C3. Markov Processes

Notes originally prepared by Judy-anne Osborn.
Editing, expansion and additions by Malcolm Brooks.

This material is not covered in the textbook by Epp. Check books
on Finite Mathematics or Discrete Mathematics in the Library,
e.g. Finite Mathematics By Maki & Thompson Chapter 8

Markov processes

Markov processes are about probabilities. We consider

- the **state** of a system, amongst a finite number of possibilities

- the **probability** of moving between states in one time-step,

- and the probable state after **many** time-steps.

- We often don't make a sharp distinction between **proportions** and **probabilities** as you will see in the examples.

  This works well for large samples but you may need to be careful with small samples.

Introductory example

adapted from 'Finite Mathematics', Maki & Thompson

A freelance computer network consultant, let's call her Cathy, is employed in weekly contracts. Each week she is either:
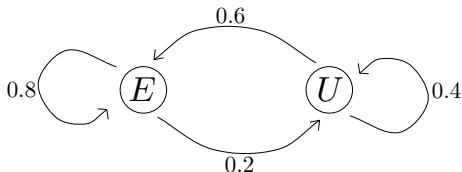
- employed (E) or
- unemployed (U).

Her records support the following assumptions:

(a) If she's employed this week, then next week she'll be employed with probability 0.8 and unemployed with probability 0.2.

(b) If she's unemployed this week, then next week she'll be employed with probability 0.6 and unemployed with probability 0.4.

System, States and Transitions

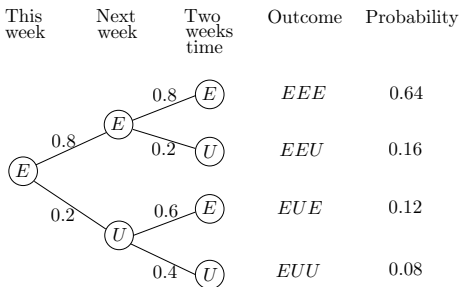We can model Cathy's situation by a Markov process:

- The **system** is Cathy herself.

- The system can be in one of two **states**:
  - E: Employed
  - U: Unemployed

- A **Transition Diagram** encodes the transition probabilities:

It is a property of a Markov Process that the probability of stepping from one state to another *only depends on the current state*.

Two time-steps

If Cathy is employed this week, what is the probability that she will be employed two weeks from now? We can use a tree:


This week	Next week	Two weeks time	Outcome	Probability
		0.8 E	EEE	0.64
	0.8 E	0.2 U	EEU	0.16
E		0.6 E	EUE	0.12
	0.2 U	0.4 U	EUU	0.08

From the tree diagram, the probability that Cathy will be employed two weeks from now is

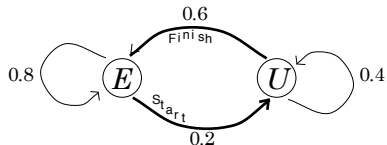$\Pr(EEE \text{ or } EUE) = \Pr(EEE) + \Pr(EUE) = 0.64 + 0.12 = 0.76.$

Two time-steps on the transition diagram

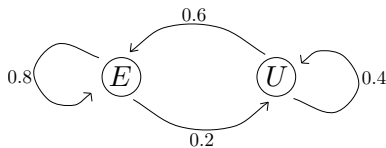Starting employed, then employment after two weeks can be shown on the transition diagram as

either
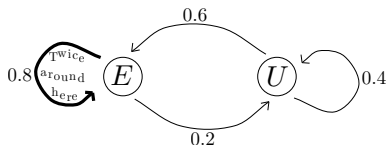
or

Transition Matrix

The information in Cathy's transition diagram

can be encoded in the **transition matrix**

$$T = \begin{array}{c} \\ E \\ U \end{array} \begin{array}{cc} \overset{\curvearrowright}{E} & U \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

State Vectors

The **state vector** x_n shows probabilities of being in each state after n time-steps.

In Cathy's case, the state vector has two entries.

- The first entry records probability of employment.
- The second entry records probability of unemployment.

Initially ($n = 0$) the only possible values for the state vector are

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if Cathy is employed, and

- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if Cathy is unemployed,

since we assumed that, in any week, Cathy was either:

100% employed and   0%   unemployed, or

  0%   employed and 100% unemployed.

