C1. Counting.

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Text Reference (Epp)	3ed: Sections	6.1-7, 7.3
	4ed: Sections	9.1-7
	5ed: Sections	9.1-7

Cardinality

This section is mostly about calculating the number of objects of some specified type; for example counting all five digit numbers with no repeated digits. Counting like this can be viewed as finding the number of members of some set, also known as finding the *size* of the set.

For a finite set A, this 'size' or 'cardinality' is just the number of members of A. However it can be defined formally as follows:

Let A be a set. Suppose there exists a bijection (one-to-one correspondence) from A to a subset of the natural numbers of the form $\{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. Then the **cardinality**, or **size** of the set A, written $|A|$, is n. Thus $|A| = n$.

Earlier in the course we saw that this definition is suitable for generalisation to infinite sets (not all infinite sets have the same cardinality.) It also points to some practical counting techniques (see next slide).

Example: numbers in an interval

- What is the cardinality of the set of natural numbers in an interval?
- Example: $S = \{150, 151, 152, ..., 160\}$
- Subtract 149 from each:

$$\begin{array}{ccccc} 150 & 151 & 152 & ... & 160 \\ \downarrow & \downarrow & \downarrow & ... & \downarrow \\ 1 & 2 & 3 & ... & 11 \end{array}$$

- We have made a bijection to the set $\{1, 2, 3, ..., 11\}$, so $|S| = 11$.

Example: numbers in an interval, generalized

- Let $S = \{a, a+1, a+2, ..., b\} \subseteq \mathbb{N}$
- A nice bijection subtracts '$a - 1'$ from each element of S. We have

$$\begin{array}{ccccc} a & a+1 & a+2 & \cdots & b \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 1 & 2 & 3 & \cdots & b-a+1 \end{array}$$

- Therefore $|S| = b - a + 1$.

A slightly harder but similar example

How many numbers from 150 to 330 inclusive are congruent to 5 mod 7?

$150 \bmod 7 = 3$ so the lowest number is $150 + 2 = 152$.

$330 \bmod 7 = 1$ so the highest number is $330 - 3 = 327$.

Now use the following composition of bijections:

$$
\begin{array}{rccccc}
 & 152 & 159 & 166 & \cdots & 327 \\
\text{subtract 5:} & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
 & 147 & 154 & 161 & \cdots & 322 \\
\text{divide by 7:} & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
 & 21 & 22 & 23 & \cdots & 46 \\
\text{subtract 20:} & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
 & 1 & 2 & 3 & \cdots & 26
\end{array}
$$

Since the composition of bijections is a bijection, the answer is 26.

Finite and infinite sets

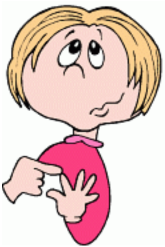
The **cardinality of the empty set** is defined to be 0. Thus $|\emptyset| = 0$. Any set S with cardinality $|S| = n \in \{0, 1, 2, ...\}$ it is said to be **finite**.

Any set S that is not finite is said to be **infinite**. We write $|S| = \infty$.

Examples:

Finite Sets	Infinite Sets
$\{1, 2, 3\}$	$\mathbb{N}$ ... natural numbers
{red, orange, yellow, green, blue, purple}	$\mathbb{Z}$ ... integers
$\{b$: b is a book in the Hancock library$\}$	$\mathbb{Q}$ ... rational numbers
$\{s$: s is a star in the Milky Way Galaxy$\}$	$\mathbb{R}$ ... real numbers
$\{\}$	$\mathcal{P}(\mathbb{R})$ ... power set of $\mathbb{R}$

Countability

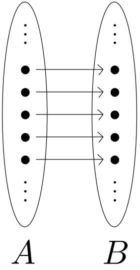


A set S is called **countable** when there is a bijection from S to a subset of the set $\mathbb{N}$ of natural numbers.

Examples:

- Any *finite* set is countable.
- The *empty set* is countable (because $\emptyset \subseteq \mathbb{N}$).
- The set $\mathbb{P}$ of all primes is countable (because $\mathbb{P} \subseteq \mathbb{N}$).
- $\mathbb{N}$ itself is countable (because $\mathbb{N} \subseteq \mathbb{N}$).
- The sets $\mathbb{N}$ and $\mathbb{P}$ are each both countable and infinite. Such sets are called **countably infinite**.

Comparing cardinalities



Generalising from the case of finite sets, we say that two sets A and B have **the same cardinality**, written $|A| = |B|$, provided that there exists a bijection (one-to-one correspondence) from A to B.

Remember that for $\phi : A \to B$ to be a bijection:

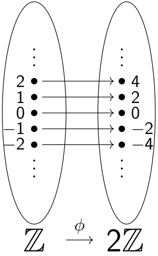
- no two arrows point to the same element of B, *i.e.*
$$\forall x, y \in A \;\; x \neq y \implies \phi(x) \neq \phi(y).$$
- each element of B must have an arrow pointing to it: *i.e.*
$$\forall b \in B \; \exists a \in A \;\; \phi(a) = b.$$

Also remember that $\phi : A \to B$ is a bijection if and only if it has an inverse $\phi^{-1} : B \to A$.
Since the inverse is also a bijection, we say that A and B have the same cardinality if and only if there is a bijection **between** them. (i.e. we don't have to specify the direction of the isomorphism).

Examples of sets with the same cardinality

1 You might expect the set $\mathbb{Z}$ of integers to have a 'bigger' cardinality than the set $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$ of even integers, since there are 'twice as many' integers as even integers.

But actually the two sets have the **same** cardinality, because the function $\phi : \mathbb{Z} \to 2\mathbb{Z} : \quad \phi(z) = 2z$ is (clearly) a bijection.

