clase 22 de agosto 2023

Susana Hernández

Invalid Date

Table of contents

Preface

This is a Quarto book.

To learn more about Quarto books visit https://quarto.org/docs/books.

1 Introduction

This is a book created from markdown and executable code.

See Knuth (1984) for additional discussion of literate programming.

2 Summary

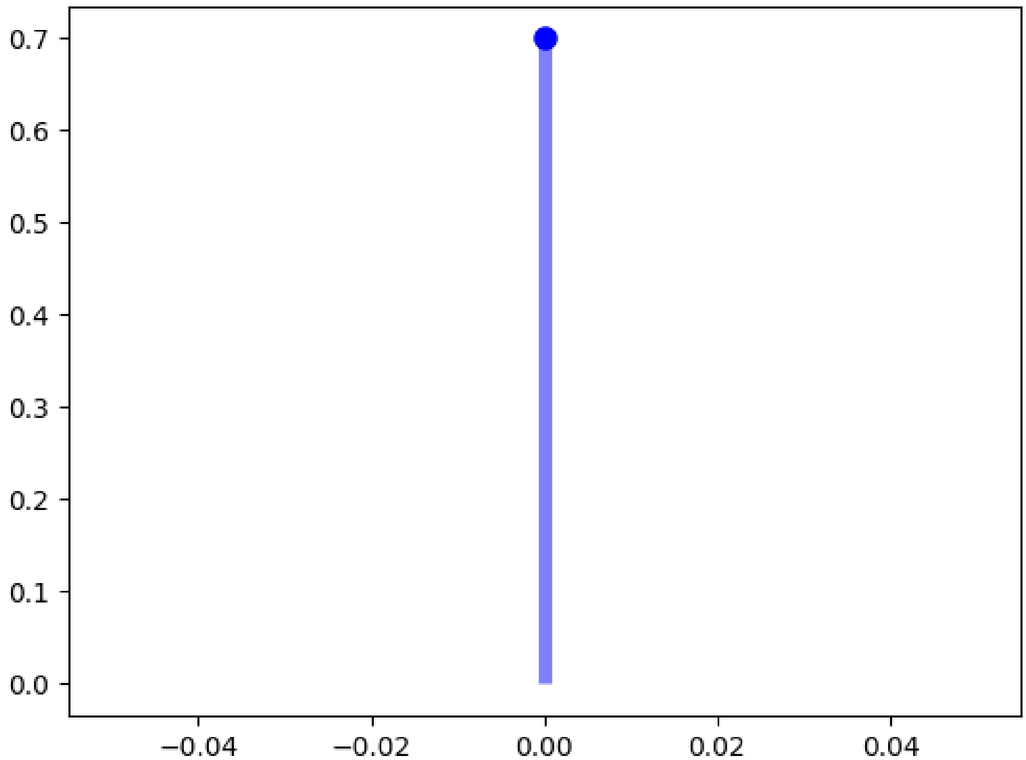
In summary, this book has no content whatsoever.

