# 1 Basic MGM algorithm

1. Solve structured inhomogeneous Poisson for $H$ on $D+B$ with BC's from original problem

$$\frac{\partial^2 H}{\partial x_i \partial x_i} = \frac{\partial F_i^n}{\partial x_i} - \left[\frac{(\nabla \cdot \mathbf{u})^{n+1} - (\nabla \cdot \mathbf{u})^n}{\Delta t}\right] \qquad \text{on } D+B$$

2. Project velocity on $D+B$

$$u_i^{*,0} = u_i^n - \Delta t \left[F_i^n + \frac{\partial H}{\partial x_i}\right] \qquad \text{on } D+B$$

3. While error in Poisson on $D$ do for $k = 0, 1, \cdots, K_{max}$

   (a) Determine BC's for ${H'}^k$

$$\begin{aligned} {H'}^k &= 0 && \text{on } \partial D \text{ (pressure BC)} \\ \frac{{H'}^k}{\partial x_i} &= 0 && \text{on } \partial D \text{ (velocity BC).} \\ {H'}^k &= \begin{cases} 0 & \text{if } k = 0 \\ \overline{H'}^{\,k-1} & \text{if } k > 0 \text{ (simple mean BC)} \end{cases} && \text{on } \partial M \\ \frac{\partial {H'}^k}{\partial x_i} &= \frac{u_i^{*,k} - 0}{\Delta t} && \text{on } \partial B \end{aligned}$$

   (b) Solve unstructured homogeneous Laplace for ${H'}^k$ on $D$ with BC's from 3.(a)

$$\frac{\partial^2 {H'}^k}{\partial x_i \partial x_i} = 0 \qquad \text{on } D$$

   (c) Project velocity on $D$

$$u_i^{*,k+1} = u_i^{*,k} - \Delta t \left[\frac{\partial {H'}^k}{\partial x_i}\right] \qquad \text{on } D$$

4. Define new pressure solution $H^n := H + {H'}^k$

$$
\begin{aligned}
u_i^{*,0} &= u_i^n - \Delta t \left[ F_i^n + \frac{\partial H}{\partial x_i} \right] \\
u_i^{*,1} &= u_i^n - \Delta t \left[ F_i^n + \frac{\partial H}{\partial x_i} \right] - \Delta t \frac{\partial {H'}^0}{\partial x_i} \\
u_i^{*,2} &= u_i^n - \Delta t \left[ F_i^n + \frac{\partial H}{\partial x_i} \right] - \Delta t \frac{\partial {H'}^0}{\partial x_i} - \Delta t \frac{\partial {H'}^1}{\partial x_i}
\end{aligned} \tag{1}
$$

### 1.1 Consider First Laplace iteration

## 2 Internal boundary conditions for MGM correction step

Below, I will list three different ways to define Dirichlet boundary values for $H'$ along mesh interfaces. To this end, let me introduce the following notation for the mean value of $H'$ in a given node $I$ at interface $\partial M$ in time step $t_n$, which is only related to the x-direction and the 2-mesh case in Figure (3) of your summary,

$$\overline{H'}_I^{\,n} := \frac{1}{2}\left[H'^{①,n}_{IBAR} + H'^{②,n}_{1}\right] \tag{2}$$

### 2.1 Simple mean value (SM)

In your section '(6.1) A simple solution' you proposed to use the mean values of the $H'$ values from the last time step $t_{n-1}$ along the interface $\partial M$

$$H'^{①,n}_{I} = H'^{②,n}_{I} = \overline{H'}_I^{\,n-1} \tag{3}$$

### 2.2 Linear Extrapolation (LE)

Due to my opinion another possible approach which probably gives a little more accuracy could be to use the mean values $\overline{H'}_I^{\,n-1}$ and $\overline{H'}_I^{\,n-2}$ of the two preceding time steps $t_{n-1}$ and $t_{n-2}$ and use linear extrapolation of these two values in time

$$H'^{①,n}_{I} = H'^{②,n}_{I} = 2\overline{H'}_I^{\,n-1} - \overline{H'}_I^{\,n-2} \tag{4}$$

Of course it is the question of initial values for this extrapolation. I started with it only after the 2nd time-step and took (SM) before. But also for (SM) there is the question of the proper initial values from my point of view. Currently, I simply use zero as initial values $H'$ values for $t_0$. However, I could also imagine to take a little more effort here (but more about this later).