Transpose of the Transition Matrix

Recall that the transition (transfer) matrix is

$$T = \begin{array}{c} \\ E \\ U \end{array} \overset{\overset{\displaystyle E \qquad U}{\curvearrowright}}{\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}}.$$

We will need

$$T' = \begin{array}{c} \\ E \\ U \end{array} \overset{\overset{\displaystyle E \qquad U}{\curvearrowleft}}{\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}}.$$

This is the **transpose** of the transition matrix.
It is very important to remember that it is always the *transpose* of
the transition matrix that is used in calculations.

Using Matrices and State Vectors

Suppose Cathy is employed in Week 0.

The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

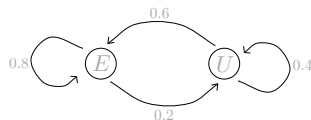After one time-step, *i.e.* next week (Week 1), her probabilities of being employed or not are given by the state vector $\mathbf{x}_1$.

This can be expressed as:

$$\mathbf{x_1} = \mathbf{T}'\mathbf{x_0}$$

$$= \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

Outcome	Prob.
EE	0.8
EU	0.2

Two time-steps

In Week 2, *i.e.* after two time-steps, Cathy's chances of work are given by the state vector $\mathbf{x}_2$.
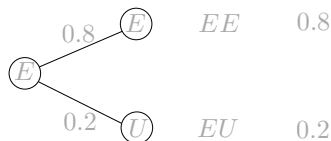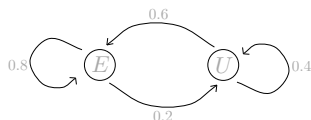
This can be calculated by:

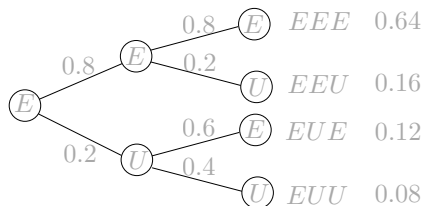$$\mathbf{x_2} = \mathbf{T}'\mathbf{x_1}$$

$$= \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 + 0.12 \\ 0.16 + 0.08 \end{bmatrix}$$

$$= \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$$

Outcome	Prob.
EEE	0.64
EEU	0.16
EUE	0.12
EUU	0.08

n time-steps

Continuing: $\mathbf{x}_3 = T'\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix} = \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix}$

$\mathbf{x}_4 = T'\mathbf{x}_3 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.752 \\ 0.248 \end{bmatrix} = \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix}$

$\mathbf{x}_5 = T'\mathbf{x}_4 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.7504 \\ 0.2496 \end{bmatrix} = \begin{bmatrix} 0.75008 \\ 0.24992 \end{bmatrix}$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Thus: $\mathbf{x}_1 = T'\mathbf{x}_0$

$\mathbf{x}_2 = T'\mathbf{x}_1 = T'T'\mathbf{x}_0 = (T')^2\mathbf{x}_0$

$\mathbf{x}_3 = T'\mathbf{x}_2 = T'T'T'\mathbf{x}_0 = (T')^3\mathbf{x}_0$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$\mathbf{x}_n = (T')^n\mathbf{x}_0$

The nth power of T'

Successive powers of $T' = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$ are:

$$(T')^2 = T'T' = \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}$$

$$(T')^3 = T'(T')^2 = \begin{bmatrix} 0.752 & 0.744 \\ 0.248 & 0.256 \end{bmatrix}$$

$$(T')^4 = T'(T')^3 = \begin{bmatrix} 0.7504 & 0.7488 \\ 0.2496 & 0.2512 \end{bmatrix}$$

$$(T')^5 = T'(T')^4 = \begin{bmatrix} 0.75008 & 0.74976 \\ 0.24992 & 0.25024 \end{bmatrix}$$

$$\vdots \qquad \vdots \qquad \vdots$$

So: $\qquad (T')^n \simeq \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$ for large values of n.

The significance of $(T')^n$ for large n

We have seen that, for Cathy, $(T')^n \simeq \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$ for large values of n.

Notice that the columns of this matrix are equal, and that

$$\begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \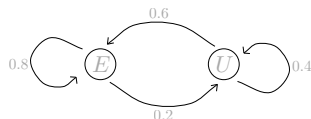end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

So, irrespective of the initial state, in the long term the state vector becomes approximately $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$. This means

No matter what, eventually Cathy will be employed 75% of the time.

The Steady State Vector

Cathy's employment situation can now be summed up by:

$$(T')^n \simeq \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$$

When, as here, the columns of $(T')^n$ tend to become all the same for large values of n, this column $\mathbf{v}$ (in this case $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$) is called a **steady state vector** because then

$$(T')^n \mathbf{u} \simeq \mathbf{v}$$

for any initial state vector $\mathbf{u}$.

