The Halting Problem

Theorem (Alan Turing, 1936): There is no algorithm that will accept any algorithm X and data set D as input and then will output "halts" or "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D

Proof: We shall use a proof by contradiction. Suppose that there is an algorithm, CheckHalt, such that if algorithm X and a data set D are input, then:

- CheckHalt(X, D) prints "halts" if X terminates in a finite number of steps when run with data set D;
- CheckHalt(X, D) prints "loops forever" if X does not terminate in a finite number of steps when run with data set D.

The Halting Problem (continued)

A sequence of characters making up an algorithm X can be regarded as a data set itself. So it is possible to call CheckHalt(X, X).

Let Test be a new algorithm that takes as input an algorithm X and so that:

- Test(X) loops forever if CheckHalt(X, X) prints "halts";
- Test(X) stops if CheckHalt(X, X) prints "loop forever".

(The existence of such an algorithm follows immediately from the existence of CheckHalt.)

Now consider what happens when we run the algorithm Test with the input Test. Either it terminates after a finite number of steps or it loops forever.

The Halting Problem (continued)

Consider first the case that Test(Test) terminates after a finite number of steps. Then CheckHalt(Test, Test) prints "halts" and so Test(Test) loops forever. This is a contradiction.

Consider next the case that Test(Test) loops forever. Then CheckHalt(Test, Test) prints "loops forever" and so Test(Test) terminates in a finite number of steps. This is also a contradiction.

In each case, we reached a contradiction. Our supposition that an algorithm such as CheckHalt exists allowed us to deduce a false statement. It is therefore false itself. □

An example to pull some ideas together

In parallel computing, multiple CPUs are connected into a network. Each CPU gets busy working on different parts of a problem. Occasionally, the CPUs need to communicate.

PROBLEM: How can we connect many CPUs into network so that:

1. We don't use too many connections, because connections cost money and take up physical space
2. When one CPU needs to communicate with another, the message does not need to be passed through too many intermediary CPUs
3. CPUs are labelled in a logical way that allows us to write easily write algorithms for getting messages around the network.

A possible solution

Connect your CPUs so as to make a hypercube.

For each $n \in \{1, 2, 3, \dots\}$, the **hypercube** H_n is the graph H_n such that

- The vertex set of H_n is $\{0, 1\}^n$; that is, the vertex set of H_n is the set of bit strings of length n
- Two vertices of H_n are adjacent if and only if they differ in exactly one bit.

Getting comfortable with H_n

When trying to become comfortable with a new family of objects, it helps to spend some time building intuition...just thinking about them.

Let's try drawing some hypercubes

The vertex set of H_1 is $\{0, 1\}$. We can identify the vertices of H_1 as points in $\mathbb{R}$ (the number line), and then draw the edge between them to get ... an interval.

The vertex set of H_2 is $\{00, 01, 10, 11\}$. We can identify the vertices of H_2 as the points $\{(0, 0), (0, 1), (1, 0), (1, 1\}$ in $\mathbb{R}^2$ (the Euclidean plane), and then draw the edges between them to get ... the frame of a square.

Getting comfortable with H_n (cont.)

The vertex set of H_3 is $\{000, 001, 010, 011, 100, 101, 110, 111\}$. We can identify the vertices of H_3 as the points

$$\{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

in $\mathbb{R}^3$, and then draw the edges between them to get ... the frame of a cube.

What do we get when represent H_4 in $\mathbb{R}^4$?

Some questions

Let $n \in \mathbb{N}$.

1. How many vertices in H_n?
2. What is the degree of each vertex in H_n?
3. How many edges in H_n?
4. What is the greatest distance between any pair of vertices in H_n?
5. Find a Hamilton circuit in H_n, if one exists.

Now suppose that a parallel computing network has the structure of H_n. In this context, what is the importance of each question and answer above?

A Hamilton Circuit in H_n

We use the n-bit reflected binary code to build a Hamilton circuit in H_n.

0, 1, 0 is a Hamilton 'circuit' on H_1.

start with rbc for $n-1$	reflect	add 0's above and 1's below
0	0	00
1	1	01
	1	11
	0	10

Note that 00, 01, 11, 10, 00 is a Hamilton circuit in H_2.

