Game Theory

End-term Report

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1 Introduction

Game theory is the study of strategic interactions between **self-interested** agents.

Definition 1.1: Self-interested Agent

An agent is known to be self-interested iff it has its own description of which states of the world it likes. Essentially, such agent has ranking of all the outcomes of the game. These rankings are quantified via **utility functions**.

Definition 1.2: Utility Function

The utility function u_i maps every possible set of actions taken by a group of agents, to a real number - the payoff of agent i.
It will be formally discussed later.

We will first start by discussing **non-cooperative game theory**, and later move towards systems where coalition, bargains or other negotiations are allowed.

Definition 1.3: Non-cooperative Game Theory

Noncooperative game theory refers to models in which each players are assumed to behave selfishly and their behaviors are directly modeled.

A common assumption in game theory is that agents (or players) are **rational** and in their actions.

Definition 1.4: Rational Agent

A player is generally assumed to be rational in the sense that:

- Each player is fully aware of all possible choices that can be made.
- Each player may form expectations about any unknowns.
- Each player deliberately chooses an action based on a process of optimization.

Essentially, a rational player will be able to determine its utility function and act towards the maximum payoff.

There are two standard representations of games, **Normal or strategic form** (where players move simultaneously) or **Extensive form** (where moves are sequential).

We will start with the normal form games.

2 Games in Normal Form

Definition 2.1: Normal Form of Games

A (finite, n-person) normal-form game is given by a tuple (N, A, u) where -

- The finite set of players N = {1, 2, . . . , n}
- The finite set of actions A_i available for the player i
- The set $A = A_1 \times A_2 \times \cdots \times A_n$ where each element $a = (a_1, a_2, \ldots, a_n) \in A$ is called an action profile
- $u = (u_1, u_2, \ldots, u_n)$ where $u_i : A \to \mathbb{R}$ is the utility (or playoff) function for player i

These games can be represented as n-dimensional matrices, where the elements of the matrix are the payoffs of each player $(u_1(a), u_2(a), \ldots, u_n(a))$ under the action profile a.

Some famous games written in their normal-form are -

2.1 The Prisoner's Dilemma

	Truth	Lie
Truth	-1, -1	-5, 0
Lie	0, -5	-3, -3

2.2 Common Payoff Game (pure coordination)

	Left	Right
Left	0, 0	1, 1
Right	1, 1	0, 0

2.3 Zero-sum Game (pure conflict)

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

2.4 Battle of Sexes

	A	B
A	1, 2	0, 0
B	0, 0	2, 1

3 Strategies in Normal-Form games

3.1 Pure Strategy profiles

When each agent plays a fixed action a_i (with a probability of 1), the action profile $a = (a_1, a_2, \dots, a_n)$ describes a pure strategy.
For example, a pure strategy in the Prisoner's Dilemma game [5] is $(\text{Truth}, \text{Truth})$ which gives the outcome $(-1, -1)$.

3.2 Mixed strategy profiles

A player with an action set A_i might not always want to play a pure action $a_i \in A_i$, but instead can play multiple actions with some positive probability.
This introduces uncertainty in games, and quantities such as the utility function of the player now needs to be calculated as the expected value of the players playoff.

Definition 3.1: Mixed Strategy

The set of mixed strategies S_i for player i is a probability distribution over the set A_i. We denote the probability of an action a_i to be played under the mixed strategy s_i by $s_i(a_i)$.

Similarly, the set $S = S_1 \times S_2 \times \cdots \times S_n$ is the set of mixed strategy profiles.

Definition 3.2: Support of a strategy

The support of a mixed strategy s_i of player i is the set of actions (or pure strategies) $\{a_i \mid s_i(a_i) > 0\}$.

Definition 3.3: Expected Utility Function

Given a normal-form game (N, A, u), the expected utility u_i for the player i of the mixed strategy profile $s = (s_1, s_2, \dots, s_n)$ is defined as

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{k=1}^{n} s_k(a_k)$$

Example of a mixed strategy profile s in the zero-sum game [6.1.2] is both players playing *Heads* with $P(\text{Heads}) = \frac{1}{2}$ and *Tails* with $P(\text{Tails}) = \frac{1}{2}$.

We can calculate the utility for Player 1 in this game by taking the expectation value -

$$u_1(s) = 1 \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) + (-1) \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) + 1 \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) + (-1) \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) = 0$$

which is naturally expected.

