MATH1005/MATH6005 Discrete Mathematical Models

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An acknowledgment of country

We acknowledge and celebrate the First Australians on whose traditional lands we live, work and study. We pay our respects to the elders past and present. In particular, we acknowledge the Ngunnawal and Ngambri people, the Traditional Owners of the land upon which the University's Acton campus is located.

Section A1: Logic

Statements: The basic unit of logic

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- Tropical storms spin clockwise in the Southern Hemisphere.
- Canberra is in New Zealand.
- $3 > 2$

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- Canberra is in New Zealand.
- $3 > 2$

Some sentences that are not statements:

- How are you going? (question)
- Canberra is a better city than Sydney. (ambiguous)
- Wake up! (instruction/advice)
- This statement is false. (neither true nor false; self-referencing)

Compound statements

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Example: Australia is in the Southern Hemisphere and tropical storms spin clockwise in the Southern Hemisphere.

Example: If 2 divides 4238 and 3 divides 4238, then 6 divides 4238.

Compound statements

Sometimes the way a compound statement is built from other statements is less clear, because our everyday use of language does not make the logical structure obvious. For example, the word ‘not’ sometimes causes confusion because of its placement.

Examples of compound statements

Q: In each of the following examples, identify clearly the way the statement is built from other statements using logical connectives:

- Australia is in the Southern hemisphere and tropical storms spin clockwise in the Southern Hemisphere.
- Australia is in the Southern Hemisphere and Canberra is not in New Zealand.
- I don't like either the one or the other.

Compound statements

Takeway: What makes for nice spoken or written language, may not necessary make the logical structure of a compound statement most clear. Worse, the everyday use of language can be ambiguous.

Mathematicians and computer scientists must agree upon the meaning of logical connectives, and must be proficient at recognising the way that compound statements are built from simpler statements, so that we can communicate effectively (human-to-human, and human-to-machine).

How to introduce a symbol for a statement

When you first learned algebra, you learned that it was useful to use variables to make abstract statements about relationships between numbers. In logic, we sometime save time and space, and clarify the logical structure of a compound statement, by introducing symbols as names for statements.

For example, we write

$$p: \text{ Australia is in the Southern Hemisphere}$$

to mean that the symbol p now represents the statement "Australia is in the Southern Hemisphere."

Statement variables and statement forms

We can also use variables to represent arbitrary statements. When a variable represents an arbitrary statement, it is called a **statement variable**.

We have symbols for various logical connectives (such as $\neg$ for 'not', $\wedge$ for 'and', $\vee$ for 'or'). We will define these properly in a moment.

An expression built using statement variables, parentheses and logical connectives is called a **statement form** if the expression becomes a statement when actual statements are substituted for the component statement variables.

Examples of statement forms

Example: Let p and q be statement variables.

The expression $q \vee \neg(p \wedge q)$ is a statement form.

The expression $p\neg \wedge q$ is not a statement form (just like $x + \times\ y$ is not an algebraic expression).

About statement forms

The value of the algebraic expression $x \times y + y$ depends on the values taken by x and y. You need to understand the multiplication and addition operations, and the order of precedence among operations, to figure out the value of the expression.

Analogously, the **truth value** of the statement form $q \vee \neg(p \wedge q)$ depends on the truth values taken by p and q. You need to understand the logical connectives $\neg$, $\wedge$, $\vee$ and the order of precedence among them to figure out the truth value of the statement form.

Truth tables

We are now ready to define logical connectives. We shall do so using truth tables. A **truth table** records the truth value taken by a statement form for each possible combination of truth values taken by the statement variables appearing in the statement form.

A logical connective: NOT

If p is a statement variable, the **negation** of p is read “not p”, denoted $\neg p$, and defined by the following truth table:

p	$\neg p$
T	F
F	T

A logical connective: AND

If p and q are statement variables, the **conjunction** of p and q is read "p and q", denoted $p \wedge q$, and defined by the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

A logical connective: OR

If p and q are statement variables, the **disjunction** of p and q is read "p or q", denoted $p \vee q$, and defined by the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

A logical connective: XOR

If p and q are statement variables, the **exclusive disjunction** of p and q is read "p or q but not both", denoted $p \oplus q$, and defined by the following truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

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If p and q are statement variables, the **exclusive disjunction** of p and q is read "p or q but not both", denoted $p \oplus q$, and defined by the following truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Note: Grammatically, the words 'and' & 'but' are both conjunctions, so are both interpreted as AND in logic. You could read $p \oplus q$ as "p or q and not both."

A logical connective: IMPLIES

If p and q are statement variables, the **conditional** of q by p is read "if p then q" or "p implies q", denoted $p \to q$, and defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

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T	F	F
F	T	T
F	F	T

There is extra vocabulary associated with conditional. In the statement form $p \to q$, we call p the **hypothesis** (or antecedent) and q the **conclusion** (or consequent).

Understanding IMPLIES

Pay careful attention to the truth table for a conditional. On first sight, some find the last two lines surprising. Spend some time making sure you know the details of the definition.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

It sometimes helps to use language to emphasize *why* a conditional is true. When the hypothesis in a conditional is false, we say that the conditional is **vacuously true**.

A logical connective: IFF

If p and q are statement variables, the **biconditional** of p and q is read "p if and only if q", denoted $p \leftrightarrow q$, and defined by the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of precedence among connectives

When evaluating a statement form:

- Obey parentheses over any precedence rule
- Evaluate $\neg$ first
- Evaluate $\wedge$, $\vee$ and $\oplus$ second. When two or more are present, parentheses may be needed
- Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

When writing statement forms, you must use parentheses to avoid ambiguity. For example, you should write $p \wedge (q \vee r)$ rather than $p \wedge q \vee r$.

