Eigen values and Eigen vectors

If A is a square matrix of order n, we can find the matrix $A-\lambda I$, where I is the n^{th} order unit matrix. The determinant of this matrix equated to zero, i.e.,

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic equation of A.

On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0 ,$$

where k^s are expressible in terms of the elements a_{ij}. The roots of this equation are called the characteristic roots or latent roots or eigen-values of the matrix A.

$\frac{dv}{dt} = 4v - 5w$, $v = 8$ at $t = 0$

$\frac{dw}{dt} = 2v - 3w$, $w = 5$ at $t = 0$

$u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$, $u(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$, $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$\frac{du}{dt} = Au$, $u = u(0)$ at $t = 0$

$v(t) = e^{\lambda t} x_1$

$w(t) = e^{\lambda t} x_2$

$u(t) = e^{\lambda t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda e^{\lambda t} x_1 = 4 e^{\lambda t} x_1 - 5 e^{\lambda t} x_2$

$\lambda e^{\lambda t} x_2 = 2 e^{\lambda t} x_1 - 3 e^{\lambda t} x_2$

$4x_1 - 5x_2 = \lambda x_1$

$2x_1 - 3x_2 = \lambda x_2$

$Ax = \lambda x$

$(A - \lambda I) x = 0$

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then the linear transformation $Y = AX$ — (1) carries the column vector X into the column vector Y by means of the square matrix A.

In practice it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (1).

Then, $\lambda X = AX$ or $AX - \lambda I X = 0$ or $[A - \lambda I] X = 0$ — (2)

The matrix equation represents n homogeneous linear equations,

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad - (3)$$

which will have a non-trivial solution only if the coefficient matrix is singular.

i.e., if $|A - \lambda I| = 0$

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A.

It has n roots and corresponding to each root, the equation (2) (or equation (3)) will have a non-zero solution, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, which is known as the eigen vector or latent vector.

Observation 1:

Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Observation 2:

If X_i is a solution for a eigen value λ_i then it follows from (2) that cX_i is also a solution, where c is an arbitrary constant. Thus the eigen vector corresponding to an eigen value is not unique, but may be any one of the vectors cX_i

Problems

1. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution:-

The characteristic equation is $|A-\lambda I|=0$.

ie, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (5-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow \quad 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow \quad (\lambda-6)(\lambda-1) = 0$

$\Rightarrow \quad \underline{\lambda = 1, \ \lambda = 6}$ are the eigen values.

If $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then

$$[A-\lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$, we have

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \left.\begin{array}{l} 4x_1 + 4x_2 = 0 \\ x_1 + x_2 = 0 \end{array}\right\}$.

or $\quad x_1 = -x_2$

let $x_2 = k$, then $x_1 = -k$.

$\therefore \quad X = \begin{bmatrix} -k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

For $\lambda = 6$, we have

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 + 4x_2 = 0$

or $\quad x_1 = 4x_2$

let $x_2 = k$, then $x_1 = 4k$

$\therefore \; X = \begin{bmatrix} 4k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$.

2. Find the eigen values and eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution:-

The characteristic equation is $|A - \lambda I| = 0$

is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(4-\lambda) - 4 = 0$

$\Rightarrow \quad 4 - \lambda - 4\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda^2 - 5\lambda = 0$

$\Rightarrow \quad \lambda(\lambda - 5) = 0$

$\Rightarrow \quad \underline{\lambda = 0, \; \lambda = 5}$ are the eigen values.

$\longrightarrow$

If $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0$, we have

$$\begin{bmatrix} 1-0 & 2 \\ 2 & 4-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{array}\right\}$$

or $x_1 + 2x_2 = 0$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{1}$$

$\Rightarrow x_1 = 2,\ x_2 = -1$

or $x_1 = 2k,\ x_2 = -k$

$\therefore X = \begin{bmatrix} 2k \\ -k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 0$

For $\lambda = 5$, we have

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{array}\right\}$$

or $2x_1 - x_2 = 0$

$$\therefore \frac{x_1}{-1} = \frac{-x_2}{2}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2}$$

$\Rightarrow x_1 = 1,\ x_2 = 2$

or $x_1 = k,\ x_2 = 2k$

$\therefore X = \begin{bmatrix} k \\ 2k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 5$

3. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:-

The characteristic equation is $|A-\lambda I|=0$

ie, $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[1(1-\lambda)-3\right] + 3\left[1-3(5-\lambda)\right] = 0$

$\Rightarrow (1-\lambda)\left[5-5\lambda-\lambda+\lambda^2-1\right] - \left[1-\lambda-3\right] + 3\left[1-15+3\lambda\right] = 0$

$\Rightarrow (1-\lambda)\left[\lambda^2-6\lambda+4\right] - \left[-\lambda-2\right] + 3\left[3\lambda-14\right] = 0$

$\Rightarrow \lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2+9\lambda-42 = 0$

$\Rightarrow -\lambda^3+7\lambda^2-36 = 0$

$\Rightarrow \lambda^3-7\lambda^2+36 = 0$

Solving, $\Rightarrow \lambda = -2, 3, 6$.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigen vector corresponding to the eigen value λ, then $[A-\lambda I]X = 0$

$\longrightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = -2$, we have

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{7\times 3 - 1\times 1} = \frac{-x_2}{1\times 3 - 3\times 1} = \frac{x_3}{1\times 1 - 3\times 7}$$

$$\Rightarrow \frac{x_1}{20} = \frac{-x_2}{0} = \frac{x_3}{-20}.$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = -2$.

For $\lambda = 3$, we have

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{-4-1} = \frac{-x_2}{-2-3} = \frac{x_3}{1-6}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ -k \\ k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 3$

For $\lambda = 6$, we have

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{4}.$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore$ $X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 6$

4. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:-

The characteristic equation is $|A-\lambda I| = 0$

ie, $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[(1-\lambda)(2-\lambda)-0] - 1[0-0] + 1[0-(1-\lambda)] = 0$

$\Rightarrow (2-\lambda)(1-\lambda)(2-\lambda) - (1-\lambda) = 0$

$\Rightarrow (1-\lambda)[(2-\lambda)(2-\lambda) - 1] = 0$

$\Rightarrow (1-\lambda)(4-2\lambda-2\lambda+\lambda^2-1) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-4\lambda+3) = 0$

$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-3) = 0$

$\Rightarrow \lambda = 1, 1, 3$ are the eigen values.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigen vector corresponding to the eigen value λ, then $[A-\lambda I]X = 0$

$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0$

$\Rightarrow \quad x_1 = -x_2 - x_3$

let $\quad x_2 = k_1$, $x_3 = k_2$

Then $\quad x_1 = -k_1 - k_2$

$\therefore \quad X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

For $\lambda = 3$, we have

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \dfrac{x_1}{2} = \dfrac{-x_2}{0} = \dfrac{x_3}{2}$

$\Rightarrow \quad \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$

$\therefore \quad X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} k \\ 0 \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

Diagonalization of a matrix

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix P, then $P^{-1}AP$ is a diagonal matrix D.
The eigenvalues of A are on the diagonal of D

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

* Any matrix with distinct eigenvalues can be diagonalized.
* The diagonalization matrix P is not unique.
* Not all matrices posses n linearly independent eigenvectors, so not all matrices are diagonalizable
* Diagonalizability of A depends on enough eigenvectors. Invertibility of A depends on nonzero eigenvalues.
* Diagonalization can fail only if there are repeated eigen values.
* The eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$ and each eigenvector of A is still an eigenvector of A^k.
  $[D^k = (P^{-1}AP)(P^{-1}AP) \dots (P^{-1}AP) = P^{-1}A^kP]$.
* Diagonalizable matrices share the same eigenvector matrix iff $AB = BA$

example $\quad A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$|A-\lambda I| = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0, 1$.

