Section A2: Sets (continued)

New sets from old

Suppose that A and B are subsets of a universe U.

The **union** of A and B, denoted $A \cup B$, is the set

$$\{x \in U \mid (x \in A) \vee (x \in B)\}.$$

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$$\{x \in U \mid (x \in A) \wedge (x \in B)\}.$$

The **difference** of B minus A, or B **without** A, denoted $B - A$ or $B \setminus A$, is the set

$$\{x \in U \mid (x \in B) \wedge (x \notin A)\}.$$

New sets from old

Suppose that A and B are subsets of a universe U.

The **complement** of A (in U), denoted A^c, is the set

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The complement of A cannot be understood unless the universe of discourse has been communicated.

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The **symmetric difference** of A and B, denoted $A \triangle B$, is the set

$$\{x \in U \mid (x \in A) \oplus (x \in B)\}.$$

Some examples

Suppose that the universe of discourse is the set $\mathbb{Z}$ and let

O be the set of odd integers
E be the set of even integers
P be the set of primes
C be the set of composite numbers.

A **composite number** is a positive integer that can be formed by multiplying two smaller positive integers.

Find simple expressions for: $O \cup E$, $O \cap E$, $E \cap P$, $O \cap P$, $P \cup C$, O^c, P^c, $E \triangle P$, $(O \triangle P) \cap \mathbb{Z}^+$

Some examples

$$
\begin{aligned}
O \cup E &= \mathbb{Z} \\
O \cap E &= \emptyset \\
E \cap P &= \{2\} \\
O \cap P &= P \setminus \{2\} \\
P \cup C &= \{2, 3, 4, 5, \ldots, \} \\
&= \{x \in \mathbb{Z} \mid x \geq 2\} \\
O^c &= E \\
P^c &= \{\ldots, -3, -2, -1, 0, 1\} \cup C \\
&= \{z \in \mathbb{Z} \mid z \leq 1\} \cup C \\
E \triangle P &= (E \cup P) \setminus \{2\} \\
(O \triangle P) \cap \mathbb{Z}^+ &= (O \cap C) \cup \{1, 2\}.
\end{aligned}
$$

Using logic to prove set identities

An **identity** is a relationship that holds not matter which substitutions are made for the variables.

Since the set operations $\cup$, $\cap$, $\setminus$, $\subseteq$, $\ ^{c}$ and $\triangle$ are defined using logical connectives, logical equivalences can be used to proved set theoretic identities.

An example

Let A, B and C be subsets of a universe U. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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Proof: Let $x \in U$.

$$
\begin{aligned}
& x \in A \cap (B \cup C) & \\
\Leftrightarrow & (x \in A) \wedge (x \in B \cup C) & \text{Defn of } \cap \\
\Leftrightarrow & (x \in A) \wedge (x \in B \vee x \in C) & \text{Defn of } \cup \\
\Leftrightarrow & ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) & \text{Distr.} \\
\Leftrightarrow & (x \in A \cap B) \vee (x \in A \cap C) & \text{Defn of } \cap \\
\Leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{Defn of } \cup \ \square
\end{aligned}
$$

Note: we use $\Leftrightarrow$ between statements here, not $\equiv$. We shall reserve $\equiv$ for when we are working with statements and statement forms considering logic only.

An “element proof.”

Let A, B and C be subsets of a universe U. Then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof:
We prove the set equality by two subset proofs. First we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

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Proof:
We prove the set equality by two subset proofs. First we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Let $x \in U$. Suppose that $x \in A \cap (B \cup C)$.

By definition of $\cap$, $x \in A$ and $x \in B \cup C$. By definition of $\cup$, $x \in B$ or $x \in C$. We consider cases.

Case $x \in B$: Since $x \in A$ and $x \in B$, $x \in A \cap B$. Hence $x \in (A \cap B) \cup (A \cap C)$.

Case $x \notin B$: Since $x \notin B$, $x \in C$. Since $x \in A$ and $x \in C$, $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$.

In all cases, $x \in (A \cap B) \cup (A \cap C)$.

Element proof (cont)

Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

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Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Let $x \in U$. Suppose that $x \in (A \cap B) \cup (A \cap C)$.

