Solution to Gaussian Elimination

Given the set of linear equations:

$$\begin{array}{rrrrrrrr} E_{1j}: & 2x_1 & + & 8x_2 & - & 5x_3 & = & 53 \\ E_{2j}: & 3x_1 & - & 6x_2 & + & 4x_3 & = & -48 \\ E_{3j}: & x_1 & + & 2x_2 & - & x_3 & = & 13 \end{array}$$

Let's look at the solution graphically first.

F .:

2:

3:

Figure 1 is the plane of E_{1j}, Figure 2 is the plane of E_{2j}, and Figure 3 is the plane of E_{3j}.

al result will be the intersection of the three planes:

One way of solving the equations:

1. Divide E_{1j} by 2; multiply E_{1j} by 3 and subtract it from E_{2j} to cancel x_1 from E_{2j}; subtract the new E_{1j} from E_{3j} to cancel x_1 from E_{3j}:

$$\begin{array}{rrrrrrrr} E_{1j}: & x_1 & + & 4x_2 & - & \frac{5}{2}x_3 & = & \frac{53}{2} \\ E_{2j}: & & - & 18x_2 & + & \frac{23}{2}x_3 & = & -\frac{255}{2} \\ E_{3j}: & & - & 2x_2 & + & \frac{3}{2}x_3 & = & -\frac{27}{2} \end{array}$$

2. Divide E_{2j} by -18; multiply E_{2j} by 2 and add it to E_{3j} to cancel x_2 from E_{3j}:

$$\begin{array}{rrrrrrrr} E_{1j}: & x_1 & + & 4x_2 & - & \frac{5}{2}x_3 & = & \frac{53}{2} \\ E_{2j}: & & & x_2 & - & \frac{23}{36}x_3 & = & \frac{85}{12} \\ E_{3j}: & & & & & \frac{2}{9}x_3 & = & \frac{2}{3} \end{array}$$

$$\begin{array}{rrrrrrrr} E_{1j}: & x_1 & + & 4x_2 & - & \frac{5}{2}x_3 & = & \frac{53}{2} \\ E_{2j}: & & & x_2 & & & = & 9 \\ E_{3j}: & & & & & x_3 & = & 3 \end{array}$$

3. Now, solve for x_3, and substitute it into E_{2j} to solve for x_2; then substitute x_2 and x_3 into E_{1j} and solve for x_1:

$$\begin{array}{rrrrr} E_{1j}: & x_1 & & & = -2 \\ E_{2j}: & & x_2 & & = 9 \\ E_{3j}: & & & x_3 & = 3 \end{array}$$

So, x_1= -2, x_2= 9 and x_3= 3.

Solving the equations in a matrix:

$$\left[\begin{array}{rrr|r} 2 & 8 & -5 & 53 \\ 3 & -6 & 4 & -48 \\ 1 & 2 & -1 & 13 \end{array}\right]$$

Two main steps are involved in this solution. The Gaussian elimination is performed first, followed by the Back-substitution:

$$\underset{\rightarrow}{E_{3j} \leftrightarrow E_{1j}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 13 \\ 3 & -6 & 4 & -48 \\ 2 & 8 & -5 & 53 \end{array}\right]$$

$$\begin{array}{c} E_{2j} - 3E_{1j} \\ E_{3j} - 2E_{1j} \\ \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 13 \\ 0 & -12 & 7 & -87 \\ 0 & 4 & -3 & 27 \end{array}\right]$$

$$\underset{\rightarrow}{E_{3j} + \frac{1}{3}E_{2j}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 13 \\ 0 & -12 & 7 & -87 \\ 0 & 0 & \frac{-2}{3} & -2 \end{array}\right]$$

- Reducing the matrix with Gaussian elimination
- Back-substitution

  Reading from the matrix:

  - finding x_3

$$\begin{array}{rcr} \frac{-2}{3}x_3 & = & -2 \\ x_3 & = & 3 \end{array}$$

  - finding x_2

$$\begin{array}{rrrrr} -12x_2 & + & 7x_3 & = & -87 \\ -12x_2 & + & 7(3) & = & -87 \\ & & 12x_2 & = & 108 \\ & & x_2 & = & 9 \end{array}$$

  - finding x_1

$$\begin{array}{rrrrrrr} x_1 & + & 2x_2 & - & x_3 & = & 13 \\ x_1 & + & 2(9) & - & 3 & = & 13 \\ & & & & x_1 & = & -2 \end{array}$$

So, x_1= -2, x_2= 9 and x_3= 3. As expected, the two different methods give the same answer.

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