Section A1: Logic (continued)

How to prove things Putting our logic to work.

How to start

Before trying to prove a statement, you should clearly identify the logical structure of the statement. Doing so allows you to understand the choices you have in choosing a logical structure for your proof.

Let's understand the logical structures that can be used to prove statements with various logical structures.

Proving $\forall$

To prove a statement of the form $\forall x\ p(x)$, one may follow this plan:

Let x be a (fixed but arbitrary) element of the predicate domain. Argue that $p(x)$ is true.

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Example
Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Working the example

We need some definitions:

Defn: An integer x is **even** if there exists an integer k such that $x = 2k$.

Defn: An integer x is **odd** if there exists an integer k such that $x = 2k + 1$.

Theorem: (The even/odd theorem)
Every integer is either even or odd; no integer is both even and odd.

We note that the theorem could be stated succinctly using our logic notation:

$$\forall x \ \ (x \text{ is even}) \oplus (x \text{ is odd})$$

where the domain of quantification is understood to be the integers.

Back to our example

Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Proof: Let x be an integer.

$\vdots$

Hence $6x^2 + 4$ is even. $\square$

What we have written above is the **logical structure of a proof**.

Back to our example

Prove the following statement: Whenever x is an integer, $6x^2 + 4$ is even.

Proof: Let x be an integer. We note that the integers are closed under multiplication and addition.

Since x is an integer and the integers are closed under multiplication, x^2 is an integer.
Since x^2 is an integer and the integers are closed under multiplication, $3x^2$ is an integer.
Since $3x^2$ is an integer and the integers are closed under addition, $3x^2 + 2$ is an integer.
Since $3x^2 + 2$ is an integer, $2(3x^2 + 2)$ is even.
But $2(3x^2 + 2) = 6x^2 + 4$.

Hence $6x^2 + 4$ is even. $\square$

Proving $\exists$

To prove a statement of the form $\exists x\ p(x)$, one may identify a particular element of the predicate domain and establish that $p(x)$ is true. Please note, it is not enough to simple state which element x of the domain has the required property, you should explain how you know that the particular element you identified has the required property (how you know that $p(x)$ is true). This is called **exhibiting an example**.

An example

Prove the following statement: The equation $x^2 - 6x + 8 = 0$ has an integer solution.

An example

Prove the following statement: The equation $x^2 - 6x + 8 = 0$ has an integer solution.

Proof: When $x = 2$, the left-hand side of the equation evaluates to $2^2 - 6(2) + 8 = 4 - 12 + 8 = 0$. Hence 2 is an integer solution to the equation. $\square$

Disproving $\forall$

To disprove a statement of the form $\forall x \ p(x)$, one should prove the statement $\exists x \ \neg p(x)$. (This is called providing a **counterexample**)

An example

Prove or disprove the following statement: For every integer x, $(x-1)^2 + (x-1)$ is positive.

An example

Prove or disprove the following statement: For every integer x, $(x-1)^2+(x-1)$ is positive.

The statement is false because $x=0$ is a counterexample. When $x=0$,

$$(x-1)^2+(x-1) = (0-1)^2+(0-1) = (-1)^2+(-1) = 1-1 = 0$$

and 0 is not positive.

Disproving $\exists$

To disprove a statement of the form $\exists x\ p(x)$, one should prove the statement $\forall x\ \neg p(x)$.

An example

Prove or disprove the following statement:

$$\exists y \ \ \forall x \ \ (y \leq x),$$

where the quantification is over the set of integers.

An example

Prove or disprove the following statement:

$$\exists y \ \ \forall x \ \ (y \leq x),$$

where the quantification is over the set of integers.

The statement is false.
(To show that the statement is false, we must show the following

$$\neg \, \exists y \ \ \forall x \ \ (y \leq x) \equiv \forall y \ \ \exists x \ \ \neg(y \leq x) \equiv \forall y \ \ \exists x \ \ (y > x))$$

Let y be an integer. Let $x = y - 1$. It is clear that $y > x$.

Proving $\to$

To prove $p \to q$ you may:

- Suppose that p is true.
- Deduce by valid and explicit reasoning that q must be true (using the truth of p along the way).

This is called **arguing directly**.

You may also:

- Suppose that $\neg q$ is true.
- Deduce by valid and explicit reasoning that $\neg p$ must be true (using the truth of $\neg q$ along the way).

This is called **arguing via the contrapositive**.

An example

Prove the following statement: For all integers x, if x is even then $x^2 + 2$ is even.