### 2.3 Taylor Series (TS)

Now, let's come to your '(6.2) Complex solution' based on a Taylor series expansion.

$$\begin{aligned}
H'^{①,n}_{IBAR} &= H'^{①,n}_{I} - \frac{1}{1!}\frac{\Delta x}{2}\frac{\partial H'^{①}}{\partial x}\bigg|_I^n + \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2\frac{\partial^2 H'^{①}}{\partial x^2}\bigg|_I^n - \cdots \\
H'^{②,n}_{1} &= H'^{②,n}_{I} + \frac{1}{1!}\frac{\Delta x}{2}\frac{\partial H'^{②}}{\partial x}\bigg|_I^n + \frac{1}{2!}\left(\frac{\Delta x}{2}\right)^2\frac{\partial^2 H'^{②}}{\partial x^2}\bigg|_I^n + \cdots
\end{aligned} \tag{5}$$

or shorter

$$\begin{aligned}
H'^{①,n}_{IBAR} &= H'^{①,n}_{I} - \frac{\Delta x}{2}\frac{\partial H'^{①}}{\partial x}\bigg|_I^n + \frac{\Delta x^2}{8}\frac{\partial^2 H'^{①}}{\partial x^2}\bigg|_I^n - \cdots \\
H'^{②,n}_{1} &= H'^{②,n}_{I} + \frac{\Delta x}{2}\frac{\partial H'^{②}}{\partial x}\bigg|_I^n + \frac{\Delta x^2}{8}\frac{\partial^2 H'^{②}}{\partial x^2}\bigg|_I^n + \cdots
\end{aligned} \tag{6}$$

For the ghost cell values $H'^{①,n}_{IBAR+1}$ and $H'^{②,n}_{0}$ we consider linear extrapolation

$$
\begin{aligned}
H'^{①,n}_{IBAR+1} &= 2H'^{①,n}_{I} - H'^{①,n}_{IBAR} \\
H'^{②,n}_{0} &= 2H'^{②,n}_{I} - H'^{②,n}_{1}
\end{aligned}
\tag{7}
$$

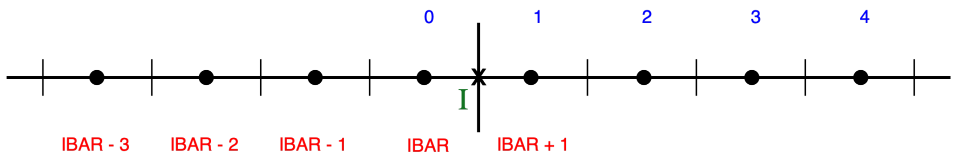


Figure 1: Decomposition in x-direction

In the following I will list the individual steps in great detail to avoid sign errors or similar in a possible implementation. First of all the following simple transformation is done based on (8)

$$
\begin{aligned}
H'^{①,n}_{I} &= H'^{①,n}_{IBAR} + \frac{\Delta x}{2}\frac{\partial H'^{①}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{8}\frac{\partial^2 H'^{①}}{\partial x^2}\bigg|^{n}_{I} + \cdots \\
H'^{②,n}_{I} &= H'^{②,n}_{1} - \frac{\Delta x}{2}\frac{\partial H'^{②}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{8}\frac{\partial^2 H'^{②}}{\partial x^2}\bigg|^{n}_{I} - \cdots
\end{aligned}
\tag{8}
$$

Now, substituting the interface values (8) into the extrapolation settings (7) in order to eliminate them

$$
\begin{aligned}
H'^{①,n}_{IBAR+1} &\approx 2\cdot\left(H'^{①,n}_{IBAR} + \frac{\Delta x}{2}\frac{\partial H'^{①}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{8}\frac{\partial^2 H'^{①}}{\partial x^2}\bigg|^{n}_{I}\right) - H'^{①,n}_{IBAR} \\
H'^{②,n}_{0} &\approx 2\cdot\left(H'^{②,n}_{1} - \frac{\Delta x}{2}\frac{\partial H'^{②}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{8}\frac{\partial^2 H'^{②}}{\partial x^2}\bigg|^{n}_{I}\right) - H'^{②,n}_{1}
\end{aligned}
\tag{9}
$$

which finally leads to

$$
\begin{aligned}
H'^{①,n}_{IBAR+1} &\approx H'^{①,n}_{IBAR} + \Delta x\frac{\partial H'^{①}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{4}\frac{\partial^2 H'^{①}}{\partial x^2}\bigg|^{n}_{I} \\
H'^{②,n}_{0} &\approx H'^{②,n}_{1} - \Delta x\frac{\partial H'^{②}}{\partial x}\bigg|^{n}_{I} - \frac{\Delta x^2}{4}\frac{\partial^2 H'^{②}}{\partial x^2}\bigg|^{n}_{I}
\end{aligned}
\tag{10}
$$