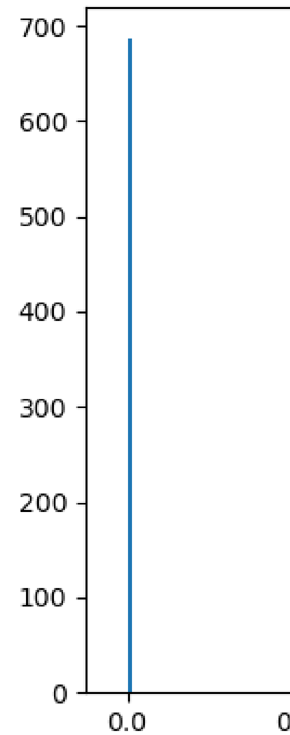
3 Tarea 1

Listing 3.1 Exploring functions to generate random variables with a Bernoulli distribution.py

```python
import numpy as np
from scipy.stats import bernoulli
import matplotlib.pyplot as plt
fig_01, ax_01 = plt.subplots(1, 1)
fig_02, ax_02 = plt.subplots(1, 1)
p = 0.3
mean, var, skew, kurt = bernoulli.stats(p, moments='mvsk')
print(mean, var, skew,kurt)

x = np.arange(bernoulli.ppf(0.01, p),
              bernoulli.ppf(0.99, p))
ax_01.plot(x, bernoulli.pmf(x, p), 'bo', ms=8, label='bernoulli pmf')
ax_01.vlines(x, 0, bernoulli.pmf(x, p), colors='b', lw=5, alpha=0.5)
r = bernoulli.rvs(p, size=1000)
ax_02.hist(r, bins=200)
plt.show()
```


