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(4) $\tau \triangleq P_2 - P_1$

(a) $\hat{\tau} = \hat{P}_2 - \hat{P}_1 = \frac{40}{50} - \frac{30}{50} = \frac{1}{5}$

$$V[\hat{\tau}] = V[\hat{P}_2] + V[\hat{P}_1] = \frac{\hat{P}_2(1-\hat{P}_2)}{n_2} - \frac{\hat{P}_1(1-\hat{P}_1)}{n_1}$$

$$= \frac{\frac{4}{5}\left(\frac{1}{5}\right)}{50} + \frac{\frac{3}{5}\left(\frac{2}{5}\right)}{50} = \frac{1}{25 \cdot 5\cancel{0}}(\cancel{1}0)$$

$$= \frac{1}{125}$$

$$\hat{se}_{\tau} = \sqrt{V[\hat{\tau}]} \simeq 0.089$$

90% CI: $\hat{\tau} \pm 1.645 \cdot \hat{se}_{\tau} = (\downarrow , \downarrow)$;)

$= 0.2 \pm 1.645 \cdot 0.089 = (0.052867, 0.347133)$

(b) Parametric Bootstrap

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$B = 10{,}000$ times

$X_1^* \sim Bin(n_1, \hat{p}_1)$

$X_2^* \sim Bin(n_2, \hat{p}_2)$

$\hat{\tau}^*[i] \leftarrow \left(\frac{X_2^*}{n_2} - \frac{X_1^*}{n_1} \right)$

$\hat{se}_{\hat{\tau}} \leftarrow stdv(\vec{\hat{\tau}}^*)$

90% CI: $\hat{\tau} \pm 1.645 \cdot \hat{se}_{\hat{\tau}} = (0.052805, 0.347195)$

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Lemma: ∘ Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

∘ $p \sim U[0,1] \Leftrightarrow f(p) = 1$.

∘ Then $f(p|x^n) \propto \text{Beta}_p(s+\alpha, n-s+\beta)$

$= \text{Beta}_p(s+1, n-s+1)$

pf: $f(p|x^n) \propto f(x^n|p) \cdot f(p)$ by Bayes thm

$$= \mathcal{L}_n(p) \cdot f(p)$$

$$= \begin{cases} p^s (1-p)^{n-s} \cdot 1, & p \in [0,1] \\ 0, & \text{o.w.} \end{cases}$$

→ note that $\text{Uniform}_p[0,1] = \text{Beta}_p(1,1)$

$= p^{\alpha-1}(1-p)^{\beta-1}$

$= p^0 (1-p)^0$

$$= \begin{cases} p^s (1-p)^{n-s} \text{Beta}(1,1), & p \in [0,1] \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} p^{s+\alpha-1} (1-p)^{n-s+\beta-1}, & p \in [0,1] \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} \text{Beta}_p(s+\alpha, n-s+\beta), & p \in (0,1) \\ 0, & \text{o.w.} \end{cases}$$

$= \text{Beta}_p(s+\alpha, n-s+\beta)$ ← $\alpha=\beta=1$ for uniform density

$= \text{Beta}_p(s+1, n-s+1)$

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4 ...

(c) $f(P_1,P_2 \mid x^{n_1}, y^{n_2}) \propto f(x^{n_1}, x^{n_2} \mid P_1, P_2) \cdot f(P_1,P_2)$

$$= h_{n_1}(P_1)\cdot 1 \cdot h_{n_2}(P_2)\cdot 1$$

$$= h_{n_1}(P_1)\, \mathrm{Beta}_{P_1}(1,1) \cdot h_{n_2}(P_2)\, \mathrm{Beta}_{P_2}(1,1)$$

$$= P_1^{S_1}(1-P_1)^{n_1-S} \cdot P_1^{\alpha-1}(1-P_1)^{\beta-1} \cdot P_2^{S_2}(1-P_2)^{n_2-S_2} P_2^{\alpha-1}(1-P_2)^{\beta-1}$$

$$= \begin{cases} P_1^{S_1+\alpha-1}(1-P_1)^{n_1-S+\beta-1} \cdot P_2^{S_2+\alpha-1}(1-P_2)^{n_2-S+\beta-1}, & P_1 \in [0,1] \text{ \& } P_2[0,1] \\ 0 & \text{, o.w} \end{cases}$$

$$= \mathrm{Beta}_{P_1}(S_1+\alpha, n_1-S_1+\beta) \cdot \mathrm{Beta}_{P_2}(S_2+\alpha, n_2-S_2+\beta)$$

$$= \mathrm{Beta}_{P_1}(S_1+1, n_1-S_1+1) \cdot \mathrm{Beta}_{P_2}(S_2+1, n_2-S_2+1)$$

B/c of indep, to sample joint, simply sample each marginal & combine.