### 2.3.1 The second derivative

As you wrote, the second derivative in x must be the negative of the second derivative in y, because we solve the Laplace problem.

But concerning your second step in the derivation under '(6.3) Second derivative' I am confused right now. There you use two terms, first a difference quotient in y and then one in x (but our goal is to replace '$IBAR+1$'). Here, I suppose we only need the first one in y such that finally holds

$$\left.\frac{\partial^2 H'^{①}}{\partial x^2}\right|_I^n = -\left.\frac{\partial^2 H'^{①}}{\partial y^2}\right|_I^n \approx -\frac{H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}}{\Delta y^2} \tag{11}$$

$$\left.\frac{\partial^2 H'^{②}}{\partial x^2}\right|_I^n = -\left.\frac{\partial^2 H'^{②}}{\partial y^2}\right|_I^n \approx -\frac{H'^{②,n}_{1,j-1} - 2H'^{②,n}_{1,j} + H'^{②,n}_{1,j+1}}{\Delta y^2} \tag{12}$$

What do you think about this? This term adds matrix entries in y-direction.

### 2.3.2 The first derivative

For the sake of completeness I also list your steps to define the first derivation. Based on the previous solutions $H'^{①,n-1}$ and $H'^{②,n-1}$, we approximate the derivative on the mesh interface by

$$\left.\frac{\partial^2 H'^{①}}{\partial x}\right|_I^n = \left.\frac{\partial^2 H'^{②}}{\partial x}\right|_I^n \approx \frac{H'^{②,n-1}_{1} - H'^{①,n-1}_{IBAR}}{\Delta x}$$

### 2.3.3 Putting everything together

Now, inserting all this stuff into (**??**) we get

$$H'^{①,n}_{IBAR+1,j} \approx H'^{①,n}_{IBAR,j} + \Delta x\left(\frac{H'^{②,n-1}_{1} - H'^{①,n-1}_{IBAR,j}}{\Delta x}\right) - \frac{\Delta x^2}{4}\left(-\frac{H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}}{\Delta y^2}\right) \tag{13}$$

$$H'^{②,n}_{0,j} \approx H'^{②,n}_{1,j} - \Delta x\left(\frac{H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}}{\Delta x}\right) - \frac{\Delta x^2}{4}\left(-\frac{H'^{②,n}_{1,j-1} - 2H'^{②,n}_{1,j} + H'^{②,n}_{1,j+1}}{\Delta y^2}\right)$$

And if I didn't make a mistake somewhere (which is not unlikely in this mess), this finally leads to

$$H'^{①,n}_{IBAR+1,j} \approx H'^{①,n}_{IBAR,j} + \left(H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}\right) + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}\right) \tag{14}$$

$$H'^{②,n}_{0,j} \approx H'^{②,n}_{1,j} - \left(H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}\right) + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{②,n}_{1,j-1} - 2H'^{②,n}_{1,j} + H'^{②,n}_{1,j+1}\right)$$

When using these terms in the matrix stencil for cells adjacent to the mesh interface, the previous '$n-1$' terms add up to the right hand side and the current '$n$' terms to the matrix itself.