A Hamilton Circuit in H_n (cont.)

start with rbc for $n-1$	reflect	add 0's above and 1's below
00	00	000
01	01	001
11	11	011
10	10	010
	10	110
	11	111
	01	101
	00	100

Note that 000, 001, 011, 010, 110, 111, 101, 100, 000 is a Hamilton circuit in H_3.

Pros and Cons

What do you think are the pros and cons of arranging your parallel computing network so that it has the structure of a hypercube?

Addendum to D2: A proof that Kruskal's Algorithm works

Text Reference (Epp) 5ed: Section 10.6

Some familiar definitions

A **tree** is a connected graph with no circuits other that the trivial ones.

A **forest** is a disjoint collection of trees.

Note that we may consider a tree to be a forest (with just one tree). We shall do this. So a forest is a graph with no circuits.

Recall: Characterising Trees (D2 Slide 91)

Theorem: Let T be a graph with n vertices. The following statements are logically equivalent:

(i) T is a tree (it is connected and has no non-trivial circuits).

(ii) T has no simple circuits and $n-1$ edges.

(iii) T is connected and has $n-1$ edges.

(iv) T is connected and every edge is a bridge.

(v) Any two vertices of T are connected by exactly one simple path.

(vi) T contains no non-trivial circuits, but the addition of any new edge (connecting an existing pair of vertices) creates a simple circuit.

Kruskal's algorithm for minimal spanning tree

Kruskal's algorithm is an obvious modification to the spanning-tree-finding algorithm we discussed previously. We simply choose the 'cheapest' possible edge at each step:

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Total weight W of this tree.

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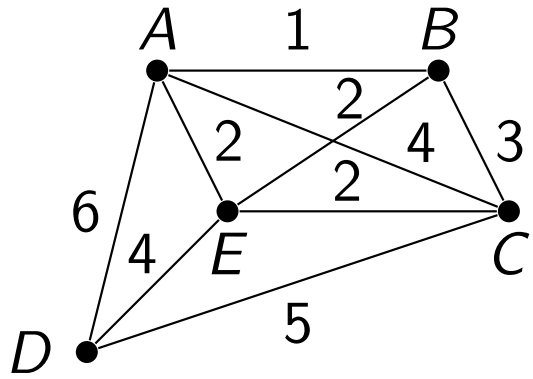
4. Repeat steps 2 and 3 until T has $n-1$ edges.

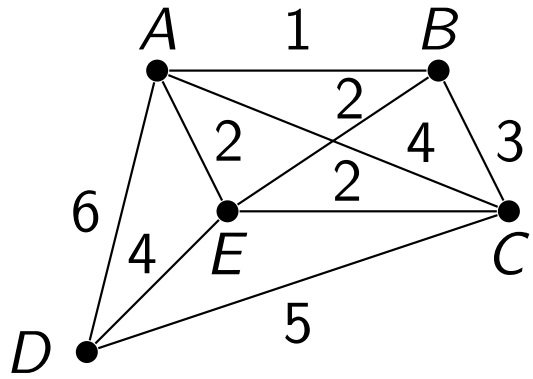
Example: Applying Kruskal's algorithm

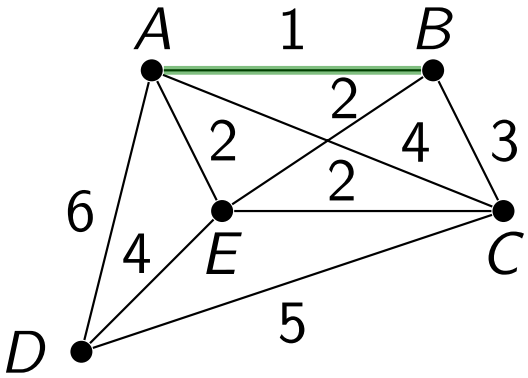
Find a minimal spanning tree for this weighted graph:



Example: Applying Kruskal's algorithm

Find a minimal spanning tree for this weighted graph:

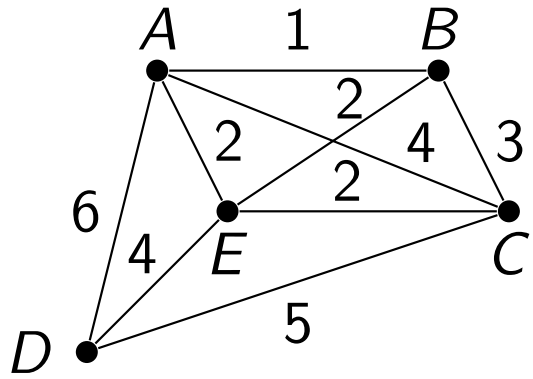


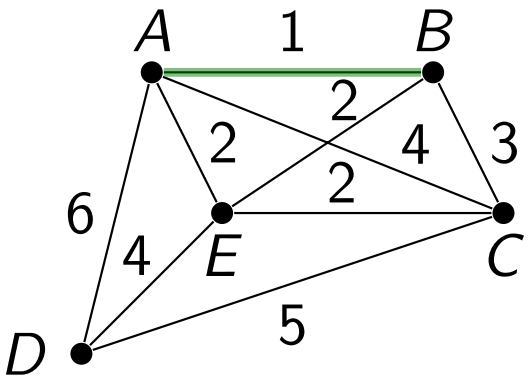


$W = 1$

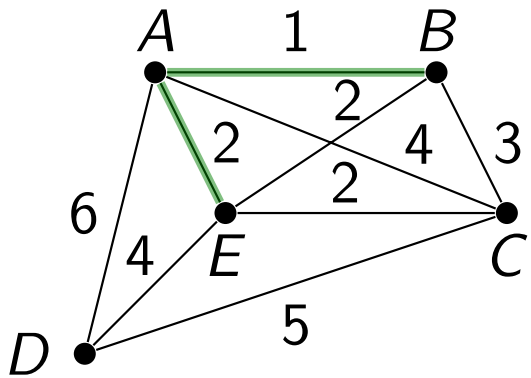
Example: Applying Kruskal's algorithm

Find a minimal spanning tree for this weighted graph:





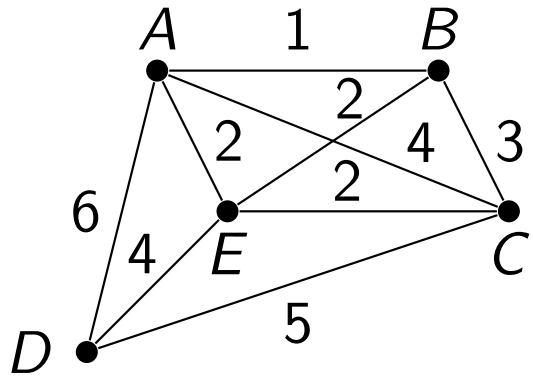
$\Longrightarrow$

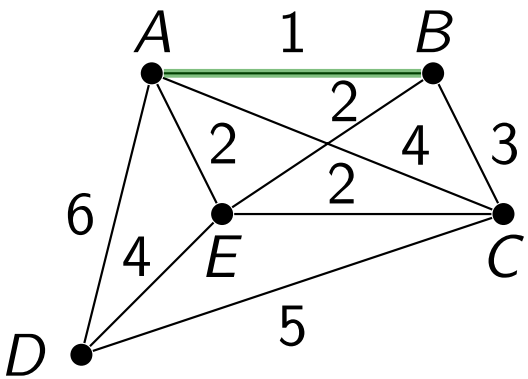


$W = 1$ $W = 3$

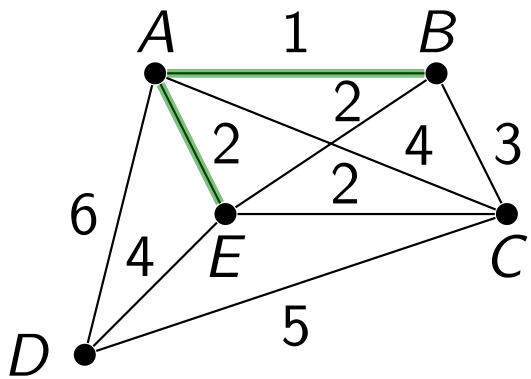
Example: Applying Kruskal's algorithm

Find a minimal spanning tree for this weighted graph:

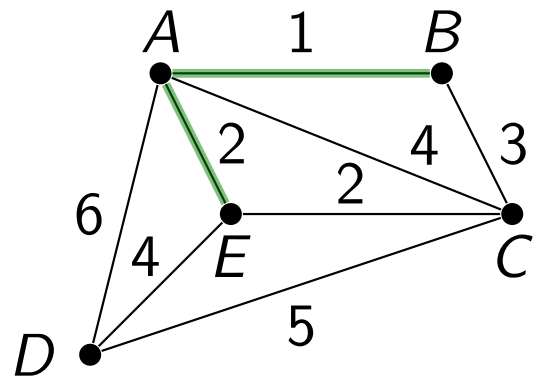




$\Longrightarrow$



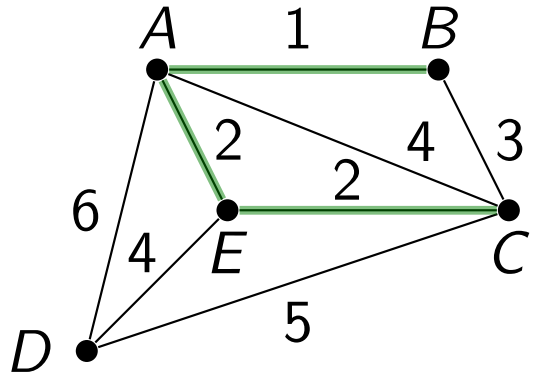
$\Longrightarrow$



$W = 1$ $W = 3$ $W = 3$

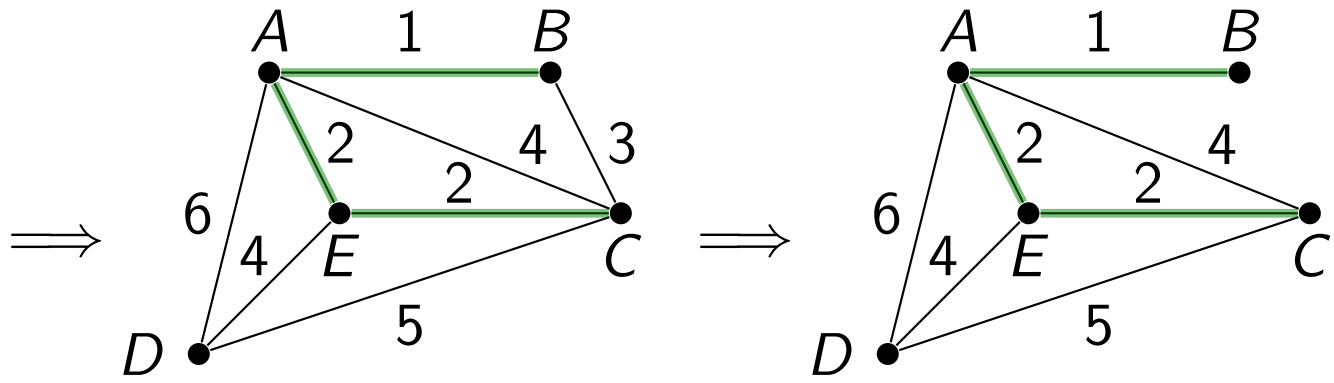
Example: Applying Kruskal's algorithm (cont.)

$\Longrightarrow$



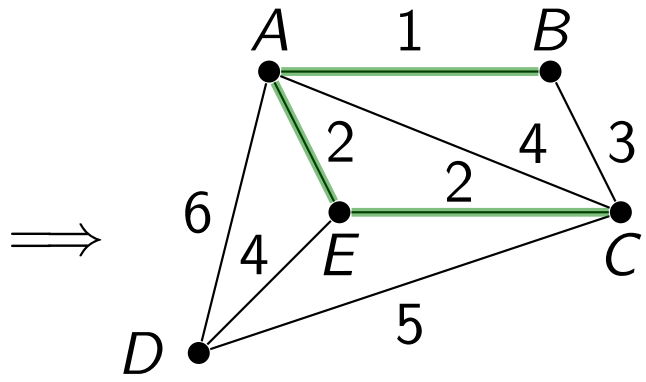
$W = 5$

Example: Applying Kruskal's algorithm (cont.)