The steady state vector is an eigenvector

The steady state vector has the property that multiplication by the transposed transition matrix does not change it, *e.g.* for Cathy:

$$T'\mathbf{v} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.60 + 0.15 \\ 0.15 + 0.10 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \mathbf{v}.$$

More generally, for *any* transition matrix T we call *any* vector $\mathbf{v}$ for which

$$T'\mathbf{v} = \mathbf{v}$$

a steady state vector for T.

Caution: A Markov process may not always reach a steady state!

Note: The definition of $\mathbf{v}$ makes it a special case of an **eigenvector**.
Courses in linear algebra cover more about eigenvectors and also numbers called **eigenvalues**.
A steady state vector has an associated eigenvalue of 1.

Some Definitions

It is time to collect and formalise the definitions made informally while discussing the example of Cathy's employment.

A **probability vector** is a vector with non-negative entries that sum to 1.

A **stochastic matrix** is a square matrix all whose rows are probaility vectors.

A (discrete) **Markov process** is a system that, at each of a sequence of time steps, can be in exactly one of a finite number k of states, with the probability of the system being in any particular state at time step $n \geq 1$ being dependent only on

(i) its state at the $(n - 1)$-th time step, and

(ii) a fixed stochastic matrix $T \in M_k(Q_+)$ called the **transition matrix** of the process.

Some Definitions (continued)

The (i, j)-entry T_{ij} of the transition matrix T specifies the probability that the system will be in the j-th state at any time step $n \geq 1$, given that it was in the i-th state at time step $n - 1$.

A **transition diagram** for a Markov process is a complete weighted directed graph with k vertices representing the states of the system and the edge from the i-th vertex to the j-th vertex labelled with the probability T_{ij}.

The **n-th state vector $\mathbf{x}_n$** is a column probability vector with k entries. The i-th entry records a probability of the system being in the i-th state at time step n.

A **steady state vector $\mathbf{v}$** is any probability vector for which $T'\mathbf{v} = \mathbf{v}$.

Using the transition matrix

The following theorem generalises to any number k of states what we saw in Cathy's example for just two states:

Theorem: Let $T = (T_{ij})_{1 \leq i,j \leq k}$ be the transition matrix for a k state Markov process with state vectors $\mathbf{x}_n$, $n \in \mathbb{N}$. Then $\forall n \geq 1$:

(i) $\boxed{\mathbf{x}_n = T'\mathbf{x}_{n-1}.}$

(ii) $\boxed{(T^n)_{ij} \text{ is the } n\text{-step } i\text{-to-}j \text{ transition probability.}}$

(iii) $\boxed{\mathbf{x}_n = (T')^n \mathbf{x}_0}$

Proof of (i): Let $\mathbf{x}_n = (p_i)_{1 \leq i \leq k}$ and $\mathbf{x}_{n-1} = (q_i)_{1 \leq i \leq k}$. Then

$$
\begin{aligned}
p_i &= \text{probability of being in state } i \text{ at step } n \\
&= \sum_{j=1}^{k} (T_{ij})(\text{prob. of being in state } j \text{ at step } n-1) = \sum_{j=1}^{k} T_{ij} q_j.
\end{aligned}
$$

The result follows by the definition of matrix multiplication.
Proofs of (ii) and (iii): These are simple corollaries to (i).

Markov Processes Have No Memory

- The state of a Markov Process at time n only depends on fixed transition probabilities and its state at time $n - 1$.

- It does not depend on the state at any earlier time. In other words, it is a *first-order* (matrix) recurrence.

- Because of this, Markov processes are said to "have no memory".

Finding steady state vectors

- One way to find a steady state vector of a Markov process is to do as we did in the example - namely multiply together enough copies of T' - or equivalently T - to see the higher powers tending to a limit.

- It does not matter whether we transpose first and then exponentiate or the other way around, since

$$(T^n)' = (T')^n$$

- There are more direct methods of finding steady state vectors, and we demonstrate these in the next example.

Another example: weather in 'Oz'

We investigate the weather on the island of Oz.
(This, it will be seen, is a rather simplified example, to illustrate
the principles without too much heavy calculation.)
Weather in Oz is not great, *e.g.* it never has two fine days in a row.

