Queueing Theory

There are many situations in daily life when a queue is formed.

example: machines waiting to be repaired,
patients waiting in a doctor's room,
cars waiting at a traffic signal.
passengers waiting to buy tickets in counters.

Queue is formed if the service required by the customer is not immediately available, that is, if the current demand for a particular service exceeds the capacity to provide the service.

Although there are many types of queueing systems, all of them can be classified and described according to the following characteristics:

1. The input (or arrival) pattern

The input describes the manner in which the customers arrive and join the queueing system. It is not possible to observe and control the actual moment of arrival of a customer for service. Hence the number of arrivals in one time period or the interval between successive arrivals is not treated as a constant, but a random variable. So the mode of arrival of customers is expressed by means of the probability distribution of the number of arrivals per unit of time or of the interarrival time. We shall deal with only those queueing systems in which the number of arrivals per unit of time has a poisson distribution with mean λ. In this case, the time between consecutive arrivals has an exponential distribution with mean $1/\lambda$.

2. The service mechanism (or pattern)

The mode of service is represented by means of the probability distribution of the number of customers serviced per unit of time or of the inter-service time. We shall deal with only those queueing systems in which the number of customers serviced per unit of time has a Poisson distribution with mean μ or equivalently the inter-service time has an exponential distribution with mean $\frac{1}{\mu}$.

3. The queue discipline

The queue discipline specifies the manner in which the customers form the queue or equivalently the manner in which they are selected for service, when a queue has been formed. The most common discipline is the FCFS (First Come First Served) or FIFO (First In First Out) as per which the customers are served in the strict order of their arrival. If the last arrival in the system is served first, we have the LCFS or LIFO discipline. If the service is given in random order, we have the SIRO discipline. In the queueing systems which we deal with, we shall assume that service is provided on the FCFS basis.

Symbolic representation of a queueing model.

Usually a queueing model is specified and represented symbolically in the form $(a/b/c):(d/e)$, where a denotes the type of distribution of the number of arrivals per unit time, b the type of distribution of the service time, c the number of servers, d the capacity of the system, viz., the maximum queue size and e the queue discipline.

Accordingly, the first four models which we will deal with will be denoted by the symbols $(M/M/1):(\infty/FIFO)$, $(M/M/s):(\infty/FIFO)$, $(M/M/1):(k/FIFO)$ and $(M/M/s):(k/FIFO)$. In the above symbols the letter M stands for Markov indicating that the number of arrivals in time t and the number of completed services in time t follow Poisson process which is a continuous time markov chain.

Difference Equations related to Poisson Queue systems

If the characteristics of a queueing system (such as the input and output parameters) are independent of time or equivalently if the behaviour of the system is independent of time, the system is said to be in steady-state. Otherwise it is said to be in transient-state.

Let $P_n(t)$ be the probability that there are n customers in the system at time t $(n>0)$.

Let λ_n be the average arrival rate when there are n customers in the system (both waiting in the queue and being served) and let μ_n be the average service rate when there are n customers in the system.

Then the differential equations are given by,

$$P_n'(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \mu_{n+1} P_{n+1}(t), \quad n \neq 0, n > 0. \quad - (1)$$

and

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t), \quad n = 0. \quad - (2)$$

In steady-state, $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t)$ and $P_0'(t)$ become zero.

Hence the differential equations (1) & (2) reduce to the difference equations,

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \quad - (3)$$

and

$$-\lambda_0 P_0 + \mu_1 P_1 = 0 \quad - (4)$$

<u>Values of P_0 and P_n for Poisson Queue systems.</u>

From (4), we have $P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad - (5)$

putting $n=1$ in (3) and using (5), we have

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 \quad - (6)$$

Similarly, we have,

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \cdots \mu_n} P_0, \text{ for } n = 1, 2 \ldots \quad (7)$$

Since the number of customers in the system can be 0 or 1 or 2 or 3 etc., which events are mutually exclusive and exhaustive, we have $\sum_{n=0}^{\infty} P_n = 1$.

ie, $P_0 + \sum_{n=1}^{\infty} P_n = 1 \Rightarrow P_0 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) P_0 = 1$

$$\therefore P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right)} \quad - (8)$$

Characteristics of infinite capacity, single server Poisson queue model 1 [(M|M|1): (∞|FIFO) model], when $\lambda_n = \lambda$ and $\mu_n = \mu$ ($\lambda < \mu$).