For $\lambda = 0$ $\quad A - 0I = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = -x_2$

$\therefore \quad X = \begin{bmatrix} k \\ -k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 1$ $\quad A - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow -\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = x_2$

$\therefore \quad X = \begin{bmatrix} k \\ k \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = -i, +i$

For $\lambda = -i$ $\quad A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \Rightarrow ix_1 - x_2 = 0 \Rightarrow x_2 = ix_1$

$\therefore \quad X = \begin{bmatrix} k \\ ik \end{bmatrix}$ or $\begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = i$ $\quad A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \Rightarrow -ix_1 - x_2 = 0 \Rightarrow x_2 = -ix_1$

$\therefore X = \begin{bmatrix} k \\ -ik \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\therefore \quad P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad P^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}$

* Diagonalize the matrix $A=\begin{bmatrix}1&1&3\\1&3&1\\3&1&1\end{bmatrix}$

Sol^n

$|A-\lambda I|=0 \Rightarrow \lambda^3-5\lambda^2-4\lambda+20=0 \Rightarrow \lambda=5,2,-2$

For $\lambda=5$

$A-5I=\begin{bmatrix}-4&1&3\\1&-2&1\\3&1&-4\end{bmatrix} \Rightarrow \frac{x_1}{7}=\frac{-x_2}{-7}=\frac{x_3}{7} \Rightarrow X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$

For $\lambda=2$

$A-2I=\begin{bmatrix}-1&1&3\\1&1&1\\3&1&-1\end{bmatrix} \Rightarrow \frac{x_1}{-2}=\frac{-x_2}{-4}=\frac{x_3}{-2} \Rightarrow X_2=\begin{bmatrix}1\\-2\\1\end{bmatrix}$

For $\lambda=-2$

$A+2I=\begin{bmatrix}3&1&3\\1&5&1\\3&1&3\end{bmatrix} \Rightarrow \frac{x_1}{14}=\frac{-x_2}{0}=\frac{x_3}{-14} \Rightarrow X_3=\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&1&1\\1&-2&0\\1&1&-1\end{bmatrix}$ $D=\begin{bmatrix}5&0&0\\0&2&0\\0&0&-2\end{bmatrix}$ $P^{-1}=\frac{1}{6}\begin{bmatrix}+2&-(-1)&+3\\-(-2)&+(-2)&-0\\+2&-(-1)&+(-3)\end{bmatrix}^T=\frac{1}{6}\begin{bmatrix}2&2&2\\1&-2&1\\3&0&-3\end{bmatrix}$

* Diagonalize the matrix $A=\begin{bmatrix}3&-2&4\\-2&6&2\\4&2&3\end{bmatrix}$

Sol^n $|A-\lambda I|=0 \Rightarrow \lambda^3-12\lambda^2+21\lambda+98=0 \Rightarrow \lambda=7,7,-2$

For $\lambda=7$

$A-7I=\begin{bmatrix}-4&-2&4\\-2&-1&2\\4&2&-4\end{bmatrix} \Rightarrow \frac{x_1}{0}=\frac{-x_2}{0}=\frac{x_3}{0}$ reducing to echelon form $\xrightarrow[R_3+R_1]{2R_2-R_1} \begin{bmatrix}-4&-2&4\\0&0&0\\0&0&0\end{bmatrix}$

$\Rightarrow -4x_1-2x_2+4x_3=0 \Rightarrow x_1=-\frac{1}{2}x_2+x_3$ $\therefore X=\begin{bmatrix}-\frac{1}{2}x_2+x_3\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}-1/2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}$

letting x_2, x_3 are free variables

$\therefore X_1=\begin{bmatrix}1\\-2\\0\end{bmatrix}$ and $X_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$

For $\lambda=-2$

$A+2I=\begin{bmatrix}+5&-2&4\\-2&8&2\\4&2&+5\end{bmatrix} \Rightarrow \frac{x_1}{36}=\frac{-x_2}{-18}=\frac{x_3}{-36} \Rightarrow X_3=\begin{bmatrix}2\\1\\-2\end{bmatrix}$

$\therefore P=\begin{bmatrix}1&1&2\\-2&0&1\\0&1&-2\end{bmatrix}$ $D=\begin{bmatrix}7&0&0\\0&7&0\\0&0&-2\end{bmatrix}$ $P^{-1}=\frac{1}{-9}\begin{bmatrix}+(-1)&-4&+(-2)\\-(-4)&+(-2)&-1\\+1&-5&+2\end{bmatrix}^T$

$=\frac{1}{9}\begin{bmatrix}1&-4&-1\\4&2&5\\2&1&-2\end{bmatrix}$

Singular Value Decomposition

Any m by n matrix A can be factored into

$A = U\Sigma V^T = $ (orthogonal) (diagonal) (orthogonal)

The columns of U (m by m) are eigenvectors of AA^T, and the columns of V (n by n) are eigenvectors of A^TA. The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Remark

For positive definite matrices, Σ is Λ and $U\Sigma V^T$ is identical to $Q\Lambda Q^T$.

