# Interpolation & Polynomial Approximation

## Lagrange Interpolating Polynomials I

Numerical Analysis (9th Edition)

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# Outline

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1. Weierstrass Approximation Theorem
2. Inaccuracy of Taylor Polynomials

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2. Inaccuracy of Taylor Polynomials
3. Constructing the Lagrange Polynomial

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# Weierstrass Approximation Theorem

## Algebraic Polynomials

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## Algebraic Polynomials

One of the most useful and well-known classes of functions mapping the set of real numbers into itself is the algebraic polynomials, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a nonnegative integer and $a_0, \ldots, a_n$ are real constants.

# Weierstrass Approximation Theorem

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## Algebraic Polynomials (Cont'd)

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- By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired.

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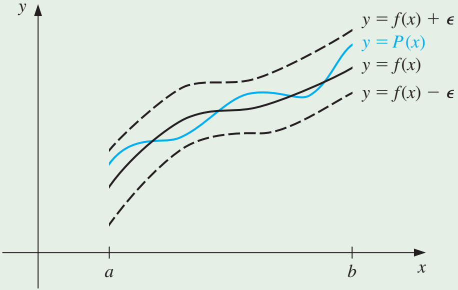
- One reason for their importance is that they uniformly approximate continuous functions.
- By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired.
- This result is expressed precisely in the Weierstrass Approximation Theorem.




## Weierstrass Approximation Theorem

Suppose that $f$ is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$

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- For these reasons, polynomials are often used for approximating continuous functions.

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# The Lagrange Polynomial: Taylor Polynomials

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- However this is not the case.
- The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.
- A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

# The Lagrange Polynomial: Taylor Polynomials

**Example: $f(x) = e^x$**

We will calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$.

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The Taylor polynomials are as follows:

# Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

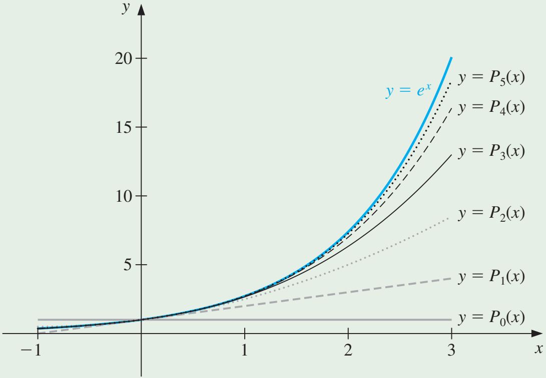
$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

# Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$




Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.

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## Example: A more extreme case

- Although better approximations are obtained for $f(x) = e^x$ if higher-degree Taylor polynomials are used, this is not true for all functions.
- Consider, as an extreme example, using Taylor polynomials of various degrees for $f(x) = \frac{1}{x}$ expanded about $x_0 = 1$ to approximate $f(3) = \frac{1}{3}$.

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Since

$$f(x) = x^{-1},\; f'(x) = -x^{-2},\; f''(x) = (-1)^2 2 \cdot x^{-3},$$

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and, in general,

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the Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n}(-1)^k(x-1)^k.$$

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| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P_n(3)$ | 1 | $-1$ | 3 | $-5$ | 11 | $-21$ | 43 | $-85$ |

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- This limits Taylor polynomial approximation to the situation in which approximations are needed only at numbers close to $x_0$.
- For ordinary computational purposes, it is more efficient to use methods that include information at various points.
- The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

# Outline

# The Lagrange Polynomial: The Linear Case

## Polynomial Interpolation

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## Polynomial Interpolation

- The problem of determining a polynomial of degree one that passes through the distinct points

$$(x_0, y_0) \qquad \text{and} \qquad (x_1, y_1)$$

is the same as approximating a function $f$ for which

$$f(x_0) = y_0 \qquad \text{and} \qquad f(x_1) = y_1$$

by means of a first-degree polynomial interpolating, or agreeing with, the values of $f$ at the given points.

