellow, green \}$

$S_1 \cup S_2 = S$

(b) $S_1 \cap S_2 = \{ blue \} \cap \{ red, yellow, green \}$

$= \phi$

$S_1 \cap S_2 = \phi$

{ {blue} , { red, yellow, green }} is partition of set S. (Satisfying for both condition).

Example 4 :

Let $A = \{a, b, c, d, e, f, g, h\}$. Consider following subsets of A.

$A_1 = \{a, b, c, d\}$, $A_2 = \{a, c, e, g, h\}$

$A_3 = \{a, c, e, g\}$, $A_4 = \{b, d\}$, $A_5 = \{f, h\}$

Determine whether each of the following is partition of A or not. Justify your answer.

(i) $\{A_1, A_2\}$ (ii) $\{A_1, A_5\}$ (iii) $\{A_3, A_4, A_5\}$ **(May 2003)**

Solution :

(i) $\{A_1, A_2\}$

(a) $A_1 \cup A_2 = \{a, b, c, d\} \cup \{a, c, e, g, h\}$

$= \{a, b, c, d, e, g, h\}$

$\neq A$

(b) $A_1 \cap A_2 = \{a\}$

$\neq \phi$

Since $\{A_1, A_2\}$ is not partition of A. Since it is not satisfying for both condition.

(ii) $\{A_1, A_5\}$

(a) $A_1 \cup A_5 = \{a, b, c, d\} \cup \{f, h\}$

$= \{a, b, c, d, f, h\}$

$\neq A$

(b) $A_1 \cap A_5 = \phi$

Since $\{A_1, A_5\}$ is not partition of A. Since e, g ∈ A does not belongs to any cell.

(iii) $\{A_3, A_4, A_5\}$

(a) $A_3 \cup A_4 \cup A_5 = \{a, c, e, g\} \cup \{b, d\} \cup \{f, h\}$

$= \{a, b, c, d, e, f, g, h\}$

$= A$

(b) $A_3 \cap A_4 \cap A_5 = \phi$

∴ It is a partition of A.

Example 5 :

Determine whether or not each of the following is a partition of the set of positive integers:

(i) $\{ \{ n : n > 5 \}, \{ n : n < 5 \} \}$

(ii) $\{ \{ n : n > 5\}, \{ 0 \}, \{ 1, 2, 3, 4, 5 \} \}$

(iii) $\{ \{ n : n^2 > 11 \}, \{ n : n^2 < 11 \} \}$

Solution :

Let $z^+ = \{ n \mid n \text{ is a positive integer} \}$

$z^+ = \{ 1, 2, 3, \ldots\ldots\ldots \}$

(i) $\{ \{ n : n > 5 \}, \{ n : n < 5 \} \}$

$\{ \{ n : n > 5 \}, \{ n : n < 5 \} \}$ is not partition of set of positive integers (z^+), since $5 \in z^+$ does not belong to any cell.

Let $X_1 = \{ n : n > 5 \}$

$= \{ 6, 7, 8, \ldots\ldots\ldots \}$

$X_2 = \{ n : n < 5 \}$

$= \{ 1, 2, 3, 4 \}$

$X_1 \cup X_2 = \{ 6, 7, 8, \ldots\ldots\ldots \} \cup \{ 1, 2, 3, 4 \}$

$= \{ 1, 2, 3, 4, 6, 7, 8, \ldots\ldots \}$

$X_1 \cup X_2 \neq Z^+$

Hence $\{ \{ n : n > 5 \}, \{ n : n < 5 \} \}$ is not portion of z^+.

(ii) { { n : n > 5}, { 0 }, { 1, 2, 3, 4, 5 } }

Let $X_1 = \{ n : n > 5 \}$

$X_2 = \{ 0 \}$

$X_3 = \{ 1, 2, 3, 4, 5 \}$

(a) $X_1 \cup X_2 \cup X_3 = \{ 6, 7, 8, \ldots \} \cup \{ 0 \} \cup \{ 1, 2, 3, 4, 5 \}$

$= \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots \}$

$X_1 \cup X_2 \cup X_3 \neq z^+$

(b) $X_1 \cap X_2 \cap X_3 = \{ 6, 7, 8, \ldots \} \cap \{ 0 \} \cap \{ 1, 2, 3, 4, 5 \}$

$= f$

{ { n : n > 5}, { 0 }, { 1, 2, 3, 4, 5 } } is not partition of z^+, since { 0 } is not subset of N.