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−0.04
−0.02
0.00
0.02
0.04


700
600
500
400
300
200
100
0
0.0

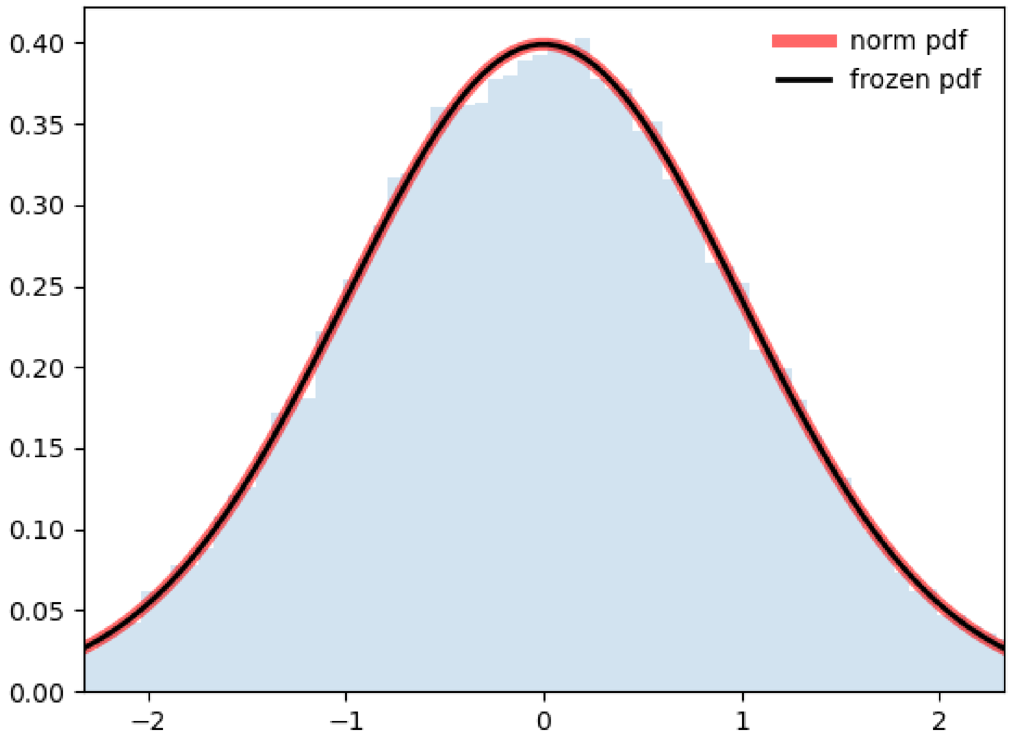
