## *Concept Review – chapter 1- LOGIC*

- $p \vee q$ (***disjunction*** of p and q): the proposition "p or q," which is true if and only if at least one of p and q is __(1)__.
- $p \rightarrow q$ (p implies q): the proposition "if p, then q," which is __(2)__ if and only if p is true and q is false.
- __(3)__ is a compound proposition that is ***always true***.
- $\neg q \rightarrow \neg p$ is logically __(4)__ to $p \rightarrow q$.
- $\forall x P(x)$ is true if and only if $P(x)$ is true for __(5)__ x in the domain.
- The proposition $p \wedge \neg q$ is __(6)__ $p \rightarrow q$.
- $\overline{p \wedge q} \equiv$ __(7)__

**Answer:** (1) true (2) false (3) tautology (4) equivalent (5) every/each/all (6) the negation of (7) $\overline{p} \vee \overline{q}$

### *Multiple choice questions*

1. Given the propositions:
   p = "you drive at more than 80km/h"
   q = "you get a speeding ticket"
   Express the ***sentence into a logical expression***.
   *"You get a speeding ticket if you drive at more than 80km/h"*

   a. $q \rightarrow p$
   b. $p \rightarrow q$
   c. $q \wedge p$
   d. $q \leftrightarrow p$

   **Answer:** $p \rightarrow q$
   **Comment:**
   - "You get a speeding ticket if you drive at more than 80km/h" $\Leftrightarrow$ q if p.
   - $p \rightarrow q$ means "q if p".

2. Given the propositions:
   p = "this computer program is correct" and
   q = "it does not produce error message during translation".
   Express the following ***sentence into a logical expression***:
   *"A necessary condition for this computer program to be correct is that it does not produce error message during translation".*

a. $q \rightarrow p$
b. $p \rightarrow q$
c. $q \wedge p$
d. $q \rightarrow p$

**Answer:** $p \rightarrow q$
**Comment:**
- Read and try to find words such as "necessary", "sufficient", "if", "only if", "if and only if", …
- $p \rightarrow q$ means "q is necessary for p".
- "A necessary condition for [A] is that [B]" means "[B] is necessary for [A]".

3. Which proposition is ***logically equivalent*** to $\neg p \rightarrow \neg(r \rightarrow \neg q)$?

a. $p \wedge q \wedge r$
b. $(p \vee r) \wedge q$
c. $p \vee r \vee q$
d. None of these

**Answer:** None of these
**Comment:**
- Keep in your mind $A \rightarrow B \equiv \neg A \vee B$
- $(r \rightarrow \neg q) \equiv \neg r \vee \neg q$
- $\neg(r \rightarrow \neg q) \equiv \neg(\neg r \vee \neg q) \equiv r \wedge q$ (De Morgan's law)
- $\neg p \rightarrow \neg(r \rightarrow \neg q) \equiv \neg p \rightarrow (r \wedge q) \equiv \neg(\neg p) \vee (r \wedge q) \equiv p \vee (r \wedge q)$.

4. Which of the following propositions is true? (x denotes a real number).

(i) $\forall x[(x \geq 1) \rightarrow (x^2 \geq 1)]$
(ii) $\forall x[(x \geq 1) \wedge (x^2 \geq 1)]$
(iii) $\forall x[(x \geq 1) \vee (x^2 \geq 1)]$

**Answer:** (i)
**Comment:**
- A counter example for (ii) is $x = -2$.
- A counter example for (iii) is $x = ½$.
- No counter example for (i).

5. Let T(x, y) = "the student x takes the class y", where x represents a student in a university, and y represents a class.
   Translate the ***logical expression into a sentence***:

   $$\forall x \exists y T(x,y)$$

   a. Some students take some classes.
   b. Every student takes at least one class.
   c. There are some classes that every student takes.
   d. Some students take every class.

   **Answer:** b
   **Comment:**
   - $\forall x$: every student/each student ➔ options (a), (d) are not true.
   - $\forall x \exists y T(x, y) \neq \exists y \forall x T(x, y)$.

6. Which of the following is the ***negation*** of the proposition:

   $$\exists x \forall y (M(x, y) \to N(x, y))?$$

   a. $\exists x \forall y \, (N(x, y) \to M(x, y))$
   b. $\exists x \forall y \, (\neg M(x, y) \to \neg N(x, y))$
   c. $\forall x \exists y \, (N(x, y) \wedge \neg M(x, y))$
   d. $\forall x \exists y \, (\neg N(x, y) \wedge M(x, y))$

   **Answer:** d
   **Comment:**
   - $\overline{\forall x P(x)} \equiv \exists x \overline{P(x)}$
   - $\overline{\exists x P(x)} \equiv \forall x \overline{P(x)}$
   - $\overline{p \to q} \equiv p \wedge \overline{q}$
   - $\overline{\exists x \forall y \left(M(x, y) \to N(x, y)\right)} \equiv \forall x \exists y \overline{\left(M(x, y) \to N(x, y)\right)}$
     $\equiv \forall x \exists y \left(M(x, y) \wedge \overline{N(x, y)}\right)$.

7. Given the premises:
   (1) If I eat spicy foods, then I have strange dreams.
   (2) I have strange dreams if there is thunder while I sleep.
   (3) I did not have strange dreams.
   Which ***conclusion*** can be drawn?

   a. I did not eat spicy foods but there was thunder while I slept.
   b. There was thunder while I slept.

c. I ate spicy foods.
d. I did not eat spicy foods and there wasn't thunder while I slept.

**Answer:** d
**Comment:**

- (1) e → d
- (2) t → d
- (3) ¬d
- Apply the rule below to (1)&(3) and to (2)&(3).

$$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$$

- We can conclude ¬e and ¬t.

8. Which proposition is ***equivalent*** to (¬p → q) ∧ (¬p → ¬q)?

(i) p
(ii) q
(iii) ¬p
(iv) F

**Answer:**
**Comment:**

- Keep in your mind A → B ≡ ¬A∨ B.
- ¬p → q ≡ ¬(¬p) ∨ q ≡ p ∨ q.
- ¬p → ¬q ≡ ¬(¬p) ∨ ¬q ≡ p ∨ ¬q.
- (¬p → q) ∧ (¬p → ¬q) ≡ (p ∨ q) ∧ (p ∨ ¬q) ≡ p ∨ (q ∧ ¬q) ≡ p ∨ (F) ≡ p.

9. Let L(x, y) be the statement "x is a friend of y", where the domain consists of all people. Translate the statement "Anybody except one person is a friend of Leo".

a. $\exists x \overline{L(x, \text{Leo})} \wedge (\forall y \neq x) L(y, \text{Leo})$

b. $\forall x \neq 1 (L(x, \text{Leo}))$

c. $\exists x \overline{L(x, \text{Leo})} \wedge (\exists y \neq x) L(y, \text{Leo})$

d. None of these.

**Answer:** a
**Comment:**

- Anybody (∀) except one person (∃) is a friend of Leo ➔ options (b) and (c) are not true.

10. Which of the following arguments is/are valid?
    (i) Any hardworking student is good in class. Tram is good in class. Therefore, Tram was working hard.
    (ii) Anyone in class understands logics. Tram is a student in class. Therefore, Tram understands logics.

    a. (i)
    b. (ii)
    c. None
    d. Both

    **Answer:** (ii)
    **Comment:**
    - $h \rightarrow g \not\equiv g \rightarrow h$.
    - If we know that P(x) is true for every x, then we can conclude that P(x) is true when x = c.

11. 
    (i) Any computer science major must take Discrete Mathematics. Anh is taking Discrete Mathematics. Therefore, Anh is a computer science major.
    (ii) Any student of FPT university lives in the dorm. Anh is living in a house. Therefore, Anh is not a student of FPT university.

    a. (i)
    b. (ii)
    c. None
    d. Both

    **Answer:** (ii)
    **Comment:**
    - $h \rightarrow g \not\equiv g \rightarrow h$.
    - (ii) is true due to the rule

$$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$$

# *Concept review – Chapter 2- SETS, FUNCTIONS, SUMS*

- $b \in A$ means b is a/an _(1)_ of the _(2)_ A.
- If every element of S is also an element of T, we say S is a _(3)_ of T.
- The _(4)_ set, denoted by $\varnothing$, is the set with _(5)_ elements.
- $A \cup B$: the _(6)_ of A and B.
- $A \cap B$: the _(7)_ of A and B.
- $A - B$ (the ***difference*** of A and B): the set containing elements that are in A _(8)_ in B.
- $A \oplus B$ (the ***symmetric difference*** of A and B): the set containing elements in _(9)_ A and B.
- A function from A to B maps _(10)_ element of A to _(11)_ element of B.
- If a function is both onto and one-to-one, it is called a _(12)_.
- If a function is a bijection, it has a/an _(13)_ function. Then, it is called an invertible function.