(iii) $\{ \{ n : n^2 > 11 \}, \{ n : n^2 < 11 \} \}$

Let $X_1 = \{ n : n^2 > 11 \} = \{4, 5, 6, \ldots \}$

$X_2 = \{ n : n^2 < 11 \} = \{ 1, 2, 3\}$

(a) $X_1 \cup X_2 = \{4, 5, 6, \ldots \} \cup \{ 1, 2, 3\}$

$= \{ 1, 2, 3, 4, 5, 6, \ldots \}$

$X_1 \cup X_2 = z^+$

(b) $X_1 \cap X_2 = \{4, 5, 6, \ldots \} \cap \{ 1, 2, 3\}$

$= \phi$

$\{\{n : n^2 > 11 \}, \{ n : n^2 < 11\}\}$ is partition of z^+ since 2 cells are disjoint and their union is z^+.

Example 6 :

Let A and B two arbitrary sets.

(i) Show that $P(A \cap B) = P(A) \cap P(B)$ or give a counter example.

(ii) Show that $P(A \cup B) = P(A) \cup P(B)$ or give a counter example. **(Dec. 200**

Solution :

(i) Let $C \in P(A \cap B)$. Then $C \subseteq A \cap B$

$\Rightarrow C \subseteq A$ and $C \subseteq B$

$\Rightarrow C \in P(A)$ and $C \in P(B)$

$\Rightarrow C \in P(A) \cap P(B)$.

$\therefore P(A \cap B) \subseteq P(A) \cap P(B)$

Conversely, Let $C \in P(A) \cap P(B)$

This implies $C \in P(A)$ and $C \in P(B)$

$\Rightarrow C \subseteq A$ and $C \subseteq B$

$\Rightarrow C \subseteq A \cap B$ i.e. $C \in P(A \cap B)$

Hence, $P(A) \cap P(B) \subseteq P(A \cap B)$

Hence, $P(A \cap B) = P(A) \cap P(B)$

(ii) Equality is not true.

Consider $A = \{ 1 \}$

$B = \{ 2 \}$

$A \cup B = \{ 1, 2 \}$

$\therefore P(A \cup B) = \{ \phi, \{ 1 \}, \{ 2 \}, \{ 1, 2\} \}$

$\therefore P(A) = \{ \phi, \{ 1 \} \}$

$P(B) = \{ \phi, \{ 2 \} \}$

$\therefore P(A) \cup P(B) = \{ \phi, \{ 1 \}, \{ 2 \} \}$

$\neq P(A \cup B)$

1.5 Power Set :

➤➤➤ [**University Exam – Dec. 96, May 2000, 2001, Dec. 2002, May 2010 !!!**]

Let A be a set, then the set of all subsets of A is called the ***Power set*** of A and is denoted by **P(A).**

Note that a power set is an example of a set of sets, that is, a set whose elements are themselves sets.

Power set of A has 2^n elements where n is number of elements in A.

1.5.1 Exercise Set 5 - (Sovled) :

Example 1 :

Let A = { 1, 2, 3 }. Determine the power set A

Solution :

Then P (A) consists of the following subsets of A :

{ }, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 } and { 1, 2, 3 } (or A)

P (A) has 2^3 = 8 elements.

$\therefore P(A) = \{ \{ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$

Example 2 :

Let set A = { a, b, c, d }. Determine the power set A.

Solution : P (A) = { { }, { a }, { b }, { c }, { d }, { a, b }, { a, c }, { a, d }, { b, c }, { b, d }, { c, d }, { a, b, c }, { a, b, d }, { a, c, d }, { b, c, d}, { a, b, c, d } }

P (A) has 2^4 = 16 elements.

Example 3 : **(May 1998)**

Determine the power P (A) of A {a, b, c}.

Solution : $P(A) = \{ \{ \}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\} \}$

P (A) has $2^3 = 8$ elements.

Example 4 :

Find the power set of the set A $\{ \alpha, \beta, \gamma \}$. **(May 19)**

Solution : $P(A) = \{ \{ \}, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\} \}$.

