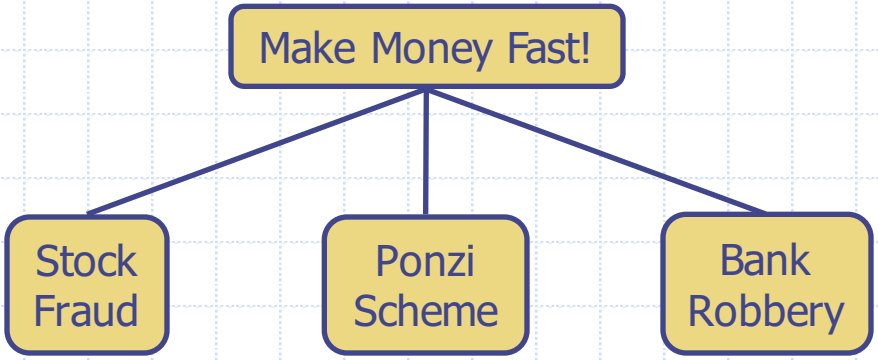
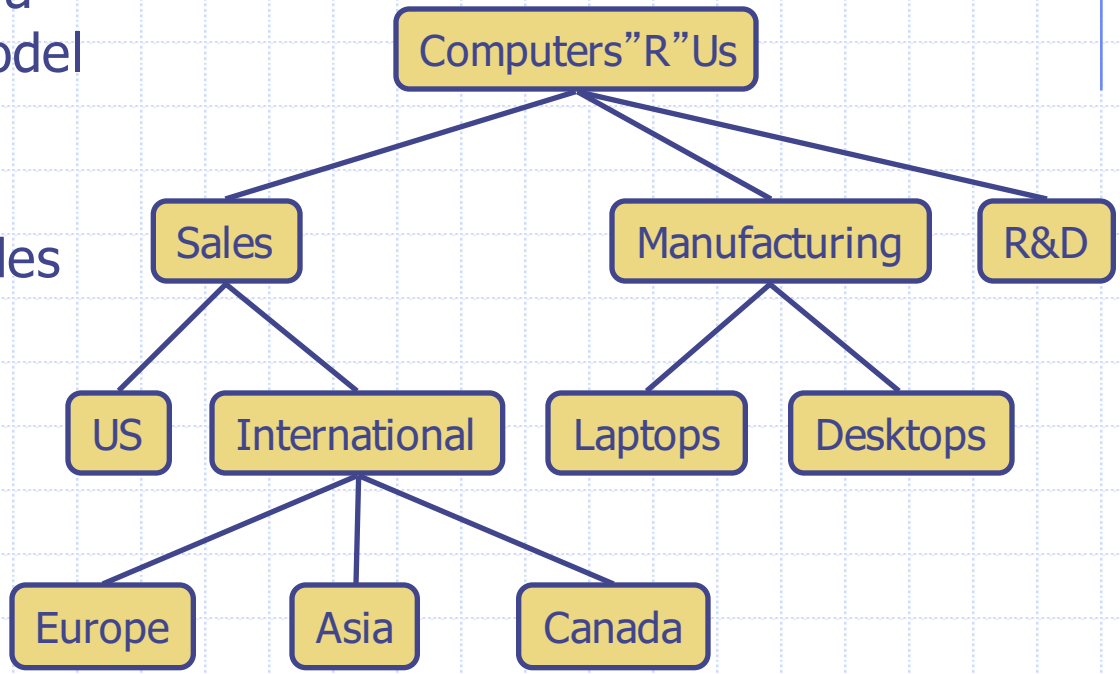
Trees