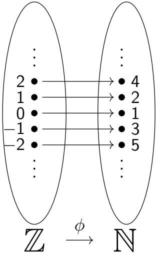


2 By tweaking the function ϕ a little we can establish the perhaps more surprising fact that $\mathbb{Z}$ also has the same cardinality as $\mathbb{N}$.

The modified version of ϕ is

$$\phi : \mathbb{Z} \to \mathbb{N} : \quad \phi(z) = \begin{cases} 2z & \text{if } z > 0 \\ 1 - 2z & \text{if } z \leq 0 \end{cases}$$

It not difficult to see that this is a bijection.



All countably infinite sets have the same cardinality

Recall that we call a set S countably infinite if (and only if) it is infinite and there is a bijection from S to a subset of $\mathbb{N}$.
In other words, S has the same cardinality as some infinite subset of $\mathbb{N}$. However

Any infinite subset S of $\mathbb{N}$ has the same cardinality as $\mathbb{N}$ itself.

Proof: Since every bijection has an inverse, it is enough to establish a bijection $\phi : \mathbb{N} \to S$. This can be done recursively as follows:

$$\begin{aligned} \phi(1) &= \text{least member of } S \\ \phi(2) &= \text{least member of } S\backslash\{\phi(1)\} \quad (2^{\text{nd}} \text{ least member}) \\ \forall n \in \mathbb{N} \;\; \phi(n+1) &= \text{least member of } S\backslash\{\phi(1), \ldots, \phi(n)\}. \end{aligned}$$

Note: It follows from the result above (and its proof) that proving that an arbitrary infinite set S (not necessarily a subset of $\mathbb{N}$) is countably infinite amounts to showing that it can be '**well-ordered**'. This means that it is possible to order the elements of S in some (perhaps ingenious) way so that S and every subset of S has a 'least' member.

Not all infinite sets have the same cardinality

The following results by Georg Cantor (1945-1918), were a breakthrough in mathematical thinking about infinite sets.

There exists a bijection $f: \mathbb{N} \to \mathbb{Q}$.

There does not exist a bijection $g: \mathbb{N} \to \mathbb{R}$.

So $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ have the same cardinality; the cardinality of $\mathbb{R}$ is different.

11.1

There exists a bijection $f : \mathbb{N} \to \mathbb{Q}$.

<u>Idea</u>: We write the elements of $\mathbb{Q}$ in a clever tabular way...

$$
\begin{array}{cccccc}
\frac{0}{1} & \frac{1}{1} & \frac{-1}{1} & \frac{2}{1} & \frac{-2}{1} & \cdots \\
\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} & \cdots \\
\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{5}{4} & \cdots \\
\vdots & & & & &
\end{array}
$$

Then move "diagonally" through the table.

11.2 If $g : \mathbb{N} \to (0,1)$, then g is not surjective.

Idea:

$f(1) = 0.d_{1,1}\, d_{1,2}\, d_{1,3} \ldots$

$f(2) = 0.d_{2,1}\, d_{2,2}\, d_{2,3} \ldots$

$f(3) = 0.d_{3,1}\, d_{3,2}\, d_{3,3} \ldots$

$\vdots$

$d_{i,j} \in \{0,1,\ldots,9\}$

Let $c = 0.c_1 c_2 c_3 \ldots$ be defined by

$$c_i = \begin{cases} 6 & \text{if } d_{i,i} \neq 6 \\ 7 & \text{if } d_{i,i} = 6. \end{cases}$$

Then $c \in (0,1)$

$c \notin$ range g.

How do we count?

How do we count? We have a collection of counting principles. When we need to count some objects, we analyse those objects until we carefully match the situation to one of the situations in which a counting principle applies.

What makes counting hard? Matching your scenario to one of the scenarios described in a counting principle takes care, as the scenarios described in the counting principles sound similar unless you read carefully.

What is your best strategy? Understand the scenarios described in the counting principles. In particular, carefully notice how the scenarios are different. Then, when you have to count something and you think that a counting principle applies, **state which counting principle you are using and explain carefully (write it out) how you know that the situation you have is like the scenario described in the counting principle**.

The structures we count

We have already discussed sets, n-tuples, lists and sequences. In a set, order is unimportant and for elements in a set there is no notion of “how many times” the element is in the set. Lists, n-tuples and sequences are different ways to talks about the same thing—order matters and elements can appear more than once unless explicitly excluded by the language. We need one more structure, one that recognises multiplicty (how many times an element appears) but not order (there is no first element, second element etc)....

Multisets

A multiset is a 'set' with multiple copies of elements allowed and acknowledged. An example is $\{c, b, a, c, a\}$, which has 2 a's, 1 b and 2 c's.

As for ordinary sets, order is irrelevant: $\{c, b, a, c, a\} = \{a, a, b, c, c\}$.

But the multiplicities **do** matter.

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}$, where,

$$\forall s \in S \quad m(s) = \text{number of copies of } s \quad \text{and} \quad r = \textstyle\sum_{s \in S} m(s).$$

So, for example, $\{c, b, a, c, a\}$ has size $r = 2 + 1 + 2 = 5$.

Principles of counting

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TO USE THIS PRINCIPLE: Count something easier, and exhibit a bijection between the set you which to count and the set you have counted.

The Pigeonhole Principle If $k+1$ or more pigeons occupy k pigeonholes, then at least one pigeonhole must contain two or more pigeons.

The Generalised Pigeonhole Principle If N objects are classified in k disjoint categories, then at least one category must contain $\lceil \frac{N}{k} \rceil$ objects. ($\lceil \frac{N}{k} \rceil$ means the least integer that is greater than or equal to $\frac{N}{k}$)

at least

Permutations There are $n!$ ways to arrange n distinct objects in a list.

Principles of counting

r-Permutations There are

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ways to select and order r out of n distinct objects.