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- Using this polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

# The Lagrange Polynomial: The Linear Case

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

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## Definition

The linear Lagrange interpolating polynomial though $(x_0, y_0)$ and $(x_1, y_1)$ is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

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Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

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So $P$ is the unique polynomial of degree at most 1 that passes through $(x_0, y_0)$ and $(x_1, y_1)$.

# The Lagrange Polynomial: The Linear Case

## Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

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## Solution

In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

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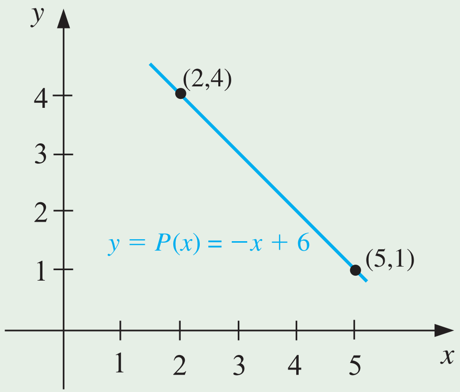
so

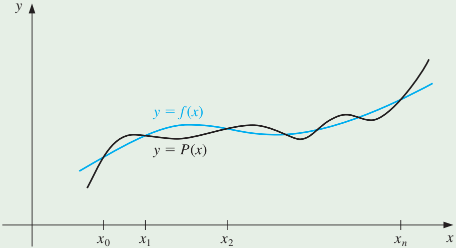
$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

# The Lagrange Polynomial: The Linear Case




The linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

# The Lagrange Polynomial: Degree $n$ Construction




To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most $n$ that passes through the $n+1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

# The Lagrange Polynomial: The General Case

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## Constructing the Degree $n$ Polynomial

- We first construct, for each $k = 0, 1, \ldots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$.

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- To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contain the term

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- To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$.
- Thus

$$L_{n,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$

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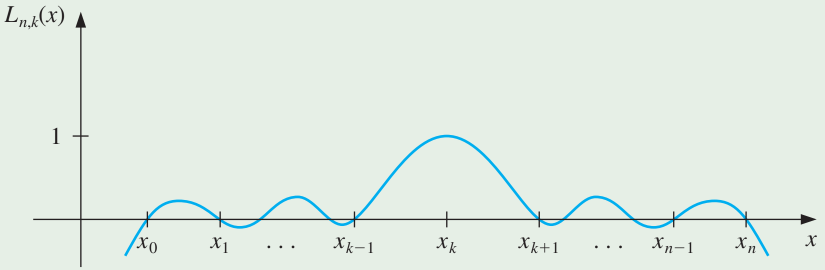

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## Theorem: *n*-th Lagrange interpolating polynomial

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and $f$ is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most $n$ exists with

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This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0, 1, \ldots, n$, $L_{n,k}(x)$ is defined as follows:

# The Lagrange Polynomial: The General Case

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## Definition of $L_{n,k}(x)$

$$\begin{aligned} L_{n,k}(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \\ &= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} \end{aligned}$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

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(a) Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

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(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

# The Lagrange Polynomial: 2nd Degree Polynomial

## Part (a): Solution

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## Part (a): Solution

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75)$$

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$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, since $f(x) = \frac{1}{x}$:

$$f(x_0) = f(2) = 1/2, \qquad f(x_1) = f(2.75) = 4/11, \qquad f(x_2) = f(4) = 1/4$$

# The Lagrange Polynomial: 2nd Degree Polynomial

## Part (a): Solution (Cont'd)

Therefore, we obtain

$$
\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.
\end{aligned}
$$

# The Lagrange Polynomial: 2nd Degree Polynomial

$$P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

## Part (b): Solution

# The Lagrange Polynomial: 2nd Degree Polynomial

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## Part (b): Solution

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

# The Lagrange Polynomial: 2nd Degree Polynomial

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Earlier, we we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$.

# Second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$