The logical structure

Let x be an integer. Prove the following statement: For all integers x, if x is even then $x^2 + 2$ is even.

Proof: Let x be an integer. We shall argue directly. Suppose that x is even.

$\vdots$

Hence $x^2 + 2$ is even. $\square$

The proof

Prove the following statement: For all integers x, if x is even then $x^2 + 2$ is even.

Proof: Let x be an integer. We shall argue directly. Suppose that x is even.

Since x is even, there exists an integer k such that $x = 2k$. Then $x^2 + 2 = (2k)^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1)$. Since k is an integer, so is $2k^2 + 1$.

Hence $x^2 + 2$ is even. $\square$

An example

Prove the following statement: For all integers x, if $x^2 + 2$ is even, then x is even.

The logical structure

Prove the following statement: For all integers x, if x^2+2 is even, then x is even.
Proof: Let x be an integer. We shall argue via the contrapositive. Suppose that x is not even.

$\vdots$

Hence x^2+2 is not even. $\square$

The logical structure II

Prove the following statement: For all integers x, if $x^2 + 2$ is even, then x is even.

Proof: Let x be an integer. We shall argue via the contrapositive. Suppose that x is not even. By the even/odd theorem, x is odd.

$\vdots$

Hence $x^2 + 2$ is odd. By the even/odd theorem, $x^2 + 2$ is not even. $\square$

The proof

Prove the following statement: For all integers x, if x^2+2 is even, then x is even.

Proof: Let x be an integer. We shall argue via the contrapositive. Suppose that x is not even. By the even/odd theorem, x is odd. So there exists an integer k such that $x = 2k+1$. Now

$$\begin{aligned} x^2+2 = (2k+1)^2+2 = (4k^2+4k+1)+2 &= 4k^2+4k+2+1 \\ &= 2(2k^2+2k+1)+1. \end{aligned}$$

Since k is an integer, so is $2k^2+2k+1$. Hence x^2+2 is odd. By the even/odd theorem, x^2+2 is not even. $\square$

Proving $\leftrightarrow$

To prove $p \leftrightarrow q$, you may first prove $p \to q$ and then prove $q \to p$

It is possible to accomplish "both directions" of proof simultaneously by arguing with biconditionals throughout your proof, but you must be careful when doing so.

An example

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

The logical structure

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.
Proof: Let x be an integer.

For the $\rightarrow$ direction, we argue directly. Suppose first that x is even.

$\vdots$

Hence $x^2 + 2$ is even.

For the $\leftarrow$ direction, we argue via the contrapositive. Now suppose x is not even.

$\vdots$

Hence $x^2 + 2$ is not even.

An example

Prove the following statement: For all integers x, x is even if and only if $x^2 + 2$ is even.

Proof: Let x be an integer.
For the $\rightarrow$ direction, we argue directly. Suppose that x is even. Since x is even, there exists an integer k such that $x = 2k$. Then $x^2 + 2 = (2k)^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1)$. Since k is an integer, so is $2k^2 + 1$. Hence $x^2 + 2$ is even.
For the $\leftarrow$ direction, we argue via the contrapositive. Suppose that x is not even. By the even/odd theorem, x is odd. So there exists an integer k such that $x = 2k + 1$. Now

$$\begin{aligned} x^2 + 2 = (2k+1)^2 + 2 = (4k^2 + 4k + 1) + 2 &= 4k^2 + 4k + 2 + 1 \\ &= 2(2k^2 + 2k + 1) + 1. \end{aligned}$$

Since k is an integer, so is $2k^2 + 2k + 1$. Hence $x^2 + 2$ is odd. By the even/odd theorem, $x^2 + 2$ is not even. $\square$

Arguing by cases

If the domain of a predicate is partitioned into subsets, you may prove a $\forall$ statement by proving it for each subset.

Example

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

The logical structure

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

Proof: Let x be an integer. By the even/odd theorem, every integer is either even or it is odd.

Consider first the case that x is even.
$\vdots$
Hence $x^2 + x + 6$ is even.

Now consider the case that x is odd.
$\vdots$
Hence $x^2 + x + 6$ is even.

In all cases, $x^2 + x + 6$ is even. $\square$

The proof

Prove the following statement: For all integers x, $x^2 + x + 6$ is even.

Proof: Let x be an integer. By the even/odd theorem, every integer is either even or it is odd.

Consider first the case that x is even. Since x is even, there exists an integer k such that $x = 2k$. Then

$$x^2 + x + 6 \;=\; (2k)^2 + (2k) + 6 \;=\; 4k^2 + 2k + 6 \;=\; 2(2k^2 + k + 3)$$

Since k is an integer, so is $2k^2 + k + 3$. Hence $x^2 + x + 6$ is even.

