...rtially ordered sets (Posets).

A relation R on a set A is called a partial order relation if R is (i) reflexive (ii) antisymmetric and (iii) transitive.

For eg. The relations $\leq, \geq, \subseteq$ are partial order relations.

Defn :- A set A together with the partial order relation R is called a partially ordered set or in brief poset and is generally denoted by (A, R) where A denotes the set and R denotes the relation.

Ex :- If S is any set and P is its power set (collection of subsets) then the relation $\subseteq$ (is a subset of) is a partial order relation on P.

Soln :- Let A, B, C be elements of P.

(i) Since $A \subseteq A$ R is reflective.

(ii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$ $\therefore$ antisymmetric

(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ $\therefore$ transitive.

$\therefore$ R is a partial order relation on P.

Ex Define a relation R on the set Z by aRb if a−b is a non negative even integer. Verify whether R is a partial order relation.

Let a, b, c be three integers.

(i) $a - a = 0$ which is a non negative even integer

$\therefore$ aRa $\therefore$ R is reflexive

(ii) If aRb and bRa then $(a-b)$ is non negative even integer and $(b-a)$ is non negative even integer. This is possible only when $a = b$ $\therefore$ antisymmetric.

(iii) aRb & bRc $\Rightarrow a - b = 2n_1$ & $b - c = 2n_2$

$a - c = (a-b) + (b-c) = 2n_1 + 2n_2 = 2(n_1 + n_2)$ $\therefore$ aRc $\therefore$ transitive.

$\therefore$ Partial order relation.

Hasse Diagram The digraph of a poset can be considerably simplified as follows.

(i) Since the relation is reflexive, we drop the loops around the vertices.

(ii) R is transitive ie if aRb and bRc then aRc, we drop the edge from a to c. Thus, we drop all edges implied by transitivity.

(iii) Finally, we arrange the whole diagram such that all arrows point upwards and then drop the arrow heads.

The resulting diagram is called the Hasse diagram.

Ex:- $A = \{1, 2, 3, 4, 12\}$ and the relation of divisibility ie aRb if $a|b$. Show that (A, R) is a poset. Also construct the diagraph of the poset and its Hasse diagram.

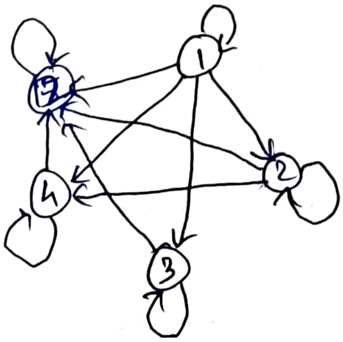
$R = \{(1,1), (1,2), (1,3), (1,4), (1,12), (2,2), (2,4), (2,12)$
$(3,3), (3,12), (4,4), (4,12), (12,12)\}$

(i) $a|a$ reflective

(ii) $a|b$ & $b|a \Rightarrow a=b$ antisymmetric

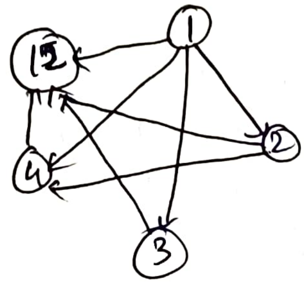
(iii) $a|b$ & $b|c \Rightarrow a|c$ transitive $\therefore$ partial order relation

$\therefore (A, R)$ is a poset. The diagraph is



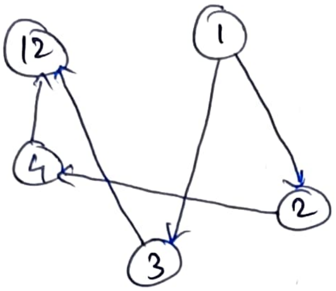
To construct Hasse diagram.

① delete the loops

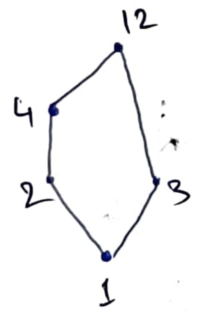


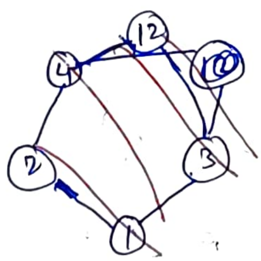
delete the edges implied by transitivity

we detete edge from 1 to 4, 1 to 12 and 2 to 12



(iii) Rearrange the diagraph if necessary, such that all edges point "upward" then drop arrow heads





This is the required Hasse diagram.