  **Answer:** (1) element/member (2) set (3) subset (4) null/empty (5) no (6) union (7) intersection (8) but not (9) exactly one of (10) each (11) exactly one (12) bijection (13) inverse

**Some multiple choice questions**

1. Let U = {a, b, c, d, e, f, g, h} be the universal set. The bit string representing the set {a, c, d, f, g} is _____.

   a. 01001001
   b. 11111100
   c. 111111
   d. 10110110

   **Answer:** d
   **Comment:**
   - U has 8 elements ➔ a bit string for a set has length 8 ➔ option (c) is not true.
   - Bit string for A:

| U | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| A | a | | c | d | | f | g | |
| Strings | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |

2. Let A = {1, 2, 3, 4, 6}, and B = {3, 4, 7} be two sets in the universal set {1, 2, 3, 4, 5, 6, 7}. Find the bit string representing the set $A \oplus B$.

a. 11111111
b. 1100111
c. 1111111
d. 1100011

**Answer:** b
**Comment:**

- U has 7 elements ➔ a bit string for a set has length 7 ➔ option (a) is not true.
- $A \oplus B = (A \cup B) - (A \cap B)$ = {elements which are in A or in B, not both}.
- $A \oplus B = \{1, 2, 6, 7\}$.

3. Let R be the set $\{(a, b) \mid a - 1 = b\}$, where a and b are in $\{-2, -1, 0, 1, 2\}$. How many elements does R have? (or in other words, what is the ***cardinality*** of R?).

a. 4
b. 3
c. 2
d. 1

**Answer:** 4
**Comment:**

- $R = \{(-1, -2), (0, -1), (1, 0), (2, 1)\}$.
- Cardinality = number of elements of R.

4. The set A has n elements, B has n+1 elements. Assume that $A \times B$ has 42 elements. Find n.

a. 6
b. 7
c. 21
d. 6 or -7
e. None of these

**Answer:** 6
**Comment:**

- $A \times B$ has $n(n+1) = 42$ elements ➔ n = 6.
- Option (d) is impossible.

5. Given the ***sequence*** $\{a_n, n = 1, 2, \ldots\}$: $a_n = 1 + 3 + \ldots + (2n-1)$.
   Find $a_{2019} - a_{2018}$.

   a. 4037
   b. 2019
   c. 2018
   d. 4038
   e. None of these

   **Answer:** 4037

   **Comment:**

   - $a_1 = 1$, $a_2 = 1 + (2\cdot1 - 1)$, $a_3 = 1 + 3 + (2\cdot3 - 1)$, etc.
   - $a_n$ = the sum of n first odd positive integers
   - $a_{2019} = 1 + 3 + \ldots + (2\cdot2018 - 1) + (2\cdot2019 - 1)$
   - $a_{2018} = 1 + 3 + \ldots + (2\cdot2018 - 1)$

6. Given the ***sequence*** $\{a_n, n = 1, 2, \ldots\}$ where $a_n$ equals the number of 1 bits in the ***binary format*** of n. Find $a_{11}$.

   a. 0
   b. 1
   c. 2
   d. 3
   e. 4

   **Answer:** 3,

   **Comment:**

   - $11 = (1011)_2$ ➔ three 1-bits ➔ a11 = 3.

7. Which statement is FALSE?
   a. $A \oplus \varnothing = A$
   b. $A \oplus A = A$
   c. $A \oplus \bar{A} = U$, where U is the universal set.

   **Answer:** $A \oplus A = A$

   **Comment:**

   - $A \oplus A = \varnothing$

8. Which rules are functions?

   a. $f: Z \to R, f(x) = 1/(x^2 - 3)$
   b. $f: Z \to Z, f(x) = 1/(x^2 - 3)$
   c. $f: Z \to Z, f(x) = 1/(x^2 - 4)$
   d. $f: Z \to R, f(x) = 1/(x^2 - 4)$

   **Answer:** a
   **Comment:**
   - Option (b): f(1) = -1/3 is not in Z.
   - Option (c), (d): f(2) = 1/0 (!), undefined.

9. Let N = {0, 1, 2, …} and f: N → N be the function $f(x) = x^2 - 2x + 1$.
   Which statement is true?
   (i) f is onto
   (ii) f is one-to-one.

   a. (i)
   b. (ii)
   c. None
   d. Both

   **Answer:** d
   **Comment:**
   - $f(x) = x^2 - 2x + 1 = (x - 1)^2$
   - f(0) = f(2) = 1 ➔ not 1-1.
   - No x such that f(x) = 3 ➔ f is not onto.

10. Let f be the function defined by f(x) = **floor**($\frac{x^2-3}{3}$).
    Find f(1), f(2) and f(5) (or f({1, 2, 5})).

    a. 0, 0, 7
    b. -1, 0, 7
    c. 0, 1, 8
    d. -1, 0, 8
    e. None of these.

    **Answer:** -1, 0, 7

**Comment:**

- f(1) = floor(-2/3) = -1.
- f(2) = floor(1/3) = 0.
- f(5) = floor(22/3) = 7.

11. Find the composite function (fog)(x) = f(g(x)), where f(x) = x + 2018 and g(x) = $x^{2018}$.

a. $(x+2018)^{2018}$
b. $x^{2018} + 2018$
c. $x^{2018}(x+2018)$
d. None of these

**Answer:** $x^{2018} + 2018$

**Comment:**

- g(x) = $x^{2018}$.
- (fog)(x) = f(g(x)) = f($x^{2018}$).
- f(x) = x + 2018 ➔ f($x^{2018}$) = $x^{2018}$ + 2018.

12. Given A = {1, 2}. Find A - P(A). (Recall that P(A) is the ***power set*** of A).

a. ∅
b. A
c. { ∅ }
d. None of these.

**Answer:** $x^{2018} + 2018$

**Comment:**

- A = {1, 2}
- Subsets of A: ∅, {1}, {2}, {1, 2}
- P(A) = { ∅, {1}, {2}, {1, 2} }
- A – P(A) = {1, 2}

13. Given A = {1, 2}, find A∩P(A).

a. A
**b. ∅**
c. { ∅ }
d. None of these

**Answer:** $x^{2018} + 2018$

**Comment:**

- A = {1, 2}
- Subsets of A: ∅, {1}, {2}, {1, 2}
- P(A) = { ∅, {1}, {2}, {1, 2} }
- A∩P(A) = { }.

14. How many elements does the set P({1, 2, 3, 4, 5, 6, 7, 8}) have?
    a. 8
    b. 16
    c. 256
    d. None of these

**Answer:** 256

**Comment:**

- P(A) has $2^n$ elements if A has n elements.

15. Compute $\sum_{i=0}^{2}\sum_{j=1}^{3}(i-j)$
    a. -1
    b. -3
    c. -9
    d. -11
    e. None of these.

**Answer:** -9

**Comment:**

- $\underbrace{(0-1)+(0-2)+(0-3)}_{i=0,\text{ j from 1 to 3}}+\underbrace{(1-1)+(1-2)+(1-3)}_{i=1,\text{ j from 1 to 3}}+\underbrace{(2-1)+(2-2)+(2-3)}_{i=2,\text{ j from 1 to 3}}=-9$

16. Find the double sum $\sum_{i=1}^{10}\sum_{j=1}^{20}(ij)$.
    a. 200
    b. 231
    c. 11550
    d. 46200
    e. None of these.

**Answer:** 11550

**Comment:**

- $$\sum_{i=1}^{10}\sum_{j=1}^{20}(ij)=\sum_{i=1}^{10}\left[i\sum_{j=1}^{20} j\right]=\sum_{i=1}^{10}\left[i(1+2+\ldots+20)\right]=\sum_{i=1}^{10}\left[i\left(\frac{20\cdot 21}{2}\right)\right]$$

  $$=210\sum_{i=1}^{10} i=210(1+2+\ldots+10)=210\cdot\frac{10\cdot 11}{2}=11550\,.$$

*Concept Review – Chapter 3 – ALGORITHMS & INTEGERS*

- f(x) is <u>_(1)_</u> if f(x) is both O(g(x)) and Ω(g(x))
- The ***linear search*** has <u>_(2)_</u> worst case time complexity.
- The ***binary search*** has <u>_(3)_</u> worst case time complexity.
- The ***bubble*** and ***insertion sorts*** have <u>_(4)_</u> worst case time complexity.
- If there is an integer c such that ac = b, we say a <u>_(5)_</u> b and write <u>_(6)_</u>.
- If a mod m = b mod m, we say a and b are <u>_(7)_</u> modulo m.
- **gcd(a, b)**: <u>_(8)_</u> of two integers a, b.
- **lcm(a, b)**: <u>_(9)_</u> of two integers a and b.
- If a and b are positive integers, then gcd(a, b)·lcm(a, b) = <u>_(10)_</u>.
- If gcd(a, b) = 1, then a and b are called <u>_(11)_</u>.
- If the function f(p) = (p + 13) mod 26 is used to ***encrypt*** a message, then <u>_(12)_</u> is used to ***decrypt*** this encoded message.
- If a and b are integers and r = a mod b, then gcd(a, b) = <u>_(13)_</u>.