Combinations There are

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

Principles of counting

Multisets (Stars and Bars) There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n. That is, there are $\binom{r+n-1}{r}$ ways to arrange a list of r stars and $n-1$ bars.

Inclusion-Exclusion If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

The Sum Rule If A is a finite set and $\{A_1, A_2, \ldots, A_m\}$ is a partition of A, then $|A| = |A_1| + |A_2| + \cdots + |A_m|$.

The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$.

Example 1

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

IDEA: A subset of A corresponds to making one of two choices for each element of A: include it in the subset or not. These choices could be recorded as a bit-string. So elements of $\mathcal{P}(A)$ can be counted by counting bit-strings...

Proof: Let A be a non-empty finite set. Let $n = |A|$. Let B be the set of n-bit strings. When representing integers, the smallest element of B represents 0 and the largest represents $2^n - 1$; it follows that $|B| = 2^n$. Since **bijections preserve cardinality**, to prove the result, it is enough to exhibit a bijection from B to $\mathcal{P}(A)$. Let $f : A \to \{1, 2, \ldots, n\}$ be a bijection. Let $g : B \to \mathcal{P}(A)$ be the function defined by the rule $b_1 b_2 \ldots b_n \mapsto \{a \in A \mid b_{f(a)} = 1\}$. We leave the reader to verify that g is a bijection. $\square$

An example illustrating the functions in Example 1

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$. Then $n = 3$ and

$$B = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Let $f : A \to \{1, 2, 3\}$ be the function defined by

$$f(\text{cat}) = 1, f(\text{dog}) = 2, f(\text{chicken}) = 3.$$

Then

$$\begin{aligned} g(011) &= \{\text{dog}, \text{chicken}\} \\ g(101) &= \{\text{cat}, \text{chicken}\} \\ g(000) &= \emptyset \\ g(100) &= \{\text{cat}\} \end{aligned}$$

The Pigeonhole Principle

Example 2

Prove the following: If there are 11 players in a soccer team that wins $12-0$, there must be at least one player in the team who scored more than once (assuming no own-goals).

Proof: We shall apply the pigeonhole principle. Imagine a score sheet in which each player's name is written on a different row. When a goal is scored, a stroke in the appropriate row records who scored it. The rows on the scoresheet are like pigeonholes, and the goals are like pigeons. Since there are 12 goals (12 pigeons) to be recorded, and only 11 rows (11 pigeonholes) on the scoresheet, at least one row must contain the record of two goals. So at least one player scored more than once. $\square$

Example 3

Prove the following: If a molecule can exist in 2 different configurations, and you have 10^9 such molecules, at least half of them must be in the same configuration.

Proof: We shall apply the generalised pigeonhole principle. We have 10^9 objects (the molecules) to be classifed into 2 disjoint categories (the configurations). By the generalised pigeonhole principle, at least one category contains $\lceil 10^9/2 \rceil$ objects (molecules). Note that $\lceil 10^9/2 \rceil \geq 10^9/2$; that is, $\lceil 10^9/2 \rceil$ is at least half of 10^9. Hence at least one configuration is taken by at least $10^9/2$ molecules. $\square$

Example 4

Show that, in any set of a thousand words, there must be at least 39 words that start with the same letter.

Proof: We shall apply the generalised pigeonhole principle. Each word can be classified by its first letter, and such categories are disjoint. Since there are 1000 words and 26 categories, at least one category (configuration) contains $\lceil 1000/26 \rceil = \lceil 38.46 \rceil = 39$ words. $\square$

Example 5 (Epp(4ed) Q9.4.33)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the role of the pigeon holes. So

$$A = \{a, b, c, d, e, f \in \mathbb{N} \ : \ a < b < c < d < e < f < 15\}$$

Pigeons: subsets of A

Pigeon holes: possible element sums of subsets of A

Now we have to count the pigeons and pigeon holes.

Example 5 (cont)

Proof: First we show that there are 62 distinct proper nonempty subsets of A (pigeons). Since $|A| = 6$, we have that $|\mathcal{P}(A)| = 2^6 = 64$. Since $\emptyset$ is empty, and A is not proper, there are 62 distinct proper nonempty subsets of A.

Now we show that there are only 60 different possible element sums among these nonempty proper subsets. A proper subset of A contains at most 5 elements. The elements of A are selected from $1, 2, \ldots, 14$. The most a sum of at most 5 elements from these integers can be is $10 + 11 + 12 + 13 + 14 = 60$. The least the sum can be is 1. Hence the element sum of a proper nonempty subset of A is between 1 and 60. (Thus there are 60 pigeonholes).

By the pigeonhole principle, at least one element sum is taken by two distinct nonempty proper subsets of A. $\square$

Example 6 (cont.)

Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

Proof: We label the 52 distinct integers $a_1, a_2, \ldots, a_{52}$. For each i such that $1 \leq i \leq 52$, let $A_i = a_i \bmod 100$. We consider two cases.
Case: There exist i, j such that $i < j$ and $A_i = A_j$.
In this case,

$$A_i = A_j \Leftrightarrow a_i \equiv a_j (\text{mod } 100) \Leftrightarrow (a_i - a_j) \text{ is divisible by } 100.$$

In this case, there are two integers whose difference is divisble by 100 and we are done.

Example 6 (cont.)

Case: There does not exist i, j such that $i < j$ and $A_i = A_j$. In this case, the integers $A_1, A_2, \ldots, A_{52}$ are distinct. We shall apply the pigeonhole principle, with the integers $A_1, A_2, \ldots, A_{52}$ playing the role of pigeons. We label 51 'pigeonholes' as shown, so that each is labelled by two not necessarily distinct two-digit numbers:

00	01	02	03		49	50
$\square$	$\square$	$\square$	$\square$	....	$\square$	$\square$
00	99	98	97		51	50

Observe that, for each box, the pair of integers labelling the box have a sum that is divisible by 100.