If the weather on a particular day is known, we cannot predict
exactly what the weather will be the next day, but we can give
probabilities of various kinds of weather as follows:

There are only three kinds: fine (F), cloudy (C) and rain (R).
Here is the behaviour:

- After F, the weather is equally likely to be C or R.
- After C, the probabilities are $\frac{1}{4}$ for F, $\frac{1}{4}$ for C and $\frac{1}{2}$ for R.
- After R, the probabilities are $\frac{1}{4}$ For F, $\frac{1}{2}$ for C and $\frac{1}{4}$ for R.

Displaying the Oz weather system

Here are three ways to show the probabilities:

As a table: Probabilities of weather tomorrow are:

Given that the weather today is:

↱	fine	cloudy	rain
fine	0	$\frac{1}{2}$	$\frac{1}{2}$
cloudy	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
rain	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

As a matrix:

$$T = \begin{array}{c} \\ F \\ C \\ R \end{array} \begin{array}{c} \phantom{\begin{bmatrix}0\end{bmatrix}} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

As a diagram:
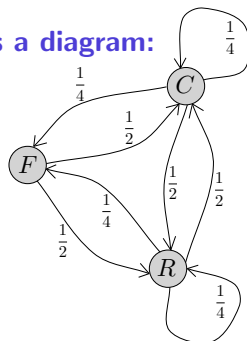
Next day in Oz, via transition matrix

- Given probabilities on Day n, we can find probabilities on Day $n+1$.

- We need a state vector. Let us call the probabilities on day n
    - Prob x that it will be fine,
    - Prob y that it will be cloudy, and
    - Prob z that it will rain.
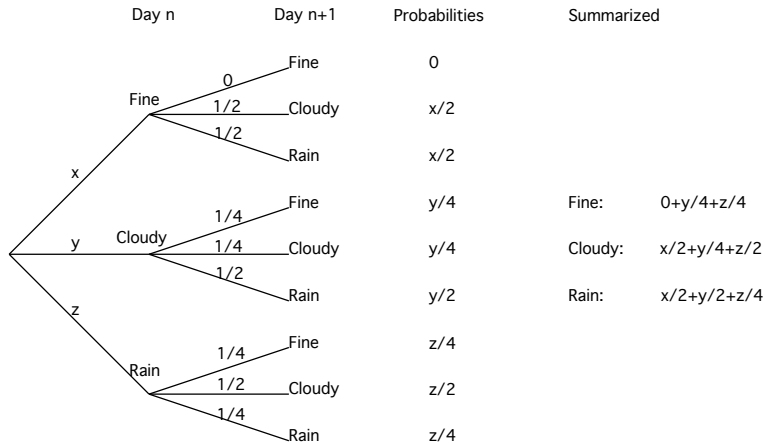
- Define $\mathbf{x}_n = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- Then, according ot the Markov process theorem:

$$\mathbf{x}_{n+1} = T' \mathbf{x}_n$$
$$= \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/4)y + (1/4)z \\ (1/2)x + (1/4)y + (1/2)z \\ (1/2)x + (1/2)y + (1/4)z \end{bmatrix}$$

Next day in Oz,via probability tree

Let's check that the probabilities obtained using the transition matrix agree with those obtained using a probability tree:

Day n	Day n+1	Probabilities	Summarized
	Fine	0	
Fine	Cloudy	$x/2$	
	Rain	$x/2$	
	Fine	$y/4$	Fine: $0+y/4+z/4$
Cloudy	Cloudy	$y/4$	Cloudy: $x/2+y/4+z/2$
	Rain	$y/2$	Rain: $x/2+y/2+z/4$
	Fine	$z/4$	
Rain	Cloudy	$z/2$	
	Rain	$z/4$	

Branch probabilities: Fine→Fine: 0, Fine→Cloudy: 1/2, Fine→Rain: 1/2; Cloudy→Fine: 1/4, Cloudy→Cloudy: 1/4, Cloudy→Rain: 1/2; Rain→Fine: 1/4, Rain→Cloudy: 1/2, Rain→Rain: 1/4. Initial: x, y, z.

Yes, the state vector $\mathbf{x}_{n+1}$ and probability tree agree. ✓

Next week in Oz

Suppose we look outside
and see that today is fine.
Then we can say that

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

That is, the probability of fine is 1, the probability of cloudy is 0 and the probability of rain is 0 today.

Then the probabilities
one week from today
(Day 7) are given by

$$\mathbf{x}_7 = (T')^7 \mathbf{x}_0.$$

Using a computer to calculate the 7th power of the matrix, we get

$$\mathbf{x}_7 = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix}^7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 819/4096 \\ 3277/8192 \\ 3277/8192 \end{bmatrix}.$$

Perhaps decimals would be more illuminating?