4 Optimal Strategies

We now look at optimality of a strategy profile from view of an outside observer.
For this we define a notion of ordering strategies.

Definition 4.1: Pareto domination

Strategy profile s *Pareto* dominates s' if $\forall\ i \in N,\ u_i(s) \geq u_i(s')$ and $\exists\ j \in N$ with $u_j(s) > u_j(s')$.

Definition 4.2: Pareto optimality

Strategy profile s is *Pareto* optimal if $\nexists\ s'$ where s' dominates s.

It is trivial to note that each game has at-least one *Pareto* optimal **pure** strategy profile.
Some games have multiple *Pareto* optimal strategies.

Example of *Pareto* optimal strategies in the Prisoner's Dilemma [5] are $(\text{Truth}, \text{Truth})$, $(\text{Truth}, \text{Lie})$ and $(\text{Lie}, \text{Truth})$.
The strategy (Lie, Lie) is dominated by $(\text{Truth}, \text{Truth})$.

5 Dominant Strategy Equilibrium

We now start by actually predicting the outcome of some games.

Let us revisit the Prisoner's Dilemma [5] again.

		Player 2: Truth	Player 2: Lie
Player 1	Truth	-1, -1	-5, 0
	Lie	0, -5	-3, -3

Consider the choices of Player 1:

- If Player 2 tells the *Truth*, it's best for Player 1 to *Lie*.
- If Player 2 tells a *Lie*, it's still best for Player 1 to *Lie*.

Hence, it is better for Player 1 to *Lie* irrespective of what Player 2 does. In this case, *Lie* is a **dominant strategy** over *Truth*.

Definition 5.1: Dominantion

Let S_{-i} denote the set of strategy profiles of all players except i.

A strategy s_i of player i **strictly dominates** another strategy s'_i if $\forall\ s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition 5.2: Dominant Strategy

A strategy s_i is the **strictly dominant** strategy of player i if s_i strictly dominates all other strategies s'_i.

With similar treatment, we see that choosing to *Lie* is the dominant strategy for both players.
Since the players are rational, both are expected to play their dominant strategies.
Hence, the outcome of the game with dominant strategy profile (Lie, Lie) is $(-3, -3)$. This is one of the strongest possible equilibrium - with dominant strategies.

This brings us to the most unintuitive part of the Prisoner's Dilemma. This game has equilibrium outcome of $(-3, -3)$ which is the only *Pareto* unoptimal outcome.
By following the best strategies, the players have obtained the worst outcome.

6 Nash Equilibrium

Enter the most awaited topic of game theory, the **Nash Equilibrium**.

We first need to define the **best response** of player i when the strategy profile of other players s_{-i} is fixed.

Definition 6.1: Best Response

Player i's best response to the strategy profile s_{-i} is a mixed strategy s_i such that

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \ \forall \ s'_i \neq s_i$$

A player can have multiple such strategies. Note that if there are at-least 2 best responses, then any combination of them is also a best response, henceforth there are infinite best responses.

Definition 6.2: Nash Equilibrium

A strategy profile s is a Nash Equilibrium in the game if $\forall \ i$ - s_i is a best response to s_{-i}.

If for all agents, we have a strict inequality on the utility functions, we further call it a **strict nash**.

We proceed to the most interesting part about Nash Equilibriums.

Theorem 6.1: Existance of Nash Equilibrium (Nash, 1951)

Every game with a finite number of players and action profiles has at least one Nash equilibrium.

A proof using *Kakutani's fixed-point theorem* is presented in this paper.

6.1 Examples of Nash Equilibrium

Let's discuss the Nash Equilibrium of the games we've previously described in 2.

6.1.1 Pure strategy profile equilibrium

As discuss in section 5, the equilibrium in **Prisoner's Dilemma** [5] is a nash equilibrium with pure strategies for each player.

For the **Common Payoff** game [2.2], it is easy to see that the only pure strategy equilibriums are (Left, Right) and (Right, Left).
Note that we cannot have any mixed strategy equilibrium as the outcome (0, 0) should not occur with some positive probability.

6.1.2 Mixed strategy nash equilibrium

In the **Zero-sum** game [6.1.2], we can see that no pure strategy gives an equilibrium.

However, we can still look for some mixed strategy equilibriums.

		Player 2	
		Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

Matching pennies (Zero-sum) game

Let us assume Player 1 adopts mixed strategy s_1 where he plays *Heads* with probability p and *Tails* with probability $1-p$.