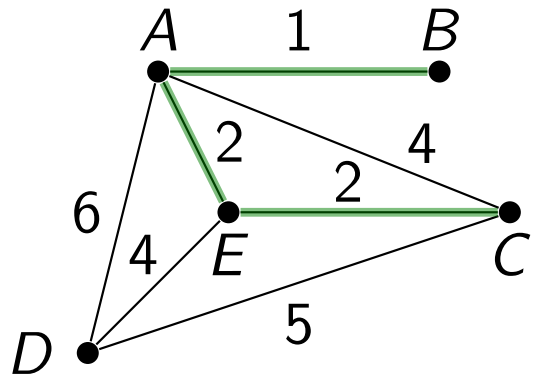


$W = 5$ $W = 5$

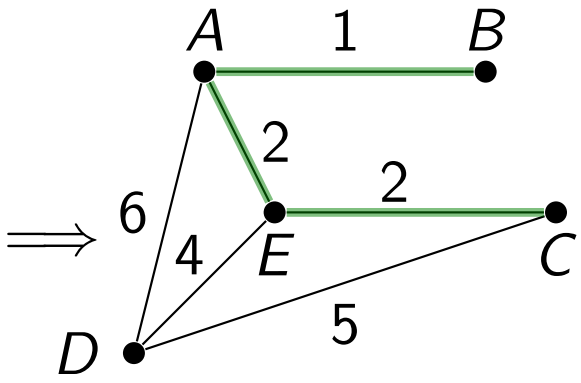
Example: Applying Kruskal's algorithm (cont.)



$W = 5$

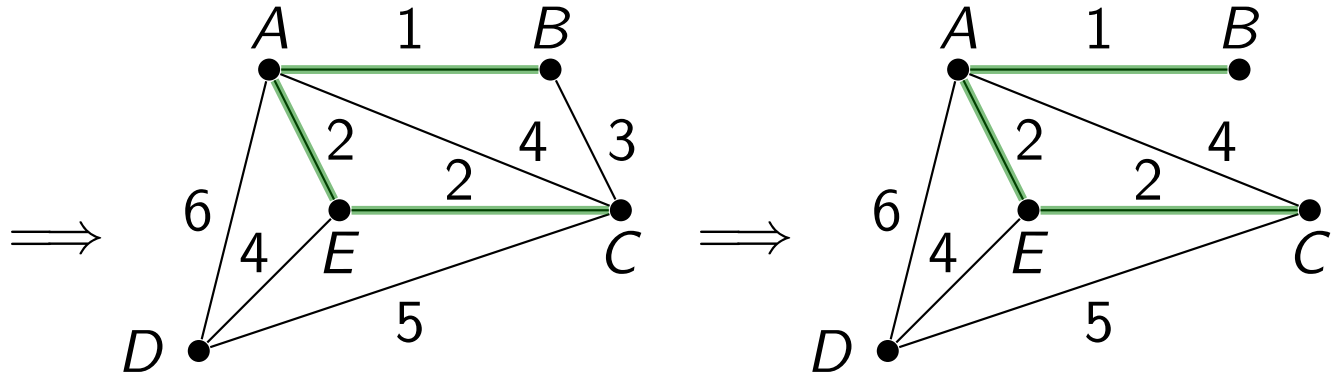


$W = 5$

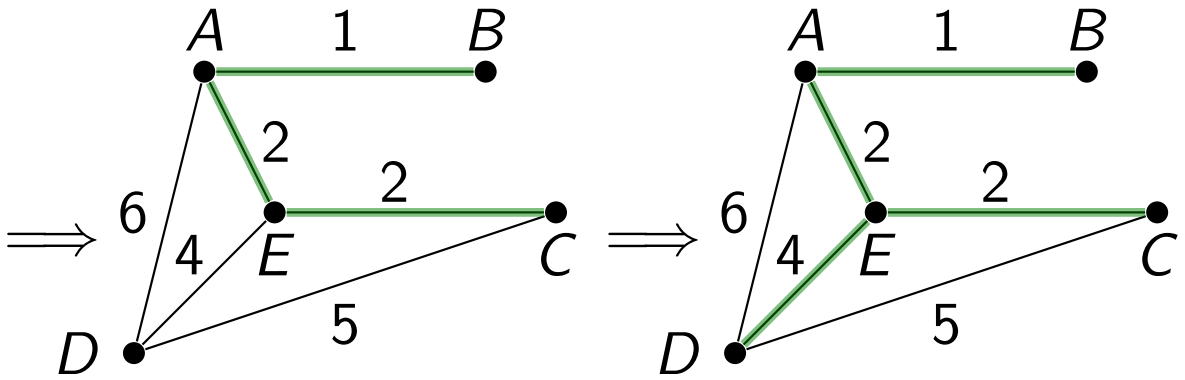


$W = 5$

Example: Applying Kruskal's algorithm (cont.)

