B2: Sequences

Text Reference (Epp)

3ed:	Sections	4.1-4, 8.1-3 (Sequences and induction), 9.3,5 (Sorting)
4ed:	Sections	5.1-4,6-8, (Sequences and induction), 11.3,5 (Sorting)
5ed:	Sections	5.1-4,6-7, (Sequences and induction), 11.3,5 (Sorting)

Sequences

Let S be a set and $I \subseteq \mathbb{Z}$. A function $a : I \to S$ is called a **sequence in** S. Special **sequence notation** is often used:

Function notation	Sequence notation
$a : I \to S$, $n \mapsto a(n)$.	$(a_n)_{n \in I} \subseteq S$

The notation $(a_n)_{n \in I}$ indicates that the function can be represented as an *ordered -tuple* or, more simply, as a *list*.

(Unlike a *set*, a list has an order, and can have repeated entries.)

Examples

- $I = \{1, 2, 3\}: \ (a_n)_{n \in I} = (a_1, a_2, a_3)$.
- $I = \mathbb{N}: \ (a_n)_{n \in I} = (a_1, a_2, a_3, \ldots)$.
- $I = \mathbb{Z}_{\geq 0}: \ (a_n)_{n \in I} = (a_0, a_1, a_2, \ldots)$.

In practice we usually leave out the parentheses and speak of "the sequence a_1, a_2, a_3" or "the sequence $a_0, a_1, a_2, \ldots \ldots$."

An agreed upon abuse of notation

The "$\subseteq S$" part of the sequence notation $(a_n)_{n \in I} \subseteq S$ indicates that the sequence members belong to S; *i.e.* that the range of the sequence function $a : I \to S$ is a subset of its codomain S.

The sequence *itself* is **not** a subset of S, since it is not a *set*.

Examples

1. Suppose n represents time (in months since January 1, 2000) and a_n is the standard savings account interest rate offered by bank X at time n. Then $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ is a sequence of interests rates since 2000 and into the future!

   For example, a_{17} is the standard savings account interest rate offered by bank X on 1 June, 2001.

Examples

2. Suppose n represents time (in months since January 1, 2000) and a_n, f_n, z_n represent the populations of amphibians, fish and zooplankton in a particular lake ecosystem at time n. Let $p_n = (a_n, f_n, z_n)$. Then $(p_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a sequence of states of the ecosystem since 2000 and into the future!

Examples

3. For each $n \in \mathbb{N}$, let a_n denote the amplitude of the harmonic of frequency $n \times f$ (where f is the fundamental frequency). Then $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}_{\geq 0}$ is a sequence of amplitudes.

4. Let U be a set of users, then $(u_n)_{n \in \{1,2,3,4,5\}} \subseteq U$ is a list of 5 users.

In examples 1, 2, 3 the indexing variable n had some intuitive meaning; in example 4 the indexing variable did not necessary have an intuitive meaning other than we have ordered the 5 interesting users into the first, second, third, fourth and fifth user.

Describing sequences: explicit definitions

An **explicit definition** of a sequence is a formula for a_n.

Examples:

1. For all $n \in \mathbb{N}$, let $a_n = 2^n$. Then

$$(a_n)_{n \in \mathbb{N}} = 2, 4, 8, 16, \ldots$$

2. Let $a_1 =$ Pierre, $a_2 =$ Julie, $a_3 =$ Paul. Then $(a_n)_{n \in \{1,2,3\}} =$ Pierre, Julie, Paul.

Describing sequences: implicit definitions

An **implicit definition** of a sequence comprises starting value(s) and a relationship between the a_n's.