If Player 2 also plays a mixed strategy, he must not see any difference in his utility while playing either of the pure strategies (otherwise he would play the strategy with maximum utility).

Hence,

$$u_2(\text{Heads}, s_1) = u_2(\text{Tails}, s_1)$$
$$(-1)\cdot p + 1\cdot(1-p) = 1\cdot p + (-1)\cdot(1-p)$$
$$p = \frac{1}{2}$$

Similar treatment for Player 2 gives that both players play with a mixed strategy of $\frac{1}{2}$ times *Heads* and $\frac{1}{2}$ times *Tails*.

The utility of both players is **0** which is expected intuitively.

6.1.3 Pure and mixed strategy nash equilibrium

The **Battle of Sexes** game [2.4] shows that both pure and mixed strategy equilibriums can be present in the same game.

Both the (A, A) and (B, B) action profiles give nash equilibriums.

By similar treatment as the previous section, a strategy with $\frac{2}{3}$ probability for the more desired action and $\frac{1}{3}$ for the other for each player is also a nash equilibrium.

6.2 Finding nash equilibriums

There are various ways to find Nash equilibriums in a game. Every class of game has a most suitable strategy of finding its equilibria.

Some common methods are -

- Best Response Analysis: For simple games, you can identify Nash equilibriums by analyzing each player's best response to the strategies of the other players.
- Iterated Elimination of Dominated Strategies (IEDS): In more complex games, you can use IEDS to iteratively eliminate dominated strategies until you reach a set of strategies that cannot be further eliminated. This set may contain a Nash equilibrium.
- Mixed Strategy Nash Equilibrium: In games with mixed strategies (where players randomize their actions), you can find Nash equilibriums by solving systems of equations representing each player's expected payoffs.
- Linear Programming: Linear programming techniques can be used to find mixed strategy Nash equilibriums in games with continuous strategy spaces and linear payoffs.
- Numerical Methods or SAT solvers: For complex games with no analytical solution, numerical methods, such as computational simulations or game-solving algorithms, can be used to approximate Nash equilibriums.
- Behavioral Game Theory: In some cases, empirical data on how real players behave can help identify Nash equilibriums by analyzing patterns of actions and strategies.

7 Correlated Equilibrium

Correlated equilibrium is a more general concept from Nash equilibrium.

Here, each agent chooses their action according to their private observation of a common public signal.

A player's signal can be correlated to value of the random public variable and the signals received by other players. In the standard games we've discussed before, we consider a degenerate case where the signals of different agents are probabilistically independent.

Definition 7.1: Correlated Equilibrium

Given an n-agent game $G = (N, A, u)$, a correlated equilibrium is a tuple (v, π, σ), where v is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n), \pi$ is a joint distribution over $v, \sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent i and every mapping $\sigma_i' : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i\left(\sigma_1\left(d_1\right), \ldots, \sigma_n\left(d_n\right)\right) \geq \sum_{d \in D} \pi(d) u_i\left(\sigma_1'\left(d_1\right), \ldots, \sigma_n'\left(d_n\right)\right)$$

That is, no player would want to deviate from his strategy (assuming others don't deviate).

Note that in the definition, we map probabilities to actions (pure strategies) rather than mixed strategies because it does not add any more generality.

We are already assigning probabilities to actions in the definition, hence encoding a mixed steategy is simply a matter of changing the σ_i.

The following theorem is quite natural

Theorem 7.1: Every nash equilibrium can be written as correlated equilibrium

For every Nash equilibrium $\sigma*$ there exists a corresponding correlated equilibrium σ.
The converse is not true.

Correlated equilibriums are good for social welfare, in the Battle of Sexes game [2.4], both agents receive a higher payoff than a mixed strategy equilibrium.

8 Perfect-Information Extensive Form

Up till now, we have been looking at games where all agents simultaneously play their moves. However, games in the real world often involve sequential actions.

To infer about the temporal structure of these games, we model them using the **extensive form**.

The word **perfect-information** comes from the assumption that all players know the structure of the game, as well as the history of the game (including moves played by other players) at all points of time.