Days 1 through 10 in Oz

Computer calculations give:

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} .250 \\ .375 \\ .375 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} .18750 \\ .40625 \\ .40625 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} .2031250 \\ .3984375 \\ .3984375 \end{bmatrix},$$

$$\mathbf{x}_5 = \begin{bmatrix} .199218750 \\ .400390625 \\ .400390625 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .1999511719 \\ .4000244141 \\ .4000244141 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .19995511719 \\ .4000244141 \\ .4000244141 \end{bmatrix},$$

$$\mathbf{x}_8 = \begin{bmatrix} .2000122070 \\ .3999938965 \\ .3999938965 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .1999969438 \\ .4000015260 \\ .4000015260 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .2000007629 \\ .3999996185 \\ .3999996185 \end{bmatrix}.$$

A steady state for the weather in Oz

These values seem to be converging to a long-term steady state of
$$S = \begin{bmatrix} .2 \\ .4 \\ .4 \end{bmatrix},$$

i.e. a probability of 0.2 of fine weather, a probability of 0.4 of cloudy weather and a probability of 0.4 of rainy weather. As they must, these probabilities sum to 1.

To check that this S really is a steady state vector, we calculate

$$T'S = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.50 & 0.25 & 0.50 \\ 0.50 & 0.50 & 0.25 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1 + 0.1 \\ 0.1 + 0.1 + 0.2 \\ 0.1 + 0.2 + 0.1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}.$$

Therefore $T'S = S.$ ✓

Finding steady state vectors

We derive another way to find steady state vectors, illustrating with weather from Oz.

Assume that $\quad S = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad$ is a steady state vector.

Then, as we know, S is an eigenvector of T'. *i.e.*

$$T'S = S$$

(In other words we have reached a stage where the probabilities don't change from day to day any more.)

Notice that we can rearrange this equation in the form

$$T'S - S = 0.$$

Remember to *think about what kinds of objects* are in this equation

$$T'S - S = 0$$

- The right-hand-side is a column vector of zeros.
- S is a column vector of unknowns (x, y and z).
- $T'S$ is the product of a square matrix and a column vector - making a column vector.

We can rewrite the equation yet again in the form

$$T'S - IS = 0$$

where I is the 3×3 identity matrix.

Finally using a distributive law, we can re-write it as:

$$(T' - I)S = 0.$$

Methods for solving the matrix equation

There are several ways to solve the equation

$$(T' - I)S = 0.$$

1. Gaussian (or Gauss-Jordan) elimination. However, Gaussian elimination is not taught nor assumed in this course. Use it if you know it, but there are other ways:

2. For 2×2 systems we can use the matrix inverse formula.

3. For larger systems we can use computer applications such as:
   Matrix Reshish: https://matrix.reshish.com
   MatrixCalc: https://matrixcalc.org/en/
   or, less conveniently but more robustly,
   WolframAlpha: https://www.wolframalpha.com/

Using Gauss-Jordan elimination

First re-write in augmented form as:

$$[T' - I | 0]$$

and then row-reduce to solve for the unknowns in S.

Since

$$T' - I = \begin{bmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1/4 & 1/4 \\ 1/2 & -3/4 & 1/2 \\ 1/2 & 1/2 & -3/4 \end{bmatrix}$$

our augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1/2 & 1/2 & -3/4 & 0 \end{array} \right]$$

Row reducing,

$$\begin{bmatrix} -1 & 1/4 & 1/4 & | & 0 \\ 1/2 & -3/4 & 1/2 & | & 0 \\ 1/2 & 1/2 & -3/4 & | & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1/4 & 1/4 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad R_2' = (-4/5)R_2$$

$$\sim \begin{bmatrix} -1 & 1/4 & 1/4 & | & 0 \\ 1 & -3/2 & 1 & | & 0 \\ 1 & 1 & -3/2 & | & 0 \end{bmatrix} \begin{matrix} \\ R_2' = 2R_2 \\ R_3' = 2R_3 \end{matrix} \quad \sim \begin{bmatrix} 1 & -1/4 & -1/4 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad R_1' = -R_1$$