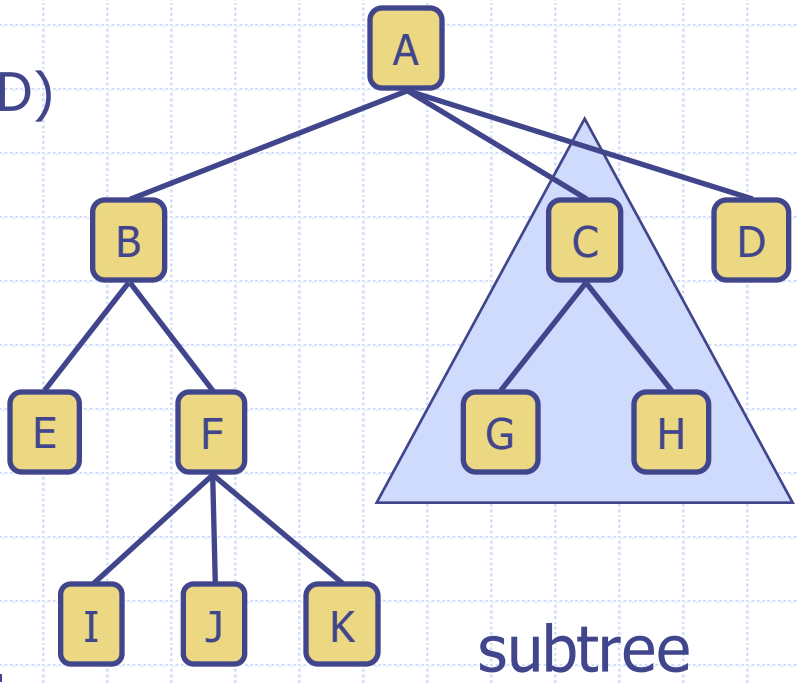
What is a Tree

- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments

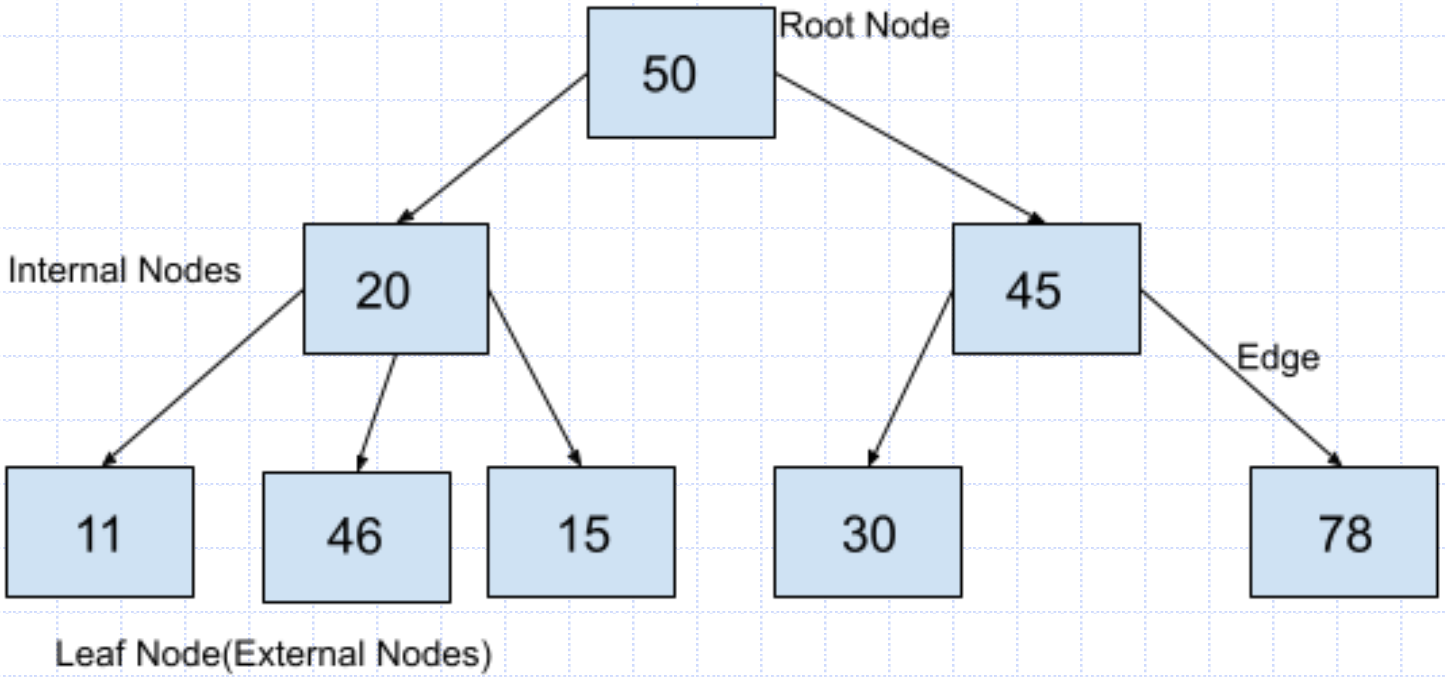


Tree Terminology

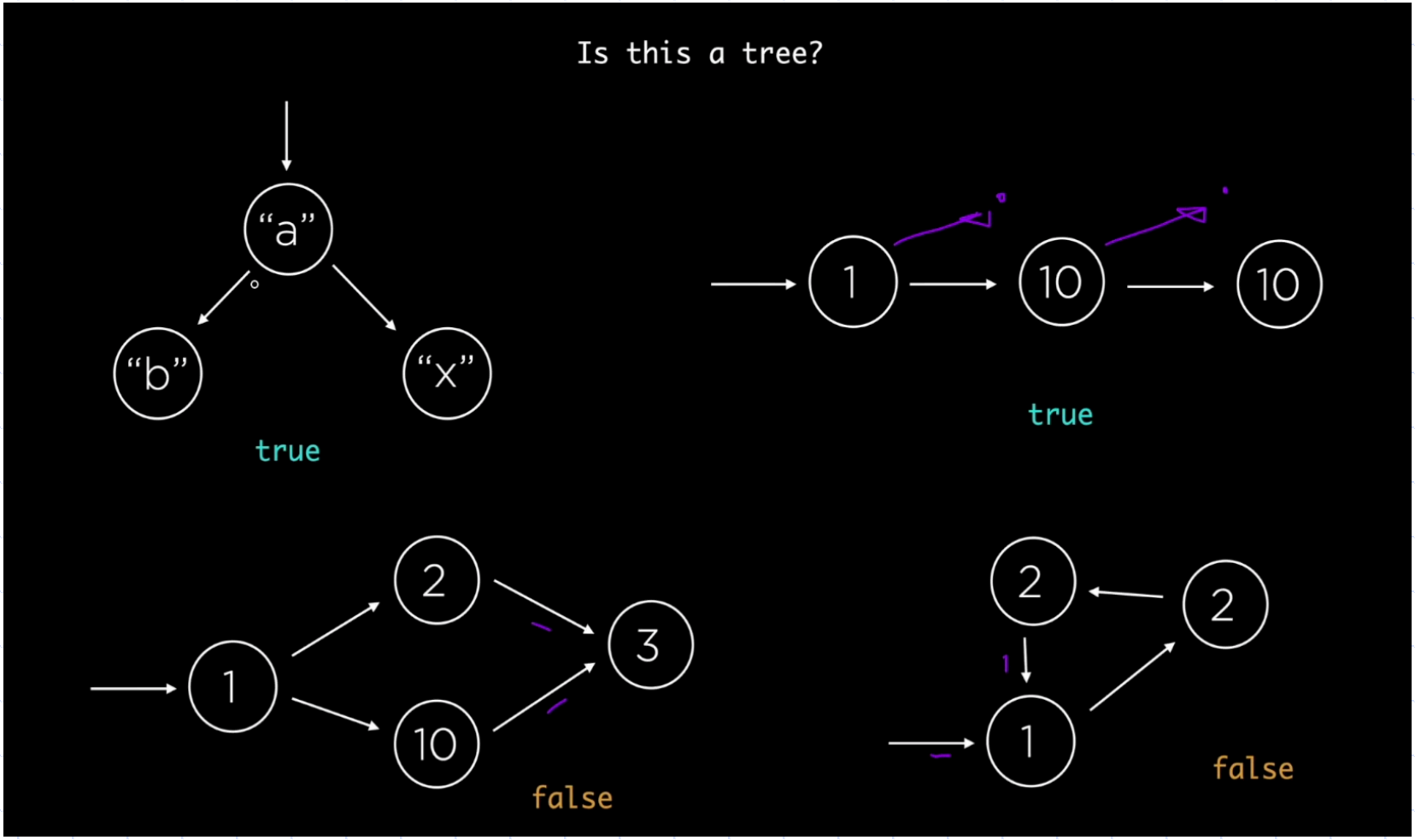
- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Depth of a node: number of ancestors
- Height of a tree: maximum depth of any node (3)
- Descendant of a node: child, grandchild, grand-grandchild, etc.
- Subtree: tree consisting of a node and its descendants