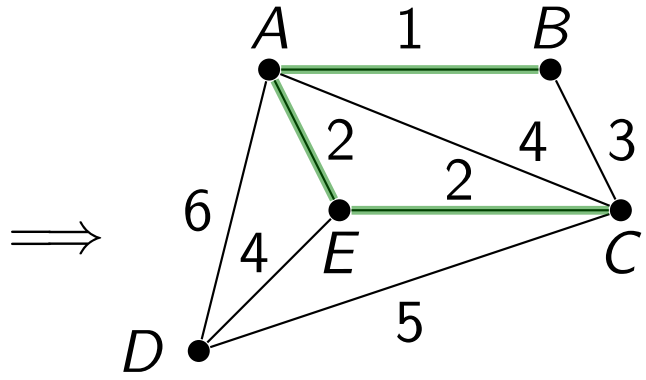


$W = 5$ $\qquad$ $W = 5$

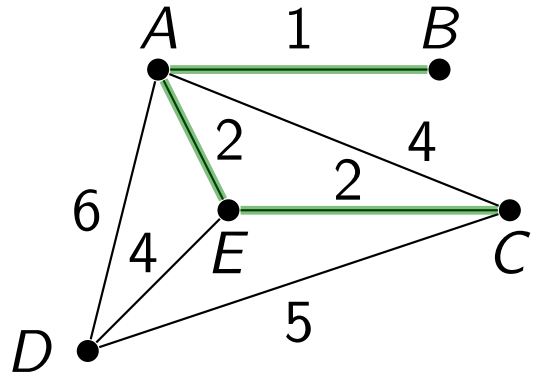


$W = 5$ $\qquad$ $W = 9$

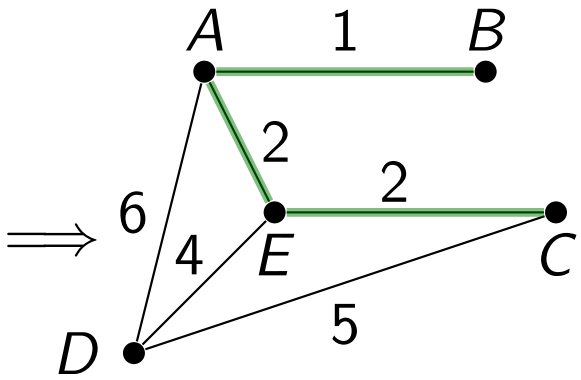
Example: Applying Kruskal's algorithm (cont.)



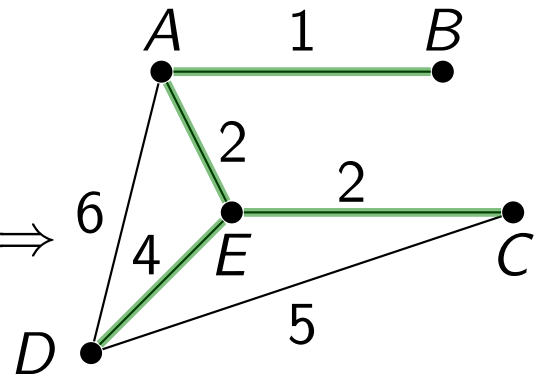
$W = 5$



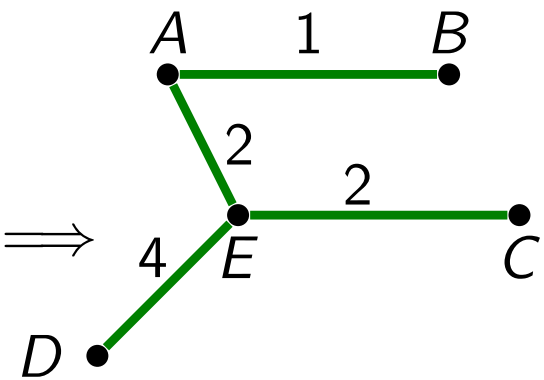
$W = 5$



$W = 5$



$W = 9$



$W = 9$

(The generated spanning tree)

Modified Kruskal's algorithm for minimal spanning tree

Input: Weighted connected graph G with n vertices.