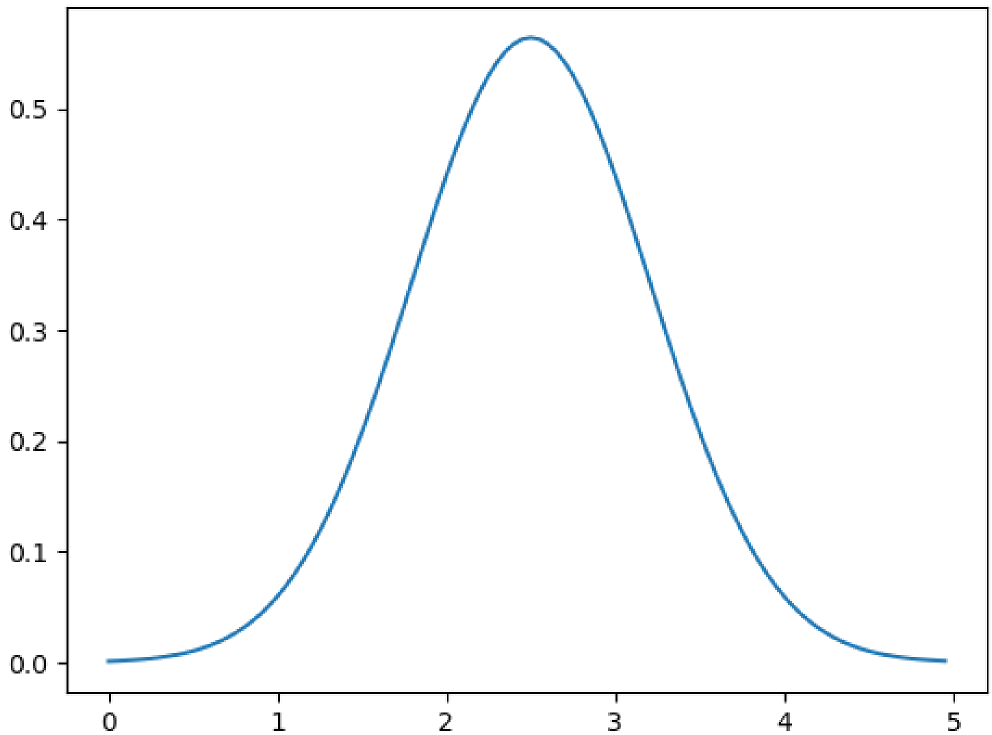
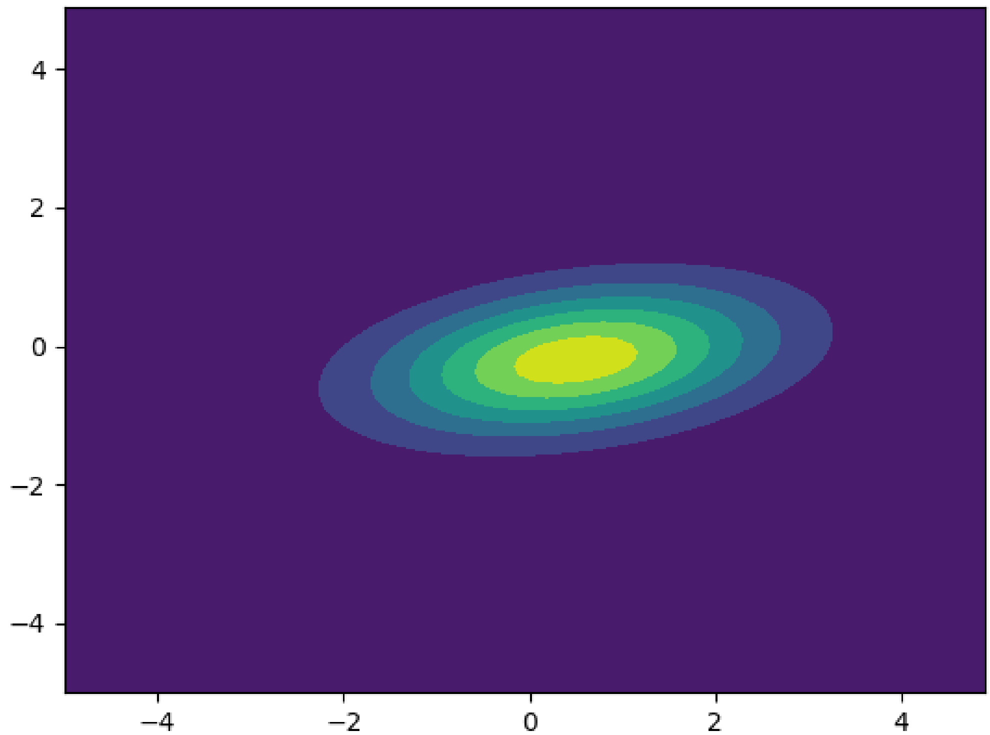
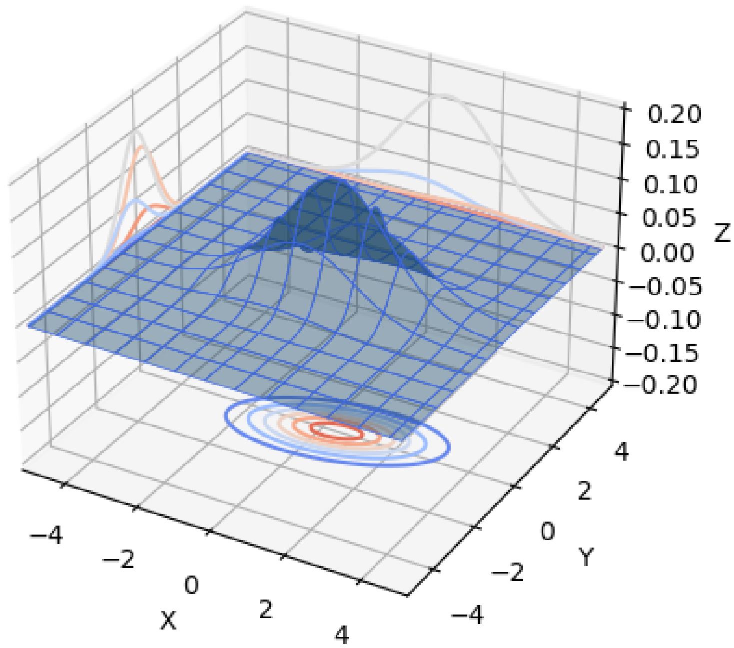


