Section A1: Logic (continued)

A logical connective: IMPLIES

Recall that the logical connective “implies” is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

This definition is not as intuitive as the definitions of the other logical connectives.

A scenario (from p. 54 of our optional text) to explain why our definition of implies is reasonable

Suppose you go to interview for a job at a store and the owner of the store makes you the following promise:

> If you show up for work Monday morning, then you will get the job.

In which of the following situations has the store owner been proven a liar?

A. You show up for work Monday morning and you get the job.
B. You show up for work Monday morning and you don't get the job.
C. You do not show up for work Monday morning and you get the job.
D. You do not show up for work Monday morning and you don't get the job.

A scenario (cont.)

The store owner did not say anything about what will happen if you don't show up for work Monday morning. Therefore, the only situation in which the owner has been proven a liar is (B).

With
p: You show up for work Monday morning
q: You get the job,
the store owner's promise is represented by $p \to q$.

The store owner is proven a liar only if the statement they made has been demonstrated to be false. Now $p \to q$ is false exactly when $\neg(p \to q)$ is true. It therefore seems reasonable to say that $\neg(p \to q)$ is true only in situation (B).

A scenario (cont.)

The scenario demonstrates that the following truth table is "reasonable":

p	q	$\neg(p \to q)$
T	T	F
T	F	T
F	T	F
F	F	F

It follows that our definition of $\to$ is "reasonable."

More vocabulary associated to $p \to q$

The statement $p \to q$ is read aloud in each of the following ways:

- p implies q
- if p then q
- p only if q
- q if p

Please note the potential for confusion.

More vocabulary associated to $p \to q$

It is helpful to have words to describe the relationships between certain implications. We have

- The **converse** of $p \to q$ is $q \to p$
- The **inverse** of $p \to q$ is $\neg p \to \neg q$
- The **contrapositive** of $p \to q$ is $\neg q \to \neg p$

Example: Consider the statement:

If you work hard, then you will pass the exam.

With

$$p : \text{You work hard}$$
$$q : \text{You pass the exam,}$$

the statement is $p \to q$.
The contrapositive is:

If you fail the exam, then you have not worked hard.

The converse is:

If you pass the exam, then you have worked hard.

The inverse is:

If you don't work hard, then you will fail the exam.

More vocabulary associated to $p \to q$

Which, if any, of the following statements concerning $p \to q$ are true:

A. The converse of the inverse is the contrapositive.
B. The inverse of the converse is the contrapositive.
C. The inverse of the contrapositive is the converse.
D. The converse of the contrapositive is the inverse.
E. The statement is logically equivalent to its contrapositive.
F. The inverse is logically equivalent to the converse.
G. The statement is not logically equivalent to its converse.

More vocabulary associated to $p \to q$

Which, if any, of the following statements concerning $p \to q$ are true:

A. The converse of the inverse is the contrapositive.
B. The inverse of the converse is the contrapositive.
C. The inverse of the contrapositive is the converse.
D. The converse of the contrapositive is the inverse.
E. The statement is logically equivalent to its contrapositive.
F. The inverse is logically equivalent to the converse.
G. The statement is not logically equivalent to its converse.

Answer: All of the above are true. Statements E, F, and G are particularly important.

Necessary and sufficient conditions

p is a **sufficient condition** for q means $p \to q$.

p is a **necessary condition** for q means $\neg p \to \neg q$.

Since $(\neg p \to \neg q) \equiv (q \to p)$, we can also say:

p is a **necessary condition** for q means $q \to p$;

p is a **necessary and sufficient condition** for q means $p \leftrightarrow q$.

In view of the above terminology, statements of the form $p \to q$ and $p \leftrightarrow q$ are often referred to as **conditional statements**.

Examples

- Being older than 16 is a necessary condition for being allowed to drive. It is not sufficient.
- Having a drivers license is a necessary and sufficient condition for being allowed to drive.
- Being a taxi driver is a sufficient condition for being allowed to drive. It is not necessary.