For other symmetric matrices, ~~Σ remains real~~ any negative eigenvalues in Λ become positive in Σ.

Remark.

U and V give orthonormal bases for all four fundamental subspaces:

first r columns of U : column space of A

last $m-r$ columns of U : left nullspace of A

first r columns of V : row space of A

last $n-r$ columns of V : nullspace of A.

The diagonal (but rectangular) matrix Σ has eigenvalues from A^TA. These positive entries (also called sigma) will be $\sigma_1, \dots, \sigma_r$. They are the singular values of A.

<u>Remark:</u>
When A multiplies a column v_j of V, it produces σ_j times a column of U. ($A = U\Sigma V^T \Rightarrow AV = U\Sigma$).

<u>Remark:</u>
Eigenvectors of AA^T and A^TA must go into the columns of U and V!

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T.$$

$\therefore$ U must be the eigenvector matrix of AA^T.
The eigenvalue matrix $\Sigma\Sigma^T$ is an $m \times m$ matrix with $\sigma_1^2, \dots, \sigma_r^2$ on the diagonal.

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$$

$\therefore$ V must be the eigenvector matrix of A^TA.
The diagonal matrix $\Sigma^T\Sigma$ has the same $\sigma_1^2, \dots \sigma_r^2$, but it is $n \times n$.

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad D \rightarrow \underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0}_{+ve}.$$

Decompose $A=\begin{bmatrix}-1\\2\\2\end{bmatrix}$ as $U\Sigma V^T$ where U and V are orthogonal matrices.

Sol^n:

$$AA^T=\begin{bmatrix}-1\\2\\2\end{bmatrix}_{3\times1}\begin{bmatrix}-1&2&2\end{bmatrix}_{1\times3}$$

$$=\begin{bmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{bmatrix}$$

$$|AA^T-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda&-2&-2\\-2&4-\lambda&4\\-2&4&4-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\left[(4-\lambda)^2-16\right]+2\left[-2(4-\lambda)+8\right]-2\left[-8+2(4-\lambda)\right]=0$$

$$(1-\lambda)\left[16-8\lambda+\lambda^2-16\right]+2\left[\cancel{-8}+2\lambda+\cancel{8}\right]-2\left[\cancel{-8}+\cancel{8}-2\lambda\right]=0$$

$$(1-\lambda)(\lambda^2-8\lambda)+4\lambda+4\lambda=0$$

$$\lambda^2-\cancel{8\lambda}-\lambda^3+8\lambda^2+\cancel{8\lambda}=0$$

$$-\lambda^3+9\lambda^2=0$$

$$-\lambda^2(\lambda-9)=0$$

$$\lambda_1=0,\ \lambda_2=0,\ \lambda_3=9$$

$\underline{\lambda=9}$

$(AA^T-\lambda I)X=0$

$$\begin{bmatrix}-8&-2&-2\\-2&-5&4\\-2&4&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\sim\begin{bmatrix}-8&-2&-2\\0&-18&18\\0&18&-18\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\Rightarrow \left.\begin{matrix}-8x_1-2x_2-2x_3=0\\-18x_2+18x_3=0\end{matrix}\right\}\ x_2=x_3,\quad x_1=-\tfrac{1}{2}x_3$$

$$X_1=\begin{bmatrix}-1\\2\\2\end{bmatrix}$$

$\underline{\lambda=0}$

$(AA^T-\lambda I)X=0$

$$\begin{bmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\sim\begin{bmatrix}1&-2&-2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\Rightarrow x_1-2x_2-2x_3=0$$

$$\Rightarrow x_1=2x_2+2x_3$$

let $\cancel{x_2=-\tfrac{1}{2}}$ $x_2=-1,\ x_3=2$ $\qquad X_2=\begin{bmatrix}2\\-1\\2\end{bmatrix}$

let $x_2=2,\ x_3=-1$ $\qquad X_3=\begin{bmatrix}2\\2\\-1\end{bmatrix}$

$\therefore U = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$

$|A^TA - \lambda I| = 0 \Rightarrow |9 - \lambda| = 0 \Rightarrow \underline{\lambda = 9}$

Then $(A^TA - \lambda I)X = 0$

$\Rightarrow \quad 0 \cdot x_1 = 0$

let $x_1 = 1$ $\quad \therefore X = [1]$

$\therefore V = [1]$ or $V^T = [1]$.