### 2.3.4 Substituting the ghost values in the matrix stencil in 2D

**Mesh 1:** For a cell in mesh 1 adjacent to the mesh interface the usual Laplace matrix stencil is

$$\frac{H'^{\textcircled{1},n}_{IBAR-1,j} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR+1,j}}{\Delta x^2} + \frac{H'^{\textcircled{1},n}_{IBAR,j-1} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR,j+1}}{\Delta y^2} = 0$$

Now, substituting the '$IBAR+1$' component based on the derivation in (14), we get

$$\frac{H'^{\textcircled{1},n}_{IBAR-1,j} - 2H'^{\textcircled{1},n}_{IBAR,j} + \left[H'^{\textcircled{1},n}_{IBAR,j} + \left(H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{IBAR,j}\right) + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{\textcircled{1},n}_{IBAR,j-1} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR,j+1}\right)\right]}{\Delta x^2} + \frac{H'^{\textcircled{1},n}_{IBAR,j-1} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR,j+1}}{\Delta y^2} = 0$$

Let's first move the previous '$n-1$' terms on the right hand side and sum up the obvious

$$\frac{H'^{\textcircled{1},n}_{IBAR-1,j} - H'^{\textcircled{1},n}_{IBAR,j} + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{\textcircled{1},n}_{IBAR,j-1} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR,j+1}\right)}{\Delta x^2} + \frac{H'^{\textcircled{1},n}_{IBAR,j-1} - 2H'^{\textcircled{1},n}_{IBAR,j} + H'^{\textcircled{1},n}_{IBAR,j+1}}{\Delta y^2} = -\frac{H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{IBAR,j}}{\Delta x^2}$$

and then resort the whole stuff

$$H'^{\textcircled{1},n}_{IBAR,j-1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) + H'^{\textcircled{1},n}_{IBAR-1,j}\left(\frac{1}{\Delta x^2}\right) + H'^{\textcircled{1},n}_{IBAR,j}\left(-\frac{1}{\Delta x^2} - \frac{1}{2\Delta y^2} - \frac{2}{\Delta y^2}\right) + H'^{\textcircled{1},n}_{IBAR,j+1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) = -\frac{H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{IBAR,j}}{\Delta x^2}$$

which finally gives the new matrix entries for the corresponding matrix line for cell $(IBAR, j)$ in mesh 1.

**Mesh 2:** For a cell in mesh 2 adjacent to the mesh interface the usual Laplace matrix stencil is

$$\frac{H'^{\textcircled{2},n}_{0,j} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{2,j}}{\Delta x^2} + \frac{H'^{\textcircled{2},n}_{1,j-1} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{1,j+1}}{\Delta y^2} = 0$$

Now, substituting the '0' component based on the derivation in (14), we get

$$\frac{\left[H'^{\textcircled{2},n}_{1,j} - \left(H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{IBAR,j}\right) + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{\textcircled{2},n}_{1,j-1} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{1,j+1}\right)\right] - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{2,j}}{\Delta x^2} + \frac{H'^{\textcircled{2},n}_{1,j-1} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{1,j+1}}{\Delta y^2} = 0$$

Let's first move the previous '$n-1$' terms on the right hand side and sum up the obvious

$$\frac{-H'^{\textcircled{2},n}_{1,j} + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{\textcircled{2},n}_{1,j-1} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{1,j+1}\right) + H'^{\textcircled{2},n}_{2,j}}{\Delta x^2} + \frac{H'^{\textcircled{2},n}_{1,j-1} - 2H'^{\textcircled{2},n}_{1,j} + H'^{\textcircled{2},n}_{1,j+1}}{\Delta y^2} = \frac{H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{IBAR,j}}{\Delta x^2}$$

and then resort the whole stuff

$$H'^{\textcircled{2},n}_{1,j-1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) + H'^{\textcircled{2},n}_{2,j}\left(\frac{1}{\Delta x^2}\right) + H'^{\textcircled{2},n}_{1,j}\left(-\frac{1}{\Delta x^2} - \frac{1}{2\Delta y^2} - \frac{2}{\Delta y^2}\right) + H'^{\textcircled{2},n}_{1,j+1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) = \frac{H'^{\textcircled{2},n-1}_{1,j} - H'^{\textcircled{1},n-1}_{1,j}}{\Delta x^2}$$

which finally gives the new matrix entries for the corresponding matrix line for cell $(1, j)$ in mesh 2.