Negating compound statements

For statement variables p and q, the following logical equivalences hold:

$$\neg(\neg p) \equiv p \tag{1}$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q \tag{2}$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \tag{3}$$
$$\neg(p \oplus q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \tag{4}$$
$$\neg(p \to q) \equiv \quad p \wedge \neg q \tag{5}$$
$$\neg(p \leftrightarrow q) \equiv \quad (p \wedge \neg q) \vee (\neg p \wedge q) \tag{6}$$

RECALL: Each of these logical equivalences may be confirmed by demonstrating that the truth tables for the left-hand side and right-hand side match in every row.

Simplification using logical equivalences

The right-hand side of each logical equivalence on the previous slide has the pleasant property that $\neg$ only appears in front of statement variables. Using these rules, we can arrange to replace any compound statement by a logically equivalent statement in which $\neg$ only appears in front of statement variables.

Example: Consider the compound statement

$$\begin{aligned} & \neg(\neg p \wedge q) \wedge (p \vee q) \\ \equiv & (\neg(\neg p) \vee \neg q) \wedge (p \vee q) && \text{Using } (2) \\ \equiv & (p \vee \neg q) \wedge (p \vee q) && \text{Using } (1) \end{aligned}$$

Using the logical equivalences on slide #38 and #53 together (there is overlap), we can simplify further.
Example:

$$
\begin{aligned}
&\neg(\neg p \wedge q) \wedge (p \vee q) && \\
\equiv&(\neg(\neg p) \vee \neg q) \wedge (p \vee q) && \text{Using } (2) \\
\equiv&(p \vee \neg q) \wedge (p \vee q) && \text{Using } (1) \\
\equiv&p \vee (\neg q \wedge q) && \text{Using a Distributive law} \\
\equiv&p \vee (q \wedge \neg q) && \text{Using a Commutative law} \\
\equiv&p \vee c && \text{Using a Negation law} \\
\equiv&p && \text{Using an Identity law}
\end{aligned}
$$

In the above, c represent a contradiction.

Arguments

Valid arguments

An **argument** is a sequence of statements, one of which (usually the last) is flagged as the **conclusion**. The other statements, called **premises**, are alleged to justify the conclusion.

Example: *The coffee argument*

If I arrive early at Uni, I have a coffee. If I have a coffee, I focus on the lecture. I do focus on the lecture because I arrive early at Uni.

Notation for the form of an argument

Consider an argument with premises $p_1, \dots, p_n$ and conclusion q. The **form** of an argument is the expression

$$[\underbrace{p_1, p_2, \dots, p_n}_{\text{List the premises}} \mathrel{.\dot{.}.} q].$$

or is written

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline q \end{array}$$

Let's see how this exposes the nature of an argument.

Structure of the coffee argument

The premises and conclusion of the coffee argument are:
p_1: If I arrive early at Uni, I have a coffee
p_2: If I have a coffee, I focus on the lecture.
p_3: I arrive early at Uni
q: I focus on the lecture

and the form of the coffee argument is:

$$[p_1, p_2, p_3 \therefore q]$$

This doesn't expose the reasoning very effectively, but if we break down the premises by recognising them as compound statements...

Structure of the coffee argument

With

a: I arrive early at Uni
b: I have a coffee
q: I focus on the lecture

the form of the argument is

$$[a \to b, b \to q, a \therefore q]$$

or

$$\begin{array}{r} a \to b \\ b \to q \\ a \\ \hline q \end{array}$$

Valid and invalid arguments

The purpose of representing the form of an argument is often to examine the validity of the argument.

An argument with the form $[p_1, p_2, \ldots, p_n \therefore q]$ is **valid** if the statement form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$$

is a tautology.

An informal interpretation of this is that an argument is valid when the truth of the conclusion is guaranteed by the truth of the premises.