Let N denote the number of customers in the queueing system (i.e., those in the queue and the one who is being served)

~~Average number~~ Then $P_0 = \dfrac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = 1 - \dfrac{\lambda}{\mu}$

and $P_n = \left(\dfrac{\lambda}{\mu}\right)^n P_0 = \left(\dfrac{\lambda}{\mu}\right)^n \left(1 - \dfrac{\lambda}{\mu}\right)$

(i) Average number L_s of customers in the system is:

$$L_s = E[N] = \sum_{n=0}^{\infty} n \times P_n$$

$$L_s = \frac{\lambda}{\mu - \lambda}$$

(ii) Average number L_q of customers in the queue or Average length of the queue is:

$$L_q = E[N-1] = \sum_{n=1}^{\infty} (n-1) P_n$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

(iii) Average number L_w of customers in nonempty queues

$$L_w = E[(N-1) \mid (N-1) > 0], \text{ since the queue is non-empty}$$

$$L_w = \frac{\mu}{\mu - \lambda}.$$

(iv) Probability that the number of customers in the system exceeds k is:

$$P(N > k) = \sum_{n=k+1}^{\infty} P_n$$

$$P(N > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

Let W_s be the continuous random variable that represents the waiting time of a customer in the system, i.e., the time between arrival and completion of service.

Let its pdf be $f(\omega)$ and let $f(\omega/n)$ be the density function of W_s subject to the condition that there are n customers in the queueing system when the customer arrives,

(v) Then probability density function of the waiting time in the system

$$f(\omega) = \sum_{n=0}^{\infty} f(\omega/n) P_n = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{n!} e^{-\mu\omega} \omega^n \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$f(\omega) = (\mu-\lambda) e^{-(\mu-\lambda)\omega}$$ (which is an exponential distribution with parameter $\mu-\lambda$)

(vi) Average waiting time of a customer in the system is:

$$E[W_s] = \frac{1}{\mu-\lambda}$$

(vii) Probability that the waiting time of a customer in the system exceeds t is $P(W_s > t) = \int_t^{\infty} f(\omega)\, d\omega$

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

Let W_q represent the time between arrival and reach of service point. Let the pdf of W_q be $g(\omega)$ and let $g(\omega/n)$ be the density function of W_q subject to the condition that there are n customers in the system or there are $(n-1)$ customers in the queue apart from one customer receiving service.

(viii) Probability density function of the waiting time W_q in the queue is:

$$g(\omega) = \sum_{n=1}^{\infty} g(\omega/n) P_n = \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!} e^{-\mu\omega} \omega^{n-1} \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$g(\omega) = \frac{\lambda}{\mu}(\mu-\lambda) e^{-(\mu-\lambda)\omega} \quad ; \omega > 0$$

$$g(\omega) = 1 - \frac{\lambda}{\mu}, \quad \omega = 0.$$

(ix) Average waiting time of a customer in the queue is:

$$E[W_q] = \frac{\lambda}{\mu}(\mu-\lambda) \int_0^{\infty} \omega e^{-(\mu-\lambda)\omega} d\omega \Rightarrow E[W_q] = \frac{\lambda}{\mu(\mu-\lambda)}$$

(x) Average waiting time of a customer in the queue, if he has to wait is:

$$E[W_q / W_q > 0] = \frac{E[W_q]}{P[W_q > 0]} \Rightarrow E[W_q / W_q > 0] = \frac{1}{\mu-\lambda}$$

Characteristics of infinite capacity, multiple server Poisson queue model II [(M|M|s) : (∞|FIFO)], when $\lambda_n = \lambda$ for all n $(\lambda < s\mu)$

(i) Values of P_0 and P_n:

For the Poisson queue system,

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \times P_0, \; n \geq 1 \qquad -(1)$$

$$P_0 = \left[1 - \sum_{n=1}^{\infty}\left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right)\right]^{-1} \qquad -(2)$$

If there is a single server $\mu_n = \mu$ for all n.

But there are s servers working independently of each other. If there ~~are~~ be less than s customers, i.e., if $n < s$, only n of the s servers will be busy and the others idle and hence the mean service rate will be $n\mu$.