There are now exactly 51 'pigeonholes', and 52 numbers in the list $A_1, \ldots, A_{52}$. So at least two of the numbers in the list $A_1, \ldots, A_{52}$ label the same pigeonhole. This means that at least two of the numbers in the list $A_1, \ldots, A_{52}$ have a sum that is divisible by 100.
$\square$

Example 7

Andy, Beth and Cai are standing in line.
In how many different orders could this queue be arranged?

Proof: Arranging Andy, Beth and Cai in a line is like arranging 3 distinct objects in a list. Thus there are $3! = 3 \times 2 \times 1 = 6$ ways to do this. $\square$

Checking our answer (unnecessary, but instructive): We can represent an "outcome" by listing three letters in order: the first letter represents the first person in the line; the second letter represents the second person in the line; and the third letter represents the third person in the line. The entire set of outcomes, containing 6 elements, is represented below:

$$\{ABC, ACB, BAC, BCA, CAB, CBA\}$$

Example 8

A pet show awards 1st, 2nd and 3rd prizes. There are 5 entrants:

Rachel the Rabbit   Charles the Chicken   Tilly the Terrier
Bob the Bilby   Karen the Kangaroo

In how many ways can the prizes be handed out?

Proof: Handing out the prizes is like selecting and ordering 3 out of 5 distinct objects. Thus there are

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 5 \times 4 \times 3 = 60$$

ways to hand out the prizes. $\square$

Checking our answer to Example 8

We can represent an "outcome" by listing three letters in order: the first letter represents the pet awarded first prize; the second letter represents the pet awarded second prize; and the third letter represents the pet awarded third prize. The entire set of outcomes, containing 60 elements, is represented below:

{ BCK, BCR, BCT, BKC, BKR, BKT,
BRC, BRK, BRT, BTB, BTC, BTR,
CBK, CBR, CBT, CKB, CKR, CKT,
CRB, CRK, CRT, CTB, CTK, CTR,
KBC, KBR, KBT, KCB, KCR, KCT,
KRB, KRC, KRT, KTB, KTC, KTR,
RBC, RBK, RBT, RCB, RCK, RCT,
RKB, RKC, RKT, RTB, RTC, RTK,
TBC, TBK, TBR, TCB, TCK, TCR,
TKB, TKC, TKR, TRB, TRC, TRK, }

Example 9

A different pet show does not give 1st, 2nd and 3rd prizes. Instead, three "ribbon-winning" pets are chosen to travel to the district show. There are 5 entrants:

Rachel the Rabbit   Charles the Chicken   Tilly the Terrier
Bob the Bilby   Karen the Kangaroo

In how many ways can the ribbon-winners be chosen?

Proof: Selecting the ribbon-winning pets is like choosing a set of 3 objects (ribbon-winning pets) from a set of 5 objects (the pets in the show). Thus there are

$$C(5,3) = \binom{5}{3} = \frac{P(5,3)}{3!} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

ways to choose the ribbon-winners. $\square$

Checking our answer to Example 9

Each "outcome" may be represented by a set containing the ribbon-winning pets. The ten sets representing the ten different outcomes are listed below:

{Rachel, Charles, Tilly},	{Rachel, Charles, Bob},
{Rachel, Charles,Karen},	{Rachel, Tilly, Bob},
{Rachel, Tilly, Karen},	{Rachel, Bob, Karen},
{Charles, Tilly, Bob},	{Charles, Tilly, Karen},
{Charles, Bob, Karen},	{Tilly, Bob, Karen}

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So, for example, $\{c, b, a, c, a\}$ has size $r = 2 + 1 + 2 = 5$.

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Principles of counting

Bijections preserve cardinality If A and B are finite sets and there exists a bijection $f : A \to B$, then $|A| = |B|$.
TO USE THIS PRINCIPLE: Count something easier, and exhibit a bijection between the set you which to count and the set you have counted.

The Pigeonhole Principle If $k + 1$ or more pigeons occupy k pigeonholes, then at least one pigeonhole must contain two or more pigeons.

The Generalised Pigeonhole Principle If N objects are classified in k disjoint categories, then at least one category must contain $\lceil \frac{N}{k} \rceil$ objects. ($\lceil \frac{N}{k} \rceil$ means the least integer that is greater than or equal to $\frac{N}{k}$)

Permutations There are $n!$ ways to arrange n distinct objects in a list.

Example 1

Prove the following: If A is a non-empty finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

IDEA: A subset of A corresponds to making one of two choices for each element of A: include it in the subset or not. These choices could be recorded as a bit-string. So elements of $\mathcal{P}(A)$ can be counted by counting bit-strings...

Proof: Let A be a non-empty finite set. Let $n = |A|$. Let B be the set of n-bit strings. When representing integers, the smallest element of B represents 0 and the largest represents $2^n - 1$; it follows that $|B| = 2^n$. Since **bijections preserve cardinality**, to prove the result, it is enough to exhibit a bijection from B to $\mathcal{P}(A)$. Let $f : A \to \{1, 2, \ldots, n\}$ be a bijection. Let $g : B \to \mathcal{P}(A)$ be the function defined by the rule $b_1 b_2 \ldots b_n \mapsto \{a \in A \mid b_{f(a)} = 1\}$. We leave the reader to verify that g is a bijection. $\square$

An example illustrating the functions in Example 1

Let $A = \{\text{cat}, \text{dog}, \text{chicken}\}$. Then $n = 3$ and

$$B = \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Let $f : A \to \{1, 2, 3\}$ be the function defined by

$$f(\text{cat}) = 1, f(\text{dog}) = 2, f(\text{chicken}) = 3.$$

Then

$$\begin{aligned}
g(011) &= \{\text{dog}, \text{chicken}\} \\
g(101) &= \{\text{cat}, \text{chicken}\} \\
g(000) &= \emptyset \\
g(100) &= \{\text{cat}\}
\end{aligned}$$

The Pigeonhole Principle

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Permutations There are $n!$ ways to arrange n distinct objects in a list.