### 2.3.5 Substituting the ghost values in the matrix stencil in 3D

**Mesh 1:** For a cell in mesh 1 adjacent to the mesh interface the usual Laplace matrix stencil is

$$
\begin{aligned}
&\frac{H'^{①,n}_{IBAR-1,j,k} - 2H'^{①,n}_{IBAR,j,k} + H'^{①,n}_{IBAR+1,j,k}}{\Delta x^2} && (15)\\
+&\frac{H'^{①,n}_{IBAR,j-1,k} - 2H'^{①,n}_{IBAR,j,k} + H'^{①,n}_{IBAR,j+1,k}}{\Delta y^2} && (16)\\
+&\frac{H'^{①,n}_{IBAR,j-1,k} - 2H'^{①,n}_{IBAR,j,k} + H'^{①,n}_{IBAR,j+1,k}}{\Delta y^2} && (17)\\
&= 0 && (18)
\end{aligned}
$$

Now, substituting the '$IBAR+1$' component based on the derivation in (14), we get

$$\frac{H'^{①,n}_{IBAR-1,j} - 2H'^{①,n}_{IBAR,j} + \left[H'^{①,n}_{IBAR,j} + \left(H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}\right) + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}\right)\right]}{\Delta x^2} + \frac{H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}}{\Delta y^2} = 0$$

Let's first move the previous '$n-1$' terms on the right hand side and sum up the obvious

$$\frac{H'^{①,n}_{IBAR-1,j} - H'^{①,n}_{IBAR,j} + \frac{\Delta x^2}{4\Delta y^2}\left(H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}\right)}{\Delta x^2} + \frac{H'^{①,n}_{IBAR,j-1} - 2H'^{①,n}_{IBAR,j} + H'^{①,n}_{IBAR,j+1}}{\Delta y^2} = -\frac{H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}}{\Delta x^2}$$

and then resort the whole stuff

$$H'^{①,n}_{IBAR,j-1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) + H'^{①,n}_{IBAR-1,j}\left(\frac{1}{\Delta x^2}\right) + H'^{①,n}_{IBAR,j}\left(-\frac{1}{\Delta x^2} - \frac{1}{2\Delta y^2} - \frac{2}{\Delta y^2}\right) + H'^{①,n}_{IBAR,j+1}\left(\frac{1}{4\Delta y^2} + \frac{1}{\Delta y^2}\right) = -\frac{H'^{②,n-1}_{1,j} - H'^{①,n-1}_{IBAR,j}}{\Delta x^2}$$

which finally gives the new matrix entries for the corresponding matrix line for cell $(IBAR, j)$ in mesh 1.

This is all done in extreme detail now, but I found it more secure than on paper, thanks to Copy&Paste. I really hope that not one million sign and component errors have crept in here. I'll focus on the code again now.

Have I understood your suggested procedure correctly by then or do you think that the above derivations are correct so far?

## 2.4 True (approximate) solution

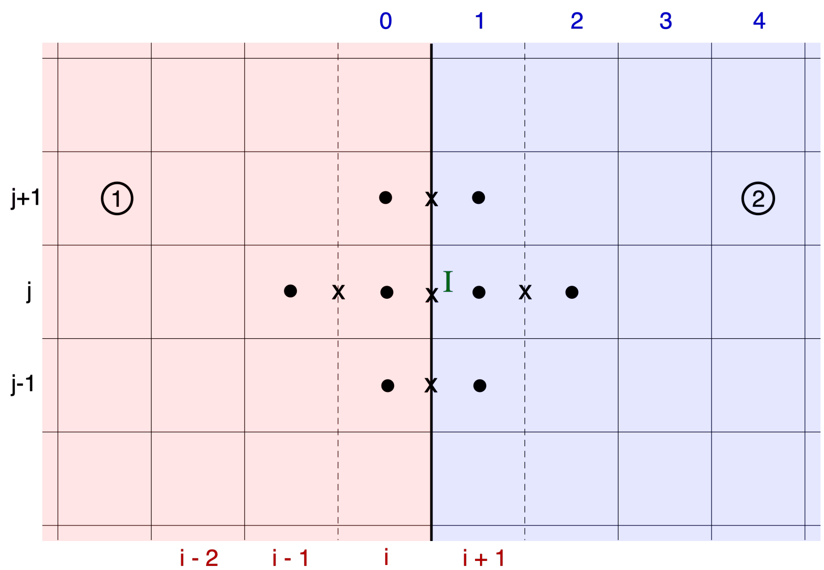


Figure 2: Interface between two adjacent meshes

Goal is to enforce $\nabla^2 H' = 0$ on $\partial M$

$$\frac{\overline{H'}_{I-1,j} - 2\overline{H'}_{I,j} + \overline{H'}_{I+1,j}}{\Delta x^2} + \frac{\overline{H'}_{I,j-1} - 2\overline{H'}_{I,j} + \overline{H'}_{I,j+1}}{\Delta y^2} = 0$$

