UNIT NO :3.3

Functions

Definition: A relation from set X to set Y is a function from set X to set Y
if for every element x in the domain, there corresponds exactly one element y in the range.

Note : The definition of a function requires that a relation must be satisfying two conditions in order to qualify as a function:

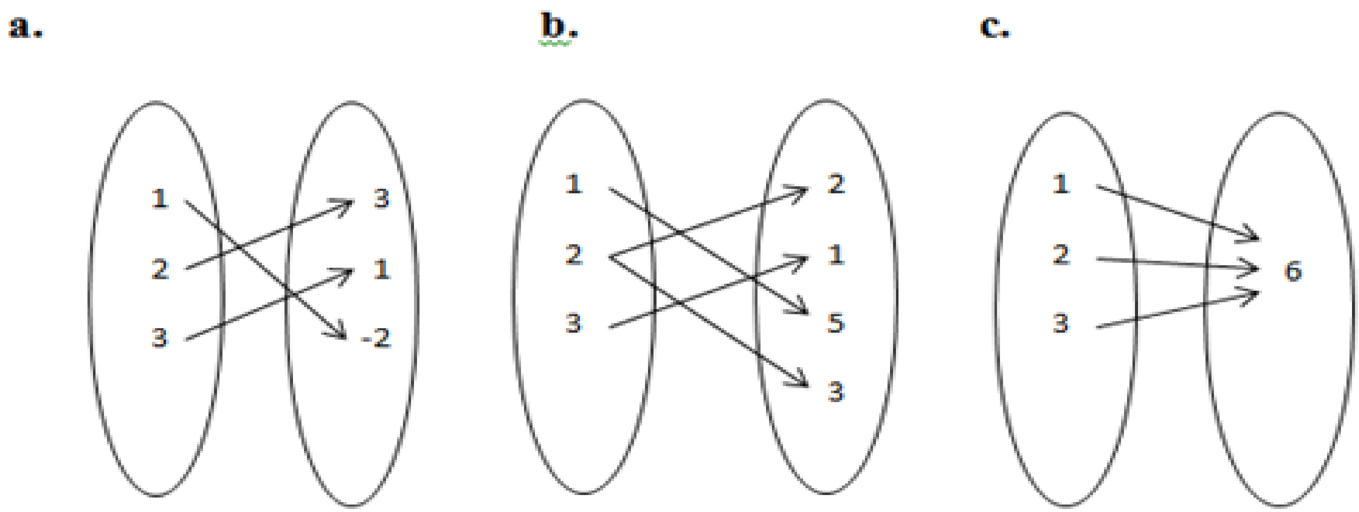
The first condition is that every $x \in X$ must be related to $y \in Y$ that is the domain of f must be X and not merely a subset of X (X is covered)

The second requirement of uniqueness can be expressed as: (Not one Many)

$(x, y) \in f$ and $(x, z) \in f \implies y = z$

Remark: Functions are sometimes also called **mappings** or **transformations**

Example Determine which of the relations are function.



In “a” Relation is a function.

In “b” Relation is not a function.

In “c” Relation is a function.

Types of Functions

1. **One-to-One** or **Injective**: A function $f: A \to B$ is called one to-one or injective if each element of B is the image of at most one element of A