If $n \geq s$, all the s servers will be busy and hence the mean service rate $= s\mu$.

i.e.,
$$\mu_n = \begin{cases} n\mu, & \text{if } 0 \leq n < s \\ s\mu, & \text{if } n \geq s \end{cases} \qquad -(3)$$

Using (3) in (1) & (2)

$\Rightarrow$
$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{if } 0 \leq n < s \qquad -(4) \\ \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{if } n \geq s \qquad -(5) \end{cases}$$

$$P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}} \qquad (6)$$

(ii) Average number of customers in the queue or average queue length is

$$L_q = E(N_q) = E(N-s) = \sum_{n=s}^{\infty}(n-s)P_n \Rightarrow L_q = \frac{1}{s.s!}\frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$$

(iii) Average number of customers in the system is:

$$E(N_s) = E(N_q) + \frac{\lambda}{\mu} = \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 + \frac{\lambda}{\mu}$$

(iv) Average time a customer has to spend in the system is:

$$E(W_s) = \frac{1}{\lambda} E(N_s) = \frac{1}{\mu} + \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$$

(v) Average time a customer has to spend in the queue is.

$$E(W_q) = \frac{1}{\lambda} E(N_q) = \frac{1}{\mu} \cdot \frac{1}{s \cdot s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$$

(vi) Probability that an arrival has to wait is:

$$P(W_s > 0) = P(N \geq s) = \sum_{n=s}^{\infty} P_n = \frac{\left(\frac{\lambda}{\mu}\right)^{s} P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$$

(vii) Probability that an arrival enters the service without waiting is $1 - P(\text{an arrival has to wait})$

$$= 1 - \frac{\left(\frac{\lambda}{\mu}\right)^{s} P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$$

(viii) Mean waiting time in the queue for those who actually wait is

$$E(W_q | W_s > 0) = \frac{E(W_q)}{P(W_s > 0)} = \frac{1}{\mu s\left(1-\frac{\lambda}{\mu s}\right)} = \frac{1}{\mu s - \lambda}$$

(ix) Probability that there will be someone waiting is

$$P(N \geq s+1) = \sum_{n=s+1}^{\infty} P_n = \frac{\left(\frac{\lambda}{\mu}\right)^{s} P_0}{s!} \cdot \frac{\left(\frac{\lambda}{\mu s}\right)}{\left(1-\frac{\lambda}{\mu s}\right)}$$

(x) Average number of customers (in non-empty queues), who have to actually wait is.

$$L_w = E[N_q | N_q \geq 1] = \frac{\left(\frac{\lambda}{\mu s}\right)}{\left(1-\frac{\lambda}{\mu s}\right)}$$

Characteristics of Finite capacity, single server Poisson queue Model III [(M|M|1):(k|FIFO) model].

(i) Values of P_0 and P_n:

For the Poisson queue system $P_n = P(N=n)$ in the steady-state is given by the difference equations,

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \;;\; n>0$$

$$-\lambda_0 P_0 + \mu_1 P_1 = 0 \;;\; n=0.$$

This model represents the ~~system~~ situation in which the system can accommodate only a finite number k of arrivals. If a customer arrives and the queue is full, the customer leaves without joining the queue.

$$\therefore \quad \mu_n = \mu, \quad n=1,2,3\ldots$$

$$\lambda_n = \begin{cases} \lambda, & n=0,1,2,\ldots(k-1) \\ 0, & n=k,k+1,\ldots \end{cases}$$

$$P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{k+1}, & \text{if } \lambda = \mu \end{cases}$$

$$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n \left[\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right], & \text{if } \lambda \neq \mu \\[2ex] \dfrac{1}{k+1}, & \text{if } \lambda = \mu \end{cases}$$

(ii) Average number of customers in the system i.e.

$$E(N) = \sum_{n=0}^{k} n P_n = \begin{cases} \dfrac{\lambda}{\mu-\lambda} - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \text{if } \lambda \neq \mu \\[2ex] \dfrac{k}{2}, & \text{if } \lambda = \mu \end{cases}$$

(iii) Average number of customers in the queue.

$$E[N_q] = E[N-1] = \sum_{n=1}^{k}(n-1)P_n = E(N) - \frac{\lambda}{\mu}$$

For this model, the overall effective arrival rate, denoted by λ' is given by $\lambda' = \mu(1-P_0)$.