This can be transformed to

$$\overline{H'}_{I,j} = \left[\frac{\overline{H'}_{I-1,j} + \overline{H'}_{I+1,j}}{\Delta x^2} + \frac{\overline{H'}_{I,j-1} + \overline{H'}_{I,j+1}}{\Delta y^2}\right] \Big/ \left[\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right]$$

where (in the simplest case with equidistant grid size of same resolution in every mesh)

$$\begin{aligned}
\overline{H'}_{I-1,j} &= \frac{1}{2}(H'^{①,n-1}_{i-1,j} + H'^{①,n-1}_{i,j}) \\
\overline{H'}_{I+1,j} &= \frac{1}{2}(H'^{②,n-1}_{1,j} + H'^{②,n-1}_{2,j}) \\
\overline{H'}_{I,j-1} &= \frac{1}{2}(H'^{①,n-1}_{i,j-1} + H'^{②,n-1}_{1,j-1}) \\
\overline{H'}_{I,j+1} &= \frac{1}{2}(H'^{①,n-1}_{i,j+1} + H'^{②,n-1}_{1,j+1})
\end{aligned} \tag{19}$$

## 2.5 Interface meets external boundary

Let's have a look to the case, where the interface meets an external boundary

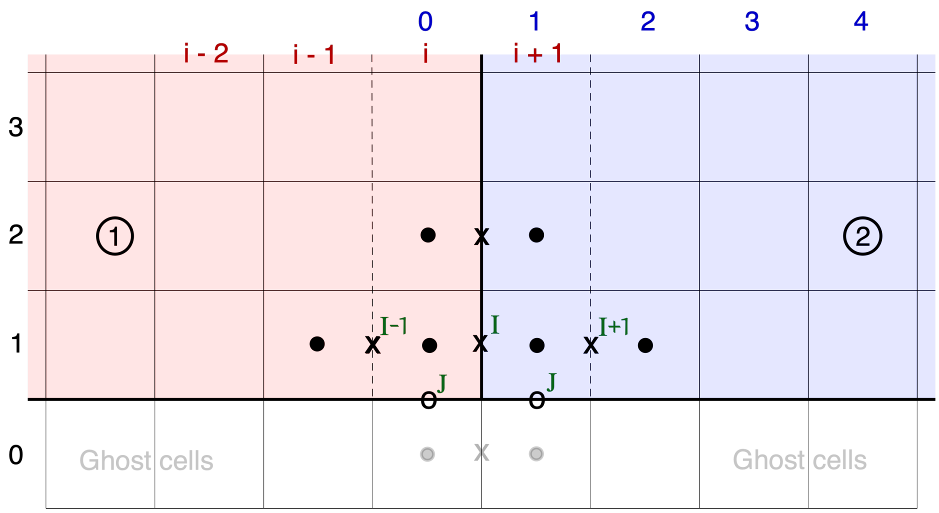


Figure 3: Interface meets an external boundary

The usual stencil in point $(I,1)$ looks like

$$\frac{\overline{H'}_{I-1,1} - 2\overline{H'}_{I,1} + \overline{H'}_{I+1,1}}{\Delta x^2} + \frac{\overline{H'}_{I,0} - 2\overline{H'}_{I,1} + \overline{H'}_{I,2}}{\Delta y^2} = 0$$

### 2.5.1 Dirichlet case

The Dirichlet boundary values in the lower $y$-direction are defined as

$$\overline{H'}_{I,0} = 2\overline{H'}_{I,J} - \overline{H'}_{I,1} \qquad \rightarrow \qquad \overline{H'}_{I,0} = -\overline{H'}_{I,1}$$

because $\overline{H'}_{I,J} = 0$ due to the homogeneous Laplace boundary. Thus, the stencil in $\overline{H'}_{I,1}$ looks like

$$\frac{\overline{H'}_{I-1,1} - 2\overline{H'}_{I,1} + \overline{H'}_{I+1,1}}{\Delta x^2} + \frac{-\overline{H'}_{I,1} - 2\overline{H'}_{I,1} + \overline{H'}_{I,2}}{\Delta y^2} = 0$$

which results in

$$\overline{H'}_{I,1} = \left[\frac{\overline{H'}_{I-1,1} + \overline{H'}_{I+1,1}}{\Delta x^2} + \frac{\overline{H'}_{I,2}}{\Delta y^2}\right] \Big/ \left[\frac{2}{\Delta x^2} + \frac{3}{\Delta y^2}\right]$$