Definition 8.1: Perfect-information game

A (finite) perfect-information game (in extensive form) a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ where

- N is the set of n players
- A is the set of all actions
- H is the set of non-terminal choice nodes
- Z is the set of terminal nodes, disjoint from H
- $\chi : H \to 2^A$ is the action function, which assigns to each choice node a set of possible actions
- $\rho : H \to N$ is the player function which assigns a player whose action is at the node
- $\sigma : H \times A \to H \cup Z$ is the successor function, which maps a choice node and an action to a new node
- $u = (u_1, u_2, \ldots, u_n)$ where $u_i : Z \to \mathbb{R}$ is the utility function for the player i on the terminal node Z

Note that in the tree if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$.

Due to this, we can clearly define the *history* of a node, which is the sequence of nodes (and actions) leading to that node.

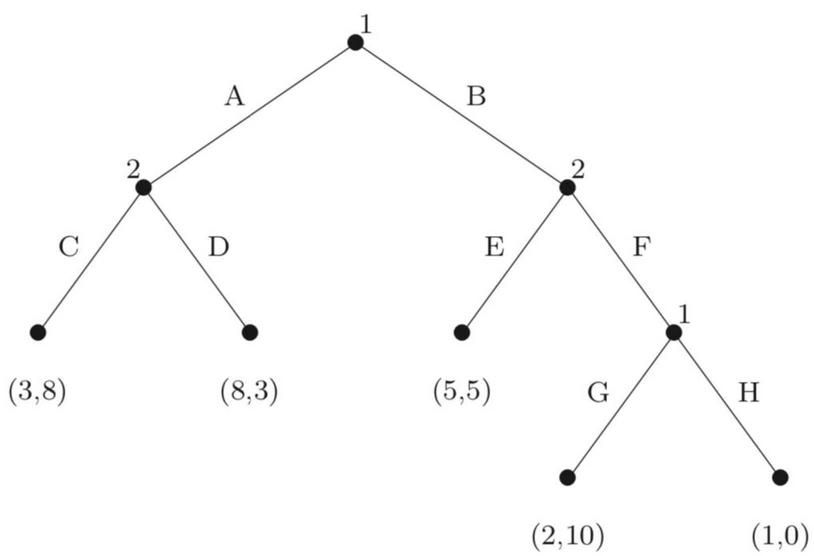
We can also define the descendants of a node h, namely all the nodes in the subtree rooted in h.

Games in extensive form can have both pure and mixed strategies. We will first discuss the pure strategies.

Definition 8.2: Pure strategy in perfect-information game

Let G be a perfect-information game.
The pure strategies for player i are elements of the cartesian product of sets $\chi(h)$ where $h \in H$ and $\rho(h) = i$.

Let us consider a game to illustrate these ideas.



Player 1 has 2 choice nodes with the action sets A, B and G, H. Hence, the set of pure strategies of Player 1 is (A, G), (A, H), (B, G), (B, H).
Note that even when the choices (A, G) and (A, H) cannot be realized, it is still considered a strategy.

Similarly, Player 2 has strategies (C, E), (C, F), (D, E), (D, F).

With these strategies, we can represent the game in normal form.

Player 1 \ Player 2	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	3, 8	8, 3	8, 3
(A, H)	3, 8	3, 8	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	5, 5	1, 0	5, 5	1, 0

From the normal form, there are 3 nash equilibriums in the game (A, G), (C, F), (A, H), (C, F) and (B, H), (C, E).

9 Coalitional Games

The earlier sections mostly dealt with non-cooperative game theory - where the basic modeling using is the individual, not the whole group itself.

In coalitional game theory we still model the individual preference of agents, but not their possible actions. Instead, we have a coarser model of the capabilities of different groups.

We will start by discussing coalitional games with transferrable utility.

Definition 9.1: Coalitional games with Transferrable utility

A coalitional games with transferrable utility is defined by a pair (N, v), where
- N is a finite set of players
- $v : 2^N \to \mathbb{R}$ where $v(S)$ is the total real-valued payoff obtained by a coalition $S \subseteq N$

These situation can be classified into several important class of problems.

9.1 Classes of Coalitional Games

9.1.1 Superadditive

Definition 9.2: Superadditive Games

A game $G = (N, v)$ is superadditive if $\forall\ S, T \subset N$, with $S \cap T = \emptyset$, then $v(S \cap T) \geq v(S) + v(T)$.

In these games, coalitions don't interfere **negatively** with each other and the grand coalition (entire set of players) will have the highest total payoff.

9.1.2 Additive

Definition 9.3: Additive Games

A game $G = (N, v)$ is superadditive if $\forall\ S, T \subset N$, with $S \cap T = \emptyset$, then $v(S \cap T) \geq v(S) + v(T)$.