Element proof (cont)

Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Let $x \in U$. Suppose that $x \in (A \cap B) \cup (A \cap C)$.

By definition of $\cup$, $x \in A \cap B$ or $x \in A \cap C$. We consider cases.

Case $x \in A \cap B$: By definition of $\cap$, $x \in A$ and $x \in B$. Since $x \in B$, $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$, $x \in A \cap (B \cup C)$.

Case $x \notin A \cap B$: Since $x \notin A \cap B$, $x \in A \cap C$. By definition of $\cap$, $x \in A$ and $x \in C$. Since $x \in C$, $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$, $x \in A \cap (B \cup C)$.

In all cases, $x \in A \cap (B \cup C)$. $\square$

Another construction

For any set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

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Q: If A has n elements, how many elements does $\mathcal{P}(A)$ have?

A: $\mathcal{P}(A)$ has 2^n elements... for reasons we will explain when we discuss counting techniques later in the course.

Cartesian products: Another way to make new sets from old

Order and multiplicity

In sets, there is no sense of the order in which elements appear and there is no idea of how many times an elements appears.

However, in many situations the order in which data appears is important, and the same data sometimes appears multiple times.

We now look at a construction that allows us to represent order and multiplicity.

Ordered n-tuples

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple** $(x_1, x_2, \ldots, x_n)$ consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples are **equal** when their elements match up exactly in order. Symbolically:

$$\begin{aligned}&(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)\\ \Leftrightarrow&(x_1 = y_1) \wedge (x_2 = y_2) \wedge \cdots \wedge (x_n = y_n).\end{aligned}$$

An ordered m-tuple and an ordered n-tuple cannot be equal if $m \neq n$.

Examples

$(a, b, c) \neq (b, c, a)$ because their first elements differ.

$(a, a, b, c) \neq (a, b, c)$ because one is an ordered 4-tuple and the other is an ordered triple.

The elements in ordered n-tuples do not need to be of the same type. For example, $(\text{cat}, \text{car}, 1, \$)$ is an ordered 4-tuple.

We are, however, usually interested in sets of ordered n-tuples where all of the elements in, say, the i-th position are of the "same type"...

Cartesian product

Given (not necessarily distinct sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, …, $a_n \in A_n$.

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as:

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$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

This last expression may be read aloud as: "the set of all ordered n-tuples with elements a_1, a_2, through, a_n such that a_1 comes from A_1, a_2 comes from A_2, through a_n comes from A_n."

A remark

The expression

$$\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

does not appear to conform to the rules of set-builder notation we laid out in the last lecture because

- the domain part introduces variables but does not specify a domain for each;
- the "predicate" does not appear to be a single predicate.

We can fix the second concern easily by making a rule that in a predicate, each comma is read and understood as "and". It is usually better to use $\wedge$.

What I have written is an entirely standard way to describe a Cartesian product, even though it seems like a poor use of set-builder notation.

Examples

Let

$$A = \{\text{cat}, \text{dog}, \text{chicken}\}$$
$$B = \{\text{yes}, \text{no}\}$$
$$C = \{100, 300\}$$

Then

$$A \times B = \{(\text{cat}, \text{yes}), (\text{cat}, \text{no}), (\text{dog}, \text{yes}), (\text{dog}, \text{no}), (\text{chicken}, \text{yes}), (\text{chicken}, \text{no})\}.$$

and

$$C \times C = \{(100, 100), (100, 300), (300, 100), (300, 300)\}.$$

Partitions: A structure for recognising that a classification works well

Motivation

A common task in any discipline (science, mathematics, philosophy, humanities, ...) is that of classifying things of a certain type into various sub-types. Thanks to our development of set theoretic tools, we have a way to formalise what it means for such a classification scheme to work really well.

Q: What properties do you think an excellent classification scheme will have?

Example

Which, if any, of the following classification schemes works well?

- We classify each integer as positive, negative or 0.
- We classify each song on the charts as pop, rock or hip-hop.
- We classify each student enrolled in this course as a mathematician or a computer scientist or a physicist.

Disjoint sets

Sets A, B are called **disjoint** when $A \cap B = \emptyset$.

