Error Propagation

# Functions of one variable, *f*(*x*)

Suppose we measure *x*, but the physical quantity of interest is some function *f*(*x*). How do we extract the error on *f*, Δ*f*, from the error on *x*, Δ*x*?

The simplest thing to do is to calculate *f*(*x*−Δ*x*) and *f*(*x*+Δ*x*). **This is perfectly legitimate**, and will almost always work (the only case in which it will return misleading results is when the function *f* turns round inside the range *x*±Δ*x*—for example, the upper limit of sin(89±3)° is 1, not sin 92°). This method is particularly useful when the dependence on *x* is complicated, when Δ*x* is not small, or when the error is asymmetric (in the above example, ).

However, in the case in which Δ*x* is small, we have   
(the modulus sign is because both Δ*f* and Δ*x* are normally defined to be positive). This is the formula that leads to such standard results as   
where in the last case Δ*x* must be in radians.

# Functions of more than one variable

The fundamental rule from statistics is that, for *uncorrelated* variables

**the variance of the sum is the sum of the variances.**

This is actually easy to prove:   
and for uncorrelated variables the covariance *Vxy* = 0 by definition.

For *Vx−y*, the only term that changes sign is the covariance, so for uncorrelated variables the variance of the difference is also the sum of the variances. For the product *xy*, consider the log: for uncorrelated variables   
and the same is true for *x*/*y*.

Using   
we get the general rule that, for *uncorrelated* variables *x*, *y*,   
(which can obviously be extended to more than two variables). If the function is difficult to dif­feren­tiate, or if Δ*x*, Δ*y* are not small, the partial derivatives can be replaced by working out *f*(*x*±Δ*x*, *y*) and *f*(*x*, *y*±Δ*y*) and finding the difference from *f*(*x*, *y*). Note that you vary **one variable at a time**: you do *not* compare *f*(*x*, *y*) with *f*(*x*+Δ*x*, *y*+Δ*y*), because doing that is assuming that your variables are completely correlated (*ρxy* = +1).

If your variables *are* correlated, you need to include the covariance term:   
(this is exactly true for linear functions of *x* and *y*, and approximately true for non-linear func­tions).

# Some standard formulae

If we assume that the variables *x* and *y* are uncorrelated and have uncertainties *σx* and *σy* res­pectively, applying the above rules leads to the results in the table below.

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| Function | Uncertainty |
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Note that (1) for trigonometric functions, *σ* must be expressed in **radians** and (2) for *f* = ln *x*, **not** for *f* = log10 *x* (you can convert ln *x* to log10 *x* by multiplying by log10 *e* = 0.4343, so in fact *σ*(log10 *x*) = 0.4343 *σx*/*x*).