Example 2

Prove the following: If there are 11 players in a soccer team that wins $12-0$, there must be at least one player in the team who scored more than once (assuming no own-goals).

Proof: We shall apply the pigeonhole principle. Imagine a score sheet in which each player's name is written on a different row. When a goal is scored, a stroke in the appropriate row records who scored it. The rows on the scoresheet are like pigeonholes, and the goals are like pigeons. Since there are 12 goals (12 pigeons) to be recorded, and only 11 rows (11 pigeonholes) on the scoresheet, at least one row must contain the record of two goals. So at least one player scored more than once. $\square$

Example 3

Prove the following: If a molecule can exist in 2 different configurations, and you have 10^9 such molecules, at least half of them must be in the same configuration.

Proof: We shall apply the generalised pigeonhole principle. We have 10^9 objects (the molecules) to be classifed into 2 disjoint categories (the configurations). By the generalised pigeonhole principle, at least one category contains $\lceil 10^9/2 \rceil$ objects (molecules). Note that $\lceil 10^9/2 \rceil \geq 10^9/2$; that is, $\lceil 10^9/2 \rceil$ is at least half of 10^9. Hence at least one configuration is taken by at least $10^9/2$ molecules. $\square$

Example 4

Show that, in any set of a thousand words, there must be at least 39 words that start with the same letter.

Proof: We shall apply the generalised pigeonhole principle. Each word can be classified by its first letter, and such categories are disjoint. Since there are 1000 words and 26 categories, at least one category (configuration) contains $\lceil 1000/26 \rceil = \lceil 38.46 \rceil = 39$ words. $\square$

Example 5 (Epp(4ed) Q9.4.33)

Let A be a set of six distinct positive integers each of which is less than 15. Show that there must be two distinct nonempty proper subsets of A whose elements when added up give the same sum.

IDEAS: The phrase '*there must be two distinct subsets*' in the second sentence of the question suggests that the subsets should play the role of the pigeons.

Then the phrase '*give the same sum*' suggests that the possible sums should play the role of the pigeon holes. So

$$A = \{a, b, c, d, e, f \in \mathbb{N} \; : \; a < b < c < d < e < f < 15\}$$

Pigeons:	subsets of A
Pigeon holes:	possible element sums of subsets of A

Now we have to count the pigeons and pigeon holes.

Example 5 (cont)

Proof: First we show that there are 62 distinct proper nonempty subsets of A (pigeons). Since $|A| = 6$, we have that $|\mathcal{P}(A)| = 2^6 = 64$. Since $\emptyset$ is empty, and A is not proper, there are 62 distinct proper nonempty subsets of A.

Now we show that there are only 60 different possible element sums among these nonempty proper subsets. A proper subset of A contains at most 5 elements. The elements of A are selected from $1, 2, \ldots, 14$. The most a sum of at most 5 elements from these integers can be is $10 + 11 + 12 + 13 + 14 = 60$. The least the sum can be is 1. Hence the element sum of a proper nonempty subset of A is between 1 and 60. (Thus there are 60 pigeonholes).

By the pigeonhole principle, at least one element sum is taken by two distinct nonempty proper subsets of A. $\square$

Example 6 (cont.)

Given a set of 52 distinct integers, show that there must be two whose sum or difference is divisible by 100.

Proof: We label the 52 distinct integers $a_1, a_2, \ldots, a_{52}$. For each i such that $1 \leq i \leq 52$, let $A_i = a_i \bmod 100$. We consider two cases.
Case: There exist i, j such that $i < j$ and $A_i = A_j$.
In this case,

$$A_i = A_j \Leftrightarrow a_i \equiv a_j \pmod{100} \Leftrightarrow (a_i - a_j) \text{ is divisible by } 100.$$

In this case, there are two integers whose difference is divisble by 100 and we are done.

Example 6 (cont.)

Case: There does not exist i, j such that $i < j$ and $A_i = A_j$. In this case, the integers $A_1, A_2, \ldots, A_{52}$ are distinct. We shall apply the pigeonhole principle, with the integers $A_1, A_2, \ldots, A_{52}$ playing the role of pigeons. We label 51 'pigeonholes' as shown, so that each is labelled by two not necessarily distinct two-digit numbers:

00	01	02	03		49	50
$\square$	$\square$	$\square$	$\square$	....	$\square$	$\square$
00	99	98	97		51	50

Observe that, for each box, the pair of integers labelling the box have a sum that is divisible by 100.

There are now exactly 51 'pigeonholes', and 52 numbers in the list $A_1, \ldots, A_{52}$. So at least two of the numbers in the list $A_1, \ldots, A_{52}$ label the same pigeonhole. This means that at least two of the numbers in the list $A_1, \ldots, A_{52}$ have a sum that is divisible by 100. $\square$

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Permutations There are $n!$ ways to arrange n distinct objects in a list.

Principles of counting

r-Permutations There are

$$P(n,r) = \frac{n!}{(n-r)!}$$

ways to select and order r out of n distinct objects.

Combinations There are

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

ways to choose a set of r objects from a set of n objects (that is, to select r out of n distinct objects when the order in which objects are selected in not important). The notation $\binom{n}{r}$ is read "n choose r."

Example 7

Andy, Beth and Cai are standing in line.
In how many different orders could this queue be arranged?