$$\therefore E[N_q] = E[N] - \frac{\lambda'}{\mu}$$

(iv) Average waiting time in the system and in the queue is:

$$E[W_s] = \frac{1}{\lambda'}E[N]$$

and

$$E[W_q] = \frac{1}{\lambda'}E[N_q]$$

Characteristics of finite queue, multiple server Poisson queue model IV [(M/M/s):(k|FIFO) model]

(i) Values of P_0 and P_n

For the Poisson queue system,

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 \,, \quad n \geq 1$$

where

$$P_0 = \left[1 + \sum_{n=1}^{k} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) \right]^{-1}$$

For this model,

$$\lambda_n = \begin{cases} \lambda \,, & \text{for } n = 0, 1, 2, \ldots, k-1 \\ 0 \,, & \text{for } n = k, k+1, \ldots \end{cases}$$

$$\mu_n = \begin{cases} n\mu \,, & \text{for } n = 0, 1, 2, \ldots s-1 \\ s\mu \,, & \text{for } n = s, s+1, \ldots \end{cases} \qquad 1 < s < k.$$

$$P_0 = \sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s}$$

$$P_n = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 \,, & \text{for } n \leq s \\ \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n \cdot P_0 \,, & \text{for } s < n \leq k \\ 0 \,, & \text{for } n > k \end{cases}$$

(ii) Average queue length or average number of customers in the queue is

$$E(N_q) = E(N-s) = \sum_{n=s}^{k} (n-s) P_n$$

$$= P_0 \cdot \left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2} \left[1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s} \right],$$

where $\rho = \frac{\lambda}{\mu s}$

(iii) Average number of customers in the system is

$$E[N] = \sum_{n=0}^{k} n P_n$$

$$= E(N_q) + \frac{\lambda'}{\mu},$$

where $\lambda' = \mu\left[s - \sum_{n=0}^{s-1} (s-n)P_n\right]$ is the effective arrival rate.

(iv) Average waiting time in the system and in the queue is

$$E[W_s] = \frac{1}{\lambda'} E[N]$$

and $$E[W_q] = \frac{1}{\lambda'} E[N_q]$$

example

Arrivals at a telephone booth are considered to be Poisson with an average time of 12 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 minutes.

(a) Find the average number of persons waiting in the system.

(b) What is the probability that a person arriving at the booth will have to wait in the queue?

(c) What is the probability that it will take him more than 10 minutes altogether to wait for the phone and complete his call?

(d) Estimate the fraction of the day when the phone will be in use.

(e) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for at least 3 minutes for phone. By how much the flow of arrivals should increase in order to justify a second booth?

(f) What is the average length of the queue that forms from time to time?

Soln $\lambda = \frac{1}{12}$, $\mu = \frac{1}{4}$

(a) $E[N] = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{\frac{1}{4}-\frac{1}{12}} = \frac{1}{2} = 0.5$

(b) $P(N>0) = \frac{\lambda}{\mu} = \frac{1}{3}$

(b) $P(W>0) = 1 - P(W=0) = 1 - P_0 = 1-\left(1-\frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \frac{1}{3}$

(c) $P(W>10) = e^{-(\mu-\lambda)\times 10} = e^{-\left(\frac{1}{12}-\frac{1}{4}\right)10} = e^{-\frac{5}{3}} = 0.1889$

(d) $\cancel{P}\ 1-P(\text{phone is idle}) = 1-P_0 = 1-\left(1-\frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{1}{3}$

(e) second phone is installed if $E[W_q] > 3$

i.e $\frac{\lambda}{\mu(\mu-\lambda)} > 3$ i.e, $\frac{\lambda_R}{\frac{1}{4}\left(\frac{1}{4}-\lambda_R\right)} > 3 \Rightarrow \lambda_R > \frac{3}{4}\left(\frac{1}{4}-\lambda_R\right) \Rightarrow \lambda_R > \frac{3}{28}$

arrival rate should increase by $\frac{3}{28}-\frac{1}{12} = \frac{1}{42}$

(f) $E(N_q|N_q>0) = E[N_q|N>1] = \cancel{\frac{E(N_q)}{P(N>1)}} = \cancel{\frac{E[N_q]}{1-P[N\le 1]}} = \cancel{\frac{E[N_q]}{1-P_0-P_1}} = \cancel{\frac{\lambda^2}{\mu(\mu-\lambda)}} \times \cancel{\frac{1}{1-\left(1+\frac{\lambda}{\mu}\right)}}$

$= \frac{\mu}{\mu-\lambda} = \frac{1/4}{1/4-1/12} = \frac{3}{2}$

example:

Customers arrive at a one-man barber shop according to a Poisson process with a mean interarrival time of 12 minutes. Customers spend an average of 10 minutes in the barber chair.