$$\forall x, x' \in A, f(x) = f(x') \Rightarrow x = x'$$

For instance, $f(x) = 2x$ from Z to Z is injective

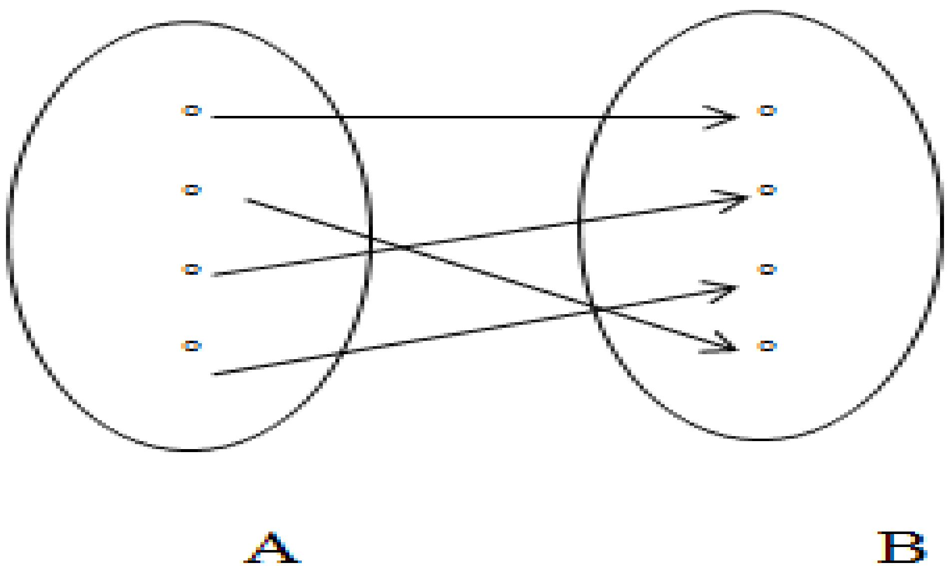


Figure One-to-one function

Types of Functions

2. Onto or **Surjective :** A function f : A → B is called onto or surjective if every element of B has preimage in A

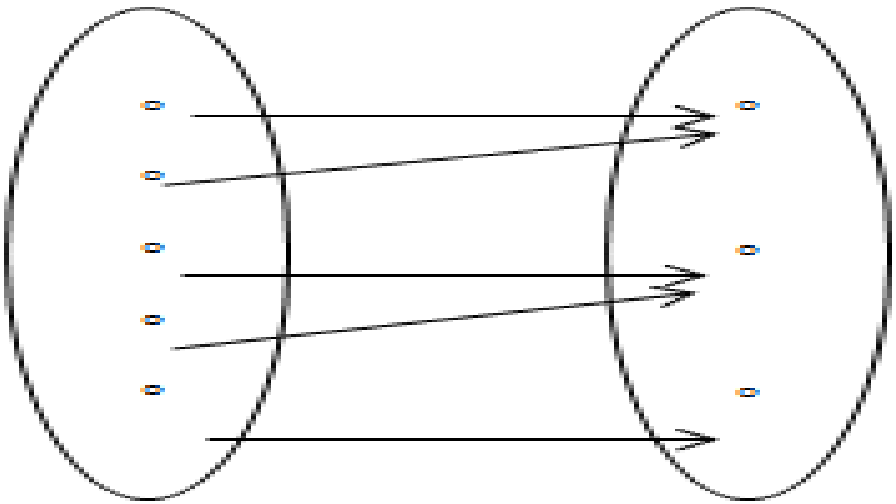
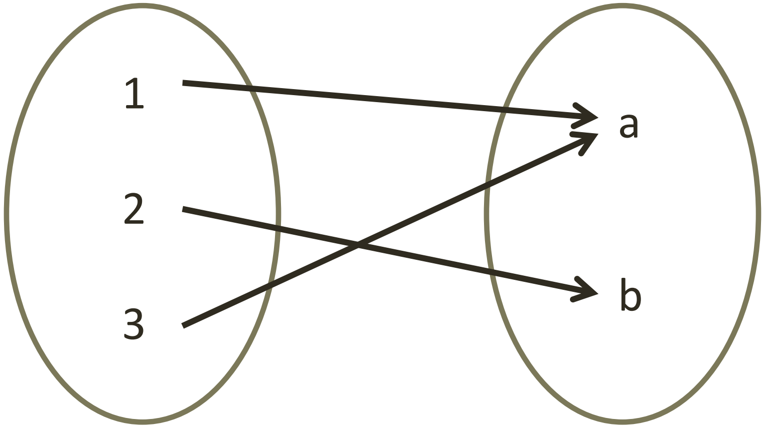


Figure : Onto function

Example Using two-element sets or three-element sets as domains and ranges, find an example of an onto function that is not one-to-one.

Notice that the function given by
$f(1) = a, f(2) = b, f(3) = a$
is an example of a function from $\{1, 2, 3\}$ to $\{a, b\}$ that is onto but not one to one.



Examples

Let f: $N \rightarrow N$, $f(x) = 5x$. Is f injective?

f is injective.

Let f: $N \rightarrow N$, $f(x) = x^2$. Is f injective?