Output: Minimal spanning tree T for G.
Total weight W of this tree.

Method:

1. Initialise T to have all the vertices of G but no edges. Initialise W to 0.
2. From the edges currently in G pick one, e, of least weight and remove it from G.
3. If adding e to T does not create a circuit in T, add e to T and add weight(e) to W.
4. Repeat steps 2 and 3 **until** G **has no edges left to consider**.

A lemma and a theorem

Lemma: Suppose that G is a connected, weighted graph. The output from Kruskal's algorithm on input G is the same as the output of the modified Kruskal's algorithm on input G.

Theorem: (The correctness of Kruskal's Algorithm) Suppose that G is a connected, weighted graph. The output of Kruskal's algorithm on input G is is a minimal spanning tree T of G.

Correctness of (modified) Kruskal's Algorithm

Proof: Suppose that G is a connected, weighted graph with n vertices and that $T = T_m$ is the subgraph of G produced when G is input to the modified Kruskal's algorithm. We shall prove the following, in sequence:

1. T_m is a forest
2. T_m is connected
3. T_m is a spanning tree of G
4. T_m is a minimal spanning tree of G.

By the Lemma on the previous slide, this proves the theorem.

Correctness of (modified) Kruskal's Algorithm

We introduce some notation to help us talk about the algorithm, and what things look like during the execution.

Let m be the number of edges in G. Let $(e_1, e_2, \ldots, e_m)$ be the edges of G, in sequence, such that

$$\forall i \in \{1, 2, \ldots, m-1\} \text{ weight}(e_i) \leq \text{weight}(e_{i+1}).$$

Without loss of generality, we may assume that the modified Kruskal's algorithm considers the edges in order $e_1, e_2, \ldots$.

Let T_0 be T as it looks after step 1 (T_0 has all the vertices of G, but no edges). For $i = 1, 2, \ldots, m$, let T_i be T as it looks after step 3 has been executed i times (that is, after e_i is considered).

Correctness of (modified) Kruskal's Algorithm

Let $p(i)$: T_i is a forest.

We use mathematical induction to prove $\forall i \in \{0, 1, \ldots, m\}\ p(i)$.

Base case: It is clear that T_0 is a graph with n vertices and 0 edges. Since T_0 has no edges, it is a forest (with n connected components). **Hence $p(0)$ holds.**

Inductive Case: Let $i \in \{0, 1, \ldots, m-1\}$, and suppose that $p(0)$, $p(1), \ldots, p(i)$ all hold. Since $p(i)$ is true, T_i is a forest. Since $i \leq m-1 < m$, step 2 of the algorithm will be executed for an $(i+1)$-th time. The edge removed from G in step 2 is e_{i+1}. Step 3 of the algorithm will be executed an $(i+1)$-th time. We consider cases, based on whether adding e_{i+1} to T_i creates a circuit.

Correctness of (modified) Kruskal's Algorithm

Consider first the case that adding e_{i+1} to T_i makes a circuit. Then $T_{i+1} = T_i$. Since T_i is a forest, T_{i+1} is a forest.

Consider next the case that adding e_{i+1} to T_i does not make a circuit. Then T_{i+1} is constructed from T_i by adding e_{i+1}. Since T_i is circuit-free, and adding e_{i+1} does not add a circuit, T_{i+1} is circuit-free. That is, T_{i+1} is a forest.

In each case, T_{i+1} is a forest. **Hence $p(i+1)$ is true.**

By the principle of mathematical induction, $\forall i \in \{0, 1, \ldots, m\}$ $p(i)$. Since $p(m)$ is true, T_m is a forest.