Proof: Arranging Andy, Beth and Cai in a line is like arranging 3 distinct objects in a list. Thus there are $3! = 3 \times 2 \times 1 = 6$ ways to do this. $\square$

Checking our answer (unnecessary, but instructive): We can represent an "outcome" by listing three letters in order: the first letter represents the first person in the line; the second letter represents the second person in the line; and the third letter represents the third person in the line. The entire set of outcomes, containing 6 elements, is represented below:

$$\{ABC, ACB, BAC, BCA, CAB, CBA\}$$

Example 8

A pet show awards 1st, 2nd and 3rd prizes. There are 5 entrants:

Rachel the Rabbit	Charles the Chicken	Tilly the Terrier
Bob the Bilby	Karen the Kangaroo	

In how many ways can the prizes be handed out?

Proof: Handing out the prizes is like selecting and ordering 3 out of 5 distinct objects. Thus there are

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 5 \times 4 \times 3 = 60$$

ways to hand out the prizes. $\square$

Checking our answer to Example 8

We can represent an "outcome" by listing three letters in order: the first letter represents the pet awarded first prize; the second letter represents the pet awarded second prize; and the third letter represents the pet awarded third prize. The entire set of outcomes, containing 60 elements, is represented below:

{ BCK,	BCR,	BCT,	BKC,	BKR,	BKT,
BRC,	BRK,	BRT,	BTB,	BTC,	BTR,
CBK,	CBR,	CBT,	CKB,	CKR,	CKT,
CRB,	CRK,	CRT,	CTB,	CTK,	CTR,
KBC,	KBR,	KBT,	KCB,	KCR,	KCT,
KRB,	KRC,	KRT,	KTB,	KTC,	KTR,
RBC,	RBK,	RBT,	RCB,	RCK,	RCT,
RKB,	RKC,	RKT,	RTB,	RTC,	RTK,
TBC,	TBK,	TBR,	TCB,	TCK,	TCR,
TKB,	TKC,	TKR,	TRB,	TRC,	TRK, }

Example 9

A different pet show does not give 1st, 2nd and 3rd prizes. Instead, three "ribbon-winning" pets are chosen to travel to the district show. There are 5 entrants:

Rachel the Rabbit   Charles the Chicken   Tilly the Terrier
Bob the Bilby   Karen the Kangaroo

In how many ways can the ribbon-winners be chosen?

Proof: Selecting the ribbon-winning pets is like choosing a set of 3 objects (ribbon-winning pets) from a set of 5 objects (the pets in the show). Thus there are

$$C(5,3) = \binom{5}{3} = \frac{P(5,3)}{3!} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

ways to choose the ribbon-winners. $\square$

Checking our answer to Example 9

Each "outcome" may be represented by a set containing the ribbon-winning pets. The ten sets representing the ten different outcomes are listed below:

{Rachel, Charles, Tilly},	{Rachel, Charles, Bob},
{Rachel, Charles,Karen},	{Rachel, Tilly, Bob},
{Rachel, Tilly, Karen},	{Rachel, Bob, Karen},
{Charles, Tilly, Bob},	{Charles, Tilly, Karen},
{Charles, Bob, Karen},	{Tilly, Bob, Karen}

A mixed example

How many distinguishable ways can the letters of the word

MILLIMICRON

be arranged?

If we were to distinguish between like letters using labels, as in

$$\mathrm{M_1 I_1 L_1 L_2 I_2 M_2 I_3 C\,R\,O\,N}$$

there would be $11! = 39\,916\,800$ different arrangements.

Now we must compensate for the over-counting induced by this distinguishing between indistinguishable arrangements.

Since MILLIMICRON has 2 M's, 3 I's and 2 L's the true answer is :

$$\frac{11!}{2!\,3!\,2!} = 1\,663\,200.$$

This example generalises both permutations and combinations.
Can you see how?

Principles of counting

Multisets (Stars and Bars) There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n. That is, there are $\binom{r+n-1}{r}$ ways to arrange a list of r stars and $n-1$ bars.

Inclusion-Exclusion If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

The Sum Rule If A is a finite set and $\{A_1, A_2, \ldots, A_m\}$ is a partition of A, then $|A| = |A_1| + |A_2| + \cdots + |A_m|$.

The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$.

'Stars and Bars'

Let s and b be positive integers. How many different strings of length $s+b$ can we form out of s stars ($\star$) and b bars ($|$)? For example, here are three different strings of length 13 formed out of 10 stars and 3 bars:

$$| \star \star \star \star | \star \star | \star \star \star \star$$
$$\star | \star \star \star \star \star \star \star | | \star \star$$
$$\star \star \star | \star \star | \star \star | \star \star \star$$

ANSWER: Consider a horizontal arrangement of cells numbered 1 through $s+b$.

1	2	3	...	$s+b$
			...	

Making a string of s stars and b bars is like choosing a set of b numbers from the set $\{1, 2, \ldots, s+b\}$, where the set of chosen numbers is the set of positions containing bars. Hence we can form $\binom{s+b}{b}$ different strings of length $s+b$ from s stars and b bars.

'Stars and Bars' example

(Epp(4ed) Q9.6.15)

For how many integers from 1 through 99 999 is the sum of their digits equal to 10?

Proof: By inserting leading zeros if necessary, all the integers to be counted can be considered to be 5-digit strings *abcde* with $a+b+c+d+e=10$.

Each of these 5-digit strings can be represented as a length-14 pattern of 10 **stars** and 4 **bars**. For example:

$\star\star|\star\star\star\star|\star|\star\star|\star$ represents 24 121.

$\star\star\star||\star\star\star\star|\star\star\star|$ represents 30 430.

$|||\star\star\star\star\star\star\star\star\star|\star$ represents 00 091.