$$\sim \begin{bmatrix} -1 & 1/4 & 1/4 & | & 0 \\ 0 & -5/4 & 5/4 & | & 0 \\ 0 & 5/4 & -5/4 & | & 0 \end{bmatrix} \begin{matrix} \\ R_2' = R_2 + R_1 \\ R_3' = R_3 + R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad R_1' = R_1 + (1/4)R_2$$

$$\sim \begin{bmatrix} -1 & 1/4 & 1/4 & | & 0 \\ 0 & -5/4 & 5/4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad R_3' = R_3 + R_2$$

$$\uparrow$$
This column
tells us we
need a parameter

Let $z = t$, $t \in \mathbb{R}$

So our original matrix equation is equivalent to

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ;$$

that is, to just the two equations

$$x - (1/2)z = 0$$
$$y - z = 0.$$

Substituting the parameter $z = t$ gives

$$x - (1/2)t = 0$$
$$y - t = 0,$$

leading to the solution $x = (1/2)t$
$$y = t$$
$$z = t.$$

Recall we were solving $(T' - I)S = 0$.

We have found that this equation has an infinite family of solutions for S in the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1/2)t \\ t \\ t \end{bmatrix}, \quad t \in \mathbb{R}$$

But a steady-state vector has an extra property: it must be a **probability vector**, *i.e.* its entries must sum to 1.

Hence we fix a unique solution by requiring that $x + y + z = 1$, *i.e.*

$$(1/2)t + t + t = 1 \implies (5/2)t = 1 \implies t = 2/5$$

so that S is given by
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}$$

– the same as we found before by exponentiating and guessing.

A short cut

A short cut to this process is to take the augmented matrix $[T' - I|0]$ as below,

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1/2 & 1/2 & -3/4 & 0 \end{array}\right]$$

throw away the last row and replace it with $[1...1|1]$, as in

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

and solve this new system to directly obtain the unique solution for S.

After row-reducing the new system we find that

$$
\left[
\begin{array}{ccc|c}
-1 & 1/4 & 1/4 & 0 \\
1/2 & -3/4 & 1/2 & 0 \\
1 & 1 & 1 & 1
\end{array}
\right]
\sim
\left[
\begin{array}{ccc|c}
1 & 0 & 0 & 1/5 \\
0 & 1 & 0 & 2/5 \\
0 & 0 & 1 & 2/5
\end{array}
\right]
$$

from which we read off the steady state vector

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix}.
$$

Can you figure out *why* this short cut works?

Solving by Computer (using Reshish)

The system of equations

$$\left[\begin{array}{ccc|c} -1 & 1/4 & 1/4 & 0 \\ 1/2 & -3/4 & 1/2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

is entered into the Reshish Matrix Calculator, using the "Gauss-Jordan Elimination" Tool :

Note that I have chosen to use "fractional" coefficients, to ensure an exact solution.

Here is how Reshish responds:

Show solution

Solution set:

$x_1 = 1/5$
$x_2 = 2/5$
$x_3 = 2/5$

Back to the first example

We have seen that to find the steady state vector S for a Markov process with transition matrix T we need to solve the linear system that results from replacing the last equation in

$$(T' - I)S = 0$$

by the equation that says that S is a probability vector.

For Cathy's employment process we had

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

and, by a 'guess and check' method, we discovered that

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}.$$

Solution by matrix inverse

Because T is 2×2, and we have a formula for the inverse of a 2×2 matrix, we can find Cathy's steady state vector directly, without Gaussian elimination or computer. There are three steps:

1. Write out the matrix equation $(T' - I)S = 0$:

$$\left(\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i.e. \; \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. Replace the second equation by $x + y = 1$:

$$\begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution by matrix inverse (conclusion)

3. Solve this system using matrix inverse:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.2 & 0.6 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-0.2 - 0.6} \begin{bmatrix} 1 & -0.6 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-0.8} \begin{bmatrix} -0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 6/8 \\ 2/8 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

New Example — Colours of flowers

A species of flower (carnations say) has three colour varieties. The relevant genetics are as shown in the table:

Colour	Genotype
Red	RR
Pink	RW
White	WW

At the nursery they are always crossed with the pink variety. What will be the long term proportions of the three varieties?

First, we need the transition probabilities. Can you work them out?