P (A) has $2^3 = 8$ elements

Example 5 :

If A $\{\phi, a\}$, then construct the sets $A \cup P(A)$, $A \cap P(A)$

Solution : $P(A) = \{\phi, \{a\}, \{\phi\}, \{\phi, a\}\}$

$A \cup P(A) = \{\phi, a, \{\phi\}, \{a\}, \{\phi, a\}\}$

$A \cap P(A) = \{\phi\}$

Example 6 :

Let $A = \{\phi\}$. Let $B = P(P(A))$.

(i) Is $\phi \in B$? $\phi \subseteq B$?

(ii) Is $\{\phi\} \in B$? $\{\phi\} \subseteq B$.

(iii) Is $\{\{\phi\}\} \in B$? $\{\{\phi\}\} \subseteq B$?

Solution : $P(A) = \{\phi, \{\phi\}\}$

$B = P(P(A)) = \{\phi, \{\phi\}, \{\{\phi\}\}, P(A)\}$

(i) The element $\phi \in B$. The empty set ϕ is always a subset of B.

(ii) Both are true, one as element and the other as subset containing the single element ϕ.

(iii) Both are true, the first as element and the second as a singleton subset containing the element $\{\phi\}$.

Example 7 :

If $A \subseteq B$, then $P(A) \subseteq P(B)$.

Solution :

Let $C \in P(A)$. Then $C \subseteq A$ which implies that $C \subseteq B$.

Hence $C \in P(B)$.

$\therefore \quad P(A) \subseteq P(B)$

Example 8 :

What is the power set of the set $\{\phi\}$? **(May 2006)**

Solution :

The set of all subset of set A is called the **Power set** of A.

Power set of the set $\{\phi\}$ is $\{\phi, \{\phi\}\}$ i.e. it has $2^1 = 2$ elements.

1.5.2 Exercise Set - 6 (Unsolved) :

1. How many integers between 1 and 2000 are divisible by 2, 3, 5 or 7?
2. Find the number of integers between 1 and 500 not divisible by '7'.
3. Under what conditions the following hold true.

   (a) $(A - B) \cup (A - C) = A$

   (b) $(A - B) \cup (A - C) = \phi$

   (c) $(A - B) \cap (A - C) = \phi$

   (d) $(A - B) \oplus (A - C) = \phi$

Ans. : (a) $A \cap B = \phi$ and $A \cap C = \phi$

(b) A is a null set or $A = B = C$ or $A \subseteq B \subseteq C$

(c) $A \subseteq B \cup C$

(d) $A \cap B = A \cap C$

4. State whether true or false.

   1. $A \cup P(A) = P(A)$
   2. $A \cap P(A) = A$
   3. $\{A\} \cup P(A) = P(A)$
   4. $\{A\} \cap P(A) = A$
   5. $A - P(A) = A$
   6. $P(A) - \{A\} = P(A)$

Ans. : (1) False (2) True (3) True (4) True (5) True (6) False

5. Let $A = \{1, \{1\}, \{2\}, 3\}$. Determine which of the following statements are true and which are false.

   (a) $1 \in A$ (b) $1 \subseteq A$ (c) $\{1\} \in A$ (d) $\{1\} \subseteq A$ (e) $\{\{1\}\} \subseteq A$

   (f) $2 \in A$ (g) $\{2\} \in A$ (h) $\{2\} \subseteq A$ (i) $\{3\} \in A$ (j) $\{3\} \subseteq A$

Ans.: (a), (c) (d), (e), (g) and (j) are true and others are false.

6. Let $A = \{x \in N : x \le 12\}$. Let $A = \{x : x \text{ is odd}\}$, $B = \{X : X > 7\}$ and $C = \{X : X \text{ is divisible by } 3\}$. Depict the sets on a Venn diagram. Hence write down the following sets in enumerated form.

   (a) $A \cap B$ (b) $B \cup C$ (c) $\bar{A}$ (d) $(A \cup \bar{B}) \cap C$

   (e) $\overline{A \cup C} \cup \bar{C}$

Ans.: (a) { 9, 11 } (b) { 3, 6, 8, 9, 10, 11, 12 }, (c) { 2, 4, 6, 8, 10, 12 }

(d) { 3, 6, 9 } (e) { 1, 2, 4, 5, 7, 8, 10, 11 }

7. Illustrate the second absorption law $A \cup (A \cap B) = A$ using Venn diagrams.

8. Show that $\overline{\bar{A} \cap B} = A \cup \bar{B}$ using the Laws of Logic.

1.6 University Questions and Answers :

Dec. 2008

Q. 1 Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(Section 1.4.1, Example 6) **(4 Marks)**

Q. 2 Explain with an example the Mutual Inclusion-Exclusion principle for three sets.

(Section 1.2.3(4)) **(4 Marks)**

Q. 3 Find how many integers between 1 and 60 are not divisible by 2, nor by 3 and nor by 5.

(Section 1.2.4, Example 16) **(6 Marks)**

Q. 4 Let A and B two arbitrary sets.

Show that $P(A \cap B) = P(A) \cap P(B)$ and also give a counter example.