Given a set of sets $\mathcal{S}$, the sets in $\mathcal{S}$ are said to be **pairwise disjoint** when

$$\forall A, B \in \mathcal{S} \;\; A \neq B \Rightarrow A \cap B = \emptyset.$$

An example

Let P be the set of prime numbers, C the set of composite numbers, and E be the set of even integers.

EXAMPLE: Let $\mathcal{A} = \{\{1\}, P, C\}$. Since

$$\{1\} \cap P = \emptyset, \{1\} \cap C = \emptyset \text{ and } P \cap C = \emptyset,$$

the sets in $\mathcal{A}$ are pairwise disjoint.

EXAMPLE: Let $\mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$. Since $(E \cap \mathbb{N}) \cap P = \{2\}$, the sets in $\mathcal{B}$ are not pairwise disjoint.

Partitions

Let S be a set and $\mathcal{A} \subseteq \mathcal{P}(S)$ (so $\mathcal{A}$ is a set, the elements of which are subsets of S). We say that $\mathcal{A}$ is a **partition** of S when each of the following statements is true:

1. $\emptyset \notin \mathcal{A}$
2. every element of S is an element of some set in $\mathcal{A}$ (that is, $\forall s \in S \ \exists A \in \mathcal{A} \ s \in A$)
3. the sets in $\mathcal{A}$ are pairwise disjoint.

Q: Do you agree or disagree that the three properties listed in the definition of a partition are a reasonable interpretation of what it means for a classification scheme (that classifies the elements of S) to be 'excellent.'

Examples

- $\mathcal{A} = \{\{1\}, P, C\}$ is a partition of $\mathbb{N}$
- $\mathcal{B} = \{\{1\}, P, E \cap \mathbb{N}\}$ is not a partition of $\mathbb{N}$ because the sets in $\mathcal{B}$ are not pairwise disjoint.
- $\mathcal{A} = \{\{1\}, P, C\}$ is not a partition of $\mathbb{Z}_{\geq 0}$ because $0 \in \mathbb{Z}_{\geq 0}$ but 0 is not in any set in $\mathcal{A}$.
- Let P, C, E, O be as above. Then $\{P \cap C, P \cap E, P \cap O\}$ is not a partition of P, because $P \cap C = \emptyset$.
- Let P, C, E, O be as above. Then $\{\{1\}, P \cap E, P \cap O, C\}$ is a partition of $\mathbb{N}$.

Russell's Paradox

Russell's paradox

Most sets are not members of themselves.

EXAMPLE: If $P = \{\text{pigeon, parrot}\}$ then $P \notin P$ (since $\{\text{pigeon, parrot}\} \neq \text{pigeon}$, and $\{\text{pigeon, parrot}\} \neq \text{parrot}$.)

But some sets **are** members of themselves.

EXAMPLE: The set of all sets. The set of all things that are not birds.

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EXAMPLE: The set of all sets. The set of all things that are not birds.

We shall call a set **regular** if it is not a member of itself.

We let $\mathcal{R}$ denote the set of all regular sets.

Question: **Is $\mathcal{R}$ itself regular?**

$$
\begin{aligned}
& \mathcal{R} \text{ not regular} \\
\Longrightarrow\ & \mathcal{R} \text{ is a member of itself} \\
\Longrightarrow\ & \mathcal{R} \in \mathcal{R} \\
\Longrightarrow\ & \mathcal{R} \in \text{the set of all regular sets} \\
\Longrightarrow\ & \mathcal{R} \text{ is regular}
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{R} \text{ is regular} \\
\Longrightarrow\ & \mathcal{R} \text{ is not a member of itself} \\
\Longrightarrow\ & \mathcal{R} \notin \mathcal{R} \\
\Longrightarrow\ & \mathcal{R} \notin \text{the set of all regular sets} \\
\Longrightarrow\ & \mathcal{R} \text{ not regular}
\end{aligned}
$$

A contradiction either way! (paradox)... Naive set theory fails!

Axiomatic Set Theory

The paradox on the previous slide was discovered by Bertrand Russell in 1901, and is known as **Russell's Paradox**.

To avoid problems such as Russell's paradox we need to move to a more formal approach. The most classical is an axiomatic system called ZFC (Zermelo-Fraenkel with choice). We will not study it in this course, but will assume some of its axioms. In particular:

Axiom 1: A set T can only be defined as a subset of a known set U. That is, the definition must have the form:

$$T \;=\; \{x \in U \mid p(x)\}$$

where the domain of predicate p includes all elements of U.