Now consider the case that x is odd. Since x is odd, there exists an integer k such that $x = 2k + 1$. Then

$$\begin{aligned} x^2 + x + 6 = (2k+1)^2 + (2k+1) + 6 &= 4k^2 + 4k + 1 + 2k + 1 + 6 \\ &= 4k^2 + 6k + 8 = 2(2k^2 + 3k + 4). \end{aligned}$$

Since k is an integer, so is $2k^2 + 3k + 4$. Hence $x^2 + x + 6$ is even.

In all cases, $x^2 + x + 6$ is even. $\square$

Proof by contradiction

To prove a statement p, you may disprove $\neg p$. One way to do this is to suppose $\neg p$, and use this fact to deduce a statement we know to be false. Since a true statement cannot imply a false statement, we must have that $\neg p$ is false. This is called a **proof by contradiction**.

An example

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

The logical structure

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

Proof: We shall use a proof by contradiction. Suppose there exist integers x and y such that $5x + 20y = 4$.

$\vdots$

Since have deduced something we know to be false, our original supposition is impossible. $\square$

The proof

Prove the following statement: No integers x and y exist for which $5x + 20y = 4$.

Proof: We shall use a proof by contradiction. Suppose there exist integers x and y such that $5x + 20y = 4$.

Dividing both sides of the equation by 5 yields $x + 4y = \frac{4}{5}$. Since x and y are integers, $x + 4y$ is an integer. Since $x + 4y$ is an integer and $x + 4y = \frac{4}{5}$, we deduce that $\frac{4}{5}$ is an integer; this is, of course, false.

Since have deduced something we know to be false, our original supposition is impossible. $\square$

Some advice

1. Before starting a proof, clearly identify the logical structure of the statement to be proved.
2. Consider your options for a logical structure that will prove the statement.
3. Write down the logical structure of your argument so that the reader knows what is going on.
4. When deciding between a direct argument and an argument via the contrapositive, try whichever direction appears to allow you to make the strongest supposition first. The same advice applies when considering a proof by contradiction or one of the other methods.

Section A2: Sets

References

The ideas about sets that we cover in the next few lectures are covered in the following sections in our optional text:

3ed: Chapter 5

4ed: Chapter 6

5ed: Sections 1.2-1.3 and Chapter 6

Sets and elements

A **set** is a collection of **elements**.

This is an intuitive statement, but cannot be considered a definition because we do not give a precise definition of "collection" or "element" (other than saying they are the things that are in sets). Even so, this is where we start.

The notation $a \in S$ is read "a is an element of S".

The notation $a \notin S$ is read "a is not an element of S".

Axiom of extensionality

A set is determined by what its elements are. No importance is placed on the order in which elements are considered, or how many times an element appears in the set. Membership is the only things that matters.

Methods for describing a set

To effectively communicate which set you are talking about, you may:

- Describe S with language that communicates the precise nature of the set
- Use set-roster notation
- Use set-builder notation

Some important sets described with language

We write $\emptyset$ for the **emptyset**. It is the set with no elements

Let $\mathbb{Z}_{\geq 0}$ denote the set of **non-negative integers**

Let $\mathbb{N}$ denote the set of **positive integers** (sometimes called the natural numbers). Note that $0 \notin \mathbb{N}$ but $0 \in \mathbb{Z}_{\geq 0}$.

Let $\mathbb{Z}$ denote the set of **integers**.

Let $\mathbb{Q}$ denote the set of **rational numbers**.

Let $\mathbb{R}$ denote the set of **real numbers**.

Specifying membership with precision

Which, if any, of the following sentences define sets:

A. Let E denote the set of Australian species that are endangered.

B. Let E denote the set of Australian species that are *officially* endangered.

C. Let E denote the set of species that are currently listed under Section 178 of the Environment Protection and Biodiversity Conservation Act 1999 (EPBC Act).

Specifying membership with precision

Which, if any, of the following sentences define sets:

A. Let P denote the set of all (computer) programs.

B. Let P denote the set of all programs written in the language C.

C. Let P denote the set of all *correct* programs written in the language C.

E. Let P denote the set of programs written in the language C that terminate in finite time.

F. Let P denote the set of programs written in the language C that accept no input from the user and will run to completion without a run-time error in finite time.