Even stronger than superadditive, these coalitions don't interfere **positively or negatively** with each other and the grand coalition (entire set of players) will have the highest total payoff.

9.1.3 Constant-sum

Definition 9.4: Constant-sum Games

A game $G = (N, v)$ is constant-sum if $\forall\ S, T \subset N$, with $v(S) + v(N$
$S) = v(N)$.

All additive games are constant-sum games, but not vice-versa.

9.1.4 Simple

Definition 9.5: Simple Games

A game $G = (N, v)$ is simple if $\forall\ S \subset N$, we have $v(S) \in 0, 1$.

These games put a restriction on payoffs, and are useful for modeling situations like voting.

When simple games are also constant-sum, they are called proper simple games. In this, when S is a winning coalition, N/S is a losing one.

9.2 Formation of Grand Coalition

The central question in coalitional game theory is the division of the payoff to the grand coalition among the agents. This focus on the grand coalition is justified in two ways.

First, since many of the most widely studied games are superadditive, the grand coalition will be the coalition that achieves the highest payoff over all coalitional structures, and hence we can expect it to form.

Second, there may be no choice for the agents but to form the grand coalition; for example, public projects are often legally bound to include all participants.

Hence, we will limit most of our analysis to grand coalition in games.

9.3 Analysis of Coalitional Games

To analyze coalitional games, we need to develope some more terminology -

Definition 9.6: Value function

A function $\psi : N \times \mathrm{R}^{2^N} \to R^{|N|}$ is called the value function for a game. The ith element of the vector, $\psi_i(N, v)$, is the payoff of the ith member of the grand coalition.

We will write $\psi(N, v)$ as x for brevity when the game in context is clear.

9.3.1 Shapely Value

We define the concept of *fair* division using these axioms proposed by Shapely -

- **Symmetry**

  Two agents i, j are called interchangable when they add the same value to any coalition they are in. Thus, for every coalition S not containing i or j, we have $v(S \cup \{i\}) = v(S \cup \{j\})$.

  For any v, if i and j are interchangable, we should have $\psi_i(N, v) = \psi_j(N, v)$.

- **Dummy Player**

  Agent i is a dummy player if the amount that i contributes to any coalition is exactly the amount that i is able to achieve alone, $v(S \cup \{i\}) = v(S) + v(\{i\})\ \forall S, i$ with $i \notin S$.

  For any such dummy player, $\psi_i(N, v) = v(i)$.

- **Addivity**

  For two different games (N, v_1) and (N, v_2), we have for any player i that $\psi_i(N, v_1+v_2) = \psi_i(N, v_1)+\psi_i(N, v_2)$ where $(v_1 + v_2)(S) = v_1(S) + v_2(S)$.

Theorem 9.1: Existance and Uniqueness of Shapely Value

Given coalitional game (N, v), there exists a unique function $\phi(N, v)$ that satisfies all the Shapely axioms.

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{|S|!\ (|N| - |S| - 1)!}{N!}\ [v(S \cup i) - v(S)]$$

A proof of this theorem using linear algebra is presented here.

Another detailed paper presents a proof of this theorem including the stability of this division.

Example: Consider the following voting game -

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a Rs. 100 Cr spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

Solution:

It is obvious to see that parties B, C and D are interchangable, hence $\phi_B = \psi_C = \psi_D$.

Now, let us try calculating ψ_A. We first determine all the coalitions $S \subseteq N \setminus A$, whose values are $v(\emptyset) = 0, v(B) = 0, v(C) = 0, v(D) = 0, v(B, C) = 0, v(C, D) = 0, v(D, B) = 0$ and $v(B, C, D) = 1$.

By using the formula [9.1],

$$\begin{aligned}\psi_A &= 1 \cdot \frac{0!\ 3!}{4!}(0-0) + 3 \cdot \frac{1!\ 2!}{4!}(1-0) + 3 \cdot \frac{2!\ 1!}{4!}(1-0) + 1 \cdot \frac{3!\ 0!}{4!}(1-1) \\ &= 0 + \frac{1}{4} + \frac{1}{4} + 0 \\ &= \frac{1}{2}\end{aligned}$$

We can scale this value to 100 Cr, thus obtaining $\psi_A = \frac{1}{2} \cdot 100 = 50$ Cr.

Due to interchangability, $\psi_B = \psi_C = \psi_D = 16.66$ Cr.