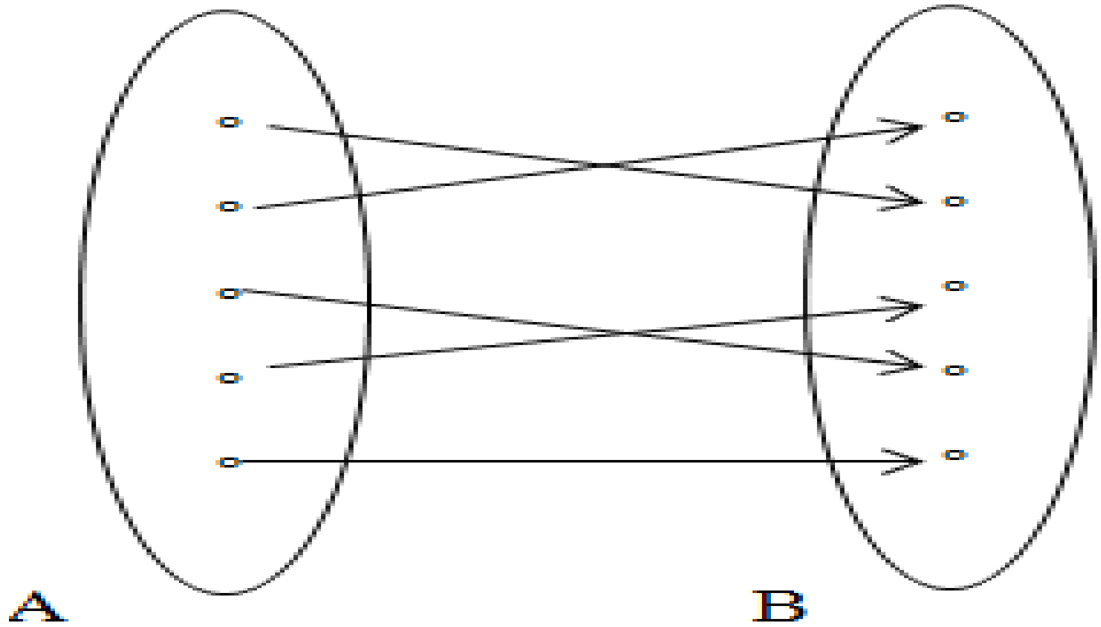
$f(x) = x^2$ is injective.

Let f: $R \rightarrow R$, $f(x) = x^2$ Is f injective ?

$f(x) = x^2$ is not injective as $(-x)^2 = x^2$

Check None of them are surjective (onto) !!!

3. One-To-One Correspondence or **Bijective:** A function $f: A \rightarrow B$ is said to be a one-to-one correspondence, or bijective, or a bijection, if it is one-to-one and onto



Example Prove that a function $f:R \rightarrow R$ defined by $f(x)=2x-3$ is a bijective function

If $f(a) = f(b)$

$\Rightarrow 2a - 3 = 2b - 3$

$\Rightarrow a = b.$

Thus $f(a) = f(b) \Rightarrow a = b$

Hence f is injective.

Let $f(x)=y$

$\Rightarrow 2x-3=y$

$\Rightarrow x=(y+3)/2$ & $x=(y+3)/2 \in R$

Thus for $y \in R$(codomain) $\exists$ $x=(y+3)/2 \in R$(domain) such that $f(x)=y$

Hence, f is surjective.

Hence, f is bijective.

Example: Is a function $f: Z \to Z$ defined by $f(x) = 2x - 3$ a bijective function?

If $f(a) = f(b)$

$\Rightarrow 2a - 3 = 2b - 3$

$\Rightarrow a = b$.

Thus $f(a) = f(b) \Rightarrow a = b$

Hence f is injective.

Let $f(x)=y$

$\Rightarrow 2x-3=y$

$\Rightarrow x = (y+3)/2$ But $x = (y+3)/2 \notin Z$

Thus for $y \in Z$(codomain) There is no $x \in Z$(domain) such that $f(x)=y$

Hence, f is Not surjective.

Hence, f is Not bijective.

Definition Inverse Function

If $f: A \to B$ is a bijective function, its inverse is the function $f^{-1}: B \to A$ such that $f^{-1}(y) = x$ if and only if $f(x) = y$

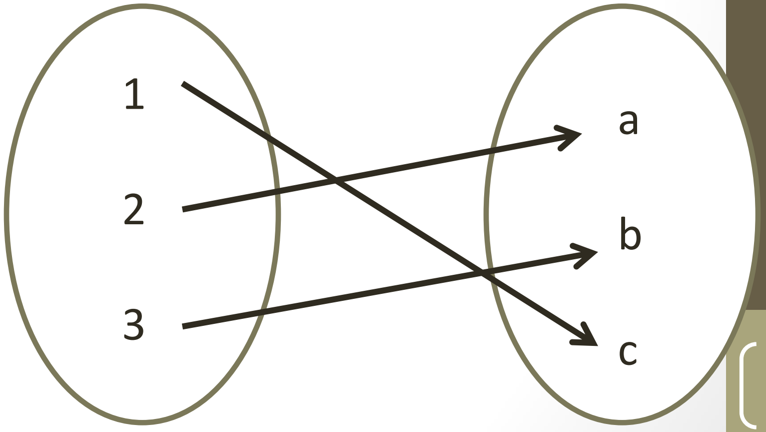
Example : let f be the function from {1, 2, 3} to {a, b, c} such that f (1) = c, f (2) = a, and f (3) = b. Is f invertible, and if it is, what is its inverse?