Set-roster notation

A set S may be specified using **set-roster notation** by:

- writing all of its elements, with elements separated by commas, and the entire collection enclosed by braces;
- writing some elements and ellipses (ellipses are $\ldots$ that read "and so on"), enclosing the entire description between braces. The elements written must establish enough of a pattern, in a way that is obvious to the reader, that it becomes clear how the pattern continues and whether or not the pattern ends at some point.

Examples

Describe each of the following sets in words.

$S = \{\text{Mr Cookie, Lewis, Goblin}\}$

$O = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

$T = \{2, 3, 5, 7, \ldots\}$

$U = \{3, 5, 7, \ldots, 19\}$

Examples

Describe each of the following sets in words.

$S = \{$Mr Cookie, Lewis, Goblin$\}$.
The names of guinea pigs that live in Adam's house.

$O = \{\dots, -3, -1, 1, 3, 5, \dots\}$.
The odd integers.

$T = \{2, 3, 5, 7, \dots\}$.
The prime numbers.

$U = \{3, 5, 7, \dots, 19\}$.
This could be "The odd primes less than 20" or "The odd integers between 2 and 20." As soon as we identify two reasonable interpretations, we know this description needs improvement to be effective.

Axiom of extensionality again

Recall: A set is determined by what its elements are. No importance is placed on the order in which elements are considered, or how many times an element appears in the set.

So $\{a, b, c\} = \{c, b, a\} = \{a, a, a, a, b, b, c\}$

Set-builder notation

Recall that a predicate is a sentence that involves at least one variable, and the domain of the variable must be specified.

To describe a set using **set-builder notation** we use a predicate $p(x)$, a predicate domain D, and we write

$$\{x \in D \mid p(x)\}$$

for "the set of all x in D for which $p(x)$ is true.".

Note: The use of $|$ to separate the specification of the domain from the predicate is not universal. Other commonly used symbols include ':' and ';'. In all cases, the symbol may be read as "such that" or "for which".

Universe of discourse

In any mathematical context, there is usually a type of object you are thinking about. We will usually consider sets that are subsets of a set U, called the **universe of discourse**, or **universal set**. This often forms the domain of the predicates we use to describe sets in a given situation.

For example, in one context the universe of discourse may be the set of positive integers, while in another it may by the set of connected graphs (we will find out about these later).

The importance of having a universe of discourse will be explored later.

An Example

Example:

$$S = \{\underbrace{x \in \mathbb{Z}}_{\text{domain}} \mid \underbrace{x^2 + 2x - 15 = 0}_{\text{predicate}}\}.$$

The expression may be read aloud as:

- "S is equal to the set of all x from the set of integers such that $x^2 + 2x - 15 = 0$."
- "S is equal to the set of all integers x such that $x^2 + 2x - 15 = 0$."
- "S is equal to the set of all integers x for which $x^2 + 2x - 15 = 0$."

Easily defined sets are not always easy

Example: Let $T = \{p \in \mathbb{Z}^+ \mid p \text{ and } p+2 \text{ are both prime}\}$.

I can, for example, tell you that $3, 11, 17 \in T$ and $6, 7 \notin T$.

If you give me any positive integer n, and leave me alone long enough, I can tell you whether or not $n \in T$.

The statement:

The set T has infinitely many elements

is known as the Twin Prime Conjecture. It has been studied since 1849. At the time of writing, no one knows whether the statement is true or false.

Subsets

Let A and B be sets. We say that A is a **subset** of B, or A is contained in B, and we write $A \subseteq B$, when every element of A is also an element of B. Symbolically,

$$A \subseteq B \Leftrightarrow \forall x\, (x \in A \to x \in B).$$

You may say that $\subseteq$ is the set theoretic analogue of IMPLIES in logic.

Understanding $A \not\subseteq B$

We often indicate that the negation of a statement is true by placing a diagonal slash through the key symbol in the statement. Now

Understanding $A \not\subseteq B$

We often indicate that the negation of a statement is true by placing a diagonal slash through the key symbol in the statement. Now

$$\begin{aligned}
A \not\subseteq B \Leftrightarrow &\quad \neg(A \subseteq B) \\
\Leftrightarrow &\quad \neg\forall x\ (x \in A \to x \in B) \\
\Leftrightarrow &\quad \exists x\ \neg(x \in A \to x \in B) \\
\Leftrightarrow &\quad \exists x\ x \in A \wedge (\neg(x \in B)) \\
\Leftrightarrow &\quad \exists x\ x \in A \wedge x \notin B.
\end{aligned}$$

Understanding $A \not\subseteq B$

We often indicate that the negation of a statement is true by placing a diagonal slash through the key symbol in the statement. Now

$$\begin{aligned}
A \not\subseteq B \Leftrightarrow \quad & \neg(A \subseteq B) \\
\Leftrightarrow \quad & \neg\forall x \; (x \in A \to x \in B) \\
\Leftrightarrow \quad & \exists x \; \neg(x \in A \to x \in B) \\
\Leftrightarrow \quad & \exists x \; x \in A \wedge (\neg(x \in B)) \\
\Leftrightarrow \quad & \exists x \; x \in A \wedge x \notin B.
\end{aligned}$$

We can interpret this intuitively: $A \not\subseteq B$ means that there is at least one element in A that is not in B.