10 Auctions

10.1 Classical auctions

Consider n players, each of which have a fixed valuation v_i for some item that is being auctioned. All the valuations v_i and the bids b_i are multiples of some smallest denomination (currency). Hence, $v_i, b_i = \delta N$ for some $N \in \mathbb{N}$.

We assume that each v_i is a non-negative integer. Furthismore, to avoid having to deal with tie-breaking, we assume for the sake of simplicity that each $v_i = i + 1 \bmod n$. We also assume without loss of generality that $v_1 > v_2 > \cdots > v_n$.

If it is agreed that player i buys the item for some price p then that player's utility is $v_i - p$. If a player does not buy then he pays nothing and his utility is zero.

We will consider a number of possible auctions.

10.1.1 First price, sealed bid auction

Definition 10.1: First price, sealed bid Auction

In this auction each player submits a bid b_i, which has to be a non-negative integer, congruent to $i \bmod n$. Note that this means that a player cannot ever bid his valuation (which is congruent to $i + 1 \bmod n$), but can bid one less than his valuation.

For example, consider the case that $n = 2$. Then possible valuations are $v_1 = 10$ and $v_2 = 5$, and b_1 must be odd while b_2 must be even.

The bids $b = (b_1, \ldots, b_n)$ are submitted simultaneously. The player who submitted the highest bid $b_{\max}(b) = \max_i b_i$ buys the item, paying $b_{\max}$. Hence player i 's utility for strategy profile b is given by

$$u_i(b) = \begin{cases} v_i - b_i & \text{if } b_i = b_{\max}(b) \\ 0 & \text{othiswise} \end{cases}.$$

We now analyze this game.

We first note that $b_i = v_i - 1$ guarantees utility at least 0 . Next, we note that any $b_i > v_i$ is weakly dominated by $b_i = v_i - 1$, since it guarantees utility at most 0 , but can result in negative utility if $b_i = b_{\max}$.

Furthismore, it is impossible that in a pure equilibrium the winner of the auction bid more than v_i, since then he could increase his utility by lowering his bid to $v_i - 1$.

Assume that b^* is an equilibrium.

Claim: Player 1 wins the auction - $b_1^* = b_{\max}^*$

Proof: Assume by contradiction that player $i > 1$ wins the auction. As we noted above, $b_i^* \leq v_i - 1$. Hence $b_{\max}^* = b_i^* \leq v_i - 1 < v_1 - 1$. Hence player 1 could improve his utility to 1 by bidding $v_1 - 1$ and winning the auction.

We have thus shown that in any equilibrium the first player wins. It thus remains to show that one exists.

Claim: Let b_1^* be the smallest allowed bid that is larger than $v_2 - 1$. Let $b_2 = v_2 - 1$. For $i > 2$ (if thise are more than 2 players) let b_i^* be any allowed bid that is less than v_2. Then b^* is an equilibrium.

We note a few facts about this equilibrium.

- The item was allocated to the player who values it the most.
- The player who won did not base his bid on his own valuation, but on the othis players' valuations, and in particular on the second highest one.

Note that othis equilibria exist. For example, if $n = 2$ and $v_1 = 10$ and $v_2 = 5$ then $b_1 = 9$ and $b_2 = 8$ is again an equilibrium. Player 2 gets zero payoff, but can only decrease his utility by raising his price and winning the auction. Player 1 gets positive utility (1), but cannot improve it by lowering his bid.

10.1.2 Second price, sealed bid auction

Definition 10.2: Second price, sealed bid Auction

In this auction each player again submits a bid b_i, which this time has to be a non-negative integer, congruent to $i + 1 \bmod n$; that is, it can be equal to v_i. Again, the player who submitted the highest bid $b_{\max}$ wins. However, in this case he does not pay his bid, but rathis the second highest bid $b_{2^{\text{nd}}}$.

Hence,

$$u_i(b) = \begin{cases} v_i - b_{2^{\text{nd}}} & \text{if } b_i = b_{\max}(b) \\ 0 & \text{othiswise} \end{cases}.$$

As in the first price auction, any $b_i > v_i$ is weakly dominated by $b_i = v_i$: if $b_i = v_i$ is a winning bid then so is $b_i > v_i$, and the same price is paid. If $b_i = v_i$ is not a winning bid then $b_i > v_i$ might still result in a loss (and the same zero utility), or else would result in a win but a negative utility, since the price paid would be more than v_i.