Example

The phrase “the set of all birds” does not define a set in ZFC. But, if we have managed to define A as the set of all animals, then “the sets of all animals that are birds” does define a set in ZFC as we can define:

A: set of all animals.

$B = \{b \in A \mid b \text{ is a bird}\} \subseteq A$.

Russell's paradox removed

In ZFC we can still say that a set S is **regular** if and only if it satisfies the condition (predicate) $S \notin S$.

However we cannot define $\mathcal{R}$ as the set of *all* regular sets.

Instead, for any *known set of sets* $\mathcal{U}$ we can define $\mathcal{R}_{\mathcal{U}}$ by

$$\boxed{\mathcal{R}_{\mathcal{U}} = \{S \in \mathcal{U} \mid S \text{ is regular}\} = \{S \in \mathcal{U} \mid S \notin S\}}$$

So we again ask: **For any set of sets $\mathcal{U}$, is $\mathcal{R}_{\mathcal{U}}$ itself regular?**

Let $\mathcal{U}$ be a known set of sets.

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Let $\mathcal{U}$ be a known set of sets. Then

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\begin{aligned}
& \mathcal{R}_{\mathcal{U}} \text{ not regular} \\
\Longrightarrow\quad & \mathcal{R}_{\mathcal{U}} \text{ is a member of itself} \\
\Longrightarrow\quad & \mathcal{R}_{\mathcal{U}} \in \mathcal{R}_{\mathcal{U}} \\
\Longrightarrow\quad & \mathcal{R}_{\mathcal{U}} \in \{ S \in \mathcal{U} \mid S \text{ is regular } \} \\
\Longrightarrow\quad & \mathcal{R}_{\mathcal{U}} \text{ is regular} \qquad \text{Contradiction!}
\end{aligned}
$$

So we again ask: **For any set of sets $\mathcal{U}$, is $\mathcal{R}_{\mathcal{U}}$ itself regular?**

Let $\mathcal{U}$ be a known set of sets. Then

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$$

$$
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& \mathcal{R}_{\mathcal{U}} \text{ regular} \\
\Longrightarrow\ & \mathcal{R}_{\mathcal{U}} \text{ is not a member of itself} \\
\Longrightarrow\ & \mathcal{R}_{\mathcal{U}} \notin \mathcal{R}_{\mathcal{U}} \\
\Longrightarrow\ & \mathcal{R}_{\mathcal{U}} \notin \{S \in \mathcal{U} \mid S \text{ is regular }\} \\
\Longrightarrow\ & (\mathcal{R}_{\mathcal{U}} \notin \mathcal{U}) \vee (\mathcal{R}_{\mathcal{U}} \text{ is not regular})
\end{aligned}
$$

So we again ask: **For any set of sets $\mathcal{U}$, is $\mathcal{R}_\mathcal{U}$ itself regular?**

Let $\mathcal{U}$ be a known set of sets. Then

$$\begin{array}{ll} & \mathcal{R}_\mathcal{U} \text{ not regular} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \text{ is a member of itself} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \in \mathcal{R}_\mathcal{U} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \in \{S \in \mathcal{U} \mid S \text{ is regular } \} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \text{ is regular} \qquad \text{Contradiction!} \end{array}$$

Since

$$\begin{array}{ll} & \mathcal{R}_\mathcal{U} \text{ regular} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \text{ is not a member of itself} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \notin \mathcal{R}_\mathcal{U} \\ \Longrightarrow & \mathcal{R}_\mathcal{U} \notin \{S \in \mathcal{U} \mid S \text{ is regular } \} \\ \Longrightarrow & (\mathcal{R}_\mathcal{U} \notin \mathcal{U}) \vee (\mathcal{R}_\mathcal{U} \text{ is not regular}) \end{array}$$

$\Big((\mathcal{R}_\mathcal{U} \text{ regular}) \Longrightarrow (\mathcal{R}_\mathcal{U} \text{ is not regular})\Big)$ is contradictory, we conclude that $\mathcal{R}_\mathcal{U}$ is regular but not in $\mathcal{U}$. There is no overall contradiction.