The transition matrix is

$$T = \begin{matrix} & \text{Red} & \text{Pink} & \text{White} \\ \text{Red} \\ \text{Pink} \\ \text{White} \end{matrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix} .$$
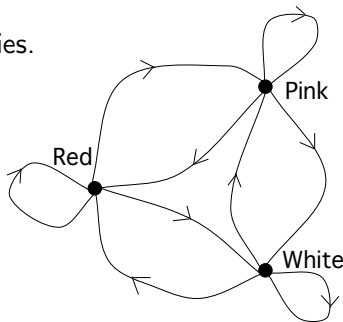
Finding the steady state

(a) $[T' - I | 0]$ is

$$
\left[
\begin{array}{ccc|c}
-0.5 & 0.25 & 0 & 0 \\
0.5 & -0.5 & 0.5 & 0 \\
0 & 0.25 & -0.5 & 0
\end{array}
\right]
$$

(b) Replacing the bottom row with all 1's gives

$$
\left[
\begin{array}{ccc|c}
-0.5 & 0.25 & 0 & 0 \\
0.5 & -0.5 & 0.5 & 0 \\
1 & 1 & 1 & 1
\end{array}
\right]
$$

Finding the steady state (cont.)

(c) Row reduction gives

$$\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \qquad \sim \left[\begin{array}{ccc|c} 1 & -0.5 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_3' = (1/4)R_3$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0.5 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1' = 2R_1 \\ R_2' = 2R_2 \end{array} \qquad \sim \left[\begin{array}{ccc|c} 1 & -0.5 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_2' = R_2 + 2R_3$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 1.5 & 1 & 1 \end{array}\right] \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 + R_1 \end{array} \qquad \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_1' = R_1 + (1/2)R_2$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right] \quad R_3' = R_3 + 3R_2 \qquad \text{yielding } S = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}.$$

Finding the steady state by computer

Alternatively, we can solve the system using the computer. For example, using Reshish:

Hence there is a unique steady state vector of

$$S = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

Fractional			
	X_1	X_2	X_3	b
1	-1/2	1/4	0	0
2	1/2	-1/2	1/2	0
3	1	1	1	1

Solution set:

$x_1 = 1/4$
$x_2 = 1/2$
$x_3 = 1/4$

So the species has a steady state in which 25% of the flowers are coloured red, 50% pink, and 25% white.

Checking the answer

The steady state vector S must be an eigenvector of T'.
Let's check:

$$T'S = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/8 + 1/8 \\ 1/8 + 1/4 + 1/8 \\ 1/8 + 1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$
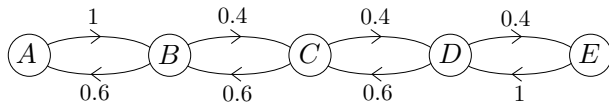
So yes, $\qquad T'S = S$. $\qquad \checkmark$

Will a Markov process always get to a steady state?

Not necessarily!

Example: chemical compounds in transition

Consider a chemical compound whose molecule can exist in any one of five states, termed A, B, C, D and E.

Each molecule frequently undergoes transitions from one state to another, always to an 'adjacent' state, according to the probabilities shown in the transition diagram.

The transition matrix for this Markov Process is

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{ccccc} A & B & C & D & E \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

A beaker full of chemical

- Now suppose we have a beaker full of this chemical. We expect the proportions of A, B, C, D, E to relate to transition probabilities.

- What proportions of the compound will be in the various states?

- To do a thorough analysis of all possible behaviours of this Markov Process, you need to study '*eigenvalues and eigenvectors*' – a reason to take a course or read a book on Linear Algebra.

- But let's see what we can figure out without those tools.

Chemical example — investigating with a computer

Suppose the beaker only contains form 'A' to start with, *i.e.*
$\mathbf{x}_0 = [1, 0, 0, 0, 0]'$. Then by computer to 6dp we find:

$$\mathbf{x}_{100} = (T')^{100}\mathbf{x}_0$$
$$= [0.415383 \, , 0.000000 \, , 0.461538 \, , 0.000000 \, , 0.123077]'$$
$$\mathbf{x}_{101} = T'\mathbf{x}_{100}$$
$$= [0.000000 \, , 0.692308 \, , 0.000000 \, , 0.307692 \, , 0.000000]'$$

and continuing in the same manner

$$\mathbf{x}_{102} = [0.415383 \, , 0.000000 \, , 0.461538 \, , 0.000000 \, , 0.123077]'$$
$$\mathbf{x}_{103} = [0.000000 \, , 0.692308 \, , 0.000000 \, , 0.307692 \, , 0.000000]'$$
$$\mathbf{x}_{104} = [0.415383 \, , 0.000000 \, , 0.461538 \, , 0.000000 \, , 0.123077]'$$
$$\mathbf{x}_{105} = [0.000000 \, , 0.692308 \, , 0.000000 \, , 0.307692 \, , 0.000000]'$$

$$\vdots \qquad\qquad\qquad \vdots$$

It appears to alternate!