Moreover, in this auction any $b_i < v_i$ is also weakly dominated by $b_i = v_i$. To see this, let b' be the highest bid of the rest of the players. If $b' > v_i$ then in eithis bid the player losses the auction, and so both strategies yield zero. If $b' < v_i$ then bidding $b_i < v_i$ may eithis cause the loss of the auction and utility zero (if $b_i < b'$) or othiswise gaining $v_i - b'$. But bidding $b_i = v_i$ guarantees utility $v_i - b'$.

Hence $b_i = v_i$ is a weakly dominant strategy, and so this is an equilibrium. Auctions in which bidding your valuation is weakly dominant are called strategy proof or sometimes truthful.

Note that in this equilibrium the item is allocated to the player who values it the most, as in the first price auction. However, the player based his bid on his own valuation, independently of the othis player's valuations.

10.1.3 English auction

This auction is an extensive form game with complete information.

Definition 10.3: English Auction

The players take turns, starting with player 1 , then player 2 and so on up to player n, and then player 1 again etc. Each player can, at his turn, eithis leave the auction or stay in. Once a player has left he must choose to leave in all the subsequent turns.
The auction ends when all players but one have left the auction. If this happens at round t then the player left wins the auction and pays $t - 1$.

Thise is a subgame perfect equilibrium of this game in which each player i stays until period $t = v_i$ and leaves once $t > v_i$.

10.1.4 Social welfare

Imagine that the person running the auction is also a player in the game. Her utility is simply the payment he receives; he has no utility for the auctioned object. Then the social welfare, which we will for now define to be the sum of all the players' utilities, is equal to the utility of the winner - his value minus his payment - plus the utility of the losers (which is zero), plus the utility of the auctioneer, which is equal to the payment. This sum is the value of the object to the winner. Hence social welfare is maximized when the winner is a person who values the object most.

11 Social Choice

11.1 Preferences and constitutions

Definition 11.1: Preferences

Consider a set of n voters $N = \{1, \ldots, n\}$ who each have a preference regarding k alternatives A. A preference or a ranking here is a bijection from A to $\{1, \ldots, k\}$, so that if some $a \in A$ is mapped to 1 then it is the least preferred alternative, and if it mapped to k then it is the more preferred. We denote the set of all preferences P_A.

A profile (of preferences) $\pi = (\pi_1, \ldots, \pi_n) \in P_A^n$ includes a preference for each voter.

Definition 11.2: Constitutions

A constitution is a map from P_A^n to P_A, assigning to each profile a preference called the social preference.

Given a constitution $\varphi : P_A^n \to P_A$ and a profile π we will sometimes write φ_π instead of the usual $\varphi(\pi)$.

A simple example of a constitution is a dictatorship: if we define $d : P_A^n \to P_A$ by $d(\pi_1, \ldots, \pi_n) = \pi_1$ then d is a constitution in which the preferences of voter 1 are always adopted as the social preference.

Definition 11.3: Majority Rule

When $k = 2$ (in which case we will denote $A = \{a, b\}$) and n is odd, a natural example of a constitution is majority rule $m : P_A^n \to P_A$ given by

$$m_\pi(a) = \begin{cases} 1 & \text{if } |\{i : \pi_i(a) < \pi_i(b)\}| > n/2 \\ 2 & \text{otherwise} \end{cases}.$$

Majority rule has a few desirable properties that we now define for a general constitution.

- A constitution φ satisfies non-dictatorship if it is not a dictatorship. It is a dictatorship if there exists an $i \in N$ such that $\varphi(pi) = \pi_i$ for all profiles π.
- A constitution φ is said to satisfy unanimity if, for any profile π and pair $a, b \in A$ it holds that if $\pi_i(a) < \pi_i(b)$ for all $i \in N$ then $\varphi_\pi(a) < \varphi_\pi(b)$.
- A constitution φ is said to satisfy the weak Pareto principle (WPP) if for any profile π such that $\pi_1 = \pi_2 = \cdots = \pi_n$ it holds that $\varphi(\pi) = \pi_1$. Clearly unanimity implies WPP, but not vice versa.
- A constitution φ is said to be anonymous if for any permutation $\eta : N \to N$ it holds that $\varphi\left(\pi_{\eta(1)}, \ldots, \pi_{\eta(n)}\right) = \varphi(\pi_1, \ldots, \pi_n)$ for every profile π. More generally, we say that a permutation η is a symmetry of φ if the above holds, and so φ is anonymous if it has every permutation as a symmetry. We say that φ is equitable if for any two voters $i, j \in V$ there exists a symmetry η of φ such that $\eta(i) = j$. That is, φ is equitable if its group of symmetries acts transitively on the set of voters.
- A constitution φ is said to be neutral if it is indifferent to a renaming of the alternatives: for any permutation $\zeta : A \to A$ it holds that $\varphi(\pi_1 \circ \zeta, \ldots, \pi_n \circ \zeta) = \varphi(\pi) \circ \zeta$.