Ans The function f is invertible because as shown in figure image set is covered and it is a one-to-one correspondence.

f^{-1} reverses the direction by f

so $f^{-1}(a) = 2$,

$f^{-1}(b) = 3$ and $f^{-1}(c) = 1$



Example let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1.$ Is f invertible, and if it is, what is its inverse?

Ans: consider If f(a) = f(b)

$\Rightarrow$a + 1 = b + 1

$\Rightarrow$ a = b.

Thus f(a) = f(b) $\Rightarrow$ a = b   Hence f is injective.

Let f(x)=y

$\Rightarrow$ x+1 = y

$\Rightarrow$x = y-1 But x = y-1$\in$ Z

Thus for y $\in$ Z(codomain) There is x = y-1 $\in$ Z(domain) such that f(x)=y   Hence, f is surjective.

Hence, f is bijective.

Now x = y-1

Consequently $f^{-1}(y) = y - 1$

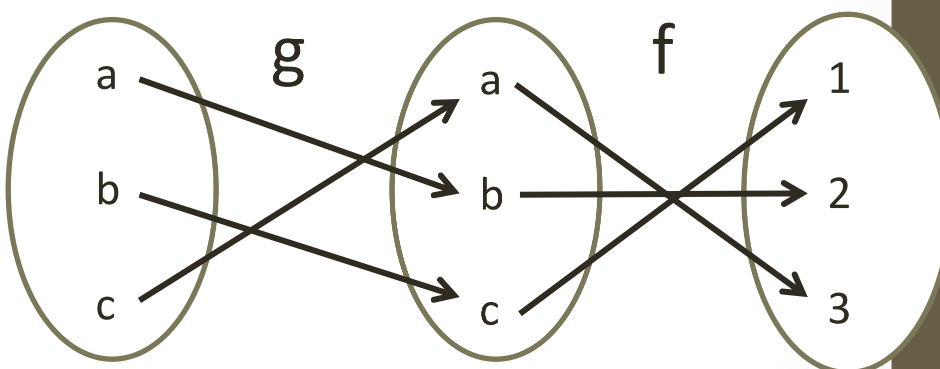
Definition : Function Composition

Given two functions $f: A \to B$ and $g: B \to C$
the composite function of f and g is the function
$g_o f: A \to C$ defined by
$(g_o f)(x) = g(f(x))$ for every x in A

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "g_o f"] & B \arrow[r, "g"] & C
\end{tikzcd}
$$

Example: Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a.
Let f be the function from the set {a, b, c} to the set {1,2, 3} such that f (a) = 3, f (b) = 2, and f (c) = 1. What is the composition of fog, and what is the composition of gof ?

Solution: Consider diagram representation of information.



The composition $f o g$ is defined by

(fog)(a) = f(g(a)) = f (b) = 2,

$$(\mathrm{fog})(\mathrm{b}) = \mathrm{f}\big(\mathrm{g}(\mathrm{b})\big)$$
$$= \mathrm{f}(\mathrm{c}) = 1,$$

$$(\mathrm{fog})(\mathrm{c}) = \mathrm{f}(\mathrm{g}(\mathrm{c})) = \mathrm{f}(\mathrm{a}) = 3.$$

Note that $g o f$ is not defined, because the range of f is not a subset of the domain of g.

Example Let f and g be the functions from the set of integers to the set of integers defined by f (p) = 2p + 3 and g(q) = 3q + 2. What is the composition of f and g? What is the composition of g and f ?

Solution:

Both the compositions $f_o\, g$ and $g_o\, f$ are defined. Moreover,

$$(f_o\, g)(x) = f(g(x)) = f(3x + 2)$$
$$= 2(3x + 2) + 3 = 6x + 7$$

and

$$(g_o\, f)(x) = g(f(x)) = g(2x + 3)$$
$$= 3(2x + 3) + 2 = 6x + 11.$$

Try putting some value of x (say 1) and verify !!!

EX

- Function $f: R - \{1\} \rightarrow R - \{3\}$ is defined as $f(x) = \frac{3x-2}{x-1}$. Prove that f is bijective
- Functions $f: R \rightarrow R$, $g: R \rightarrow R$ are defined as $f(x) = 5x + 3$,

$g(x) = 1 + 3x$ then find fog ,fof ,gof & $gogof$