Probability Theory, Linear Algebra and Queuing Theory: Assignment - 1

1. If $\mathbb{R}$ is the field of real numbers and V is the set of vectors in a plane, further if addition of vectors is the internal binary composition in V and the multiplication of elements of $\mathbb{R}$ with those of V as the external composition, prove that $V(\mathbb{R})$ is a vector space.

2. Let $\mathbb{R}$ be the field of real numbers and let P_3 be the set of all polynomials of degree at most 3, over the field $\mathbb{R}$. Prove that P_3 is a vector space over the field $\mathbb{R}$.

3. If a mass m is placed at the end of a spring and if the mass is pulled downward and released, the mass-spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form $y(t) = c_1 cos\omega t + c_2 sin\omega t$, where ω is a constant that depends on the spring and the mass. Show that the set of all functions described by the above function is a vector space.

4. Let H $= \{a - 3b, b - a, a, b\}$ where a and b are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$.

5. Let V be a vector space of all 2 x 2 matrices over reals. Determine whether W is a sub-space of V or not, where
(i) W consists of all matrices with non-zero determinant.
(ii) W consists of all matrices A such that $A^2 = A$.
(iii) W $= \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbb{R}\}$.

6. Let V be a vector space of real valued derivable functions on $(0, \infty)$, then show that the set $S = \{x^2e^x, xe^x, (x^2 + x - 1)e^x\}$ is linearly independent.

7. In a P_2 let $v_1 = 2t^2 + t + 2$; $v_2 = t^2 - 2t$; $v_3 = 5t^2 - 5t + 2$; $v_4 = -t^2 - 3t - 2$. Determine if the vector $u = t^2 + t + 2$ belongs to the span$\{v_1, v_2, v_3, v_4\}$. (Here P_2 is the vector space of all polynomials of degree at the most n over the field of real numbers)

8. Find the basis for the subspace spanned by the vectors
$v_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$, $v_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$, $v_4 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$, $v_5 = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$, $v_6 = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T$.

9. Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$.

10. Let V be a space of 2 x 2 matrices over $\mathbb{R}$ and let W be the sub-space generated by
$$\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}.$$

Show that (i) $\left\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\right\}$ forms a basis set.
(ii) $dimW = 2$.

11. Find the bases for the row space and column space of the matrices
$U = \begin{bmatrix} 1 & 5 & -2 & 3 & 5 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, (ii) A $= \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$

12. Determine a basis for the null space and left null space of the matrix A $= \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

13. Find bases for the four fundamental sub-spaces of the matrices
(i) $A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 2 & -3 & 3 & 3 & 4 \\ 4 & -6 & 9 & 5 & 9 \\ 2 & -3 & -3 & 4 & -1 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$.

14. Which of the following mappings are linear transformations
(i) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$.
(ii) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, T(x, y, z) = (x^2 + xy, xy, yz)$.

15. Find a linear transformation for
(i) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$, such that $T(1, 1) = (1, -1, 1, -1)$ and $T(1, -1) = (1, 1, 1, 1)$.
(ii) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that $T(-1, 1) = (-1, 0, 2)$ and $T(2, 1) = (1, 2, 1)$.

16. Find the range space, null space, rank and nullity of T and verify the rank-nullity theorem for,
(i) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y, 2x - 3z, -2x + 3z)$.
(ii) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, defined by $T(x, y, z) = (x + 3y, 2x - 2z)$.
(iii) $T : V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$, defined by $T(e_1) = (1, 0, 2), T(e_2) = (1, 1, 0), T(e_3) = (1, -1, 4)$.
(iv) $T : V_2(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$, defined by $T(1, 2) = (4, 3, 3), T(3, 2) = (8, 1, 5)$.