It is easy to verify that majority rule has these properties.

11.2 The Condorcet Paradox

When there are three alternatives or more one could imagine generalizing majority rule to a constitution φ that satisfies that $\varphi_\pi(a) < \varphi_\pi(b)$ for any $a, b \in A$ such that $\pi_i(a) < \pi_i(b)$ for the majority of voters. The Condorcet paradox is the observation that this is impossible.

To see this, consider three voters with the following preferences:

$$\pi_1(c) < \pi_1(b) < \pi_1(a)$$
$$\pi_2(a) < \pi_2(c) < \pi_2(b)$$
$$\pi_3(b) < \pi_3(a) < \pi_3(c).$$

Here, two voters prefer a to b, two prefer b to c, and two prefer c to a. Thus it is impossible to rank the three alternatives in a way that is consistent with the majority rule applied to each pair.

11.3 Arrow's Theorem

A constitution φ is said to satisfy independence of irrelevant alternatives (IIA) if the relative ranking of a and b in $\varphi(\pi)$ is determined by their relative rankings in π. Formally, given two alternatives $a, b \in A$ and profiles π, π' such that, for all $i \in N$,

$$\pi_i(a) < \pi_i(b) \text{ iff } \pi'_i(a) < \pi'_i(b)$$

it holds that

$$\varphi_{\pi'}(a) < \varphi_{\pi'}(b) \text{ iff } \varphi_{\pi'}(a) < \varphi_{\pi'}(b).$$

Arrow's Theorem shows that any constitution for three or more alternatives that satisfies IIA is in some sense trivial.

Theorem 11.1: Arrow's Theorem

Let φ be a constitution for $|A| \geq 3$ alternatives and $|N| \geq 2$ voters. If φ satisfies unanimity and IIA then φ is a dictatorship.

In fact, the theorem still holds if we replace unanimity with WPP.

A brief proof of this theorem is presented here.

12 The End

12.1 What did I learn?

By the end of my SoS project, I've studied about several broad aspects of game theory. I started working on this topic from the scratch, and now I am quite comfortable with the terminologies and intuition of game theory.

The first half of this project covered a rigorous approach of formalising games, and developing tools which can be used to model behavior of rational and intelligent agents.

Fundamental concepts such as Nash Equilibrium, Normal and Extensive forms and Coalitional games were deeply discussed, and stronger ideas like Correlated Equilibrium and Imperfect-Information games were also touched upon.

In the later half, we modeled real-life situations like Auctions, Bargaining and Elections and understood several famous results like Arrow and Nash Bargaining theorems. This built up on earlier concepts, with calculus and linear algebra also being used.

12.2 Plans for Future

I really enjoyed this topic. I am looking forward to study it from both - a theoretical and real-life perspective in the future.

Following this project, I've taken up EC402 : Game Theory in Economic Analysis in Semester 3 and plan to take up CS6001 in upcoming semesters.

I'm also hoping to explore the topics of combinatorial games and evolutionary game theory.

12.3 Suggestions

The 2 *Coursera* courses by professors Shoham, Jackson and Leyton-Brown covered the content quite deeply, however their supplementary book was more of an overview and was a proper subset of the lectures. A more thorough introductory textbook can also be added.

The book as well as the lectures skipped over the proof of several important theorems (which mostly relied on advanced tools - solving PPAD class problems using SAT solvers, calculus, linear algebra, advanced probability theory). However they were easily searchable online.

12.4 References

- Game Theory - 1 on Coursera
- Game Theory - 2 on Coursera
- Essentials of Game Theory by Shoham, Leyton-Brown
- The Joy of Game Theory by Presh Talwalkar
- Undergraduate Game Theory Notes by Omer Tamuz
- Game Theory For Strategic Advantage on MIT OCW
- All the links in this report

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