### 2.5.2 Neumann case

The Neumann boundary values in the lower $y$-direction are defined as

$$\frac{\overline{H'}_{I,0} - \overline{H'}_{I,1}}{\Delta y} = 0 \qquad \rightarrow \qquad \overline{H'}_{I,0} = \overline{H'}_{I,1}$$

Thus, the stencil in $\overline{H'}_{I,1}$ looks like

$$\frac{\overline{H'}_{I-1,1} - 2\overline{H'}_{I,1} + \overline{H'}_{I+1,1}}{\Delta x^2} + \frac{\overline{H'}_{I,1} - 2\overline{H'}_{I,1} + \overline{H'}_{I,2}}{\Delta y^2} = 0$$

which results in

$$\overline{H'}_{I,1} = \left[ \frac{\overline{H'}_{I-1,1} + \overline{H'}_{I+1,1}}{\Delta x^2} + \frac{\overline{H'}_{I,2}}{\Delta y^2} \right] \Big/ \left[ \frac{2}{\Delta x^2} + \frac{1}{\Delta y^2} \right]$$

## 2.6 Multiple meshes meet

We still have to think of the treatment of cases where multiple meshes meet together. Please have a look at the node in the yellow mesh 4 which is not available by the usual WALL(IW) structures on mesh 1. Here, I think, we should use an alternative treatment or exchange the related values by a global data exchange (which would be possible, but would cost this exchange).

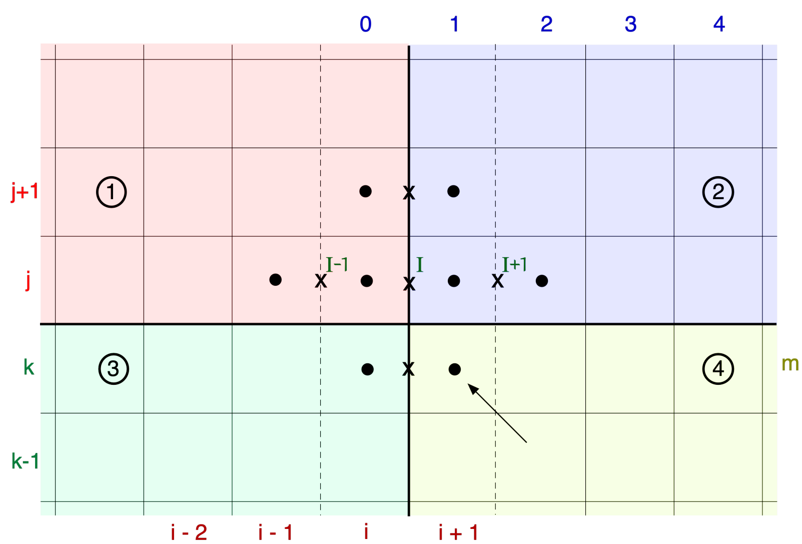


Figure 4: Interfaces in case of 4 adjacent meshes

$$\overline{H'}_{I,j} = \left[ \frac{\overline{H'}_{I-1,j} + \overline{H'}_{I+1,j}}{\Delta x^2} + \frac{\overline{H'}_{I,j-1} + \overline{H'}_{I,j+1}}{\Delta y^2} \right] \Big/ \left[ \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} \right]$$

with

$$\overline{H'}_{I-1,j} = \frac{1}{2}(H'^{①,n-1}_{i-1,j} + H'^{①,n-1}_{i,j})$$

$$\overline{H'}_{I+1,j} = \frac{1}{2}(H'^{②,n-1}_{1,j} + H'^{②,n-1}_{2,j})$$

$$\overline{H'}_{I,j+1} = \frac{1}{2}(H'^{①,n-1}_{i,j+1} + H'^{②,n-1}_{1,j+1})$$

$$\overline{H'}_{I,j-1} = \frac{1}{2}(H'^{③,n-1}_{i,k} + H'^{④,n-1}_{1,m})$$

(20)