Linear Algebra, Probability and Queuing Theory (MMA202T)

Improvement Test

1. $\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 1 & 3 & k \end{vmatrix} = 0 \Rightarrow 2(-k-3) - 1(3k-1) + 3(9+1) = 0$
$\Rightarrow -2k - 6 - 3k + 1 + 30 = 0 \Rightarrow k = +5$

2. zero vector $0 = 0 + 0i$, inverse element $-a-bi$

3. $S = \{HH, HT, TH, TT\}$

X = no of heads

Y = tail on first flip

X \ Y	0	1
0	0	1/4
1	1/4	1/4
2	1/4	0

4. $U = X+Y,\ V = X-Y \Rightarrow X = \frac{U+V}{2},\ Y = \frac{U-V}{2} \Rightarrow J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

5. $\int_{x=0}^{2}\int_{y=0}^{2} cxy\,dy\,dx = 1 \Rightarrow \int_{x=0}^{2}\left[cx\frac{y^2}{2}\right]_0^2 dx = 1 \Rightarrow \int_{x=0}^{2} 2cx\,dx = 1 \Rightarrow 2c\left[\frac{x^2}{2}\right]_0^2 = 1$
$\Rightarrow 4c = 1 \Rightarrow c = 1/4$

1a. $T: R^3 \to R^2$, $T(1,0,2) = (0,3)$, $T(1,-1,0) = (1,3)$, $T(2,1,0) = (5,0)$

$T = \begin{bmatrix} 0 & 1 & 5 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 5 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1/2 \\ 1/3 & -2/3 & -1/6 \\ 1/3 & 1/3 & -1/6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ (1) (2) (1)

$\therefore T(x,y,z) = (2x+y-z,\ x-2y+z)$ (1)

1b. $W = \left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle|\ a,b,c \in R\right\}$ Let $\alpha = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}$, $\beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in W$, $c \in R$.

$\alpha + \beta = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix} \in W$ (2) $c.\alpha = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cc_1 \end{bmatrix} \in W$ (2) $\therefore W$ is a subspace of $M_{2\times 2}$

2a $\Sigma = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix} \Rightarrow \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = 4, 1.$ (2)

$\lambda = 4$: $\Sigma - 4I = \begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $e = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (1)

$\lambda = 1$: $\Sigma - I = \begin{bmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $e = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. (1)

The axes of the contours are $\pm c\alpha_2\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ & $\pm c\alpha_1\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ (2)

2b

X_1 \ X_2	2	4
1	0.24	0.06
3	0.56	0.14

X_1	1	3
$P(X_1)$	0.3	0.7

(1)

X_2	2	4
$P(X_2)$	0.8	0.2

(1)

$R_X = \begin{bmatrix} 6.6 & 5.76 \\ 5.76 & 6.4 \end{bmatrix}$ (2)

3. $A = \begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & 3 & 2 & -3 \\ -1 & 2 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 4 \\ 0 & 5 & 5 & -10 \\ 0 & 5 & 5 & -10 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 4 \\ 0 & 5 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (2)

Basis of $R(A) = \{(2,1,-1,4), (1,3,2,-3)\}$ (1) Basis of $C(A) = \{(2,1,-1), (1,3,2)\}$ (1)

$Ax=0 \rightarrow \left.\begin{array}{l} 2x_1 + x_2 - x_3 + 4x_4 = 0 \\ 5x_2 + 5x_3 - 10x_4 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = x_3 - 3x_4 \\ x_2 = -x_3 + 2x_4 \end{array}$ (2) $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Basis of $N(A) = \{(1,-1,1,0), (-3,2,0,1)\}$ (1)

$A^T = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 3 \\ 4 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (2)

$A^T y = 0 \Rightarrow \begin{array}{l} 2y_1 + y_2 - y_3 = 0 \\ 5y_2 + 5y_3 = 0 \end{array} \Rightarrow y_2 = -y_3,\ y_1 = y_3$ $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (1)

Basis of $N(A^T) = \{(1,-1,1)\}$ (1)

4.

X \ Y	0	1	2
0	0.52	0.20	0.04
1	0.14	0.02	0.01
2	0.06	0.01	0

(2)

(i)

x	0	1	2
P(x)	0.76	0.17	0.07

(1)

y	0	1	2
P(y)	0.72	0.23	0.05

(1)

$E(X) = 0.31$ (1), $E(Y) = 0.33$ (1), $E(XY) = 0.06$ (1)

$\mathrm{Cov}(XY) = E(XY) - E(X)E(Y) = -0.0423$ (1)

X	0	1	2
$P(X / Y=2)$	$\frac{0.04}{0.05}$	$\frac{0.01}{0.05}$	$\frac{0}{0.05}$

$\approx$

0	1	2
$\frac{4}{5}$	$\frac{1}{5}$	0
0.8	0.2	0

(2)

5. $f(x,y) = 2e^{-x-y},\ x<y$

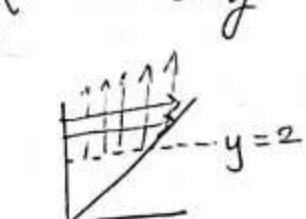
(i) $f(x,y) \geq 0\ \forall x,y$ (1), $\int_{x=0}^{\infty}\int_{y=x}^{\infty} 2e^{-x-y}\,dy\,dx = \int_{x=0}^{\infty} \left[-2e^{-x-y}\right]_x^{\infty} dx$

$= \int_{x=0}^{\infty} 2e^{-2x}\,dx = \left.-e^{-2x}\right]_0^{\infty} = 1$ (3) $\therefore f(x,y)$ is a valid density function.

(ii) $P(X<2) = \int_{x=0}^{2}\int_{y=x}^{\infty} 2e^{-x-y}\,dy\,dx$ (1) $= \int_{x=0}^{2} \left[-2e^{-x-y}\right]_x^{\infty} dx = \int_{x=0}^{2} 2e^{-2x}\,dx$

$= \left.-e^{-2x}\right]_{x=0}^{2} = 1 - e^{-4} = 0.9817$ (2)

(iii) $P(Y>2) = \int_{y=2}^{\infty}\int_{x=0}^{y} 2e^{-x-y}\,dx\,dy$ (1) $= \int_{y=2}^{\infty} \left[-2e^{-x-y}\right]_{x=0}^{y} dy = \int_{y=2}^{\infty} (2e^{-y} - 2e^{-2y})\,dy$



$= \left.-2e^{-y} + e^{-2y}\right]_2^{\infty} = +2e^{-2} - e^{-4} = 0.2524$ (2)