B times

$P_1^* \sim \mathrm{Beta}(S_1+1, n_1-S_1+1)$

$P_2^* \sim \mathrm{Beta}(S_2+1, n_2-S_2+1)$

$\tau^*[i] \leftarrow P_2^* - P_1^*$

$\bar{\tau} \leftarrow \mathrm{mean}(\vec{\tau}^*) = \frac{1}{B}\sum_i^B \tau^*[i]$

$\cancel{\hat{se}_{\tau} \leftarrow \mathrm{std}(\vec{\tau}^*) = \left(\frac{1}{B}\sum_i^B (\tau^*[i]-\bar{\tau})^2\right)^{1/2}}$

90% int $\leftarrow$ (percentile($\vec{\tau}^*$, 5%), percentile($\vec{\tau}^*$, 95%))

4

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(d) $\psi(p_1, p_2) = \log\left(\frac{\frac{p_1}{1-p_1}}{\frac{p_2}{1-p_2}} \right)$

$\hat{\psi} = \psi(\hat{p}_1, \hat{p}_2) = \log\left(\frac{\hat{p}_1/(1-\hat{p}_1)}{\hat{p}_2(1-\hat{p}_2)} \right)$

$\hat{\nabla}\psi = \begin{bmatrix} \partial\psi/\partial p_1 \\ \partial\psi/\partial p_2 \end{bmatrix} = \begin{bmatrix} 1/(p_1 - p_1^2) \\ 1/(p_2^2 - p_2) \end{bmatrix} = \begin{bmatrix} 25/6 \\ -25/4 \end{bmatrix}$

$\ell_n = s_1 \log(p_1) + (n_1 - s_1)\log(1-p_1) + s_2 \log(p_2) + (n_2 - s_2)\log(1-p_2)$

$\frac{\partial^2 \ell_n}{\partial p_1^2} = \frac{-s_1}{p_1^2} + \frac{(n_1 - s_1)}{(1-p_1)^2}$ $\qquad$ $\frac{\partial^2 \ell_n}{\partial p_2^2} = \frac{-s_2}{p_2^2} + \frac{(n_2 - s_2)^2}{(1-p_2)^2}$

$E_{\binom{p_1}{p_2}}[\ \ ] = \frac{-n_1 p_1}{p_1^2} + \frac{(n_1 - n_1 p_1)}{(1-p_1)^2}$ $\qquad$ $E_{\binom{p_1}{p_2}}[\ \ ] = \frac{-n_2}{p_2} + \frac{n_2}{(1-p_2)}$

$= \frac{-n_1}{p_1} + \frac{n_1}{(1-p_1)}$

@mle $\hat{p}_2 = \frac{4}{5}$ $\qquad = \frac{-n_2}{\hat{p}_2} + \frac{n_2}{(1-\hat{p}_2)}$

$\hat{p}_1 = \frac{3}{5}$ $\qquad$ @mle $= \frac{-n_1}{\hat{p}_1} + \frac{n_1}{(1-\hat{p}_1)}$

$= \frac{-375}{2}$

$= -250 \cdot 5/6 = \frac{-125 \cdot 5}{3}$

$= \frac{-625}{3}$

$\hat{I}_n = -\begin{bmatrix} -625/3 & 0 \\ 0 & -375/2 \end{bmatrix} \longrightarrow \hat{J}_n = \hat{I}_n^{-1} = \frac{1}{1875}\begin{bmatrix} 9 & 0 \\ 0 & 10 \end{bmatrix}$

$\hat{se}_\psi^2 = (\hat{\nabla}\psi^T)\hat{J}_n(\nabla\psi) = \begin{bmatrix} \frac{25}{6}, & \frac{-25}{4} \end{bmatrix} \frac{1}{1875}\begin{bmatrix} 9 & \\ & 10 \end{bmatrix}\begin{bmatrix} 25/6 \\ -25/4 \end{bmatrix} = \frac{7}{24}$

$\hat{se}_\psi = \sqrt{\frac{7}{24}} \simeq 0.54$

90% CI: $\hat{\psi} \pm 1.645\, \hat{se}_\psi$

(e) posterior mean: $\bar{\psi} \triangleq \iint_{A} \psi(p_1, p_2) f(p_1, p_2 \mid x^{n_1}, y^{n_2}) \, dp_1 dp_2$

$\to$ use indep (again) to sample.

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B times

$$\begin{aligned} & p_1^* \sim \text{Beta}_{p_1}(\qquad\qquad) \\ & p_2^* \sim \text{Beta}_{p_2}(\qquad\qquad) \\ & \psi^*[i] \leftarrow \psi(p_1^*, p_2^*) \end{aligned}$$

posterior mean $\longrightarrow$ $\bar{\psi} \leftarrow \frac{1}{B} \sum_{i=1}^{B} \psi^*[i]$

90% interval $\longrightarrow$ $\begin{cases} a \leftarrow \text{percentile}(\psi^*, 5\%) \\ b \leftarrow \text{percentile}(\psi^*, 95\%) \end{cases}$