There are $\binom{14}{4}$ ways to arrange a list of 10-stars and 4-bars. But five of the patterns have ten stars in a row and so don't count (ten is not a digit). So the number of integers is

$$\binom{14}{4}-5=\frac{14\times13\times12\times11}{4\times3\times2\times1}-5=996.\square$$

Counting 'Multisets'

What is a 'multiset'?

It's a 'set' with multiple copies of elements allowed and acknowledged.
An example is $\{c, b, a, c, a\}$, which has 2 a's, 1 b and 2 c's.
As for ordinary sets, order is irrelevant: $\{c, b, a, c, a\} = \{a, a, b, c, c\}$.
But the multiplicities **do** matter.

Formally, a **size-r multiset** is a set S together with a 'multiplicity function' $m : S \to \mathbb{N}$, where,

$$\forall s \in S \quad m(s) = \text{number of copies of } s \quad \text{and} \quad r = \sum_{s \in S} m(s).$$

So, for example, $\{c, b, a, c, a\}$ has size $r = 2 + 1 + 2 = 5$.

Counting multisets

How many different size-r multisets can be formed from members of a set S of cardinality n?
IDEA: A size r-multiset formed from members of a set S of cardinality n can be represent by a pattern of r stars and $n-1$ bars.

For example if $S=\{a,b,c,d\}$ and $r=5$ then $\{c,b,a,c,a\}$ is represented by $\star\star|\star|\star\star|$ $(m(a)=2, m(b)=1, m(c)=2, m(d)=0)$.

Which multisets, selectected from S, are represented by the following arrangements?

$\|\|\star\star\star\|\star\star$	$\|\star\star\star\|\star\star\|$	$\star\star\|\star\|\star\|\star$
$\{c,c,c,d,d\}$	$\{b,b,b,c,c\}$	$\{a,a,b,c,d\}$

There are $\binom{r+n-1}{r}$ size-r multisets with members from a set of size n.

Multisets example

(Epp(4ed) Q9.6.6)

If n is a positive integer, how many 5-tuples of integers from 1 through n can be formed in which the elements of the 5-tuple are written in non-increasing order?

For $n = 9$ some 5-tuples are $(8, 6, 4, 2, 1)$, $(9, 3, 3, 2, 2)$, $(6, 6, 6, 6, 6)$.

There is a bijection (one-to-one correspondence) between the set of all these 5-tuples and the set of all size-5 multisets chosen from $\{1, \ldots, n\}$, because the r 'members' of the multiset can only be arranged in one way in non-increasing order.

So by stars-and-bars, there are $\binom{5+n-1}{5} = \binom{n+4}{5}$ of these 5-tuples.

For example for $n = 3$ there are $\binom{7}{5} = \binom{7}{2} = 21$ such 5-tuples:

33333 33332 33331 33322 33321 33311 33222 33221 33211 33111
32222 32221 32211 32111 31111 22222 22221 22211 22111 21111 11111

New counts from old

- The Sum Rule
- The Product Rule
- Inclusion-Exclusion

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The Product Rule If $A_1, A_2, \ldots, A_m$ are finite sets, then $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|$.

The Sum Rule

If sets A and B are finite and *disjoint* then the cardinality of their union $A \cup B$ is the sum of the their individual cardinalities, i.e.

$$\boxed{A \cap B = \emptyset \implies |A \cup B| = |A| + |B|.}$$

More generally, if $\{A_1, A_2, ..., A_m\}$, $m \in \mathbb{N}$, is a *partition* of the finite set A then

$$\boxed{|A| = |A_1| + |A_2| + ... + |A_m|.}$$

Example:
For $U = \{-10, \ldots, 10\} \subseteq \mathbb{Z}$ and $S = \{n \in U \ : \ |20 - n^2| > 10\}$, find $|S|$.

Observe that $|20 - n^2| > 10 \iff n^2 < 10 \ \vee \ n^2 > 30$.

So $S = \{-3, -2, \ldots, 2, 3\} \cup \{6, 7, \ldots, 10\} \cup \{-10, -9, \ldots, -6\}$.

Hence $|S| = 7 + 5 + 5 = 17$.

The Product Rule

For finite sets A and B the cardinality of their cartesian product $A \times B$ is the product of the their individual cardinalities, i.e.

$$\boxed{|A \times B| = |A| \times |B|.}$$

More generally, for finite sets $A_1, A_2, ..., A_m$, $m \in \mathbb{N}$,

$$\boxed{|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|.}$$

Example:

A regular **cubic die, D6,** has face set $C = \{1, 2, 3, 4, 5, 6\}$.

An **octahedral die, D8,** has face set $O = \{1, 2, \dots, 8\}$.
A **dodecahedral die, D12,** has face set $D = \{1, 2, \dots, 12\}$.

The number of possible outcomes from throwing the three dice together is $|C \times O \times D| = |C| \times |O| \times |D| = 6 \times 8 \times 12 = 576.$

The product rule via an outcome construction procedure

Often the product rule is implemented by designing a procedure by which we construct one of the objects to be counted, and then counting the number of different possible outcomes from each step of the procedure. When you do this you must be careful that each outcome can be constructed in only one way.

Suppose that a web-banking password is always 8 characters long and it always comprises two upper case letters, one digit, and 5 lower case characters. How many different passwords can be created that follow these rules?

Counting passwords

A: We construct a password in 10 steps.
In step 1, we choose the two positions in which the upper case letters will be placed. There are $\binom{8}{2}$ ways to make this choice.

In step 2, we choose one integers from the remaining 6 positions. This are $\binom{6}{1}$ ways to make this choice.

In step 3, we choose an upper case letter to go in the first place for an upper case letter. There are 26 ways to make this choice.

In step 4, we choose an upper case letter to go in the second place for an upper case letter. There are 26 ways to make this choice.

In step 5, we choose a digit to go in the digit place. There are 10 ways to make this choice.