(Section 1.4.1, Example 6) **(6 Marks)**

May 2009

Q. 5 Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$,

(Section 1.4.1, Example 6) **(4 Marks)**

Q. 6 Among the integers 1 and 300, **(6 Marks)**

(i) How many of them are divisible by 3, 5 or 7 and are not divisible by 3 nor by 5 nor by 7

(ii) How many of them are divisible by 3 but not by 5 nor by 7 ?

(Section 1.2.4 ,Example 18)

Dec. 2009

Q. 7 For sets A, B, C prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(Section 1.3.3 , Example 1) **(4 Marks)**

May 2010

Q. 8 Define the following terms with the example : **(4 Marks)**

(i) Disjoint set **(Section 1.1.3.10)**

(ii) Symmetric difference **(Section 1.2.1(5))**

Q. 9 Define following - Power Set. **(Sections 1.5)** **(2 Marks)**

Dec. 2010

Q. 10 Among the integers 1 to 1000 :

(i) How many of them are not divisible by 3, nor by 5, nor by 7?

(ii) How many are not divisible by 5 and 7 but divisible by 3?

(Section 1.2.4, Example 21) **(10 Marks)**

May 2011

Q. 11 A survey of 500 television watchers produced the following information :

285 watch football games

195 watch hockey games

115 watch basketball games

45 watch football and basketball games

70 watch football and hockey games

50 watch hockey and basketball games

50 do not watch any of the 3 kinds of games.

(i) How many people in the survey watch all 3 kinds of games ?

(ii) How many people watch exactly one of the sports ?

(Section 1.2.4, Example 23) **(10 Marks)**

Dec. 2011

Q. 12 A survey on a sample of 25 new cars being sold of a local auto dealer was conducted to see which of three popular options, air conditioning A, radio R, and power windows W, were already installed. The survey found,

15 had air conditioning

12 had radio

11 had power windows

5 had air conditioning and power window

9 had air conditioning and radio

4 had radio and power windows

5 had all three options.

Find the number of cars having,

(i) only one of these options

(ii) radio and power windows but not air conditioning

(iii) none of these options. **(6 Marks)**

(Section 1.2.4, Example 11) **(4 Marks)**

Q. 13 Explain the following terms with suitable example :

(i) Disjoint set **(Section 1.1.3.10)**

(ii) Symmetric difference **(Section 1.2.1(5))**

(iii) Partition set **(Section 1.4)**

(iv) Cartesian product. **(Section 1.2.1(6))**

May 2012

Q. 14 Prove that If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$. **(5 Mark**

(Section 1.1.4, Example 3)

Q. 15 In a survey of 60 people, it was found that

25 read Business India

26 reads India Today.

26 read Times of India.

11 read both Business India and India Today.

09 read both Business India and Times of India.

08 read both India Today and Times of India.

08 read none of the three.

(i) How many read all three ?

(ii) How many read exactly one ? **(8 Marks**

(Section 1.4.1, Example 8)

Dec. 2012

Q. 16 Out of 250 candidates who failed in an examination, it was revealed that 128 failed i mathematics, 87 in physics and 134 in aggregate. 31 failed in mathematics and in Physics 54 failed in the aggregate and in mathematics, 30 failed in the aggregate and in physics Find how many candidates failed.

(i) in all the three subjects.

(ii) in mathematics but not in physics.

(iii) in the aggregate but not in mathematics.

(iv) in physics but not in aggregate or in mathematics. **(Section 1.2.4, Example 27)**

May 2013

Q. 17 In a survey of 60 people, it was found that 25 read Newsweek magazine, 26 read Time and 26 read Fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune, 8 read no magazine at all. **(6 Marks**

(a) Find the number of people who read all three magazines.

(b) Determine the number of people who read exactly one magazine.

(Section 1.4.1, Example 8)