However starting with a beaker half full of A and half of B, *i.e.*
$\mathbf{x}_0 = [0.5, 0.5, 0, 0, 0]'$, and again using formulae

$$\mathbf{x}_n = (T')^n \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}_{n+1} = T' \mathbf{x}_n$$

repeatedly we get

$$\mathbf{x}_{100} = [0.207692\,, 0.346154\,, 0.230769\,, 0.153846\,, 0.061539]'$$
$$\mathbf{x}_{101} = [0.207692\,, 0.346154\,, 0.230769\,, 0.153846\,, 0.061539]'$$
$$\mathbf{x}_{102} = [0.207692\,, 0.346154\,, 0.230769\,, 0.153846\,, 0.061539]'$$
$$\vdots \qquad\qquad\qquad \vdots$$

This looks like a steady state!

So this Markov Process is different to those we used to model
employment, weather in Oz, and flower-colours because

eventual behaviour depends on where you start!

Steady state(s) for a beaker of chemical?

We can solve for the steady state to find out if it is unique.

We need to solve

$$T'S = S$$

for $S = [x_1, x_2, x_3, x_4, x_5]'$ subject to additional constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

We use the 'short cut' method:

(a) First construct $[T' - I | 0]$.

(b) Then replace the last row with all 1's.

(c) Then solve by Gaussian elimination or computer.

Steady state(s) for a beaker of chemical?

(a) $[T' - I \,|\, 0]$ is

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0.4 & -1 & 0 \end{array} \right]$$

(b) Replace the last row with all 1's

$$\left[\begin{array}{ccccc|c} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]$$

55

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} R_1' = -R_1 \\ R_2' = (-5/2)R_2 \\ R_3' = (-5/2)R_3 \\ R_4' = (-5/2)R_4 \end{array}$$

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & -1 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 1.6 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} R_2' = R_2 + R_1 \\ \\ \\ R_5' = R_5 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4' = R_4 + (5/2)R_5 \end{array}$$

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.4 & -1 & 1 & 0 \\ 0 & 0 & 3.4 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_3' = R_3 + R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} \\ \\ R_3' = R_3 + (3/2)R_4 \end{array}$$

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 6.1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4' = R_4 + R_3 \\ R_5' = R_5 + (8.5)R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 45/130 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} \\ R_2' = R_2 + (3/2)R_3 \end{array}$$

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 16.25 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ R_5' = R_5 + (16.25)R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 27/130 \\ 0 & 1 & 0 & 0 & 0 & 45/130 \\ 0 & 0 & 1 & 0 & 0 & 15/65 \\ 0 & 0 & 0 & 1 & 0 & 10/65 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} R_1' = R_1 + (5/3)R_2 \end{array}$$

$$\sim \begin{bmatrix} -1 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4/65 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ R_5' = (4/65)R_5 \end{array}$$

Steady state for a beaker of chemical - by computer

Decimal ⌄	X_1	X_2	X_3	X_4	X_5	b
1	-1	0.6	0	0	0	0
2	1	-1	0.6	0	0	0
3	0	0.4	-1	0.4	0	0
4	0	0	0.4	-1	1	0
5	1	1	1	1	1	1

This confirms the unique steady-state solution found by row reduction on the previous slide:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 27/130 \\ 45/130 \\ 15/65 \\ 10/65 \\ 4/65 \end{bmatrix} = \begin{bmatrix} 0.2077 \\ 0.3462 \\ 0.2308 \\ 0.1538 \\ 0.0615 \end{bmatrix}.$$

Solution set:

$x_1 = 27/130$
$x_2 = 9/26$
$x_3 = 3/13$
$x_4 = 2/13$
$x_5 = 4/65$

So the steady-state proportions of the five forms of the chemical are:

A: 20.77%,　B: 34.62%,
C: 23.08%,　D: 15.38%,
E: 6.15%.

A steady state for a beaker of chemical - conclusion

We found that **provided** the beaker reaches a steady-state, then proportions of the various forms of the chemical remain stable at

$A : 20.77\%,\quad B : 34.62\%,\quad C : 23.08\%,\quad D : 15.38\%,\quad E : 6.15\%.$

Individual molecules DO change their form, but at a rate such that overall, proportions remain as above.

END OF SECTION C3