Figure 3.1: Figura 3


0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5


4
2
0
−2
−4
−4
−2
0
2
4



0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
Z
−4
−2
0
2
4
X
−4
−2
0
2
4
Y

Listing 3.2 `Exploring functions to generate random variables with a Gaussian distribution.py`

Listing 3.3 `Revising multivariate Gaussian.py`

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import axes3d
from scipy.stats import multivariate_normal

x = np.linspace(0, 5, 100, endpoint=False)
y = multivariate_normal.pdf(x, mean=2.5, cov=0.5);

fig1 = plt.figure()
ax = fig1.add_subplot(111)
ax.plot(x, y)
# plt.show()

x, y = np.mgrid[-5:5:.1, -5:5:.1]
pos = np.dstack((x, y))
rv = multivariate_normal([0.5, -0.2], [[2.0, 0.3], [0.3, 0.5]])
fig2 = plt.figure()

ax2 = fig2.add_subplot(111)
ax2.contourf(x, y, rv.pdf(pos))
# plt.show()

ax = plt.figure().add_subplot(projection='3d')
ax.plot_surface(
    x,
    y,
    rv.pdf(pos),
    edgecolor='royalblue',
    lw=0.5,
    rstride=8,
    cstride=8,
    alpha=0.4
)
ax.contour(x, y, rv.pdf(pos), zdir='z', offset=-.2, cmap='coolwarm')
ax.contour(x, y, rv.pdf(pos), zdir='x', offset=-5, cmap='coolwarm')

ax.contour(x, y, rv.pdf(pos), zdir='y', offset=5, cmap='coolwarm')