  **<u>Answer:</u>** (1) Θ(g(x)) (2) O(n) (3) O(logn) (4) $O(n^2)$ (5) divides (6) a | b (7) congruent (8) greatest common divisor (9) least common multiple (10) ab (11) relatively prime (12) $f^{-1}(p) = (p – 13) \bmod 26$ (13) gcd(b, r)

***Some multiple choice questions***

1. Given $f(x) = x^2$ and g(x) = 3x + 2018. Consider the statements:

   (i) f(x) is O(g(x))

   (ii) g(x) is O(f(x))

   Which statement is TRUE?

   a. (i)
   b. (ii)
   c. Both
   d. None

   **<u>Answer:</u>** (ii)

   **<u>Comment:</u>**

   - $g(x) = 3x + 2018 \sim x$ is $O(x^2)$

2. Find the least integer k such that $\frac{(\sqrt{x^8+x^4+1}\ +1)(logx+3)}{x^2+1}$ is $O(x^k)$.

   a. 1
   b. 2
   c. 3
   d. 4
   e. None of these

**Answer:** 3

**Comment:**

- $\frac{(\sqrt{x^8+x^4+1}+1)(logx+3)}{x^2+1} \sim \frac{(\sqrt{x^8})(logx)}{x^2} = x^2logx$.
- To easily estimate O-notation, keep in your mind the order: 1 < logx < x < xlogx < $x^2$ < $x^2$logx < $x^3$ < … < $2^x$ < $3^x$ … (for x large enough)

3. Which function is Ω(xlogx)?
   a. 2018x
   b. $x^2$/2018
   c. $2018^{2018}$
   d. None of the others

   **Answer:** $x^2$/2018

   **Comment:**

   - $x^2$/2018 ~ $x^2$ is Ω(xlogx).

4. Given f(x) = $3x^2$ + 12x + 2 and g(x) = $x^3$
   Which statement is True?
   a. f(x) is O(g(x))
   b. f(x) is Ω(g(x))
   c. f(x) is Θ(g(x))
   d. None of these

   **Answer:** f(x) is O(g(x))

   **Comment:**

   - f(x) = $3x^2$ + 12x + 2 ~ $x^2$
   - g(x) = $x^3$

5. Suppose a mod 4 = 3 and b mod 8 = 7, find ab ***mod*** 4.
   a. 1
   b. 2
   c. 3
   d. 21
   e. 9

   **Answer:** 1

   **Comment:**

   - b = 8k + 7 ➔ b = 4(2k) + 4 + 3 ➔ b mod 4 = 3
   - ab mod 4 = ((a mod 4)(b mod 4)) mod 4 = (3·3) mod 4 = 9 mod 4 = 1.

6. Which of the following integers are ***congruent to 13 modulo 7***?
   a. -6
   b. -1
   c. 1

d. 7
e. None of these

**Answer:** -1

**Comment:**

- x ***congruent to 13 modulo 7*** $\Leftrightarrow$ $x \equiv 13 \pmod 7$
- $a \equiv b \pmod m \Leftrightarrow m \mid (a - b) \Leftrightarrow (a - b) \mid\mid m$
- $-6 - 13 = -19 \nmid 7$ ➔ option (a) is not true.
- $-1 - 13 = -14 \mid\mid 7$ ➔ option (b) is true.

7. Consider the ***algorithm:***

*procedure giaithuat($a_1$, $a_2$, …, $a_n$ : integers)*
*count:= 0*
*for i:= i to n do*
    *if $a_i > 0$ then count: = count + 1*
*print(count)*

Give the **best big-O complexity** for the algorithm above.

a. O(n)
b. O(logn)
c. O(1)
d. O($n^2$)
e. None of these

**Answer:** O(n)

**Comment:**

- Pay attention to “for loops” (or while loops).
- No “other loops” (for or while) inside the “for loop”.

8. Consider the algorithm:

procedure GT($a_1$, $a_2$, …, $a_n$ : integers)
count:= 0
for i:= 2 to n do
    for j:= 1 to i-1 do
        if $a_i = a_j$ then count = count +1
print(count)

Give the **best big-O complexity** for the algorithm above.

a. O(n)
b. O(logn)
c. O(1)
d. O($n^2$)
e. None of these

**Answer:** O($n^2$)

**Comment:**

- Pay attention to “for loops” (or while loops).

- There is one “for loop” (via index j) inside the first for loop (via index i).

9. Find the ***base 7 expansion*** of 186
   a. 354
   b. 331
   c. 413
   d. 271
   e. None of these

   **Answer:** 354

   **Comment:**
   - $(186) = (a_3a_2a_1a_0)_7$
   - $a_0$ = 186 mod 7 = 4 ➔ options b, c and d are not true.
   - 186 div 7 = 26
   - $a_1$ = 26 mod 7 = 5
   - 26 div 7 = 3
   - $a_2$ = 3 mod 7 = 3
   - 3 div 7 = 0 ➔ stop

10. Find the ***binary format*** of $(2010)_3$.
   a. 11110
   b. 111010
   c. 100111
   d. 11100
   e. None of these

   **Answer:** 111010

   **Comment:**
   - $(2010)_3 = 58 = (111010)_2$
   - $(2011)_3 = 2 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 1 \cdot 3^0 = 58$.

11. Suppose ***pseudo-random numbers*** are produced by using:
   $x_{n+1} = (3x_n + 5) \bmod 7$.
   If $x_3 = 5$, find $x_2$ and $x_4$.
   a. $x_4 = 6, x_2 = 3$
   b. $x_4 = 6, x_2 = 0$
   c. $x_4 = 9, x_2 = 3$
   d. $x_4 = 3, x_2 = 5$
   e. None of these

   **Answer:** $x_4 = 6, x_2 = 0$

   **Comment:**
   - You can test to find possible options.
   - $x_{n+1} = (3x_n + 5) \bmod 7$.

- Assume $x_2 = 3$, it follows that $x_3 = (3x_2 + 5) \bmod 7 = (3 \cdot 3+5) \bmod 7 = 14 \bmod 7 = 0$ ➔ options a, c are not true.
- $x_3 = 5$ ➔ $x_4 = (3x_3 + 5) \bmod 7 = 20 \bmod 7 = 6$ ➔ option d is not true.

12. Use the function f(x) = (x + 17) mod 26 to ***encrypt*** the message CV.

    a. CM
    b. TM
    c. FA
    d. TV
    e. None of these

    **Answer:** TM
    **Comment:**
    - A = 0, B = 1, C = 2, …, V = 21, …, Z = 25.
    - f(x) = (x + 17) mod 26
    - f(C) = f(2) = 19 = T
    - f(V) = f(21) = (21 + 17) mod 26 = 12 = M.

13. Suppose the password for a quiz ***has been encoded*** using the function f(p) = (p + 13) mod 26.
    If the result is SE, find the quiz password.
    a. GP
    b. FR
    c. HR
    d. DM
    e. None of these
    **Answer:** FR
    **Comment:**
    - f(p) = (p + 13) were used to encrypt the quiz password.
    - We use $f^{-1}(p) = (p – 13) \bmod 26$ to decrypt “SA” to obtain the password.
    - SE = 18 4
    - $f^{-1}(18) = (18 – 13) \bmod 26 = 5$ = F
    - $f^{-1}(4) = (4 – 13) \bmod 26 = -9 \bmod 26 = 17$ = R.

14. Which pair of integers are ***relatively prime***?
    a. (17, 51)
    b. (5, 24)
    c. (11, 121)
    d. (37, 111)

e. None of the others

**Answer:** (5, 24)

**Comment:**

- Two integers a and b are called relatively prime iff gcd(a, b) = 1.
- gcd(17, 51) = 3.
- gcd(11, 121) = 11.
- gcd(37, 111) = 37.
- gcd(5, 24) = 1.

15. If a, b are positive integers such that gcd(a, b) = 37 and ab = 111111, find lcm(a, b).

a. 4111107
b. 3003
c. 37
d. 1
e. None of the others

**Answer:**

**Comment:**

- gcd(a, b)·lcm(a, b) = ab ➔ lcm(a, b) = ab/gcd(a, b) = 111111/37 = 3003.

16. Consider the ***bubble sort algorithm***:

bubblesort($a_1$, $a_2$, ..., $a_n$: integers)
for i:= 1 to n-1
    for j:= 1 to n-i
        if $a_{j+1} < a_j$ then swap($a_j$, $a_{j+1}$)

If the input is [3, 1, 4, 5, 2], what is the result after the second pass (i = 2)?

a. [1, 2, 3, 4, 5]
b. [1, 3, 2, 4, 5]
c. [2, 1, 3, 4, 5]
d. [1, 3, 2, 5, 4]
e. None of these

**Answer:** [1, 3, 2, 4, 5]

**Comment:**

- The initial list is [3, 1, 4, 5, 2].
- Some changes when i = 1: [3, 1, 4, 5, 2] (j = 1)➔ [1, 3, 4, 5, 2] (j = 2)➔ [1, 3, 4, 5, 2] (j = 3)➔ [1, 3, 4, 5, 2] (j = 4)➔ [1, 3, 4, 2, 5].
    - After the 1st pass (i = 1), we have the list [1, 3, 4, 2, 5].
- Some changes when i = 2: [1, 3, 4, 2, 5] (j = 1)➔ [1, 3, 4, 2, 5] (j = 2)➔ [1, 3, 4, 2, 5] (j = 3)➔ [1, 3, 2, 4, 5].

