Interpolation & Polynomial Approximation

Lagrange Interpolating Polynomials II

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

Outline

1. Interpolating Polynomial Error Bound

Outline

Outline

1. Interpolating Polynomial Error Bound
2. Example: 2nd Lagrange Interpolating Polynomial Error Bound
3. Example: Interpolating Polynomial Error for Tabulated Data

Outline

1. Interpolating Polynomial Error Bound
2. Example: 2nd Lagrange Interpolating Polynomial Error Bound
3. Example: Interpolating Polynomial Error for Tabulated Data

The Lagrange Polynomial: Theoretical Error Bound

Theorem

The Lagrange Polynomial: Theoretical Error Bound

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$.

The Lagrange Polynomial: Theoretical Error Bound

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

The Lagrange Polynomial: Theoretical Error Bound

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (1/6)

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (1/6)

Note first that if $x = x_k$, for any $k = 0, 1, \ldots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields the result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (1/6)

Note first that if $x = x_k$, for any $k = 0, 1, \ldots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields the result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

If $x \neq x_k$, for all $k = 0, 1, \ldots, n$, define the function g for t in $[a, b]$ by

$$\begin{aligned} g(t) &= f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)} \\ &= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} \end{aligned}$$

The Lagrange Polynomial: Theoretical Error Bound

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Error Bound: Proof (2/6)

The Lagrange Polynomial: Theoretical Error Bound

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Error Bound: Proof (2/6)

Since $f \in C^{n+1}[a, b]$, and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$, we have

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - P(x)] \cdot 0 = 0$$

The Lagrange Polynomial: Theoretical Error Bound

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Error Bound: Proof (3/6)

The Lagrange Polynomial: Theoretical Error Bound

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Error Bound: Proof (3/6)

We have seen that $g(x_k) = 0$. Furthermore,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x - x_i)}{(x - x_i)} \\ &= f(x) - P(x) - [f(x) - P(x)] = 0 \end{aligned}$$

The Lagrange Polynomial: Theoretical Error Bound

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Error Bound: Proof (3/6)

We have seen that $g(x_k) = 0$. Furthermore,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x - x_i)}{(x - x_i)} \\ &= f(x) - P(x) - [f(x) - P(x)] = 0 \end{aligned}$$

Thus $g \in C^{n+1}[a, b]$, and g is zero at the $n + 2$ distinct numbers $x, x_0, x_1, \ldots, x_n$.

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (4/6)

Since $g \in C^{n+1}[a, b]$, and g is zero at the $n + 2$ distinct numbers $x, x_0, x_1, \ldots, x_n$, by Generalized Rolle's Theorem ▸ Theorem there exists a number ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$.

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (4/6)

Since $g \in C^{n+1}[a,b]$, and g is zero at the $n+2$ distinct numbers $x, x_0, x_1, \ldots, x_n$, by Generalized Rolle's Theorem ▸ Theorem there exists a number ξ in (a,b) for which $g^{(n+1)}(\xi) = 0$. So

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi} \end{aligned}$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (4/6)

Since $g \in C^{n+1}[a,b]$, and g is zero at the $n+2$ distinct numbers $x, x_0, x_1, \ldots, x_n$, by Generalized Rolle's Theorem ▸ Theorem there exists a number ξ in (a,b) for which $g^{(n+1)}(\xi) = 0$. So

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi} \end{aligned}$$

However, $P(x)$ is a polynomial of degree at most n, so the $(n+1)$st derivative, $P^{(n+1)}(x)$, is identically zero.

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (5/6)

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (5/6)

Also, $\displaystyle\prod_{i=0}^{n} \frac{t - x_i}{x - x_i}$ is a polynomial of degree $(n + 1)$, so

$$\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^{n}(x - x_i)} \right] t^{n+1} + \text{(lower-degree terms in } t\text{)},$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (5/6)

Also, $\displaystyle\prod_{i=0}^{n} \frac{t - x_i}{x - x_i}$ is a polynomial of degree $(n+1)$, so

$$\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^{n}(x - x_i)} \right] t^{n+1} + (\text{lower-degree terms in } t),$$

and

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} = \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)}$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (6/6)

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (6/6)

We therefore have:

$$\begin{aligned} 0 &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi} \\ &= f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)} \end{aligned}$$

The Lagrange Polynomial: Theoretical Error Bound

Error Bound: Proof (6/6)

We therefore have:

$$\begin{aligned} 0 &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)} \right]_{t=\xi} \\ &= f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)} \end{aligned}$$

and, upon solving for $f(x)$, we get the desired result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Outline

1. Interpolating Polynomial Error Bound

2. Example: 2nd Lagrange Interpolating Polynomial Error Bound

3. Example: Interpolating Polynomial Error for Tabulated Data

Lagrange Interpolating Polynomial Error Bound

Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, ▸ Original Example we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$.

Lagrange Interpolating Polynomial Error Bound

Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, Original Example we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

Lagrange Interpolating Polynomial Error Bound

Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, ▸ Original Example we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

Note

We will make use of the theoretical result ▸ Theorem written in the form

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $n = 2$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2) = -\frac{1}{\xi(x)^4}(x - 2)(x - 2.75)(x - 4)$$

for $\xi(x)$ in $(2, 4)$.