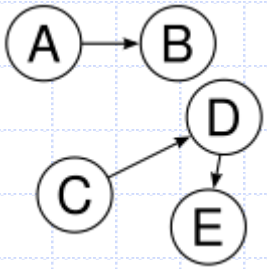
Tree Terminology

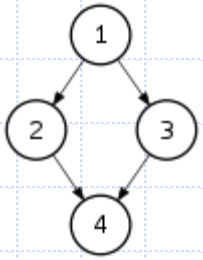


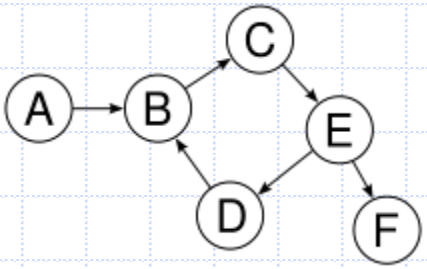
What is a Tree

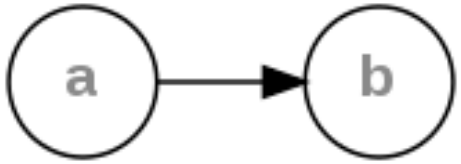


What is a Tree









Tree ADT

- We use positions to abstract nodes
- Generic methods:
  - Integer len()
  - Boolean is_empty()
  - Iterator positions()
  - Iterator iter()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterator children(p)
  - Integer num_children(p)
- Query methods:
  - Boolean is_leaf(p)
  - Boolean is_root(p)
- Update method:
  - element replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

Abstract Tree Class in Python

```
1  class Tree:
2    """Abstract base class representing a tree structure."""
3
4    #--------------------------- nested Position class ---------------------------
5    class Position:
6      """An abstraction representing the location of a single element."""
7
8      def element(self):
9        """Return the element stored at this Position."""
10       raise NotImplementedError('must be implemented by subclass')
11
12     def __eq__(self, other):
13       """Return True if other Position represents the same location."""
14       raise NotImplementedError('must be implemented by subclass')
15
16     def __ne__(self, other):
17       """Return True if other does not represent the same location."""
18       return not (self == other)           # opposite of __eq__
19
```

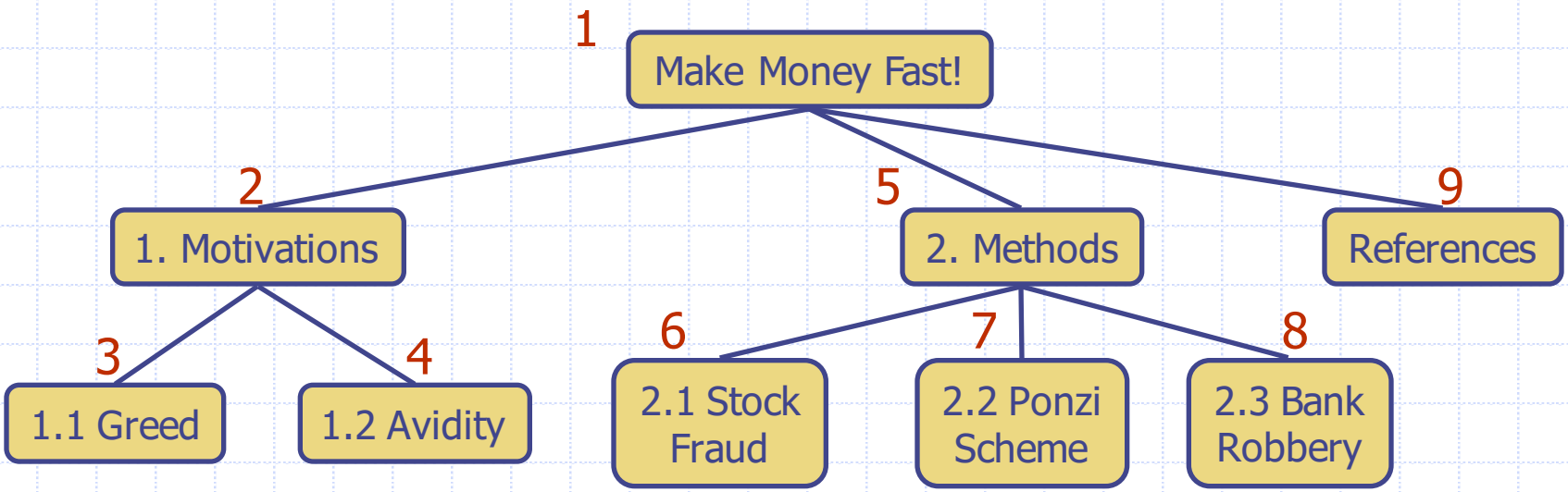
```
20   # ---------- abstract methods that concrete subclass must support ----------
21   def root(self):
22     """Return Position representing the tree's root (or None if empty)."""
23     raise NotImplementedError('must be implemented by subclass')
24
25   def parent(self, p):
26     """Return Position representing p's parent (or None if p is root)."""
27     raise NotImplementedError('must be implemented by subclass')
28
29   def num_children(self, p):
30     """Return the number of children that Position p has."""
31     raise NotImplementedError('must be implemented by subclass')
32
33   def children(self, p):
34     """Generate an iteration of Positions representing p's children."""
35     raise NotImplementedError('must be implemented by subclass')
36
37   def __len__(self):
38     """Return the total number of elements in the tree."""
39     raise NotImplementedError('must be implemented by subclass')
```

```
40   # ---------- concrete methods implemented in this class ----------
41   def is_root(self, p):
42     """Return True if Position p represents the root of the tree."""
43     return self.root( ) == p
44
45   def is_leaf(self, p):
46     """Return True if Position p does not have any children."""
47     return self.num_children(p) == 0
48
49   def is_empty(self):
50     """Return True if the tree is empty."""
51     return len(self) == 0
```

Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a node is visited before its descendants
- Application: print a structured document

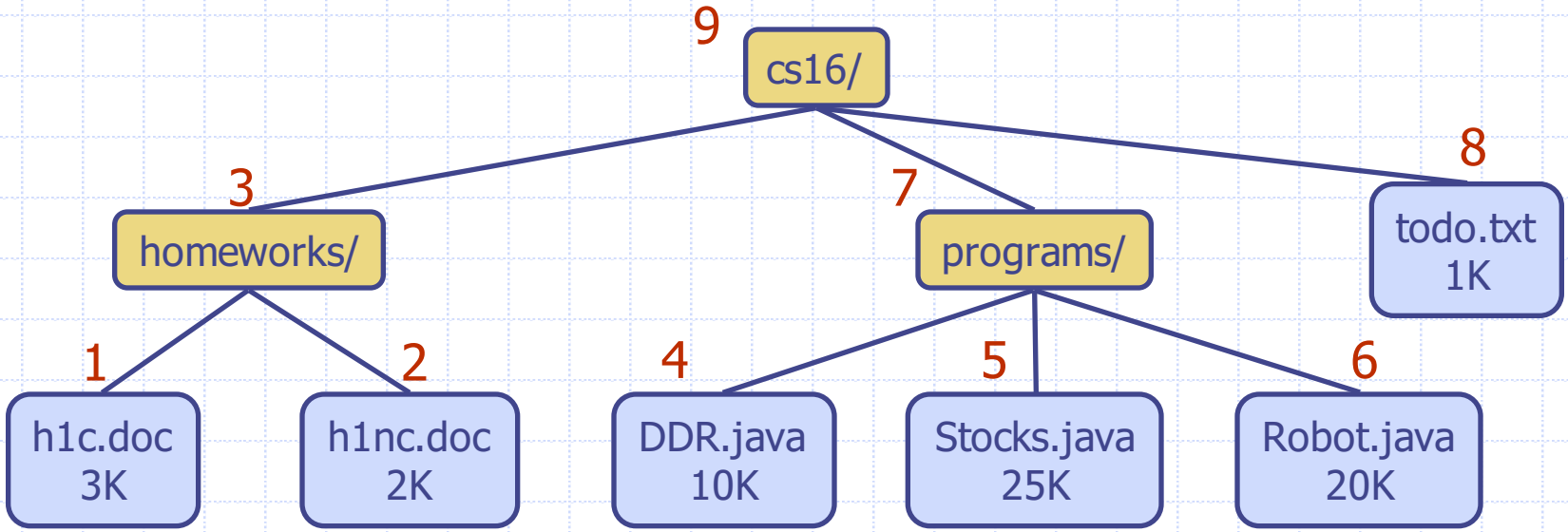
Algorithm *preOrder*(*v*)
    visit(*v*)
    for each child *w* of *v*
        preorder (*w*)



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants
- Application: compute space used by files in a directory and its subdirectories

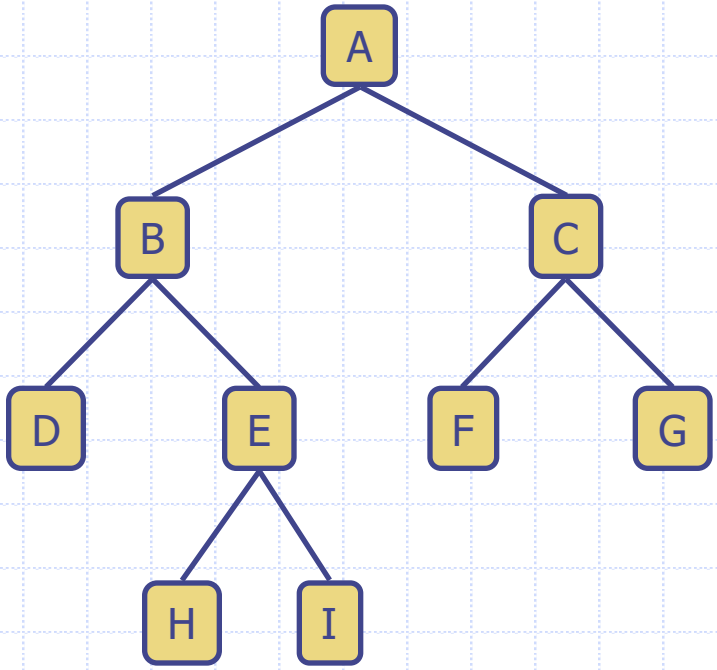
Algorithm *postOrder*(*v*)
  for each child *w* of *v*
    postOrder (*w*)
  visit(*v*)