- After the 2nd pass ($i = 2$), we obtain the list [1, 3, 2, 4, 5].

17. Consider the ***Linear search*** algorithm:

procedure linear_search($x$: integer, $a_1, a_2, \ldots, a_n$: distinct integers)
$i := 1$
while ($i \leq n$ and $x \neq a_i$ )
    $i := i + 1$
if $i \leq n$ then location := $i$
else location := 0
return location

Given the sequence $a_n$: 3, 1, 5, 7, 4, 6
How many comparisons ($\leq$ or $\neq$) required to locate the position of 5?

a. 7
b. 9
c. 11
d. 13
e. None of these

**Answer:** 7
**Comment:**

- $n = 6$, $x = 5$.
- $i := 1$ ➔ ($1 \leq 6$ and $5 \neq 3$ ) is true ➔ $i: = i + 1 = 2$.
- $i := 2$ ➔ ($2 \leq 6$ and $5 \neq 1$ ) is true ➔ $i: = i + 1 = 3$.
- $i := 3$ ➔ ($3 \leq 6$ and $5 \neq 5$ ) is FALSE ➔ exit the while loop
- $i = 3 \leq 6$ ➔ location = 3.
- Conclusion: 7 comparisons ($\leq$, $\neq$).

# *Chapter 4 – Induction & Recursion*

### *Review questions*

1. Find a ***recursive definition*** for $a_n = 1 + (-1)^n$, n = 0, 1, 2, ...

   a. $a_0 = 2$ and $a_n = a_{n-2}$, for n > 0.
   b. $a_0 = 2$, $a_1 = 0$ and $a_n = a_{n-2}$, for n > 1.
   c. $a_0 = 0$, $a_1 = 2$ and $a_n = a_{n-2}$, for n > 1.

   **Answer:** b

   **Comment:**

   - $a_n = 1 + (-1)^n$ ➔ $a_0 = 1 + (-1)^0 = 2$ ➔ option (c) is not true.
   - From the definition given by option a, we cannot calculate $a_1$ ➔ option (a) is not true.
   - Note that $a_0 = 2$, $a_1 = 0$, $a_2 = 2 = a_0$, $a_3 = 0 = a_1$, $a_4 = 2 = a_2$, etc. ➔ $a_n = a_{n-2}$.

2. Find a ***recursive definition*** for the set of positive integers NOT divisible by 3.
   a. $1 \in S$,
      and $x \in S \rightarrow x+1 \in S$, $x+2 \in S$.
   b. $1 \in S$,
      and $x \in S \rightarrow x+3 \in S$.
   c. $1 \in S$, $2 \in S$,
      and $x \in S \rightarrow x+3 \in S$.

   **Answer:** c

   **Comment:**

   - Some positive integers not divisible by 3: 1, 2, 4, 5, 7, 8, …
   - From the definition given by option (a): $x \in S \rightarrow x+2 \in S$ ➔ $3 \in S$ (!) ➔ option (a) is not true.
   - We cannot generate the number 2 by applying the definition in option (b) ➔ option (b) is not true.

3. Study the set S of bit strings ***defined recursively*** by:
      String 1 belongs to S
      String 11x belongs to S if string x belongs to S.
   Which statement is true?
   (i) $11111 \in S$

(ii) $111111 \in S$

a. (i)
b. (ii)
c. Both
d. None

**Answer:** (i)

**Comment:**

- $x \in S \xrightarrow{\text{definition}} 11x \in S$. This means, add “11” before the string x you have.
- $1 \in S \xrightarrow{\text{definition}} 111 \in S \xrightarrow{\text{definition}} 11111 \in S \xrightarrow{\text{definition}} 1111111 \in S \xrightarrow{\text{definition}} \ldots$

4. Let f(n) = f(n/3) + 2 and f(1) = 3, where n is divisible by 3.
   Find f(27).
   a. 9
   b. 7
   c. 5
   d. 6
   e. None of these.

   **Answer:** 9

   **Comment:**

   - f(n) = f(n/3) + 2.
   - f(3) = f(3/3) + 2 = f(1) + 2 = 5.
   - f(9) = f(9/3) + 2 = f(3) + 2 = 7.
   - f(27) = f(27/3) + 2 = f(9) + 2 = 9.

5. Find a ***recursive definition*** for the sequence f(n) = n, for n = 1, 2, 3, …
   a. f(1) = 1, f(n) = n+ f(n-1) for n>1.
   b. f(n) = f(n-1) +1 for all $n \geq 1$.
   c. f(1) = 1 and f(n) = f(n-1) +1 for all n > 1.
   d. f(1) = 1, f(n) = n for all n>1.
   e. None of the others.

   **Answer:** c

   **Comment:**

   - Option (d), f(n) = n is not a recursive definition ➔ not true.
   - Option (b), no basis step, no one knows the value of f(1) ➔ option (b) is not true.

- Option (a) ➔ f(2) = 2 + f(1) = 3. But we want to define f(n) = n, the one that f(2) = 2 ➔ option (a) is not true.
- f(n) = n ➔ f(n-1) = n – 1 ➔ f(n) = f(n-1) + 1.

6. Consider the ***recursive algorithm***:
   procedure *giaithuat*(n : positive integer, a: real number)
   if n = 1 then *giaithuat*(n,a): = a
   else *giaithuat*(n, a) = *giaithuat*(n-1, a) + a
   What is the output if n = 4, a = 2.5?

   a. 8
   b. 16
   c. 10
   d. None of these.

   **Answer:** 10

   **Comment:**

   - *giaithuat*(1, 2.5) = 2.5 ( the case with n = 1)
   - *giaithuat*(n, a) = *giaithuat*(n-1, a) + a if n > 1
   - *giaithuat*(2, 2.5) = *giaithuat*(1, 2.5) + 2.5 = 2.5 + 2.5 = 5
   - *giaithuat*(3, 2.5) = *giaithuat*(2, 2.5) + 2.5 = 5 + 2.5 = 7.5
   - *giaithuat*(4, 2.5) = *giaithuat*(3, 2.5) + 2.5 = 7.5 + 2.5 = 10.

7. Consider the set A of bit strings ***defined recursively*** by

   1∈A

   if x∈A, then x11∈A

Which of the following strings is in A?

a. The empty string λ, the string with no symbols.
b. String 11
c. String 111
d. String 1111

   **Answer:** string 111

   **Comment:**

   - x∈A → x11∈A.

- $1 \in A \rightarrow 111 \in A$
- $111 \in A \rightarrow 11111 \in A$
- etc.

8. To prove the statement " 6 divides $n^3$ - n for all integers $n \geq 0$", the ***mathematical induction*** method is used as the following:
   (1) The statement is true for n = 0.
   (2) Suppose _____, the statement is true, that is, "6 divides $k^3 - k$".
   (3) We have, $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k + 1) = k^3 - k + 3(k^2 + k)$.

   As 6 divides $k^3 - k$ (assumption, step 2) and $3(k^2 + k)$ is a multiple of 6, we conclude that $(k+1)^3 - (k+1)$ is also a multiple of 6.

   (4) By induction, 6 divides $n^3 - n$ for all integers $n \geq 0$.

   Fill in the blank at step (2).

   a. there exist an integer $k \geq 0$.
   b. for every integer $k \geq 0$.
   c. there are some integers $k \geq 0$.

   **Answer:** for every integer $k \geq 0$

9. which statements are true? (n is any positive integer).

   (i) $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = n^3$

   (ii) $1! + 2! + ... + n! = (n+1)! - 1$

   a. (i)
   b. (ii)
   c. None
   d. Both

   **Answer:** None

   **Comment:**

   - Examine (i) with n = 2: $1^2 + (2\cdot2-1)^2 = 2^3 \Leftrightarrow 10 = 8$ (!) ➔ option (i) is not true.
   - Examine (ii) with n = 3: $1! + 2! + 3! = (3+1)! - 1 \Leftrightarrow 11 = 23$ (!) ➔ option (ii) is not true.

9. Find f (2018) if f(n) = -f(n - 3) and f(0) = 1, f(1) = 4, f(2) = 6.

   a. 1

b. 4
c. 6
d. -1
e. -4
f. -6

**Answer:** 6

**Comment:**

- $f(n) = -f(n - 3)$
- $f(n-3) = -f(n-6)$
- $f(n) = -f(n - 3) = -(-f(n-6)) = f(n-6)$
- $f(2018) = f(2018 – 6) = f(2012)$
- $f(2012) = f(2012 – 6) = f(2006)$
- $f(2006) = f(2000) = \ldots = f(8) = f(2) = 6$.
- Note that 2018 mod 6 = 2.

10. Give a ***recursive definition*** of the set $A = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$.
Which of the following definitions is/are true?
(i) $2 \in A$; if $x \in A$ then $x+3 \in A$ or $x – 3 \in A$.
(ii) $-1 \in A$; if $x \in A$ then $x + 3 \in A$ or $x – 3 \in A$.

a. (i)
b. (ii)
c. None
d. Both

**Answer:** Both

**Comment:**

- Start with basic element (2 or -1) other elements can be generated by adding or subtracting by 3.