The coffee argument again

a	b	q	$((a \to b) \wedge (b \to q) \wedge a) \to q$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The coffee argument again

a	b	q	$((a \to b) \wedge (b \to q) \wedge a) \to q$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The truth table demonstrates that the statement form is a tautology, and hence the argument is valid.

The coffee argument again

a	b	q	$((a \to b) \land (b \to q) \land a) \to q$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Time saving tips:

- If all premises are T, we need the conclusion T.
- Once we find a false premise, we can stop work on the row.

Another Example

Consider the following argument:
If fair elections are held and the unemployment rate is below 10%, the civil unrest disappears. But civil unrest is on the rise, so the elections were unfair!

Let's examine the structure. With:

a: Fair elections are held
b: The unemployment rate is below 10%
c: Civil unrest disappears

the argument has the form

$$[a \wedge b \to c, \neg c \therefore \neg a]$$

The civil unrest argument again

a	b	c	$((a \wedge b \to c) \wedge (\neg c)) \to \neg a$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The civil unrest argument again

a	b	c	$((a \wedge b \to c) \wedge (\neg c)) \to \neg a$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Since the fourth line of the truth table evaluated to F, the statement form is not a tautology and the argument is invalid.

Fixing the civil unrest argument

Consider the following argument:
If fair elections are held and the unemployment rate is below 10%, then civil unrest disappears. But civil unrest is on the rise, so the elections were unfair or the unemployment rate is at least 10%.

Let's examine the structure. With:

a: Fair elections are held
b: The unemployment rate is below 10%
c: Civil unrest disappears

the amended argument has the form

$$[a \wedge b \to c, \neg c \therefore (\neg a \vee \neg b)]$$

The validity of the civil unrest argument

a	b	c	$((a \wedge b \to c) \wedge (\neg c)) \to (\neg a \vee \neg b)$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The validity of the civil unrest argument

a	b	c	$((a \wedge b \to c) \wedge (\neg c)) \to (\neg a \vee \neg b)$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The truth table demonstrates that the statement form is a tautology and the amended argument is valid.

Some standard valid arguments

Modus ponens	Modus tollens	Division into cases
$\begin{array}{r} p \to q \\ p \\ \hline q \end{array}$ **VALID**	$\begin{array}{r} p \to q \\ \neg q \\ \hline \neg p \end{array}$ **VALID**	$\begin{array}{r} p \vee q \\ p \to r \\ q \to r \\ \hline r \end{array}$ **VALID**
The simplest argument form. The others are based on this.	An implication **is** equivalent to its contrapositive.	At least one of the cases holds, and each case alone implies the conclusion.

Some standard invalid arguments

Converse error	Inverse error
$$\begin{array}{r} p \to q \\ q \\ \hline p \end{array}$$ **INVALID**	$$\begin{array}{r} p \to q \\ \neg p \\ \hline \neg q \end{array}$$ **INVALID**
An implication is **not** equivalent to its converse.	An implication is **not** equivalent to its inverse.

Identify the structure and determine validity

Example:
If interest rates rise, so does unemployment.
Interests rates are rising.
Therefore unemployment is rising.

Example:
If you studied hard for the exam you will have passed.
You did pass.
So you must have studied hard for the exam.

Identify the structure and determine validity

Example:
If Ned robbed a bank in Euroa, then
he was in Victoria.
But he was in New South Wales.

So Ned didn't rob the bank.

Identify the structure and determine validity

Example:
The user has changed their status or they have uploaded pictures.
If status has been changed, the page needs to be refreshed.
If pictures have been uploaded, then the page needs to be refreshed too.
So the page needs to be refreshed.

Identify the structure and determine validity

Example:
If we don't ban plastic bags, pollution will increase.
We have banned plastic bags, so pollution will not increase.

Some final notes

The validity/invalidity of an argument says nothing about whether or not the premises are true. An argument can be valid even though some of the premises are false. The validity of the argument merely assures us that if the premises are true, then the conclusion will be true. (See Modus ponens).