Def^n Hasse Diagram : A Hasse diagram of a poset (A, R) is a figure in which

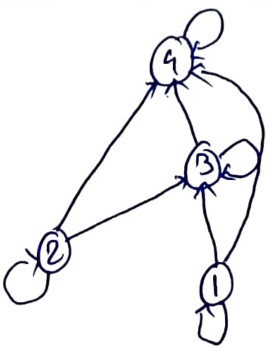
(i) the vertices represent the element's of A

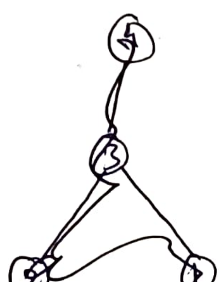
(ii) there is an upward line from x to y whenever xRy and $x \neq y$

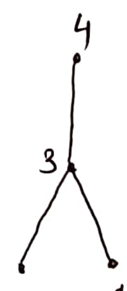
(iii) the figure has least number of segments that accomplish the property (ii)

Ex:- let $A = \{1, 2, 3, 4\}$ $R = \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

Draw the diagraph and then the Hasse diagram.







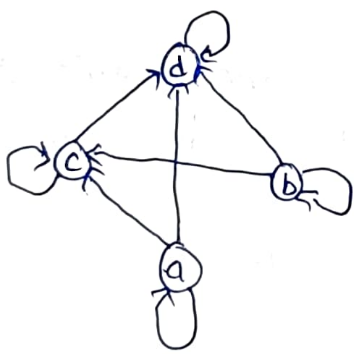
(3) Let $A = \{a, b, c, d\}$ and R be the relation on A whose matrix is

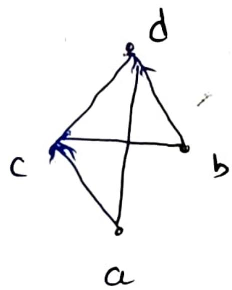
$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

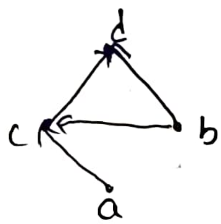
Show that R is a partial order. Construct the Hasse diagram of R.

$R = \{(a,a), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$

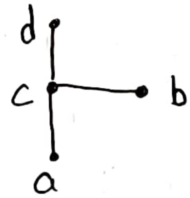
The diagraph of R







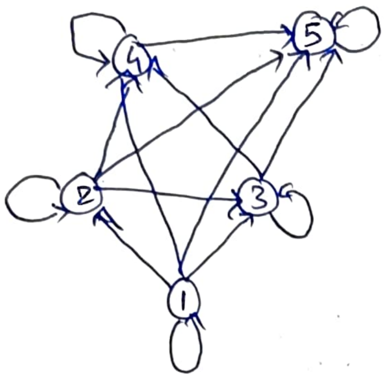
The Hasse diagram

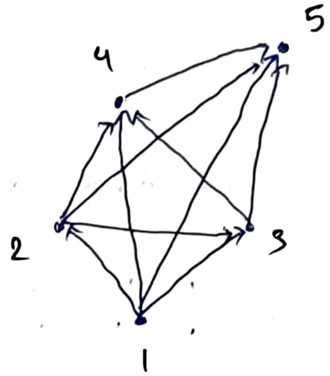


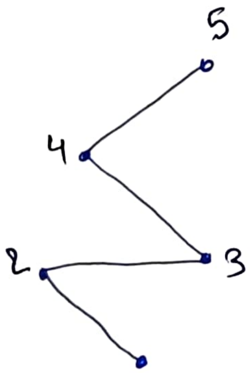
(4) Determine the Hasse diagram of the relation on $A = \{1, 2, 3, 4, 5\}$ whose matrix is shown below

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The diagraph is



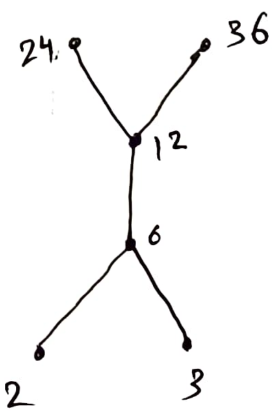




This is called a chain.

(5) Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be the relation 'is divisible by' i.e. aRb means $a|b$. obtain the relation matrix and draw the Hasse diagram.

$$
M_R = \begin{array}{c} \\ 2 \\ 3 \\ 6 \\ 12 \\ 24 \\ 36 \end{array}
\begin{array}{c} \begin{array}{cccccc} 2 & 3 & 6 & 12 & 24 & 36 \end{array} \\
\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}
$$



⑥ Draw the Hasse diagram for the following set $\{(a,b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$

The matrix of the relation is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 6 & 8 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$