# Concept Review – Chapter 5-7. COUNTING

1. How many different ***functions*** from {a, {a}, b, {a, b}} to {x, y, {z}, y}?
   a. $3^4$
   b. $4^3$
   c. 12
   d. Only one
   e. $4^4$

   **Answer:** $3^4$

   **Comment:**
   - {a, {a}, b, {a, b}} has 4 elements.
   - {x, y, {z}, y} has 3 elements.

2. How many different ***one-to-one functions*** from {a, b, c} to {00, 01, 10, 11}?
   a. $3^4$
   b. $4^3$
   c. $4 \cdot 3 \cdot 2$
   d. Only one

   **Answer:** $4 \cdot 3 \cdot 2$

   **Comment:**
   - 3 steps to set up a function: (1) choose output for a, (2) choose output for b, (3) choose output for c.
   - "One-to-one" means the output values are different.
   - Use product rule to count ➔ $4 \cdot 3 \cdot 2$ one-to-one functions.

3. Suppose that a "word" is any string of **seven letters** of the alphabet, with repeated letters allowed.
   How many words begin with A **or** B **and** end with A **or** B?
   a. $2.26^5$
   **b. $4.26^5$**
   c. $26^6$
   d. None of these
4. Suppose that a "word" is any string of **seven letters** of the alphabet, with repeated letters allowed. How many words begin with a vowel **or** end with a vowel?
   a. $5 \cdot 26^6 + 5 \cdot 26^6 - 25 \cdot 26^5$
   b. $5 \cdot 5 \cdot 26^5$
   c. $2 \cdot 5 \cdot 26^5$
   d. $5 \cdot 26^6 + 5 \cdot 26^6$

**Answer:** $5 \cdot 26^6 + 5 \cdot 26^6 - 25 \cdot 26^5$

**Comment:**

- Case 1: Vowel * * * * * * ➔ $5 \cdot 26^6$ words.
- Case 2: ******vowel ➔ $5 \cdot 26^6$ words.
- Case 3: vowel*****vowel ➔ $5 \cdot 26^5 \cdot 5$ words.
- Note that case 1 includes case 3 and case 2 also includes case 3.
- Use inclusion-exclusion principle: case 1 + case 2 – case 3.

5. A club with 20 women and 17 men needs to choose three different members to be president, vice president, and treasurer. In how many ways is this possible if women will be chosen as president and vice president and a man as treasurer?
   a. $20 \cdot 19 \cdot 17$
   b. $37 \cdot 36 \cdot 35$
   c. $20 \cdot 17 \cdot 16$
   d. None of these

   **Answer:** $20 \cdot 19 \cdot 17$

   **Comment:**

   - Three steps: (1) choose a woman as a president, (2) choose a woman as a vice president, (3) choose a man as a treasurer.
   - Use product rule ➔ $20 \cdot 19 \cdot 17$ ways.

6. Find the number of **subsets** of {1, 2, 3, 4, 5, 6} that contain 3.
   a. $2^6$
   b. $2^5$
   c. 5
   d. None of these

   **Answer:** $2^5$

   **Comment:**

   - Six steps: (step 1) 2 ways (Y/N) to consider number 1, (step 2) 2 ways (Y/N) to consider number 2, (step 3) one way (Y) to consider number 3, …, (step 6) 2 ways (Y/N) to consider number 6.
   - Use product rule ➔ $2^5$ ways to choose a subset of {1, 2, 3, 4, 5, 6} that contain 3.

7. Suppose a restaurant serves a "special dinner" consisting of soup, salad, entree, dessert, and beverage. The restaurant has five kinds of soup, three kinds of salad, ten entrees, five desserts, and four beverages. How many different special dinners are possible? (Two special dinners are different if they differ in at least one selection.)
   a. $5 \cdot 3 \cdot 10 \cdot 5 \cdot 4$
   b. $5 + 3 + 10 + 5 + 4$

c. $2^5$
d. None of these
**Answer:** 5·3·10·5·4
**Comment:**
- Five steps to serve a "special dinner": step 1 - soup (5 kinds), step 2 – salad (3 kinds), step 3 – entrée (10 entrées), step 4 – dessert (5 desserts), step 5 – beverage (5 beverages).
- Use product rule ➔ 5·3·10·5·4 ways to serve.

8. A professor teaching a Discrete Math course gives a multiple choice quiz that has ten questions, each with four possible responses: a, b, c, d. How many ways for a student to complete the test? (Assume that no answers are left blank.)
a. 40
b. $10^4$
c. $4^{10}$
d. None of these
**Answer:** $4^{10}$
**Comment:**
- Ten steps: step 1 – answer the 1st question (4 ways), step 2 – answer the 2nd question (4 ways), …, step 10 – answer the 10th question (4 ways).
- Use product rule ➔ 410 ways complete the test.

9. How many integers between 100 and 1000 inclusive that are **divisible** by 7 or 13?
a. 208
b. 189
c. 188
d. None of these

**Answer:** 189
**Comment:**
- Case 1: integers in [100, 1000] that are divisible by 7 (128 such integers).
- Case 2: integers in [100, 1000] that are divisible by 13 (69 such integers).
- Case 3: integers in [100, 1000] that are divisible by both 7 and 13 (8 such integers).
- Case 1 includes case 3 and case 2 also includes case 3.
- Use **inclusion-exclusion principle** (case 1 + case 2 – case 3) ➔ Result = 128 + 69 – 8.

10. Suppose $f(n) = 4f(n/2)$ if $n$ is divisible by 2, and $f(1) = 2$. Find $f(8)$.
a. 226
b. 128

c. 256
d. 64
e. None of these

**Answer:** 128

**Comment:**

- $f(n) = 4f(n/2)$
- $f(2) = 4f(2/2) = 4f(1) = 8$
- $f(4) = 4f(4/2) = 4f(2) = 32$
- $f(8) = 4f(8/2) = 4f(4) = 128$
- f is defined recursively.

11. You take a job that pays $55,000 annually. How much do you earn *15* years from now if you receive a three percent raise each year?

a. $55{,}000 \cdot 1.03^{15}$
b. $55{,}000 \cdot 0.03^{15}$
c. $55{,}000 \cdot 1.13^{15}$
d. $55{,}000^{1.03}$
e. None of these

**Answer:** $55{,}000 \cdot 1.03^{15}$

**Comment:**

- Use the model $Pn = P_{n-1} + rP_{n-1}$ or $P_n = (1+r)P_{n-1}$.
- $P_1 = (1+r)P_0$, $P2 = (1+r)P_1 = (1+r)^2P_0$
- $P_3 = (1+r)P_2 = (1+r)^3P_0$
- $P_n = P_0(1+r)^n$, where $r = 3\%$, $n = 15$, $P_0 = \$55{,}000$.

# *Concept Review* – Chapter 8 - Relations

- A binary relation from a set A to a set B is a _(1)_ of _(2)_.
- A binary relation on a set A is a _(3)_ of _(4)_.
- There are _(5)_ different relations on a set with n elements.
- (a, b) $\in$ R means a _(6)_ b.
- (a, b) $\in$ R if and only if (b, a) $\in$ _(7)_.
- (a, b) $\in$ $\overline{R}$ if and only if (a, b) $\notin$ _(8)_.
- If R is a relation on the set A and for every element a in A, (a, a) $\in$ R, then R is called (9)_.
- If there are some element a in A such that (a, a) $\in$ R, then R is _(10)_.
- If there are some a, b, c in A such that aRb, bRc and _(11)_, then R is **not *transitive***.
- A relation on a set A is called a/an _(12)_ if it is reflexive, symmetric and transitive.

**Answer:** (1) subset (2) A×B (3) subset (4) A×A (5) $2^{n^2}$
(6) aRb (7) $R^{-1}$ (8) R (9) reflexive (10) not irreflexive (11) $a \not{R} c$
(12) equivalence relation

### *Multiple choice questions*

1. If R = {(a, b), (b, c), (c, c)}, how many ordered pairs are there in $R^2$?
   a. 1 b. 2 c. 3 d. 4 e. none of these

   **Answer:** 3

   **Comment:**
   - $R^2$ is R∘R.
   - (a, b)$\in$R and (b, c)$\in$R ➔ (a, c)$\in R^2$, etc.
   - $R^2$ = {(a, c); (c, c); (b, c)}.
2. A company makes four kinds of products. Each product has a size code, a weight code, and a shape code. The following table shows these codes:

| Size Code | Weight Code | Shape Code |
|---|---|---|
| 42 | 27 | 42 |
| 27 | 38 | 13 |
| 13 | 12 | 27 |
| 42 | 38 | 38 |

   Which of the three codes is a ***primary key***?
   a. size code
   b. weight code
   c. shape code
   d. No primary key

**Answer:** shape code

3. If X = (Fran Williams, 617885197, MTH 202, 248B West), find the projection $P_{1,3}(X)$.
   a. (Fran Williams, MTH 202)
   b. (617885197, 248B West)
   c. (Fran Williams, 617885197)
   d. None of these

   **Answer:** (Fran Williams, MTH 202)

   **Comment:**

   - The projection $P_{1,3}(X)$ helps us to keep the 1st and 3rd components of X and delete the other components in the 5-tuples X.