Example: A truth table for a statement form

Q: Write a truth table for $(p \wedge q) \vee (\neg p \wedge \neg r)$.

First we note that:

- Our table needs a row for each combination of truth values among the statement variables. Since there are three statement variables in the statement form, our table will need $2^3 = 8$ rows.
- We need to obey the order of precedence among logical connectives as we evaluate the truth value of the statement form.

We shall present two ways to lay out our work.

Method 1: Extra columns

p	q	r	$p \wedge q$	$\neg p$	$\neg r$	$\neg p \wedge \neg r$	$(p \wedge q) \vee (\neg p \wedge \neg r)$
T	T	T	T	F	F	F	T
T	T	F	T	F	T	F	T
T	F	T	F	F	F	F	F
T	F	F	F	F	T	F	F
F	T	T	F	T	F	F	F
F	T	F	F	T	T	T	T
F	F	T	F	T	F	F	F
F	F	F	F	T	T	T	T

An advantage of this method is that you leave a record of your working.

A disadvantage is that is can take a lot of room on the page.

Method 2: Align T/F under connectives

p	q	r	$(p \wedge q) \vee (\neg p \wedge \neg r)$
T	T	T	T $\mathbf{T}$ F F F
T	T	F	T $\mathbf{T}$ F F T
T	F	T	F $\mathbf{F}$ F F F
T	F	F	F $\mathbf{F}$ F F T
F	T	T	F $\mathbf{F}$ T F F
F	T	F	F $\mathbf{T}$ T T T
F	F	T	F $\mathbf{F}$ T F F
F	F	F	F $\mathbf{T}$ T T T

- Be careful because your work leaves less of a record of how you did it.
- Bold, circle or highlight the column with the final result.

Statements and statement forms

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EXAMPLE: Consider the statement "$\sqrt{2}$ is negative or non-negative." The statement is an instance of the statement form $p \vee q$ with

$$p : \sqrt{2} \text{ is negative}$$
$$q : \sqrt{2} \text{ is not negative.}$$

The statement is also an instance of the statement form $p \vee \neg p$.

Another example

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EXAMPLE: Consider the statement "Tropical storms spin clockwise in the Southern hemisphere." The statement is an instance of the statement form $p \to q$ with

$$p: \text{ We are in the Southern Hemisphere}$$
$$q: \text{ Tropical storms spin clockwise.}$$

Do you agree that the following statements mean the same thing?

- Tropical storms spin clockwise in the Southern hemisphere
- Tropical storms spin clockwise if we are in the Southern hemisphere
- If we are in the Southern Hemisphere then tropical storms spin clockwise.

Tautologies

A statement form that is true regardless of the truth values of its variables is called a **tautology**. A statement whose form is a tautology is also called a **tautology**.

We can show that a statement form is a tautology by making a truth table. We can show that a statement is a tautology by making a truth table for a corresponding statement form.

Please note that a statement may be true even though it is not a tautology.

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Please note that a statement may be true even though it is not a tautology.

EXAMPLE: The statement "6 is an even integer" is true, but it is not a tautology. The statement "6 is even or 6 is not even" is true and it is a tautology.

Showing a statement form is a tautology

Example: The statement form $p \vee \neg p$ is a tautology, as shown by:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Showing a statement form is a tautology

Example: The statement form $p \vee \neg p$ is a tautology, as shown by:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Example: The statement "$\sqrt{2}$ is negative or non-negative" is a tautology because it is an instance of the statement form $p \vee \neg p$ with

$$p : \sqrt{2} \text{ is negative,}$$

and the statement form $p \vee \neg p$ is a tautology.

Contradiction

A statement form that is false regardless of the truth values of its variables is called a **contradiction**. A statement whose form is a contradiction is also called a **contradiction**.

We can show that a statement form is a contradiction by making a truth table. We can show that a statement is a contradiction by making a truth table for the corresponding statement form.

Logical equivalence

When two statement forms f, g have identical truth tables we say they are **logically equivalent** and write $f \equiv g$.

(ALTERNATIVE DEFINITION: When f, g are statement forms and $f \leftrightarrow g$ is a tautology, we say that f and g are **logically equivalent** and write $f \equiv g$.)

Knowing some logical equivalences allows us to replace complicated statement forms by simpler, logically equivalent, statement forms.

Some well-known logical equivalences are shown on the next slide.

From page 49 of your optional text

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Note: The text uses $\sim$ for negation instead of $\neg$.

Example: An equivalent form of XOR

Q: Justify the earlier claim that $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$.

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We compute a truth table:

p	q	$p \oplus q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

Because the entries in column 3 and column 7 agree in every row, the truth table above establishes that
$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$.

Example: Another equivalent form for XOR

Q: Show that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

Example: Another equivalent form for XOR

Q: Show that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

We compute a truth table:

p	q	$p \oplus q$	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	F	T	T
F	F	F	T	T	F	F	F

Because the entries in column 3 and column 8 agree in every row, the truth table above establishes that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$.

Suggested activities

- Ensure you know how to watch videos and download slides from the ECHO360 system.
- Master the vocabulary introduced in this lecture and practice making truth tables for complex statement forms.
- Look at the materials in the week 1 section on Wattle, especially the "Lecture 0" slides.
- Look at the practice problems A1.
- Find the ebook of our optional text in the ANU library catalogue and see what you think. Find exercises at the end of §2.1 related to this lecture.