Section A1: Logic (continued)

How to prove things Putting our logic to work.

How to start

Before trying to prove a statement, you should clearly identify the logical structure of the statement. Doing so allows you to understand the choices you have in choosing a logical structure for your proof.

Let's understand the logical structures that can be used to prove statements with various logical structures.

Proving $\forall$

To prove a statement of the form $\forall x \, p(x)$, one may follow this plan:

Let x be a (fixed but arbitrary) element of the predicate domain. Argue that $p(x)$ is true.

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Example
Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Working the example

We need some definitions:

Defn: An integer x is **even** if there exists an integer k such that $x = 2k$.

Defn: An integer x is **odd** if there exists an integer k such that $x = 2k + 1$.

Theorem: (The even/odd theorem)
Every integer is either even or odd; no integer is both even and odd.

We note that the theorem could be stated succinctly using our logic notation:

$$\forall x \;\; (x \text{ is even}) \oplus (x \text{ is odd})$$

where the domain of quantification is understood to be the integers.

Back to our example

Example

Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Proving $\exists$

To prove a statement of the form $\exists x\ p(x)$, one may identify a particular element of the predicate domain and establish that $p(x)$ is true. Please note, it is not enough to simple state which element x of the domain has the required property, you should explain how you know that the particular element you identified has the required property (how you know that $p(x)$ is true). This is called **exhibiting an example**.

An example

Prove the following statement: The equation $x^2 - 6x + 8 = 0$ has an integer solution.

Disproving $\forall$

To disprove a statement of the form $\forall x \; p(x)$, one should prove the statement $\exists x \; \neg p(x)$. (This is called providing a **counterexample**)

An example

Prove or disprove the following statement: For every integer x, $(x-1)^2 + (x-1)$ is positive.

Disproving $\exists$

To disprove a statement of the form $\exists x\ p(x)$, one should prove the statement $\forall x\ \neg p(x)$.

An example

Prove or disprove the following statement:

$$\exists y \;\; \forall x \;\; (y \leq x),$$

where the quantification is over the set of integers.

Proving $\rightarrow$

To prove $p \rightarrow q$ you may:

- Suppose that p is true.
- Deduce by valid reasoning that q must be true (using the truth of p along the way).

This is called **arguing directly**.

You may also:

- Suppose that $\neg q$ is true.
- Deduce by valid reasoning that $\neg p$ must be true (using the truth of $\neg q$ along the way).

This is called **arguing via the contrapositive**.

An example

Prove the following statement: For all integers x, if x is even then $x^2 + 2$ is even.

An example

Prove the following statement: For all integers x, if $x^2 + 2$ is even, then x is even.

Proving $\leftrightarrow$

To prove $p \leftrightarrow q$, you may first prove $p \to q$ and then prove $q \to p$

It is possible to accomplish "both directions" of proof simultaneously by arguing with biconditionals throughout your proof, but you must be careful when doing so.

An example

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

Arguing by cases

If the domain of a predicate is partitioned into subsets, you may prove a $\forall$ statement by proving it for each subset.

Example

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

Proof by contradiction

To prove a statement p, you may disprove $\neg p$. One way to do this is to suppose $\neg p$, and use this fact to deduce a statement we know to be false. Since a true statement cannot imply a false statement, we must have that $\neg p$ is false. This is called a **proof by contradiction**.

An example

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

Some advice

1. Before starting a proof, clearly identify the logical structure of the statement to be proved.
2. Consider your options for a logical structure that will prove the statement.
3. Write down the logical structure of your argument so that the reader knows what is going on.
4. When deciding between a direct argument and an argument via the contrapositive, try whichever direction appears to allow you to make the strongest supposition first. The same advice applies when considering a proof by contradiction or one of the other methods.