(a) What is the expected number of customers in the barber shop and in the queue?

(b) Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.

(c) How much time can a customer expect to spend in the barber's shop?

(d) Management will provide another chair and hire another barber, when a customer's waiting time in the shop exceeds 1.25hrs. How much must the average rate of arrivals increase to warrant a second barber?

(e). What is the average time customers spend in the queue?

(f) what is the probability that the waiting time in the system is greater than 30 minutes?

(g) Calculate the percentage of customers who have to wait prior to getting into the barber's chair.

(h) what is the probability that more than 3 customers are in the system?

Sol^n $\lambda = \frac{1}{12}$, $\mu = \frac{1}{10}$

(a) $E[N_s] = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{\frac{1}{10}-\frac{1}{12}} = 5$, $E[N_q] = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(\frac{1}{12})^2}{\frac{1}{10}(\frac{1}{10}-\frac{1}{12})} = 4.17.$

(b) $P(\text{walk straight into chair}) = P(\text{no customers in the system}) = P_0 = 1-\frac{\lambda}{\mu} = 1-\frac{1/12}{1/10} = \frac{1}{6}$
$= 16.67\%$

(c) $P[W_s] = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{10}-\frac{1}{12}} = 60$ min.

(d) $E[W_s] > 1.25\text{hrs}$ $\quad \frac{1}{\mu-\lambda_2} > 75 \Rightarrow \lambda_2 > \mu - \frac{1}{75} = \frac{1}{10}-\frac{1}{75} = \frac{13}{150}$

average arrival rate must increase by $\frac{13}{150}-\frac{1}{12} = \frac{1}{300}$.

(e) $E[W_q] = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/12}{\frac{1}{10}(\frac{1}{10}-\frac{1}{12})} = 50$ min

(f) $P[W_s > 30] = e^{-(\mu-\lambda)30} = e^{-(\frac{1}{10}-\frac{1}{12})30} = 0.6065$

(g) $P[W>0] = 1-P[w=0] = 1-P_0 = 1-(1-\frac{\lambda}{\mu}) = \frac{\lambda}{\mu} = \frac{1/12}{1/10} = 0.8333 \approx 83.33\%$

(h) $P[N_s>3] = (\frac{\lambda}{\mu})^{3+1} = (\frac{\lambda}{\mu})^4 = (\frac{1/12}{1/10})^4 = 0.4823$

* If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and if it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket, ⑧

(a) can he expected to be seated from the start of the picture)

(b) what is the probability that he will be seated for the start of the picture?

(c) How early must he arrive in order to be 99% sure of being seated for the start of the picture?

Sol^n. $\lambda = 6 \quad \mu = \frac{60}{7.5} = 8$

(a) $E[W_s] = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = 0.5$

Since the person arrive 2 minutes before the picture starts and waiting time + time from purchase to reach the seat

$= 0.5 + 1.5 = 2$. He can just be seated.

(b) $P[\text{~~total~~ time} \leq 2\text{min}] = 1 - P[\text{~~total~~ time} > 2]$

$= 1 - P[W_s > \frac{1}{2}]$

$= 1 - e^{-(\mu-\lambda)\times\frac{1}{2}}$

$= 1 - e^{-(8-6)\times\frac{1}{2}}$

$= 0.63$

(c) $P[W_s < t] = 0.99$

$\Rightarrow P[W_s > t] = 0.01$

$e^{-(\mu-\lambda)t} = 0.01$.

$e^{-2t} = 0.01$

$\Rightarrow -2t = \ln(0.01)$

$-2t = -4.6$

$\Rightarrow t = 2.3$ min

He should arrive $2.3 + 1.5 = 3.8$ min early to be 99% sure of being seated from the start of the picture.

A duplicating machine maintained for office use is operated by an office assistant who earns Rs 5 per hour. The time to complete each job varies according to an exponential distribution with mean 6 minutes. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an 8-hour day is used as a base, determine
(a) the percentage idle time of the machine
(b) the average time a job is in the system
(c) the average earning per day of the assistant.