Proper containment

We say that A is a **proper subset** of B, or A is properly contained in B, and write $A \subsetneq B$, when every element of A is in B but there is at least one element of B that is not in A. Symbolically:

$$A \subsetneq B \Leftrightarrow (A \subseteq B) \wedge (B \nsubseteq A)$$

You are well acquainted with the following proper containments:

$$\mathbb{N} \subsetneq \mathbb{Z}_{\geq 0} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

BEWARE: Some mathematicians use $\subset$ for $\subsetneq$. I don't do this because others mathematicians use $\subset$ for $\subseteq$.

Set equality

Let A and B be sets. We say that A **equals** B, written $A = B$, when $A \subseteq B$ and $B \subseteq A$. Symbolically,

$$\begin{aligned} A = B &\Leftrightarrow (A \subseteq B) \wedge (B \subseteq A) \\ &\Leftrightarrow \forall x\, (x \in A \leftrightarrow x \in B) \\ &\Leftrightarrow \forall x\, (x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A) \end{aligned}$$

You may say that $=$ is the set theoretic analogue of IFF in logic.

An Example

A set may be described in more than one way. Some ways say more about the set, perhaps by saying why it is interesting, while others make it more obvious which things are elements of the set. For example:

$$\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}$$

The first description tells me why the set is interesting in a given context, while the second description gives me a particularly clear understanding of the which integers are in the set and which integers are not in the set.

Set equality as a type of problem

Many problems in mathematics (and computer science) can be framed as: we know one way to describe a particular set because we know what makes the set interesting, but we want to know another way to describe the same set that makes membership easier to recognise or understand. Thus we wish to show a set equality.

For example: The problem of solving the equation $x^2 + 2x - 15 = 0$ over the domain of integers is essentially the following:
Show that

$$\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}?$$

Proof To show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}$ we will show:

- $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$; and
- $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

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First we show that $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

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First we show that $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Let $x \in \{-5, 3\}$. We consider cases.

Case $x = -5$: Since $(-5)^2 + 2 \times (-5) - 15 = 25 - 10 - 15 = 0$, $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Case $x = 3$: Since $3^2 + 2 \times 3 - 15 = 9 + 6 - 15 = 0$, $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Proof To show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} = \{-5, 3\}$ we will show:

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Case $x = 3$: Since $3^2 + 2 \times 3 - 15 = 9 + 6 - 15 = 0$, $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

In all cases, $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$. Hence $\{-5, 3\} \subseteq \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

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Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Since $x^2 + 2x - 15 = 0$, we have that $(x + 5)(x - 3) = 0$. If a product of two real numbers is zero, at least one of the numbers is zero. We consider cases.

Now we show that $\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.

Let $x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\}$.

Since $x^2 + 2x - 15 = 0$, we have that $(x + 5)(x - 3) = 0$. If a product of two real numbers is zero, at least one of the numbers is zero. We consider cases.

Case $x + 5 = 0$: Then $x = -5$ and $x \in \{-5, 3\}$.

Case $x + 5 \neq 0$: Then $x - 3 = 0$. Hence $x = 3$ and $x \in \{-5, 3\}$.

In all cases, $x \in \{-5, 3\}$. Hence
$\{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \subseteq \{-5, 3\}$.$\square$

Alternate proof

Proof: Let x be an integer. Then

$$
\begin{aligned}
& x \in \{x \in \mathbb{Z} \mid x^2 + 2x - 15 = 0\} \\
\Leftrightarrow & x^2 + 2x - 15 = 0 \\
\Leftrightarrow & (x+5)(x-3) = 0 \\
\Leftrightarrow & (x+5=0) \vee (x-3=0) \\
& \text{(because a product of real numbers is zero iff} \\
& \text{at least one of the numbers is zero)} \\
\Leftrightarrow & (x=-5) \vee (x=3) \\
\Leftrightarrow & x \in \{-5, 3\}. \qquad \square
\end{aligned}
$$