Examples: Let $(a_n)_{n\in\mathbb{N}}$ be the sequence such that:

$$\begin{cases} a_1 = 2, \text{ and} \\ \forall n \in \mathbb{N}\ a_{n+1} = 2a_n. \end{cases}$$

This defines the sequence

$$(a_n)_{n\in\mathbb{N}} \ = \ 2, 4, 8, 16, \ldots,$$

Another example

Let $(a_n)_{n\in\mathbb{N}}$ be the sequence such that:

$$\begin{cases} a_1 = 0, \\ a_2 = 1, \text{ and} \\ \forall n \in \{2, 3, 4, \ldots\}\ a_{n+1} = -a_n + a_{n-1}. \end{cases}$$

Defines the sequence

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= -1+0=-1 \\ a_4 &= -(-1)+1=2 \\ a_5 &= -2+(-1)=-3 \\ \vdots\ & \quad \vdots \end{aligned}$$

Proofs about sequences

Mathematical induction

Let $P(n)$ be a predicate with variable $n \in \mathbb{N}$.
How to prove that $\forall n \in \mathbb{N} \;\; P(n)$?

METHOD 1:

Introduce a fixed but arbitrary variable: Let $n \in \mathbb{N}$.
you are now working with a fixed but arbitrary value of n.

Deduce $P(n)$ from what you know: *Insert mathemagic here.*

Victory lap: Since $P(n)$ holds for a fixed but arbitrary choice $n \in \mathbb{N}$, $P(n)$ holds for all $n \in \mathbb{N}$. *No one write this, but this is why the method works.*

Method 2:

The basis step Prove $P(1)$.

The inductive step Prove

$$\forall n \in \mathbb{N} \quad \Big((P(1) \wedge P(2) \wedge P(3) \wedge ... \wedge P(n)) \Rightarrow P(n+1)\Big)$$

Let $n \in \mathbb{N}$. Suppose that all of the statements $P(1)$, $P(2)$, ..., $P(n)$ are true. *Now deduce* $P(n+1)$ *making use somewhere of one or more of the facts* $P(1), \ldots, P(n)$.

The victory lap By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(This is also known as *strong* mathematical induction.)

From implicit to explicit definitions; Example 1

A sequence is defined implicitly by

$$\begin{cases} \boldsymbol{a_1 = 3,} \\ \forall n \in \mathbb{N}\ \boldsymbol{a_{n+1} = 3a_n} \end{cases}$$

Find an explicit definition.

From implicit to explicit definitions; Example 1

A sequence is defined implicitly by

$$\begin{cases} \boldsymbol{a_1 = 3,} \\ \forall n \in \mathbb{N}\ \boldsymbol{a_{n+1} = 3a_n} \end{cases}$$

Find an explicit definition.
First generate some values:

$a_1 = 3$, $a_2 = 9$, $a_3 = 27$, $a_4 = 81$,...

Now we make a claim/hypothesis/informed guess:
$\forall n \in \mathbb{N}\ a_n = 3^n$.

Proof that the claim is correct

We shall prove the claim using mathematical induction. Let

$$P(n): \qquad a_n = 3^n$$

Basis step: We compute

LHS of $P(1) = a_1 = 3$ (by the definition of the sequence);
RHS of $P(1) = 3^1 = 3$.

Hence $P(1)$ is true.

(LHS is an abbreviation for "left-hand side", and RHS is an abbreviation for "right-hand side.")

Inductive step: Let $n \in \mathbb{N}$. Suppose that $P(1)$, $P(2)$, ..., $P(n)$ are all true. Then

$$\begin{aligned}
&\text{LHS of } P(n+1) \\
&= a_{n+1} \\
&= 3a_n \quad \text{(from the implicit definition)} \\
&= 3(3^n) \quad \text{(using } P(n)\text{)} \\
&= 3^{n+1} \\
&= \text{RHS of } P(n+1)
\end{aligned}$$

Hence $P(n+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$. $\square$

Geometric sequences

Given a set of integers $K = \{n \in \mathbb{Z} \mid n \geq k\}$, a sequence $(a_n)_{n \in K} \subseteq \mathbb{R}$ is a **geometric sequence** when there exist $a, r \in \mathbb{R}$ such that

$$\begin{cases} a_k = a, \text{ and} \\ \forall k \in K \quad a_{k+1} = r a_k \end{cases}$$

We call a the **first term** and r the **common ratio** of the geometric sequence.

A geometric sequence can also be defined explicitly:

$$\forall n \in K \quad a_n = a r^{n-k}.$$

Another example

A sequence is defined implicitly by

$$\begin{cases} b_1 = 0, \\ \forall n \in \mathbb{N} \;\; b_{n+1} = b_n + 5. \end{cases}$$

Can we get an explicit definition?