9 is an eigenvalue of both AA^T and A^TA.

And rank of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is $r = 1$.

$\therefore \Sigma$ has only $\sigma_1 = \sqrt{9} = 3$. $\quad \therefore \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ The SVD of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$.

Obtain the SVD of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Sol^n:

$$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|AA^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) - 1 = 0 \Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}$$

$\lambda_1 = \frac{3 - \sqrt{5}}{2}$

$(AA^T - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 - \left(\frac{3-\sqrt{5}}{2}\right) & 1 \\ 1 & 1 - \left(\frac{3-\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{1+\sqrt{5}}{2} x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -\frac{2}{1+\sqrt{5}} x_2$$

let $x_2 = \frac{1+\sqrt{5}}{2} \Rightarrow x_1 = -1$

$$X = \begin{bmatrix} -1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$$

$\alpha = \frac{1+\sqrt{5}}{2}$

$\lambda_2 = \frac{3+\sqrt{5}}{2}$

$(AA^T - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 - \left(\frac{3+\sqrt{5}}{2}\right) & 1 \\ 1 & 1 - \left(\frac{3+\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \frac{1-\sqrt{5}}{2} x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -\frac{2}{1-\sqrt{5}} x_2$$

let $x_2 = \frac{1-\sqrt{5}}{2} \Rightarrow x_1 = -1$

$$X = \begin{bmatrix} -1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \beta \end{bmatrix}$$

$\beta = \frac{1-\sqrt{5}}{2}$

$$U = \begin{bmatrix} \frac{-1}{\sqrt{1+\alpha^2}} & \frac{-1}{\sqrt{1+\beta^2}} \\ \frac{\alpha}{\sqrt{1+\alpha^2}} & \frac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$$

As $A^TA = AA^T$ $\quad V^T = \begin{bmatrix} \frac{-1}{\sqrt{1+\alpha^2}} & \frac{\alpha}{\sqrt{1+\alpha^2}} \\ \frac{-1}{\sqrt{1+\beta^2}} & \frac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$

Obtain the SVD of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times3}$

$$AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$|AA^T - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda-1)(\lambda-3) = 0$

$\lambda = 1, \; \lambda = 3$

$\lambda_1 = 3$ $\quad (AA^T - \lambda I)X = 0$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = -x_2$

let $x_2 = 1, \; x_1 = -1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$

$(AA^T - \lambda I)X = 0$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow x_1 = x_2$

let $x_2 = 1, \; x_1 = 1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times3} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$|A^TA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)[(2-\lambda)(1-\lambda) - 1] + 1[-(1-\lambda)] = 0$

$\Rightarrow (1-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 1) - 1 + \lambda = 0$

$(1-\lambda)(\lambda^2 - 3\lambda + 1) - 1 + \lambda = 0$

$\lambda^2 - 3\lambda + \cancel{1} - \lambda^3 + 3\lambda^2 - \cancel{\lambda} - \cancel{1} + \cancel{\lambda} = 0$

$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$

$-\lambda(\lambda^2 - 4\lambda + 3) = 0$

$\lambda_1 = 0, \; \lambda_2 = 1, \; \lambda_3 = 3$

$\lambda_1 = 0$

$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 - x_2 = 0 \quad x_1 = x_2$

$x_2 - x_3 = 0 \quad x_2 = x_3$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$-x_2 = 0$ $\quad x_2 = 0$

$-x_1 + x_2 - x_3 = 0$ $\quad x_1 = -x_3$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\sim \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$-2x_1 - x_2 = 0$

$x_2 + 2x_3 = 0$

$x_2 = -2x_1$

$\begin{bmatrix} 1 \\ -2 \\ +1 \end{bmatrix}$

$$U = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{2\times2}$$

$$V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \quad V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}_{3\times3}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$