Linear Algebra, Probability and Queuing Theory (MMA202IT) CIE 1

Quiz

1. $\begin{vmatrix} 2t^2 & t & 2 \\ t^2 & -2t & 0 \\ kt^2 & -t & 2 \end{vmatrix} = 0 \Rightarrow 2t^2(-4t) - t(2t^2) + 2(-t^3+2kt^3) = 0$ (1)

$\Rightarrow -8t^3 - 2t^3 - 2t^3 + 4kt^3 = 0 \Rightarrow 4k = 12 \quad \underline{k=3}$ (1) (1)

2. Zero vector is 1 (1) and inverse of α is $\frac{1}{\alpha}$ (1)

3. $\begin{bmatrix} 1 & 1 & : & 2 \\ 1 & 2 & : & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & : & 2 \\ 0 & 1 & : & 1 \end{bmatrix}$ (1) $\quad c_1 + c_2 = 2,\ c_2 = 1,\ c_1 = 1$ (1) $\Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in$ colspace of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

4. $y = (4,2)$, $u = (1,1)$ $\quad \hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{6}{2}(1,1) = (3,3)$ (1) (1)

$z = y - \hat{y} = (4,2) - (3,3) = (1,-1)$ (1)

5. $A^TA = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad A^TAx = A^Tb \Rightarrow x = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (1) (1)

1a. $V = \{a + b\sqrt{2} \mid a, b \in \mathbb{R}\}$ (1)

Let $\alpha = a_1 + b_1\sqrt{2}$, $\beta = a_2 + b_2\sqrt{2}$, $\gamma = a_3 + b_3\sqrt{2} \in V$, $c, c' \in \mathbb{R}$

(i) $\alpha + \beta = (a_1+a_2) + (b_1+b_2)\sqrt{2} \in \mathbb{R}$, (ii) $(\alpha+\beta)+\gamma = ((a_1+a_2)+a_3) + ((b_1+b_2)+b_3)\sqrt{2} = \alpha + (\beta+\gamma)$ (0.5)

(iii) $\alpha + e = (a_1 + b_1\sqrt{2}) + 0 = \alpha = e + \alpha$ (iv) $\alpha + (-\alpha) = (a_1+b_1\sqrt{2}) + (-a_1 - b_1\sqrt{2}) = 0 = (-\alpha) + \alpha$

$\therefore e = 0 = 0 + 0\sqrt{2}$ (1) (vi) $c\alpha = ca_1 + cb_1\sqrt{2} \in V$ $\quad -\alpha = -a_1 - b_1\sqrt{2}$ is the inverse of α (1)

(v) $\alpha + \beta = (a_1+a_2) + (b_1+b_2)\sqrt{2} = \beta + \alpha$ (0.5) (vii) $c(\alpha+\beta) = (ca_1 + ca_2) + (cb_1 + cb_2)\sqrt{2} = c\cdot\alpha + c\cdot\beta$ (0.5)

(viii) $(c+c')\cdot\alpha = ca_1 + c'a_1 + (cb_1 + c'b_1)\sqrt{2} = c\cdot\alpha + c'\cdot\alpha$ (0.5) (ix) $(cc')\alpha = (c\cdot c')a_1 + (c\cdot c')b_1\sqrt{2} = c\cdot(c'\cdot\alpha)$ (0.5)

(x) $1\cdot\alpha = 1\cdot a_1 + 1\cdot b_1\sqrt{2} = \alpha$ (0.5) $\therefore$ 1 is the unit element.

1b. $\mathbb{P}_2 = \{a_0 + a_1x + a_2x^2\}$ $\quad \alpha = a_0 + a_1x + a_2x^2$, $\beta = b_0 + b_1x + b_2x^2 \in \mathbb{P}_2$, $c \in \mathbb{R}$.

(i) $\alpha + \beta = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \in \mathbb{P}_2$ (2)

(ii) $c\alpha = ca_0 + ca_1x + ca_2x^2 \in \mathbb{P}_2$ (2) $\therefore \mathbb{P}_2$ is a subspace of $\mathbb{P}_n$

2a. $\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & -5 \\ 2 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 2 & 4 \\ 1 & -2 & 0 & 2 \\ 5 & -5 & 2 & 10 \\ 1 & 3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 & 4 \\ 0 & -5 & -2 & 0 \\ 0 & -15 & -6 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 & 4 \\ 0 & -5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore$ The matrices are L.D (3)

$\left\{\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}\right\}$ are L.I (1) are is the basis (1) & dimension of subspace is 2 (1)

2b. $(1,2)(2,3)(3,4)(4,3)$ $\quad X^TX = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$ (1) $\quad X^Ty = \begin{bmatrix} 12 \\ 32 \end{bmatrix}$ (1) $\quad \beta = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 12 \\ 32 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.4 \end{bmatrix}$ (1)

$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 3 \end{bmatrix}$ (1)

3. $$\begin{bmatrix}1&1&1&1\\3&1&1&3\\5&0&2&3\end{bmatrix} \sim \begin{bmatrix}1&1&1&1\\0&-2&-2&0\\0&-5&-3&-2\end{bmatrix} \sim \begin{bmatrix}1&1&1&1\\0&-2&-2&0\\0&0&4&-4\end{bmatrix}$$ (2)