In step 6, we choose a lower case letter to go in the first place for a lower case letter. There are 26 ways to make this choice.

Couning passwords (cont.)

In step 7, we choose a lower case letter to go in the second place for a lower case letter. There are 26 ways to make this choice.

In step 8, we choose a lower case letter to go in the third place for a lower case letter. There are 26 ways to make this choice.

In step 9, we choose a lower case letter to go in the fourth place for a lower case letter. There are 26 ways to make this choice.

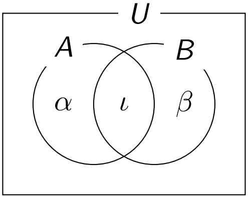
In step 10, we choose a lower case letter to go in the fifth place for a lower case letter. There are 26 ways to make this choice.

By the product rule, we can create

$$\binom{8}{2} \times \binom{6}{1} \times 26 \times 26 \times 10 \times 26^5$$

different passwords.□

Inclusion-Exclusion

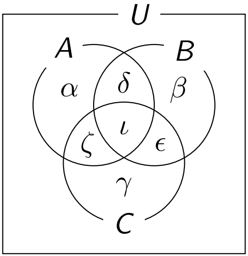


If A and B are finite sets which *may not be disjoint* the sum rule has to be modified to the **inclusion-exclusion rule:**

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

This is because the plain sum rule $|A \cup B| = |A| + |B|$ includes the intersection $A \cap B$ twice, so it has to be excluded once:

$$|A \cup B| = \alpha + \iota + \beta = (\alpha + \iota) + (\iota + \beta) - \iota = |A| + |B| - |A \cap B|.$$



The inclusion-exclusion rule can be generalised to deal with more than two sets, but it quickly gets very messy.

Can you figure out how to extend the rule to deal with just three sets A, B and C?

Bit-String Example of Inclusion-Exclusion

How many bytes start with '1' or end with '00'?

IDEA: Count the bytes that start with 1, then count the bytes that end in 00, then count the bytes that start with 1 and end with 0, then apply Inclusion-Exclusion.

Task 1: Construct a byte that starts with '1'.

- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)
- There are two ways to choose the third bit (0 or 1)

  ⋮

- There are two ways to choose the eighth bit (0 or 1)

Product Rule: Task 1 can be done in $1 \times 2^7 = 128$ ways.

Bit-String Example of Inclusion-Exclusion

Task 2

Task 2: Construct a byte that ends with '00'.

- There are two ways to choose the first bit (0 or 1)
- There are two ways to choose the second bit (0 or 1)

  ⋮

- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 2 can be done in $2^6 \times 1^2 = 64$ ways.

Bit-String Example of Inclusion-Exclusion

Task 3

Is the answer $128+64 = 196$? **NO!**

That would be overcounting.
We have to subtract off the cases we counted twice.

Task 3: Construct a string of length 8 that both starts with '1' and ends with '00'.

- There is one way to choose the first bit (1)
- There are two ways to choose the second bit (0 or 1)

$\vdots$

- There are two ways to choose the sixth bit (0 or 1)
- There is one way to choose the seventh bit (0)
- There is one way to choose the eighth bit (0)

Product Rule: Task 3 can be done in $1 \times 2^5 \times 1^2 = 32$ ways.

Bit-String Example of Inclusion-Exclusion

Conclusion

Finally, the number of ways to construct a bit string of length 8 that starts with '1' or ends with '00' is equal to:

- the number of ways to do task 1, **plus**
- the number of ways to do task 2, **minus**
- the number of ways to do both at the same time (task 3), *i.e.*

$$128 + 64 - 32 = 160.$$

Note: An alternative, and quite different, way to solve this problem is to use **complementary counting**; *i.e.* calculate $|S^c|$ where S is the set of strings we are interested in and U is the set of all (8-bit) strings. Then $|S| = |U| - |S^c| = 2^8 - 1 \times 2^5 \times 3 = 256 - 96 = 160$.

Can you see how to get the $1 \times 2^5 \times 3$?

Combinations and binomial coefficients

There are $\boldsymbol{C(n,r)} = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$ ways to choose a set of r objects from a set of n candidates.
I.e. A set of cardinality n has $\frac{n!}{r!(n-r)!}$ subsets of cardinality r.

The subsets are called ***r*-combinations**.

We say 'n choose r' for $C(n,r)$ and often write it $\dbinom{n}{r}$.

These numbers $\binom{n}{r}$ arise as coefficients in the algebraic expansion of the n-th power of the 'binomial' $(x+y)$ and are consequently also known as **binomial coefficients**. The expansion is

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Two important properties of $\binom{n}{r}$

1. $\forall r, n \in \mathbb{N}^\star$ $0 \leq r \leq n \implies$ $\boxed{\binom{n}{r} = \binom{n}{n-r}.}$ *e.g.* $\binom{5}{3} = \binom{5}{2}$.

   $\mathbb{Z}_{\geq 0}$

   Proof: Choosing the r elements of a subset S of U, with $|U| = n$, is exactly equivalent to choosing the $n-r$ elements of U to be left out.

2. $\forall r, n \in \mathbb{N}^\star$ $0 < r \leq n \implies$ $\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.}$

   $\mathbb{Z}_{\geq 0}$

   e.g. $\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$.

   (Pascal's Triangle Identity)

   Proof: Let u be a fixed member of U, with $|U| = n$.
   Subsets S of U with $|S| = r$ are of two types; those that don't contain u and those that do.
   There are $\binom{n-1}{r}$ of the first kind, since the r members of S are chosen from the $n-1$ members of $U \backslash u$.
   There are $\binom{n-1}{r-1}$ of the second kind, since the $r-1$ members of $S \backslash u$ are also chosen from the $n-1$ members of $U \backslash u$.

END OF SECTION C1