Soln $\lambda = 5$ /hour $\mu = \frac{60}{6} = 10$ /hour

(a) $P(\text{machine is idle}) = P(N=0) = P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{10} = \frac{1}{2} = 50\%$

(b) $E[W_s] = \frac{1}{\mu - \lambda} = \frac{1}{10-5} = 0.2$ hrs

(c) Average no of jobs done per hour $= E[N_s] = \frac{\lambda}{\mu - \lambda} = \frac{5}{10-5} = 1$

Average earning per day $= E[N_s] \times 8 \text{ hours} \times 5 \text{ Rs/hour}$
$= 1 \times 8 \times 5$
$=$ Rs 40.

At what average rate must a clerk in a supermarket work in order to ensure a probability of 0.90 that the customer will not wait longer than 12 minutes? It is assumed that there is only one counter at which customers arrive in a Poisson fashion at an average rate of 15 per hour and that the length of the service by the clerk has an exponential distribution.

Soln $\lambda = 15$/hour $\mu = \mu_r$ /hour.

$P(W_q \le \frac{12}{60} \text{min}) = P(W_q \le \frac{1}{5}) = 0.90. \Rightarrow P(W_q > \frac{1}{5}) = 0.1$
$P(W_q > 0.2)$

i.e $\int_{0.2}^{\infty} g(w)\,dw = 0.1 \Rightarrow \int_{0.2}^{\infty} \frac{\lambda}{\mu_r}(\mu_r - \lambda) e^{-(\mu_r - \lambda)w}\,dw = \left[\frac{\lambda(\mu_r-\lambda)}{\mu_r} \frac{e^{-(\mu_r-\lambda)w}}{-(\mu_r-\lambda)}\right]_{0.2}^{\infty} = 0.1.$

$\Rightarrow \left[-\frac{\lambda}{\mu_r} e^{-(\mu_r - \lambda)w}\right]_{0.2}^{\infty} = 0.1 \Rightarrow -\frac{\lambda}{\mu_r} \times 0 + \frac{\lambda}{\mu_r} \times e^{-(\mu_r - \lambda) \times 0.2} = 0.1$

$\Rightarrow \frac{15}{\mu_r} e^{-(\mu_r - 15)0.2} = 0.1 \Rightarrow e^{-(\mu_r - 15)0.2} = 0.0067\mu_r \Rightarrow -(\mu_r - 15)0.2 = \ln 0.0067 + \ln \mu_r$

$\Rightarrow 0.2\mu_r + \ln \mu_r = 3 - \ln 0.0067 \Rightarrow 0.2\mu_r + \ln \mu_r = 8$
$\Rightarrow \mu_r = 24$ per hour.

* In a single server queueing system with Poisson input and exponential service times, if the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 h and the maximum possible number of calling units in the system is 2, find $P_n (n \geq 0)$, average number of calling units in the system and in the queue and average waiting time in the system and in the queue. (9)

Sol^n. $\lambda = 3/hr$ $\quad \mu = \frac{1}{0.25} = 4/hr$ $\quad k = 2$.

As $\lambda \neq \mu$, $\quad P_0 = \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1 - \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^3} = 0.4324$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}\right] = \left(\frac{3}{4}\right)^n \left[\frac{1 - \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^3}\right] = 0.4324(0.75)^n$$

$$E[N] = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{3}{4-3} - \frac{3\left(\frac{3}{4}\right)^3}{1 - \left(\frac{3}{4}\right)^3} = 0.8108$$

$$E[N_q] = E[N] - \frac{\lambda'}{\mu} = 0.8108 - \frac{\mu(1 - P_0)}{\mu} = 0.8108 - 1 + 0.4324 = 0.2432$$

$$E[W_s] = \frac{1}{\lambda'} E[N] = \frac{1}{\mu(1-P_0)} E[N] = \frac{1}{4(1-0.4324)} \times 0.8108 = 0.3571$$

$$E[W_q] = \frac{1}{\lambda'} E[N_q] = \frac{1}{\mu(1-P_0)} E[N_q] = \frac{1}{4(1-0.4324)} \times 0.2432 = 0.1071$$

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* The local one-person barber shop can accomodate a maximum of 5 people at a time (4 waiting and 1 getting hair-cut). Customers arrive according to a Poisson distribution and mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (exponential service time).