Another example

A sequence is defined implicitly by

$$\begin{cases} b_1 = 0, \\ \forall n \in \mathbb{N} \;\; b_{n+1} = b_n + 5. \end{cases}$$

Can we get an explicit definition? First generate some values:

$b_1 = 0$, $b_2 = 5$, $b_3 = 10$, $b_4 = 15$,..

Another example

A sequence is defined implicitly by

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Can we get an explicit definition? First generate some values:

$b_1 = 0$, $b_2 = 5$, $b_3 = 10$, $b_4 = 15$,..
Claim: $\forall n \in \mathbb{N}$ $\boldsymbol{b_n = 5(n-1)}$.

Another example

A sequence is defined implicitly by

$$\begin{cases} b_1 = 0, \\ \forall n \in \mathbb{N} \;\; b_{n+1} = b_n + 5. \end{cases}$$

Can we get an explicit definition? First generate some values:

$b_1 = 0$, $b_2 = 5$, $b_3 = 10$, $b_4 = 15$,..
Claim: $\forall n \in \mathbb{N}$ $\boldsymbol{b_n = 5(n-1)}$.

We shall use mathematical induction to prove the claim...

Let

$$P(n): \qquad b_n = 5(n-1).$$

Basis step:

LHS of $P(1) = b_1 = 0$ (by the definition of the sequence);
RHS of $P(1) = 5(1-1) = 1 \times 0 = 0$.

Hence $P(1)$ is true.

Inductive step: Let $n \in \mathbb{N}$. Suppose that $P(1)$, $P(2)$, …, $P(n)$ are all true. Then

$$\begin{aligned}
&\text{LHS of } P(n+1) \\
&= b_{n+1} \\
&= b_n + 5 \quad \text{(from the implicit definition)} \\
&= 5(n-1) + 5 \quad \text{(using } P(n)\text{)} \\
&= 5n - 5 + 5 \\
&= 5n \\
&= \text{RHS of } P(n+1)
\end{aligned}$$

Hence $P(n+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$. $\square$

Sum and products of terms

Terms of a sequence can be summed: $a_1 + a_2 + a_3 + ...$ or multiplied: $a_1 \times a_2 \times a_3 \times$ .... We use the special notation

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + ... + a_k,$$

$$\prod_{n=1}^{k} a_n = a_1 \times a_2 \times a_3 \times ... \times a_k.$$

Examples

1. $$\sum_{n=1}^{10} n = 1 + 2 + 3 + 4 + ... + 9 + 10 = 55.$$

2. $$\sum_{n=0}^{7} 2^n = 1 + 2 + 4 + 8 + ... + 128 = 255.$$

3. $$\prod_{n=1}^{5} n = 1 \times 2 \times 3 \times 4 \times 5 = 5! = 120.$$

4. $$\prod_{n=1}^{8} n^2 = 4 \times 9 \times 16 \times ... \times 64 = 1\,625\,702\,400.$$

Another example

Prove the following:

$$\forall n \in \mathbb{N} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Let

$$P(n): \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Basis step:

$$\text{LHS of } P(1) = \sum_{i=1}^{1} i = 1 \text{ (by the defn of } \sum);$$

$$\text{RHS of } P(1) = \frac{1(1+1)}{2} = \frac{1 \times 2}{2} = 1$$

Hence $P(1)$ is true.

Inductive step: Let $n \in \mathbb{N}$. Suppose that $P(1)$, $P(2)$, ..., $P(n)$ are all true. Then

$$\begin{aligned}
\text{LHS of } P(n+1) &= \sum_{i=1}^{n+1} i \\
&= \left(\sum_{i=1}^{n} i\right) + (n+1) \text{ (by the defn of } \textstyle\sum) \\
&= \frac{n(n+1)}{2} + (n+1) \quad \text{(using } P(n)) \\
&= \frac{n(n+1) + 2(n+1)}{2} = \frac{n^2 + n + 2n + 2}{2} \\
&= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2} = \text{RHS of } P(n+1)
\end{aligned}$$

Hence $P(n+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$. $\square$