Basis of $R(A) = \{(1,1,1,1),(3,1,1,3),(5,0,2,3)\}$ (1)

Basis of $C(A) = \{(1,3,5),(1,1,0),(1,1,2)\}$ (1)

$$AX = 0 \Rightarrow \left.\begin{aligned} x_1+x_2+x_3+x_4&=0\\ -2x_2-2x_3&=0\\ 4x_3-4x_4&=0\end{aligned}\right\}$$ let x_4 be the f.v $\Rightarrow x_3 = x_4,\ x_2 = -x_3 = -x_4$ (2)

$x_1 = -x_2 - x_3 - x_4 = x_4 - x_4 - x_4 \Rightarrow x_1 = -x_4$

$\therefore \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_4\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$ $\therefore$ Basis of $N(A) = \{(-1,-1,1,1)\}$ (1)

$$A^T = \begin{bmatrix}1&3&5\\1&1&0\\1&1&2\\1&3&3\end{bmatrix} \sim \begin{bmatrix}1&3&5\\0&-2&-5\\0&-2&-3\\0&0&-2\end{bmatrix} \sim \begin{bmatrix}1&3&5\\0&-2&-5\\0&0&2\\0&0&-2\end{bmatrix} \sim \begin{bmatrix}1&3&5\\0&-2&-5\\0&0&2\\0&0&0\end{bmatrix}$$ (2)

$A^TY=0 \Rightarrow y_1+3y_2+5y_3=0,\ -2y_2-5y_3=0,\ 2y_3=0$

$\Rightarrow y_3=0, y_2=0, y_1=0$ $\therefore \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore N(A^T) = \{(0,0,0)\}$ (1)

4a. $T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x,y) = (x+3y,\ 2x-2y)$

$\alpha=(x_1,y_1),\ \beta=(x_2,y_2) \in \mathbb{R}^2$ (i) $T(\alpha+\beta) = T(x_1+x_2,\ y_1+y_2)$ (2)

$= ((x_1+x_2+3(y_1+y_2)),\ 2(x_1+x_2)-2(y_1+y_2))$

$= T(\alpha)+T(\beta)$ (2)

(ii) $T(c.\alpha) = T(cx_1, cy_1) = (cx_1+3cy_1,\ 2cx_1-2cy_1) = c.T(\alpha)$ $\therefore$ T is a L.T

4b. $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T(1,1,1)=(2,2,0),\ T(1,2,1)=(4,3,1)\ T(2,1,0)=(4,1,3)$

$$A\begin{bmatrix}1&1&2\\1&2&1\\1&1&0\end{bmatrix} = \begin{bmatrix}2&4&4\\2&3&1\\0&1&3\end{bmatrix} \Rightarrow A = \begin{bmatrix}2&4&4\\2&3&1\\0&1&3\end{bmatrix}\begin{bmatrix}1/2&-1&3/2\\-1/2&1&-1/2\\1/2&0&-1/2\end{bmatrix} = \begin{bmatrix}1&2&-1\\0&1&1\\1&1&-2\end{bmatrix}$$ (1) (1)

$\therefore T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$ (1)

$$\begin{bmatrix}1&2&-1\\0&1&1\\1&1&-2\end{bmatrix} \sim \begin{bmatrix}1&2&1\\0&1&1\\0&-1&-1\end{bmatrix} \sim \begin{bmatrix}1&2&-1\\0&1&1\\0&0&0\end{bmatrix}$$ Basis of $R(T) = \{(1,0,1),(2,1,1)\}$ (1)

$Ax=0 \Rightarrow x+2y-z=0;\ y+z=0$ $y=-z,\ x=3z$ $\therefore N(T) = \{(3,-1,1)\}$ (1) (1)

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5. $\begin{bmatrix}-1&6&6\\3&-8&3\\1&-2&6\\1&-4&-3\end{bmatrix}$ $x_1=(-1,3,1,1),\ x_2=(6,-8,-2,-4),\ x_3=(6,3,6,-3)$

$u_1 = (-1,3,1,1)$ (1)

$u_2 = (6,-8,-2,-4) - \frac{(-36)}{12}(-1,3,1,1) = (3,1,1,-1)$ (2)

$u_3 = (6,3,6,-3) - \frac{6}{12}(-1,3,1,1) - \frac{30}{12}(3,1,1,-1) = (-1,-1,3,-1)$ (3)

$$Q = \begin{bmatrix}-1/2\sqrt3 & 3/2\sqrt3 & -1/2\sqrt3\\ 3/2\sqrt3 & 1/2\sqrt3 & -1/2\sqrt3\\ 1/2\sqrt3 & 1/2\sqrt3 & 3/2\sqrt3\\ 1/2\sqrt3 & -1/2\sqrt3 & -1/2\sqrt3\end{bmatrix} \quad R = \begin{bmatrix}6/\sqrt3 & -18/\sqrt3 & 3/\sqrt3\\ 0 & 6/\sqrt3 & 15/\sqrt3\\ 0 & 0 & 6/\sqrt3\end{bmatrix}$$ (2) (2)