4. Suppose R = {(a, b), (b, b), (c, a)} and S = {(a, c), (b, c)} are relations on {a, b, c}. Construct R∘S.
   a. {(a, c); (b, c)}
   b. {(a, a); (b, a)}
   c. {(a, a); (b, a); (b, c)}
   d. None of these

   **Answer:** {(a, a); (b, a)}

   **Comment:**

   - If xSy, yRz, then x(R∘S)z.
   - (a, c)∈S and (c, a)∈R ➔ (a, a)∈( R∘S)

5. Let R = {(x, y) | x – y = 5 or y = x + 5} and S = {(x, y) | x – y = 5} be relations on the set of integers. Which relation(s) is(are) **symmetric**?

   a. R
   b. S
   c. None
   d. Both

   **Answer:** R

   **Comment:**

   - Consider the relation S: (8, 3)∈S (due to 8 – 3 = 5) but (3, 8) ∉S (due to 3 – 8 ≠ 5) ➔ S is not symmetric ➔ not b, d.
   - Consider R: you can see if (8, 3) ∈R (due to 8 – 3 = 5 – the 1st condition) then (3, 8) ∈R (due to 8 = 3 + 5 – the 2nd condition)

- If $(x, y) \in R$ then $x - y = 5$ or $y = x + 5$ ➔ $y = x + 5$ or $x - y = 5$ ➔ $y - x = 5$ or $x = y + 5$ ➔ $(y, x) \in R$ ➔ R is symmetric.

6. Given a matrix of a relation R

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the statements:

(i) R is anti-symmetric

(ii) R is reflexive

Which one is true?

a. (i)
b. (ii)
c. None
d. Both

**Answer:** Both

**Comment:**

- $a[i, i] = 1$ for every index i ➔ R is reflexive.
- Cannot find $i \neq j$ such that $a[i, j] = 1$ and $a[j, i] = 1$ ➔ R is anti-symmetric.

7. Given a matrix of a relation R

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Consider the statements:

(i) R is reflexive

(ii) R is antisymmetric

Which one is true?

a. (i)
b. (ii)
c. None
d. Both

**Answer:** None

**Comment:**

- a[2, 2] = 0 for every index i ➔ R is not reflexive.
- a[4,3] = a[3,4] ➔ not anti-symmetric.

8. If $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, determine if R is (a) reflexive (b) symmetric (c) antisymmetric.

a. Yes, no, yes
b. Yes, no, no
c. No, yes, yes
d. No, no, yes
e. None of the others

**Answer:** Yes, no, no

**Comment:**

- a[i, i] = 1 for every index i ➔ R is reflexive ➔ not c, d.
- a[3,2] = a[2,3] ➔ not anti-symmetric ➔ not a.
- a[3,1] ≠ a[1,3] ➔ not symmetric.

9. Which of these binary relations on the set {1, 2, 3, 4} are ***equivalence relations?***

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (1, 3), (4, 4)}

S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

a. S
b. R
c. Both
d. None

**Answer:** S

**Comment:**

- Check for reflexive property: (3, 3)∉R ➔ R is not reflexive ➔ R is not an equivalence relation.
- S is reflexive, symmetric and transitive ➔ S is an equivalence relation.

10. How many ordered pairs in the ***equivalence relation*** corresponding to the ***partition*** {a}, {b, d}, {c} of {a, b, c, d}?

a. 3
b. 4
c. 5
d. 6
e. None of the others

**Answer:** 6
**Comment:**

- {a}, {b, d}, {c} are classes of this equivalence relation.
- The relation must contain (a, a), (b, b), (c, c), (d, d) (due to reflexive property)
- The relation must contain (b, d), (d, b) (due to symmetry and b, d are in the same class).

11. How many different binary relations from {a, b, c} to {1, 2}?

a. 6
b. 8
c. 9
d. $2^6$
e. None of these

**Answer:** $2^6$
**Comment:**

- Every binary relation from {a, b, c} to {1, 2} is a subset of {a, b, c} × {1, 2}.
- {a, b, c} × {1, 2} = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}.
- Count the ways to choose a subset of the set {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}.
- By product rule in basic counting techniques, there $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$ ways.

12. How many different binary relations on {a, b} that contains (a, b)?

a. 1
b. 8
c. 4
d. 16
e. None of these

**Answer:** 8
**Comment:**

- Every binary relation on the set {a, b} is a subset of {a, b} × {a, b}.

- $\{a, b\} \times \{a, b\} = \{(a, a), (a, b), (b, a), (b, b)\}$.
- Count the ways to choose a subset of the set $\{(a, a), (a, b), (b, a), (b, b)\}$.
- The subset (binary relation) you choose must contain (a, b).
- By product rule in basic counting techniques, there $2 \cdot 1 \cdot 2 \cdot 2 = 8$ ways.

# *Concept review – Chapter 9- GRAPHS*

- A/an <u>  (1)  </u> graph is an undirected graph with no multiple ***edges*** or loops.
- Two ***vertices*** are called <u>  (2)  </u> if there is an ***edge*** between them.
- An edge is <u>  (3)  </u> with a ***vertex*** if the vertex is an endpoint of that edge.
- <u>  (4)  </u> of a vertex in an ***undirected graph*** is the number of edges ***incident*** with this vertex with ***loops*** counted twice.
- A/an <u>  (5)  </u> is a ***path*** that does not contain an ***edge*** more than once.
- A/an <u>  (6)  </u> is a ***path of length n*** $\geq 1$ that begins and ends at the same vertex.
- A/an <u>  (7)  </u> graph is the ***undirected graph*** with n vertices where each pair of vertices is connected by an edge.
- Graph $K_{m,n}$ has <u>  (8)  </u> edges and <u>  (9)  </u> vertices.
- Graph $W_n$ has <u>  (10)  </u> edges and <u>  (11)  </u> vertices.
- A matrix representing a graph using the adjacency of vertices is called <u>  (12)  </u> of the graph.
- A matrix representing a graph using the ***incidence*** of edges and vertices is called <u> (13)</u> of the graph.
- A graph with vertex set that can be partitioned into subsets $V_1$ and $V_2$ so that each edge connects a vertex in $V_1$ and a vertex in $V_2$ is called a/an <u>  (14)  </u> graph.
- A ***circuit*** that contains ***every edge*** of a graph exactly once is called a/an <u>  (15)  </u> .
- A simple ***path*** in a graph that passes through ***each vertex*** exactly once is called a/an <u> (16).</u>

**<u>Answer:</u>** (1) simple (2) adjacent (3) incident (4) degree (5) simple path
(6) circuit (7) connected (8) m.n(9) m + n (10) 2n (11) n +1
(12) adjacency matrix (13) incidence matrix (14) bipartite (15) Euler circuit (16) Hamilton path

## *Multiple choice questions*

1. How many ***edges*** are there in an undirected graph with ***degree sequence*** 5, 5, 4, 3, 2, 1, 1, 1, 0?

   a. 21
   b. 20
   c. 10
   d. 11
   e. No such a graph.

   **<u>Answer:</u>** None
   **<u>Comment:</u>**

- The sum of degrees of all vertices in a undirected graph = 2*(the number of edges)
- (5 + 5 + 4 + 3 + 2 + 1 + 1 + 1 + 0) = 2*(the number of edges)

2. Which ***degree sequence*** corresponds to a ***simple graph***?

   (i) 5, 4, 3, 2, 1
   (ii) 5, 5, 4, 2, 1, 1

   a. (i)
   b. (ii)
   c. None
   d. Both

   **Answer:** None
   **Comment:**
   - Study (i), 3 vertices of odd degrees ➔ no such a graph.
   - Two vertices of degree 5 ➔ each of other vertices (4 vertices) has at least degree 2 ➔ (ii) is impossible to be a degree sequence of a simple graph (a graph with no loop, no multiple edge).

3. Which graph corresponds to the following ***adjacency matrix***?

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

   a. $K_6$
   b. $C_6$
   c. $W_6$
   d. $Q_3$
   e. None of these

   **Answer:** $K_6$
   **Comment:**
   - An adjacency matrix has the size of n×n, where n is the number of vertices.
   - The size of the matrix is 6×6 ➔ the graph has 6 vertices.
   - $Q_3$ has 8 vertices, $W_6$ has 7 vertices.
   - From the matrix, vertex 1 is adjacent to each of vertices {2, 3, 4, 5, 6} ➔ the matrix is not for $C_6$.

4. How many 1-entries in the ***adjacency matrix*** of graph $K_{3,5}$?

   a. 15
   b. 30
   c. 8
   d. None of these

   **Answer:** 30
   **Comment:**
   - $K_{3,5}$ is a complete bipartite graph with 8 vertices.
   - $K_{3,5}$ has 3*5 = 15 edges.
   - Number of 1s = 2*number of edges.
   - Adjacency matrix of $K_{3,5}$ is

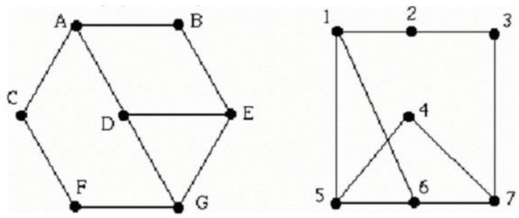
$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

5. $C_n$ is ***bipartite*** if and only if

   a. n is an integer.
   b. n is an even integer.
   c. n is an odd integer.
   d. No such n.