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2) = -\frac{1}{\xi(x)^4}(x-2)(x-2.75)(x-4)$$

for $\xi(x)$ in $(2, 4)$. The maximum value of $\frac{1}{\xi(x)^4}$ on the interval is $\frac{1}{2^4} = 1/16$.

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x - 2)(x - 2.75)(x - 4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

Because

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7),$$

the critical points occur at

$$x = \frac{7}{3} \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108} \qquad \text{and} \qquad x = \frac{7}{2} \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16}$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (3/3)

Hence, the maximum error is

$$\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)|$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (3/3)

Hence, the maximum error is

$$\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)|$$

$$\leq \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16}$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (3/3)

Hence, the maximum error is

$$\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)|$$

$$\leq \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16}$$

$$= \frac{3}{512}$$

The Lagrange Polynomial: 2nd Degree Error Bound

Solution (3/3)

Hence, the maximum error is

$$\begin{aligned}
& \max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)| \\
\leq \quad & \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16} \\
= \quad & \frac{3}{512} \\
\approx \quad & 0.00586
\end{aligned}$$

Outline

Example: Tabulated Data

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

- Suppose that a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$.

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

- Suppose that a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$.
- Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

- Suppose that a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$.
- Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.
- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

- Suppose that a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$.
- Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.
- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in [0,1], and suppose j satisfies $x_j \leq x \leq x_{j+1}$.

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

- Suppose that a table is to be prepared for the function $f(x) = e^x$, for x in $[0, 1]$.
- Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.
- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in [0,1], and suppose j satisfies $x_j \leq x \leq x_{j+1}$. The error bound theorem ▸ Theorem implies that the error in linear interpolation is

$$|f(x) - P(x)| = \left| \frac{f^{(2)}(\xi)}{2!}(x - x_j)(x - x_{j+1}) \right| = \frac{|f^{(2)}(\xi)|}{2} |(x - x_j)||(x - x_{j+1})|$$

Use of the Interpolating Polynomial Error Bound

Solution (1/3)

The step size is h, so $x_j = jh$, $x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!}|(x - jh)(x - (j+1)h)|.$$

Use of the Interpolating Polynomial Error Bound

Solution (1/3)

The step size is h, so $x_j = jh$, $x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!} |(x - jh)(x - (j+1)h)|.$$

Hence

$$\begin{aligned} |f(x) - P(x)| &\leq \frac{\max_{\xi \in [0,1]} e^{\xi}}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)| \\ &\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)|. \end{aligned}$$

Use of the Interpolating Polynomial Error Bound

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j + 1)h)$, for $jh \leq x \leq (j + 1)h$.

Use of the Interpolating Polynomial Error Bound

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \le x \le (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2\left(x - jh - \frac{h}{2}\right),$$

Use of the Interpolating Polynomial Error Bound

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \leq x \leq (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2\left(x - jh - \frac{h}{2}\right),$$

the only critical point for g is at $x = jh + \frac{h}{2}$, with

$$g\left(jh + \frac{h}{2}\right) = \left(\frac{h}{2}\right)^2 = \frac{h^2}{4}$$

Use of the Interpolating Polynomial Error Bound

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \leq x \leq (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2\left(x - jh - \frac{h}{2}\right),$$

the only critical point for g is at $x = jh + \frac{h}{2}$, with

$$g\left(jh + \frac{h}{2}\right) = \left(\frac{h}{2}\right)^2 = \frac{h^2}{4}$$

Since $g(jh) = 0$ and $g((j+1)h) = 0$, the maximum value of $|g'(x)|$ in $[jh, (j+1)h]$ must occur at the critical point.

Use of the Interpolating Polynomial Error Bound

Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Use of the Interpolating Polynomial Error Bound

Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Consequently, to ensure that the the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \leq 10^{-6}. \quad \text{This implies that} \quad h < 1.72 \times 10^{-3}.$$

Use of the Interpolating Polynomial Error Bound

Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Consequently, to ensure that the the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \leq 10^{-6}. \quad \text{This implies that} \quad h < 1.72 \times 10^{-3}.$$

Because $n = \frac{(1-0)}{h}$ must be an integer, a reasonable choice for the step size is $h = 0.001$.

Questions?

Reference Material

Generalized Rolle's Theorem

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If

$$f(x) = 0$$

at the $n + 1$ distinct numbers $a \leq x_0 < x_1 < \ldots < x_n \leq b$, then a number c in (x_0, x_n), and hence in (a, b), exists with

$$f^{(n)}(c) = 0$$

Return to Error Bound Theorem

The Lagrange Polynomial: Theoretical Error Bound

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

Return to Second Lagrange Interpolating Polynomial Example

Return to Tabulated data example with $f(x) = e^x$

The Lagrange Polynomial: 2nd Degree Polynomial

Example: $f(x) = \frac{1}{x}$

Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

Solution (Summary)

$$
\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k) L_k(x) \\
&= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}
\end{aligned}
$$

Return to Second Lagrange Interpolating Polynomial Example