Binary Trees

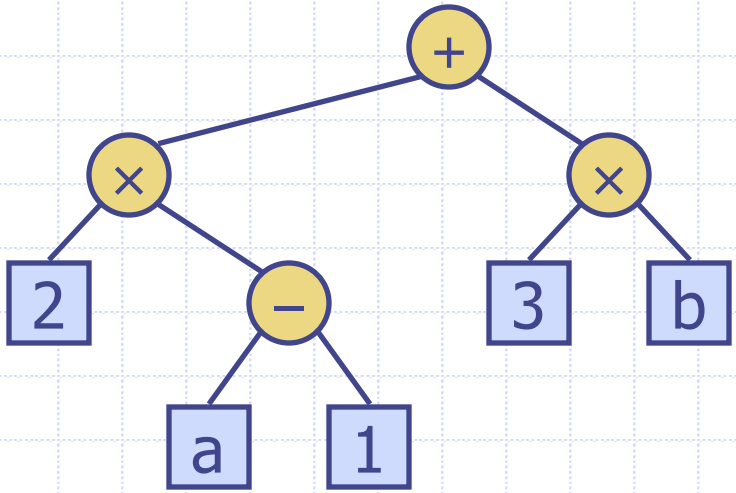
- A binary tree is a tree with the following properties:
  - Each internal node has at most two children (exactly two for proper binary trees)
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a binary tree

- Applications:
  - arithmetic expressions
  - decision processes
  - searching



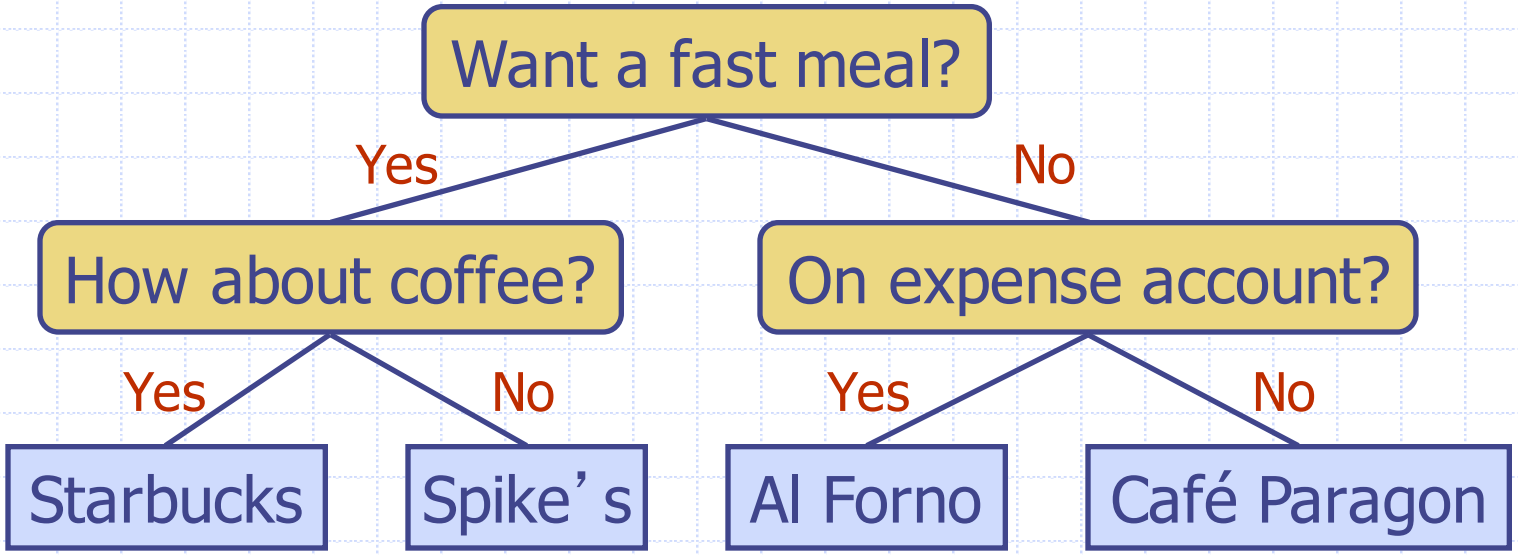
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



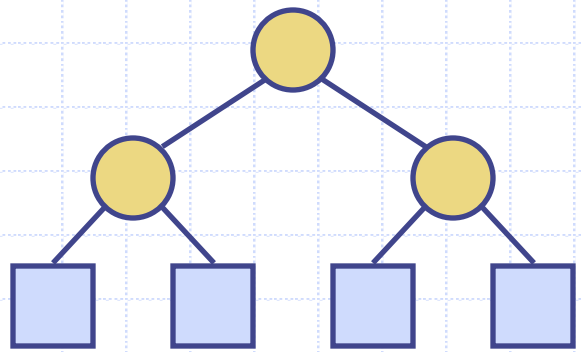
Decision Tree

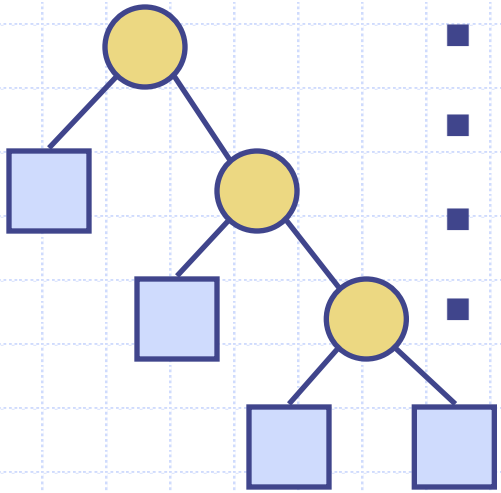
- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision



Properties of Proper Binary Trees

- Notation
  - n number of nodes
  - e number of external nodes
  - i number of internal nodes
  - h height





- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$

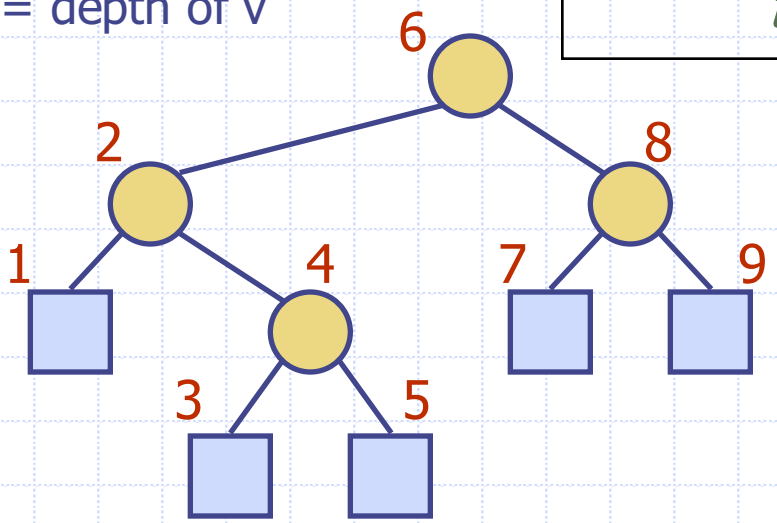
BinaryTree ADT

- The BinaryTree ADT extends the Tree ADT, i.e., it inherits all the methods of the Tree ADT
- Additional methods:
  - position left(p)
  - position right(p)
  - position sibling(p)
- Update methods may be defined by data structures implementing the BinaryTree ADT

Inorder Traversal

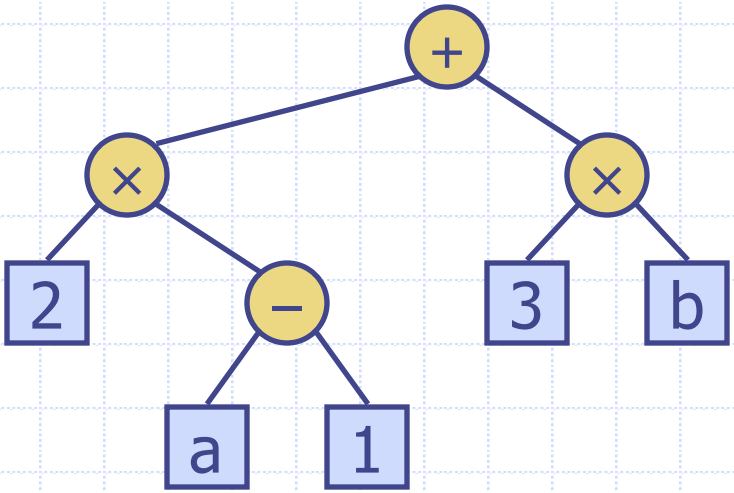
- In an inorder traversal a node is visited after its left subtree and before its right subtree
- Application: draw a binary tree
  - x(v) = inorder rank of v
  - y(v) = depth of v

Algorithm *inOrder(v)*
  if *v* **has a left child**
    inOrder (left (v))
  visit(v)
  if *v* **has a right child**
    inOrder (right (v))



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree



Algorithm *printExpression*(*v*)

if *v* **has a left child**

print("(")

inOrder (*left*(*v*))

print(*v.element* ())

if *v* **has a right child**

inOrder (*right*(*v*))

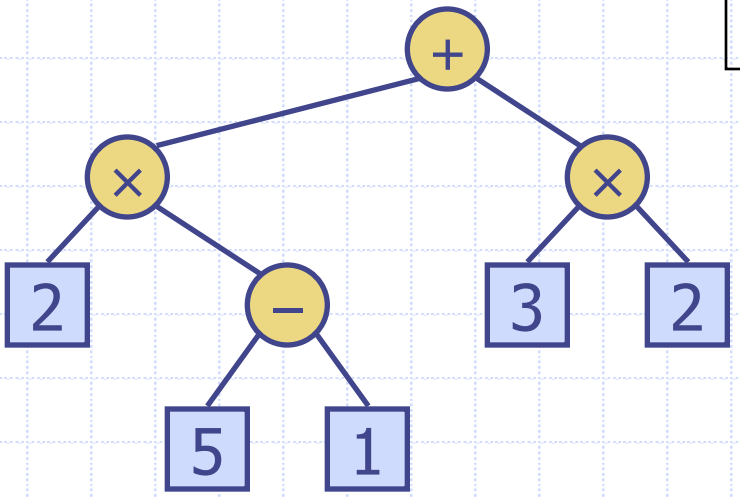
print (")")

((2 × (a − 1)) + (3 × b))

Evaluate Arithmetic Expressions

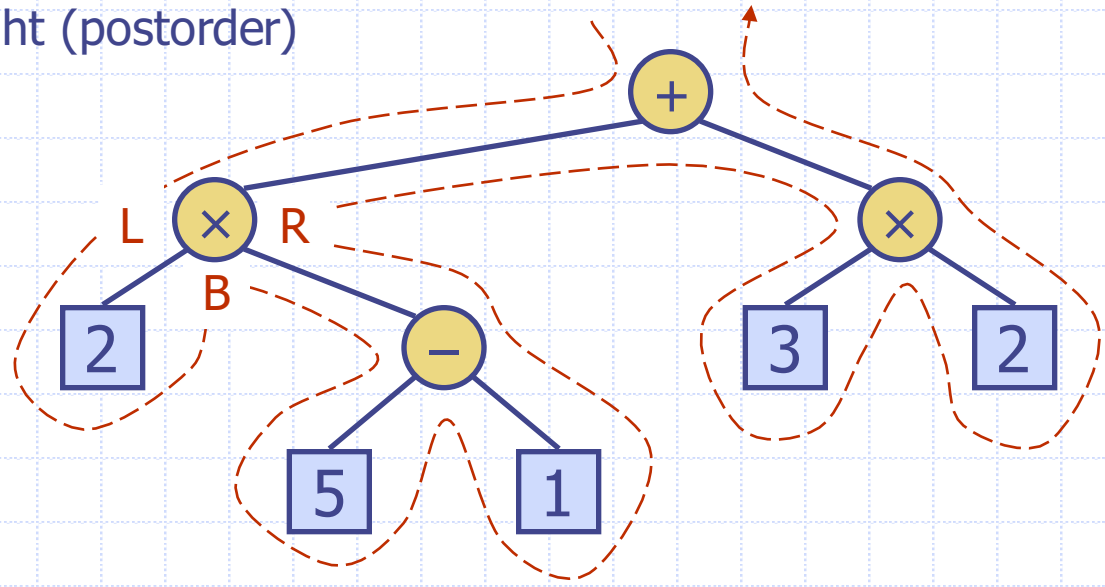
- Specialization of a postorder traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

```
Algorithm evalExpr(v)
    if is_leaf(v)
        return v.element()
    else
        x ← evalExpr(left(v))
        y ← evalExpr(right(v))
        ◊ ← operator stored at v
        return x ◊ y
```