Correctness of (modified) Kruskal's Algorithm

Next we show that T_m is a tree. **Since we know that T_m is a forest, we need only show that T_m is connected. We shall use a proof by contradiction. Suppose that T_m is not connected.** Let $F_1, F_2, \ldots, F_s$ be the connected components of G. Let $v_1 \in V(F_1)$ and let $v_2 \in V(F_2)$. Since G is connected, there exists a path α in G from v_1 to v_2. Since v_2 is not in $V(F_1)$, α visits vertices not in $V(F_1)$. Let v_3 be the first vertex visited by α that is not in $V(F_1)$. Let e_j be the edge traversed by α to reach v_3 for the first time. Then one endpoint of e_j is in F_1, and the other endpoint is not. Since the end-points of e_j lie in distinct connected components of T_m, and T_m is a forest, adding e_j to T_m will not create a circuit. Since T_{j-1} is a subgraph of T_m, adding e_j to T_{j-1} does not create a circuit. It follows that e_j will added to T_{j-1} when executing step 3 for the j-th time. But then $e_j \in T_m$. This is a contradiction. **Since our supposition that T_m is not connected allowed us to deduce a contradiction, it must be false. Hence T_m is connected.**

Correctness of (modified) Kruskal's Algorithm

Since T_m is a connected forest it is a tree. Since T_0 contains every vertex from G, T_m does too. Hence T_m is a spanning tree of G.

Correctness of (modified) Kruskal's Algorithm

Now we show that T_m is a mimimal spanning tree. We shall use a proof by contradiction. Suppose that T is not a minimal spanning tree. Let S be a minimal spanning tree for G such that the number of edges that T_m and G have in common is maximal among all minimal spanning trees (no minimal spanning tree of G shares more edges with T_m than S does—some might share different edges, but none share more). Since S is a minimal spanning tree for G and T_m is not, the set $C = \{e \in E(T_m) \mid e \notin E(S)\}$ is not empty. Let k be the minimum integer such that $e_k \in C$. Since $e_k \notin E(S)$, adding e_k to S would add a circuit. Let e_ℓ be one of the edges in that circuit such that $e_\ell \notin T_m$ (such an edge must exist because T_m is circuit-free). Let S' be the graph obtained from S by adding e_k (creating a connected graph with a circuit that contains e_ℓ) and then removing e_ℓ. Removing an edge from a circuit in a connected graph leaves a connected graph, hence S' is connected. Since S' is a connected graph with n vertices and $n-1$ edges, it is a tree.

Correctness of (modified) Kruskal's Algorithm

Suppose that weight(e_ℓ) < weight(e_k). Then $\ell < k$, and e_ℓ would have been considered for inclusion in T before e_k. By our choice of k, all of the edges in T_{k-1} are in S; hence all of the edges in $T_{\ell-1}$ are in S. Since $e_\ell \in E(S)$ and S is circuit-free and $T_{\ell-1}$ is a subgraph of S, adding e_ℓ to $T_{\ell-1}$ does not create a circuit. Hence e_ℓ will be added to T when step 3 is exceuted for the ℓ-th time. Hence $e_\ell \in T_m$. But we chose $e_\ell \notin T_m$. **This contradiction shows that weight(e_ℓ) $\geq$ weight(e_k).**

Since S' is constructed from S by removing e_ℓ and adding e_k, and weight(e_ℓ) $\geq$ weight(e_k), we have that weight(S) $\geq$ weight(S'). Since S is a minimal spanning tree, weight(S') = weight(S). It follows that S' is a minimal spanning tree of G. But S' shares one more edge with T than S does, contradicting our choice of S. **Since our supposition that T is not a minimal spanning tree allowed us to deduce a contradition, it must be false.** □

Learning Outcomes common to MATH1005/6005

Upon successful completion, students will have the knowledge and skills to:

1. Recall, invent, interpret examples of motivation for mathematical constructs used in discrete mathematics as models of processes in the world.

2. Recognise, define, explain and use terminology and notation from discrete mathematics.

3. Identify the logical structure of a statement, and then identify the logical structure of an argument that may be used to prove or disprove the statement.

4. Perform mathematical calculations in discrete mathematics using methods presented in the course.

5. Write simple proofs/construct explicit counterexamples for statements relating to discrete mathematics topics covered in the course.