Error Propagation

Functions of one variable, $f(x)$

Suppose we measure x, but the physical quantity of interest is some function $f(x)$. How do we extract the error on f, Δf, from the error on x, Δx?

The simplest thing to do is to calculate $f(x-\Delta x)$ and $f(x+\Delta x)$. **This is perfectly legitimate**, and will almost always work (the only case in which it will return misleading results is when the function f turns round inside the range $x\pm\Delta x$—for example, the upper limit of $\sin(89\pm3)°$ is 1, not $\sin 92°$). This method is particularly useful when the dependence on x is complicated, when Δx is not small, or when the error is asymmetric (in the above example, $\sin(89 \pm 3)° = 0.99985^{+0.00015}_{-0.00228}$).

However, in the case in which Δx is small, we have

$$\frac{\Delta f}{\Delta x} \simeq \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|$$

(the modulus sign is because both Δf and Δx are normally defined to be positive). This is the formula that leads to such standard results as

$$\begin{aligned}\Delta(x^2) &= 2x\Delta x,\\ \Delta(\ln x) &= \frac{\Delta x}{x},\\ \Delta(\sin x) &= \Delta x \cos x,\end{aligned}$$

where in the last case Δx must be in radians.

Functions of more than one variable

The fundamental rule from statistics is that, for *uncorrelated* variables

the variance of the sum is the sum of the variances.

This is actually easy to prove:

$$\begin{aligned}V_{x+y} &= \left\langle \left((x_i + y_i) - (\bar{x} + \bar{y})\right)^2 \right\rangle\\ &= \langle (x_i + y_i)^2 - 2(x_i + y_i)(\bar{x} + \bar{y}) + (\bar{x} + \bar{y})^2 \rangle\\ &= \left\langle \left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right) + \left(y_i^2 - 2y_i\bar{y} + \bar{y}^2\right) + 2(x_iy_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\bar{y}) \right\rangle\\ &= \overline{x^2} - \bar{x}^2 + \overline{y^2} - \bar{y}^2 + 2(\overline{xy} - \bar{x}\bar{y}) = V_{xx} + V_{yy} + 2V_{xy},\end{aligned}$$

and for uncorrelated variables the covariance $V_{xy} = 0$ by definition.

For V_{x-y}, the only term that changes sign is the covariance, so for uncorrelated variables the variance of the difference is also the sum of the variances. For the product xy, consider the log: for uncorrelated variables

$$\begin{aligned}V_{\ln x + \ln y} &= V_{\ln x} + V_{\ln y}\\ \text{i.e.}\quad \left(\Delta(\ln x + \ln y)\right)^2 &= \left(\Delta(\ln x)\right)^2 + \left(\Delta(\ln y)\right)^2\\ \text{i.e.}\quad \left(\frac{\Delta(xy)}{xy}\right)^2 &= \left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2\end{aligned}$$

and the same is true for x/y.

Using

$$\frac{\Delta f}{\Delta x} \simeq \left|\frac{\partial f}{\partial x}\right|$$

we get the general rule that, for *uncorrelated* variables x, y,

$$(\Delta f)^2 \simeq \left|\frac{\partial f}{\partial x}\right|^2 (\Delta x)^2 + \left|\frac{\partial f}{\partial y}\right|^2 (\Delta y)^2$$

(which can obviously be extended to more than two variables). If the function is difficult to differentiate, or if Δx, Δy are not small, the partial derivatives can be replaced by working out $f(x\pm\Delta x, y)$ and $f(x, y\pm\Delta y)$ and finding the difference from $f(x, y)$. Note that you vary **one variable at a time**: you do *not* compare $f(x, y)$ with $f(x+\Delta x, y+\Delta y)$, because doing that is assuming that your variables are completely correlated ($\rho_{xy} = +1$).

If your variables *are* correlated, you need to include the covariance term:

$$(\Delta f)^2 \simeq \left|\frac{\partial f}{\partial x}\right|^2 (\Delta x)^2 + \left|\frac{\partial f}{\partial y}\right|^2 (\Delta y)^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}V_{xy}$$

(this is exactly true for linear functions of x and y, and approximately true for non-linear functions).

Some standard formulae

If we assume that the variables x and y are uncorrelated and have uncertainties σ_x and σ_y respectively, applying the above rules leads to the results in the table below.

Function	Uncertainty
$f = x \pm y$	$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$
$f = xy$	$\sigma_f = xy\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$
$f = x/y$	$\sigma_f = \frac{x}{y}\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$
$f = x^n$	$\frac{\sigma_f}{f} = n\frac{\sigma_x}{x}$
$f = x^n \times y^m$	$\frac{\sigma_f}{f} = \sqrt{\left(n\frac{\sigma_x}{x}\right)^2 + \left(m\frac{\sigma_y}{y}\right)^2}$
$f = x^n \pm y^m$	$\sigma_f = \sqrt{(nx^{n-1}\sigma_x)^2 + \left(my^{m-1}\sigma_y\right)^2}$
$f = \ln x$	$\sigma_f = \sigma_x/x$
$f = e^x$	$\sigma_f/f = \sigma_x$
$f = \sin x$	$\sigma_f = \sigma_x \cos x$
$f = \cos x$	$\sigma_f = \sigma_x \sin x$

Note that (1) for trigonometric functions, σ must be expressed in **radians** and (2) $\sigma_f = \sigma_x/x$ for $f = \ln x$, **not** for $f = \log_{10} x$ (you can convert $\ln x$ to $\log_{10} x$ by multiplying by $\log_{10} e = 0.4343$, so in fact $\sigma(\log_{10} x) = 0.4343\ \sigma_x/x$).