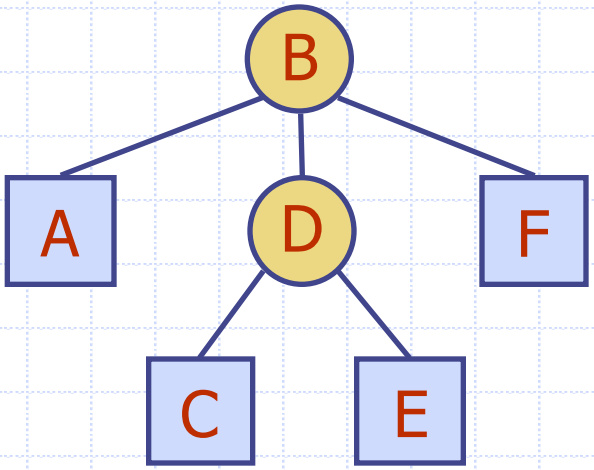
Euler Tour Traversal

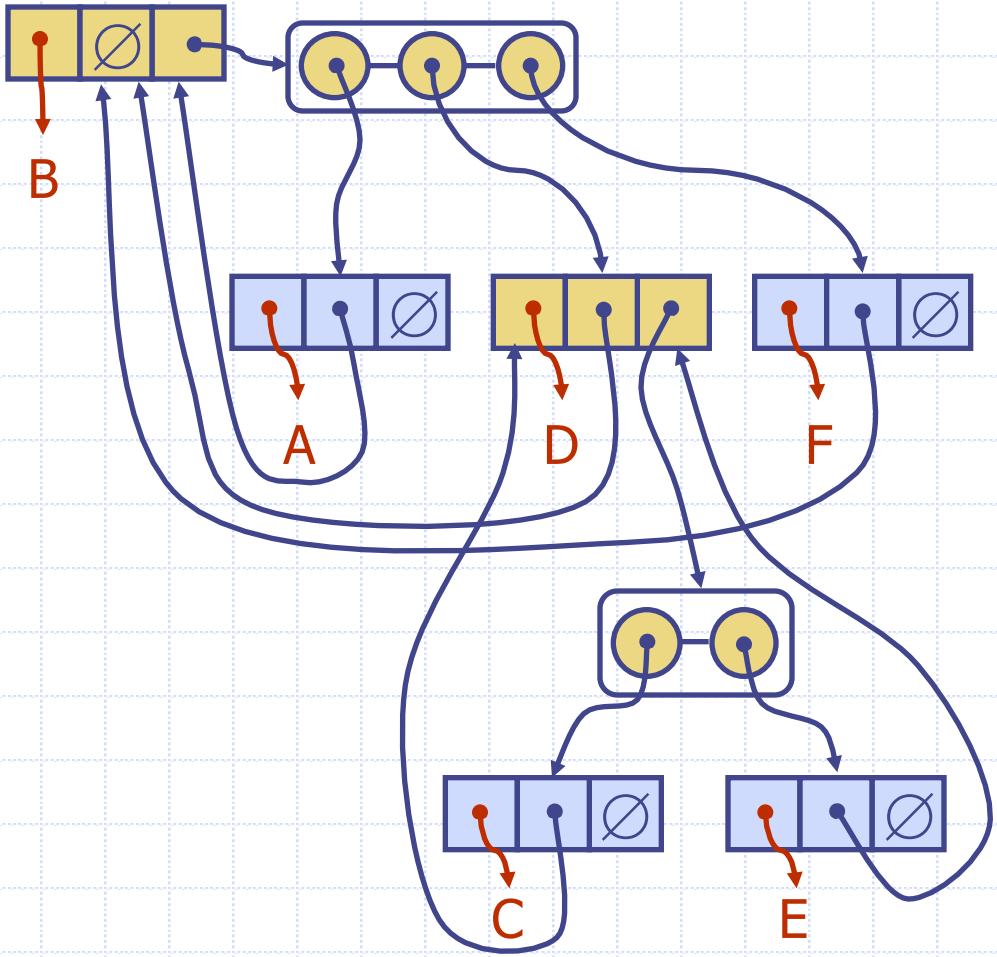
- Generic traversal of a binary tree
- Includes a special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



Linked Structure for Trees

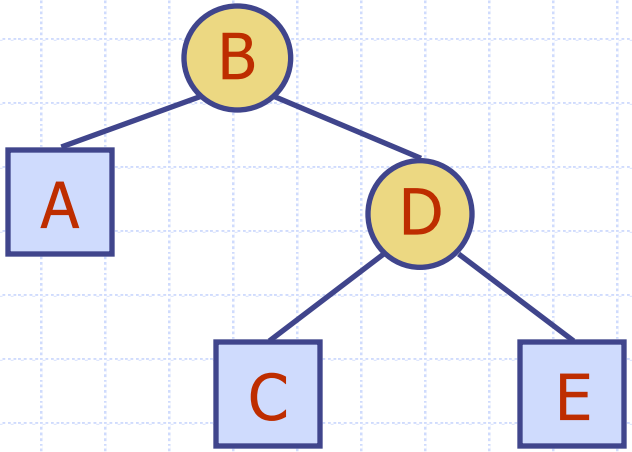
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT

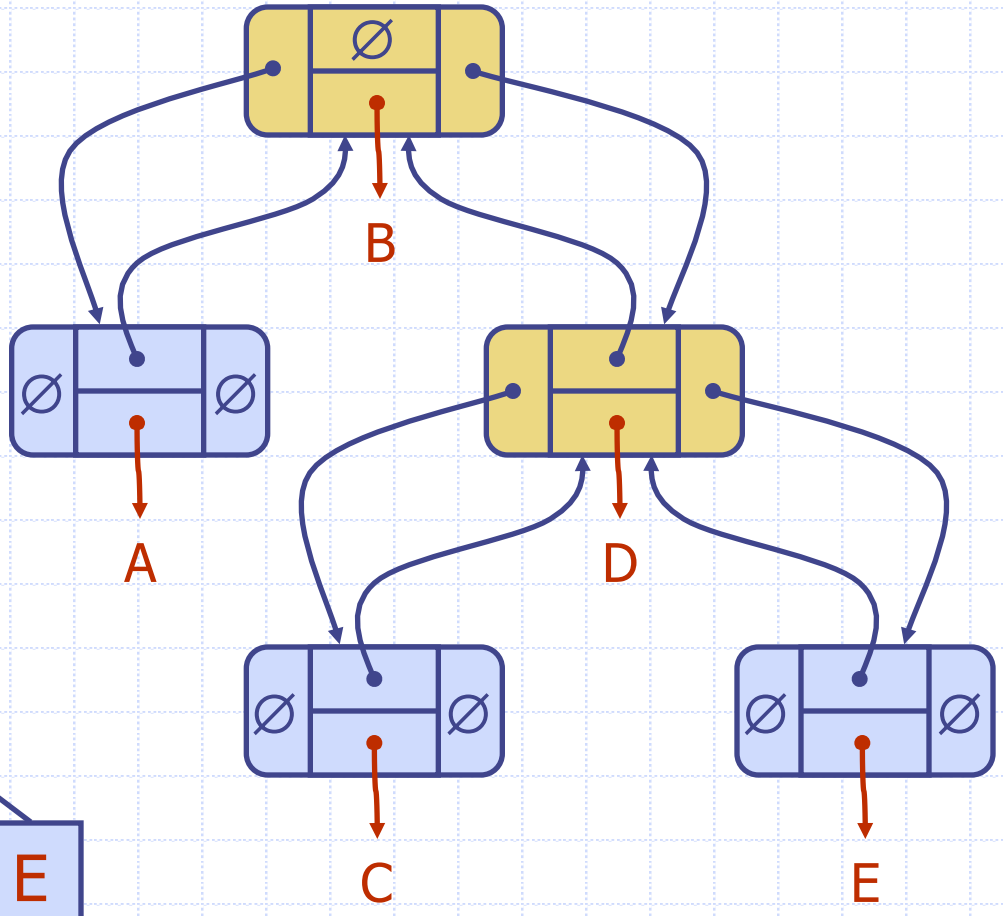




Linked Structure for Binary Trees

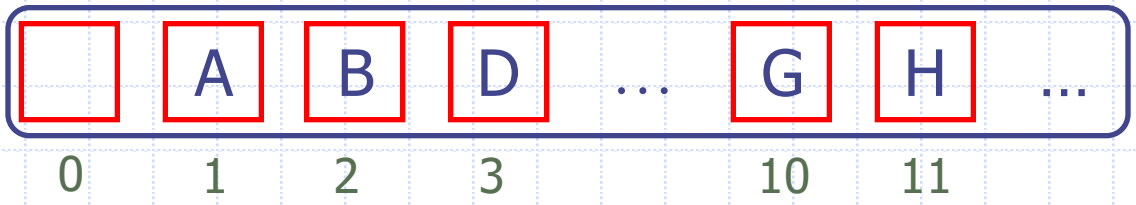
- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT

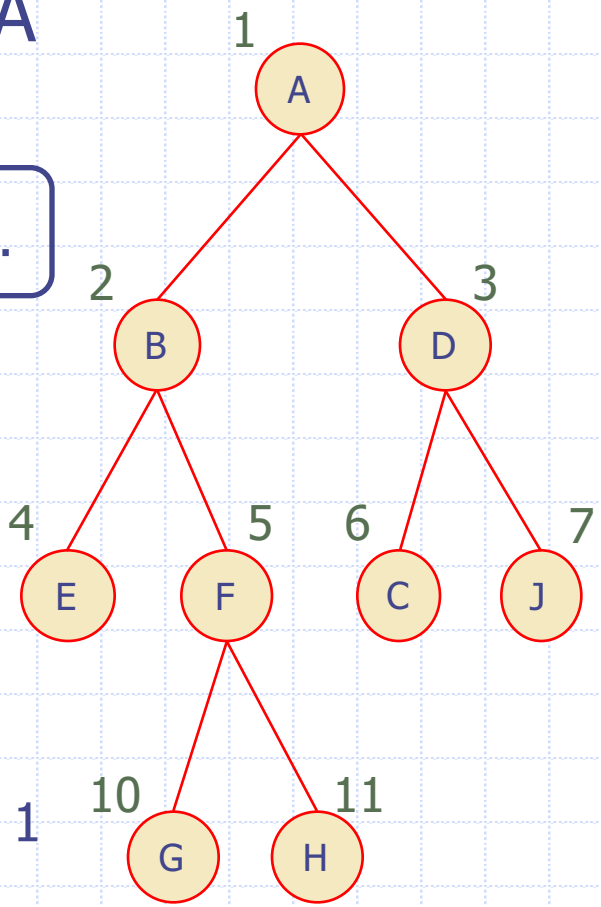




Array-Based Representation of Binary Trees

- Nodes are stored in an array A





- Node v is stored at A[rank(v)]
  - rank(root) = 1
  - if node is the left child of parent(node), rank(node) = 2 · rank(parent(node))
  - if node is the right child of parent(node), rank(node) = 2· rank(parent(node)) + 1