ax.set(
    xlim=(-5, 5),
    ylim=(-5, 5),
    zlim=(-0.2, 0.2),
    xlabel='X',
    ylabel='Y',
    zlabel='Z'
)
plt.show()
```

4 Tarea 2

Sea $Y_{\delta,h}(t)$ una caminata aleatoria. Demuestre que para δ y h pequeño tenemos

$$E\exp[i\lambda Y_{\delta,h}(t)] \approx \exp\left[-\frac{t\lambda^2 h^2}{2\delta} - \frac{t\lambda^4 h^4}{12\delta}\right]$$

Demostración:

Considere una caminata aleatoria que comienza en 0 con saltos h y $-h$ igualmente probables en los momentos δ, 2 δ,..., donde h y δ son números positivos. Más precisamente, sea $\{X_n\}_{n=1}^{\infty}$ una sucesión de elementos aleatorios independientes e idénticamente distribuidos. variables con

$$P[X_i = h] = P[X_i = -h] = \frac{1}{2}, \forall i,$$

Sea $Y_{\delta,h}(0) = 0$ y pongamos

$$Y_{\delta,h}(n\delta) = X_1 + X_2 + \cdots + X_n.$$

Para $t > 0$, defina $Y_{\delta,h}(t)$ mediante linealización, es decir, para $n\delta < t < (n+1)\delta$, defina

$$Y_{\delta,h}(t) = \frac{(n+1)\delta - t}{\delta} Y_{\delta,h}(n\delta) + \frac{t - n\delta}{\delta} Y_{\delta,h}((n+1)\delta).$$

Calculemos la función característica de $Y_{\delta,h}(t)$, donde $\lambda \in \mathbb{R}$ fijo y sea $t = n\delta$ así, $n = t/\delta$. Entonces se tiene que

$$\begin{aligned} E\exp\left[i\lambda Y_{n,\delta}(t)\right] &= \prod_{j=1}^{n} Ee^{i\lambda X_j}, \text{ por ser variables independientes,} \\ &= (Ee^{i\lambda X_j})^n, \text{ por ser idénticamente distribuidas,} \\ &= \frac{1}{2}(e^{i\lambda h} + e^{-i\lambda h})^n, \\ &= (\cos(\lambda h))^n, \\ &= (\cos(\lambda h))^{t/\delta}, \end{aligned} \tag{4.1}$$

$$. \tag{4.2}$$

Por otro lado, sea $u = [\cos(\lambda h)]^{1/\delta} \Rightarrow \ln(u) = \frac{1}{\delta}\ln[\cos(\lambda h)]$.

Usando la expansión de Taylor de $\cos(x)$ se tiene que

$$\cos(\lambda h) \approx 1 - \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^4}{4!},$$

entonces

$$\begin{aligned}
\ln(\cos(\lambda h)) &\approx \ln\left[1 - \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^4}{4!}\right] \\
&\approx -\frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^4}{4!} - \frac{1}{2}\left(-\frac{\lambda^2 h^2}{2!} + \frac{\lambda^4 h^4}{4!}\right)^2 \\
&= -\frac{\lambda^2 h^2}{2!} + \frac{\lambda^4 h^4}{4!} - \frac{1}{2}\left(\frac{\lambda^4 h^4}{4} - \frac{\lambda^6 h^6}{24^2} + \frac{\lambda^8 h^8}{24}\right) \\
&= -\frac{\lambda^2 h^2}{2} + \frac{\lambda^4 h^4}{24} - \frac{\lambda^4 h^4}{8} - \frac{\lambda^6 h^6}{(2)24^2} + \frac{\lambda^8 h^8}{48} \\
&= -\frac{\lambda^2 h^2}{2} - \frac{\lambda^4 h^4}{12} - \frac{\lambda^6 h^6}{(2)24^2} + \frac{\lambda^8 h^8}{48}
\end{aligned} \tag{4.3}$$

para una h pequeña, se satisface que,

$$-\frac{\lambda^6 h^6}{(2)24^2} + \frac{\lambda^8 h^8}{48} \approx 0$$

Por lo tanto, $\ln(\cos(\lambda h)) \approx -\frac{\lambda^2 h^2}{2} - \frac{\lambda^4 h^4}{12}$.\ Así, para δ y h pequeña, se tiene que $\ln u \approx \frac{1}{\delta}\left(-\frac{\lambda^2 h^2}{2} - \frac{\lambda^4 h^4}{12}\right)$.\ Entonces

$$u \approx \exp\left[\frac{1}{\delta}\left(-\frac{\lambda^2 h^2}{2} - \frac{\lambda^4 h^4}{12}\right)\right] \tag{4.4}$$

Entonces por la ecuación (Equation **??**)

$$E\exp\left[i\lambda Y_{n,\delta}(t)\right] \approx \exp\left[-\frac{t\lambda^2 h^2}{2\delta} - \frac{t\lambda^4 h^4}{12\delta}\right] \tag{4.5}$$

Calculando el limite

$$\lim_{\delta\to 0} E\left[\exp\left(i\lambda Y_{n,\delta}(t)\right)\right] = \lim_{\delta\to 0}\exp\left[-t\left(\left[\frac{h^2}{\delta}\right]\left(\frac{\lambda^2}{2} - \frac{\lambda^4 h^2}{24}\right)\right)\right],$$

Asumamos que $\delta \to 0$, $h \to 0$ pero $h^2/\delta \to \infty$. Entonces $\lim_{\delta\to 0} Y_{\delta,h}(t)$ no existe. Por otro lado, consideremos la siguiente renormalización,

$$\begin{aligned} E\exp\left[i\lambda Y_{n,\delta}(t) + \frac{th^2\lambda^2}{2\delta}\right] &= E\left[\exp(i\lambda Y_{n,\delta}(t))\exp\left(\frac{th^2\lambda^2}{2\delta}\right)\right] \\ &= \exp\left(\frac{th^2\lambda^2}{2\delta}\right)E\exp\left[i\lambda Y_{n,\delta}(t)\right] \\ &\approx \exp\left(\frac{th^2\lambda^2}{2\delta}\right)\exp\left[-\frac{t\lambda^2h^2}{2\delta} - \frac{t\lambda^4h^4}{12\delta}\right] \\ &= \exp\left(-\frac{t\lambda^4h^4}{12\delta}\right) \end{aligned} \tag{4.6}$$

Así, si $\delta, h \to 0$ de tal manera que $h^2/\delta \to \infty$ y $h^4/\delta \to 0$, entonces

$$\lim_{\delta\to 0} E\left[\exp\left(i\lambda Y_{n,\delta}(t) + \frac{th^2\lambda^2}{2}\right)\right] = \lim_{\delta\to 0}\exp\left(\frac{(\lambda h)^4}{24\delta}\right) = 1$$

5 Tarea 3

Si $X \sim N(\mu, \sigma)$ entonces $\left(\dfrac{X-\mu}{\sigma}\right) \sim N(0,1)$.

Calculemos la función característica de la variable $\dfrac{X-\mu}{\sigma}$,

$$\begin{aligned}
\varphi_{\frac{X-\mu}{\sigma}}(t) &= E\left[e^{it\left(\frac{X-\mu}{\sigma}\right)}\right] \\
&= E\left[e^{\left(\frac{itX}{\sigma}-\frac{it\mu}{\sigma}\right)}\right] \\
&= e^{-\frac{it\mu}{\sigma}} E\left[e^{\left(\frac{itX}{\sigma}\right)}\right] \\
&= e^{-\frac{it\mu}{\sigma}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-\mu)^2-2itx\sigma}{\sigma^2}} dx \qquad (5.1)
\end{aligned}$$

Observemos que,

$$\begin{aligned}
\frac{(x-\mu)^2-2itx\sigma}{\sigma^2} &= \frac{x^2-2x\mu+\mu^2-2itx\sigma}{\sigma^2} \\
&= \frac{x^2}{\sigma^2} - \frac{2x\mu}{\sigma^2} + \frac{\mu^2}{\sigma^2} - \frac{2itx\sigma}{\sigma^2} \\
&= \frac{x^2}{\sigma^2} - \frac{2x}{\sigma}\left(\frac{\mu+it\sigma}{\sigma^2}\right) + \frac{\mu^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \left(\frac{\mu+it\sigma}{\sigma}\right)^2 + \frac{\mu^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \frac{2it\sigma\mu}{\sigma^2} - \frac{(it\sigma)^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \frac{2it\mu}{\sigma} + t^2. \qquad (5.2)
\end{aligned}$$

Sustituyendo (**??**) en (**??**), resulta