   **Answer:** n is an even integer
   **Comment:**
   - Let use two colors to color every vertex of $C_n$.
   - Try with $C_4$, $C_5$ to see a conclusion.

6. Are these two graphs ***isomorphic***?

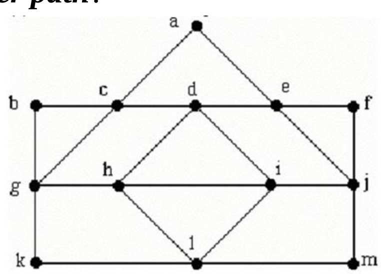



a. Two graphs are isomorphic.
b. Two graphs are not isomorphic. The first graph has a triangle, the second one has no.
c. Yes, they have the same number of vertices and edges.
d. Two graphs are not isomorphic. Two graphs do not have the same shapes.

**Answer:** Two graphs are isomorphic
**Comment:**

- A bijection: d-6 e-5 g-1 f-2 c-3 a-7 b-4
- Try to find a "special thing" that one graph has but the other graph has not.

7. Consider the graph shown below.
   (i) Does it have an ***Euler circuit***?
   (ii) Does it have an ***Euler path***?

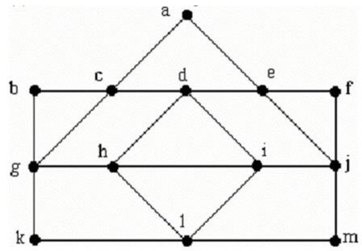



a. (i) Yes (ii) Yes
b. (i) Yes (ii) No
c. (i) No (ii) Yes
d. (i) No (ii) No

**Answer:** Yes – Yes
**Comment:**

- Every vertex of this graph has even degree ➔ yes for (i).
- Yes for (i) ➔ Yes for (ii)

8. Consider the graph shown below.
   (i) Does it have a ***Hamilton circuit***?
   (ii) Does it have a ***Hamilton path***?

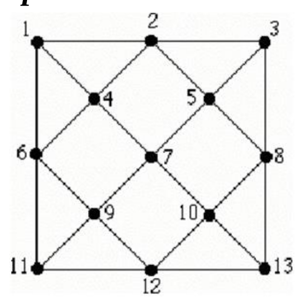



a. (i) Yes (ii) Yes
b. (i) Yes (ii) No
c. (i) No (ii) Yes
d. (i) No (ii) No

**<u>Answer:</u>** No – Yes
**<u>Comment:</u>**

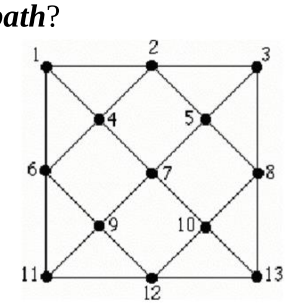
- Yes for (i) ➔ Yes for (ii)
- A Hamilton path: h-i-d-e-a-c-b-g-k-l-m-j-f

9. Consider the graph shown below.
   (i) Does it have a ***Hamilton circuit***?
   (ii) Does it have a ***Hamilton path***?




a. (i) Yes (ii) Yes
b. (i) Yes (ii) No
**c.** (i) **No (ii) Yes**
d. (i) No (ii) No

10. Consider the graph shown below.
   (i) Does it have an ***Euler circuit***?
   (ii) Does it have an ***Euler path***?




a. (i) Yes (ii) Yes

b. (i) Yes (ii) No
c. (i) No (ii) Yes
d. (i) No (ii) No

**Answer:** No – No
**Comment:**
- Yes for (i) ➔ Yes for (ii)
- No for (ii) ➔ No for (i)
- If there is at least one vertex of odd degree ➔ No for (i)
- The graph has 4 vertices of odd degree ➔ No for (ii)

11. Find the length of an ***Euler circuit*** in the graph $K_9$ or state that $K_9$ has no an Euler circuit.

a. 18
b. 9
c. 45
d. 36
e. $K_9$ has no an Euler circuit.

**Answer:** 36
**Comment:**
- $K_9$: a simple graph with 9 vertices of degree 8.
- EVERY VERTEX has EVEN DEGREE ➔ $K_9$ has an Euler circuit.
- The length of a path = number of edges.
- $K_9$ has $C_9^2 = 36$ edges.
- An Euler circuit is a circuit that passes through every edge exactly once ➔ An Euler circuit in $K_9$ passes through 36 edges of $K_9$, exactly once for each.

12. Which graphs given by ***adjacency matrices*** below are ***bipartite***?

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

G H

a. G
b. H
c. Both
d. None

**Answer:** H
**Comment:**

- Color each of 4 vertices of the graph, start form an arbitrary vertex (e.g., vertex 1).
- Use two colors (Red, Blue) (or two states 0, 1).
- Adjacent vertices ➔ different colors/states.
- Check out the adjacency of vertices of the same color.
- Let consider the graph G:

| vertex | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Color/state | Red | Blue | Red | Blue |

So, {2, 4} are vertices of the same color.
Let check out the adjacency of vertices 2 and 4 ➔ vertex 2 and vertex 4 are adjacent ➔ G is NOT bipartite.

13. List all positive integers m and n such that $K_{m,n}$ has a ***Hamilton circuit***.

a. m = n.
b. m = n > 1.
c. |m – n| = 1.
d. m and n are even.
e. None of these

**Answer:** m = n > 1
**Comment:**

- There is NO criteria to know whether a graph has a Hamilton circuit.

14. Find the length of an ***Euler circuit*** in the graph $W_7$ or state that it has no an Euler circuit.

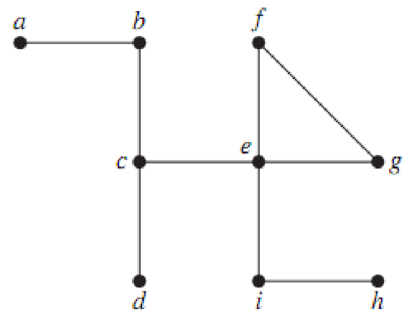
a. 7
b. 8
c. 14
d. $W_7$ has no an Euler circuit.

**Answer:** d
**Comment:**

- To have an Euler circuit, a graph must be connected and EVERY VERTEX of this graph has EVEN DEGREE.

- A graph with at least one vertex of ODD DEGREE ➔ has no an Euler circuit.
- **$W_7$:** a wheel with a center (degree 7) and 7 other vertices (degree 3).

15. How many ***cut vertices*** does the graph below have?

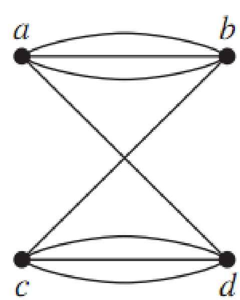



a. 0
b. 1
c. 2
d. 3
e. 4

**Answer:** 4
**Comment:**

- A vertex is called **cut vertex** if deleting this vertex makes a **disconnected** graph.
- Cut vertices are b, c, e, i.

16. How many ***paths of length 2*** *from a to c* in the graph below?

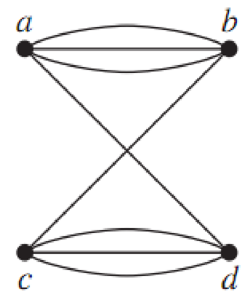



a. 3
b. 6
c. 9
d. 12

**Answer:** 6
**Comment:**

- Let A be the **adjacency matrix** of the given graph.
- To count the number of path of **length 2**, you have to compute $\mathbf{A^2}$.
- You can only do this: multiply <row a> of A by <column c> of A.

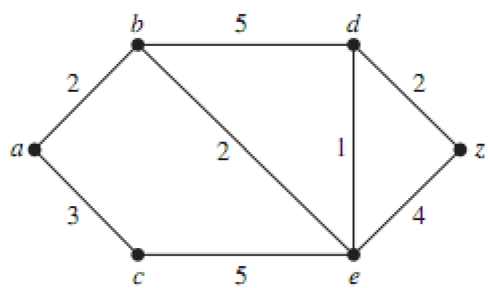
17. How many ***paths of length 3*** from b to c?




a. 27
b. 9
c. 28
d. 32
e. None of these

**<u>Answer:</u>** 28
**<u>Comment:</u>**
- Let A be the **adjacency matrix** of the given graph.
- To count the number of path of **length 3**, you have to compute $\mathbf{A^3}$.
- To see the result, read the number at the (row b, column c)-entry.
- You can only do this: multiply <row b> of $A^2$ (respectively, A) by <column c> of A (respectively, $A^2$).

18. Use ***Dijkstra's Algorithm*** to find the ***shortest path length*** from a to z in the following ***weighted graph***. How many vertices are not used?




a. 1
b. 2
c. 3
d. 4

e. None of these

**Answer:** 1

**Comment:**

- The shortest path is a-b-e-d-z and b is not used.

# *Concept review – Chapter 10 - TREES*

- A ***tree*** is a connected undirected graph with no _(1)_ .
- An ***m-ary tree*** is a tree with the property that every ***internal vertex*** has _(2)_ children.
- A ***full m-ary tree*** is a tree with the property that every ***internal vertex*** has _(3)_ children.
- A tree with n ***nodes*** has _(4)_ ***edges***.
- A ***full m-ary tree*** with i ***internal vertices*** has _(5)_ ***nodes*** and _(6)_ ***leaves***.
- A code that has the property that the code of a character is never a ***prefix*** of the code of another character is called a _(7)_.
- _(8)_ is a listing of the vertices of an ordered rooted tree defined recursively - the root is listed, followed by the first subtree, followed by the other subtrees in the order they occur from left to right.
- _(9)_ is the form of an expression obtained from a ***preorder traversal*** of the tree representing this expression.
- _(10)_ is the form of an expression obtained from a post-order traversal of the tree representing this expression.

**Answer:** (1) simple circuit (2) exactly 2 (3) exactly m (4) n – 1 (5) mi + 1 (6) (mi + 1 – i) (7) prefix code (8) Preorder traversal (9) Prefix notation/form (10) Postfix notation/form

## *Multiple choice questions*

1. A ***full binary tree*** with 99 ***internal vertices*** has ____ ***edges***.
   a. 99
   b. 198
   c. 100
   d. 199
   e. 98

   **Answer:** 198
   **Comment:**
   - Number of edges = number of nodes – 1 = n – 1.
   - “Binary” means m = 2.
   - Solve the system n = i + *l* (for every tree) and n = mi + 1 (for full m-ary tree only) where n (*l*, i) stands for number of nodes (leaves, internal nodes).

2. A ***full 5-ary tree*** with 45 ***leaves*** has ____ ***internal vertices***.

a. 11
b. 12
c. 13
d. 14
e. None of these

**Answer:** 11
**Comment:**

- Solve the system n = i + *l* (for every tree) and n = mi + 1 (for full m-ary tree only) where n (*l*, i) stands for number of nodes (leaves, internal nodes).

3. Construct a ***binary search tree*** for the words: TIME, AND, TIDE, WAIT, FOR, NO, MAN. How many ***comparisons*** are used to locate the word "MAN"?

a. 3
b. 4
c. 5
d. 6
e. None of these

**Answer:** 6
**Comment:**

- Start by adding the word "TIME" to the tree.
- The word "MAN" is also needed to compare to itself.

4. How many comparisons are used to locate the word "ending" in the ***binary search tree*** for the words of the sentence "A bad beginning makes a bad ending"?

a. 2
b. 3
c. 4
d. 5
e. None of the others

**Answer:** 5
**Comment:**

- Start by adding the word "A" to the tree.
- The same position for the same word (A, bad).

- The word “ending” is also needed to compare to itself.

5. Which codes are ***prefix codes***?

| | | | | |
|---|---|---|---|---|
| (i) | A: 010 | B: 101 | C: 1101 | D: 1011 |
| (ii) | A: 01 | B: 111 | C: 101 | D: 1101 |

   a. (i)
   b. (ii)
   c. None
   d. Both

   **<u>Answer:</u>** (ii)

   **<u>Comment:</u>**

   - Study the code scheme (i), 101 is for B and 101 is also a **prefix** of **101**1, the one is for D.

6. Which codes are ***prefix codes***?

| | | | | |
|---|---|---|---|---|
| (i) | A: 01 | B: 101 | C: 1101 | D: 0011 |
| (ii) | A: 11 | B: 1011 | C: 1101 | D: 1110 |

   a. (i)
   b. (ii)
   c. None
   d. Both

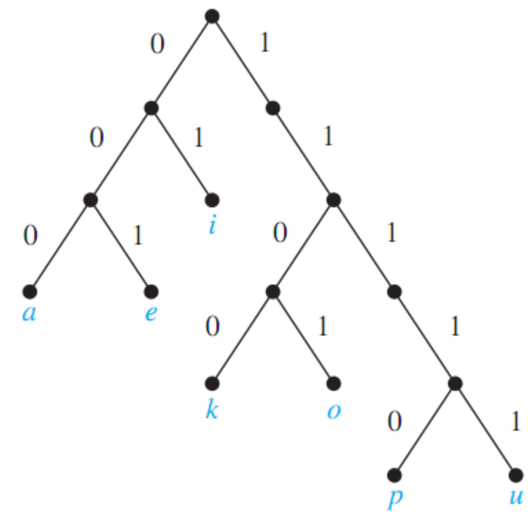
   **<u>Answer:</u>** (i)
   **<u>Comment:</u>**
   - Study the code scheme (ii), 11 is for A and 11 is also a **prefix** of **11**01, the one is for C.

7. Consider the coding scheme: t: 001, e: 101, a: 11, n: 0111.
   What is the message represented by 10111001?
   a. tea
   b. eat
   c. net
   d. an
   e. None of these

   **<u>Answer:</u>** eat
   **<u>Comment:</u>**

- Start "reading" 10111001 from left to right
- 101 is for e, 11 is for a, 001 is for t.

8. What is the code for the word "keep" if the coding scheme is represented by the following tree?




   a. 1000010011110
   b. 1010001001111
   c. 00100111101101
   d. 110000100111110
   e. None of these

   **Answer:** 110000100111110 (d)

   **Comment:**

   - Start from the **root** to go to the **leaf node**.
   - From the tree above, 1100 is for k, 001 is for e, 1110 is for p.

9. What is the *average number of bit* required to encode the word "nobody" using ***Huffman coding algorithm***?
   a. 8/3
   b. 2
   c. 7/3
   d. 2.5
   e. None of these

   **Answer: 7/3**

   **Comment:**

   - First, count the frequency of each character and list them in an increasing order (n(1) b(1) d(1) y(1) o(2)).
   - Construct a binary tree using Huffman algorithm.

10. What is the *average number of bits* required for each letter when using ***Huffman coding algorithm*** to encode the word “success”?
    a. 13/7
    b. 15/7
    c. 2
    d. 16/7
    e. None of these

    **Answer: 13/7**

    **Comment:**

    - First, count the frequency of each character and list them in an increasing order (u(1) e(1) c(2) s(3)).
    - Construct a binary tree using Huffman algorithm.

11. Use ***Huffman coding*** to encode these symbols with given frequencies: a: 0.20, b: 0.10, c: 0.15, d: 0.25, e: 0.30. What is the average number of bits required to encode a character?
    a. 3.15
    b. 2.45
    c. 2.25
    d. 3.45
    e. None of these

    **Answer: 2.25 bits/symbol**

    **Comment:**

    - Always choose 2 trees of least weights from the forest.
    - After constructing the binary tree for the given data, the number of bits for each character is given as below: 2 bits for a, 3 bits for b, 3 bits for c, 2 bits for d and 2 bits for e.
    - Average number of bits for each character can be calculated as follow:
      Ave = sum(number of bits of a character * frequency of this character) =
      2*0.20 + 3*0.10 + 3*0.15 + 2*0.25 + 2*0.30 = 2.25.

12. Find the value of the expression + - 7 * 2 1 / 3 1
    a. 4
    b. 6
    c. 8
    d. 10
    e. None of these

    **Answer:** 8

    **Comment:**

- An operator (+) is in the first position, it follows that the expression is in prefix form.
- From **right to left**, find a triple in form **<operator> <number 1> <number 2>** (for example, / 3 1) to compute **<number 1> <operator> <number 2>**.

13. Find the value of the expression 5 4 - 3 6 3 / ↑ 7 - *.
    a. 2
    b. 4
    c. 6
    d. 8
    e. None of these

    **Answer:** 2

    **Comment:**

    - The expression is in **postfix form**, because the last symbol is an operator (*).
    - From **left to right**, find a triple in form **<number 1> <number 2> <operator>** (for example, 5 4 - ) to compute **<number 1> <operator> <number 2>**.

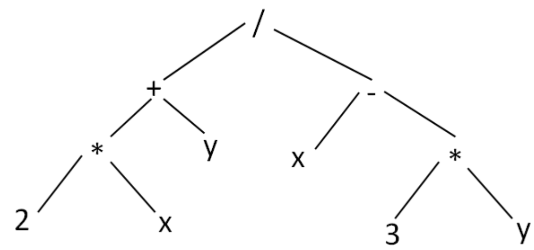
14. Find the ***postfix form*** of the infix expression (2*x + y)/(x – 3*y).

    a. 2 x * y + x 3 y * - /
    b. 2 * x y + x 3 / y * /
    c. / + * 2 x y - x * 3 y
    d. None of these

    **Answer:** a

    **Comment:**

    - The expression includes only one / ➔ option (b) is not true. In the **postfix form**, the root node (an operator) is visited last. So, the last symbol is always an operator (+, -, *, /, …) ➔ option (c) is not true.
    - In the expression (2*x + y)/(x – 3*y), the last operator is /, and therefore the postfix form ends with /.
    - The **expression tree** is shown as below


/
+
-
*
y
x
*
2
x
3
y