(a) what percentage of time is the barber idle?

(b) what fraction of the potential customers are turned away?

(c) what is the expected number of customers ~~are~~ waiting for a hair-cut?

(d) How much time can a customer expect to spend in the barber shop?

Sol[n] $\lambda = 5/\text{hr}$ $\mu = 4/\text{hr}$, $k=5$. $\lambda \neq \mu$.

(a) $P(\text{barber is idle}) = P(N=0) = P_0 = \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^{6}}$

$= 0.0888.$

$\approx 9\%.$

(b) $p(\text{customer turned away}) = P(N>5) \neq$

$= \left(\frac{\lambda}{\mu}\right)^{5}\left[\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right] = \left(\frac{5}{4}\right)^{5}\left[\dfrac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^{6}}\right]$

$= 0.2710$

(c) $E[N_q] = E[N] - (1-P_0)$

$= \dfrac{\lambda}{\mu-\lambda} - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} - 1 + P_0$

$= \dfrac{5}{4-5} - \dfrac{6\left(\frac{5}{4}\right)^{6}}{1-\left(\frac{5}{4}\right)^{6}} - 1 + 0.0888$

$3.1317 - 1 + 0.0888$

$= 2.2205$

(d) $E[W_s] = \dfrac{1}{\lambda'} E[N] = \dfrac{1}{\mu(1-P_0)} \times 3.1317 = \dfrac{1}{4(1-0.0888)} \times 3.1317$

$= 0.8592.$

* There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour, (10)

(a) what fraction of the time all the typists will be busy?

(b) what is the average number of letters waiting to be typed?

(c) what is the average time a letter has to spend for waiting and for being typed?

(d) what is the probability that a letter will take longer than 20 min waiting to be typed and being typed?

Sol^n $\lambda = 15/\text{hour}$, $\mu = 6/\text{hour}$, $s = 3$.

(a) all typists busy

$$P(N \geq 3) = \frac{\left(\frac{\lambda}{\mu}\right)^3 P_0}{3!\left(1-\frac{\lambda}{\mu \times 3}\right)} \qquad P_0 = \frac{1}{\left\{\sum_{n=0}^{2}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{3!\left(1-\frac{\lambda}{\mu\times 3}\right)}\left(\frac{\lambda}{\mu}\right)^3\right\}}$$

$$= \frac{\left(\frac{15}{6}\right)^3 \times P_0}{6\times\left(1-\frac{15}{6\times 3}\right)} \qquad = \frac{1}{\left\{1+\frac{15}{6}+\frac{1}{2}\times\left(\frac{15}{6}\right)^2\right\} + \left\{\frac{1}{6\left(1-\frac{15}{6\times3}\right)}\left(\frac{15}{6}\right)^3\right\}}$$

$$= \frac{1}{22.25}$$

$$= \cancel{2.6}\ 2.5^3 \times 0.0449 \qquad = 0.0449.$$

$$= 0.7016$$

(b) $$E[N_q] = \frac{1}{3\times 3!}\frac{\left(\frac{\lambda}{\mu}\right)^{3+1}}{\left(1-\frac{\lambda}{\mu\times 3}\right)^2}\times P_0 = \frac{1}{18}\times\frac{\left(\frac{15}{6}\right)^4}{\left(1-\frac{15}{6\times3}\right)^2}\times 0.0449$$

$$= 3.5078$$

(c) $$E[W_s] = \frac{1}{\lambda}E[N_s] = \frac{1}{\lambda}\left[E(N_q)+\frac{\lambda}{\mu}\right] = \frac{1}{15}\left[3.5078+\frac{15}{6}\right] =$$

$$= 0.4005$$

(d)

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* A supermarket has two girls attending to sales at the counters. If the service time for each customer is exponental with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour,

(a) what is the probability that a customer has to wait for service?

(b) what is the expected percentage of idle time for each girl?

(c) If the customer has to wait ~~in~~ the queue, what is the expected length of his waiting time?

$s=2,\ \lambda = \frac{10}{60} = \frac{1}{6},\ \mu = \frac{1}{4}$

Sol^n

(a) $P(w_s > 0) = P(N \geq 2) = \dfrac{\left(\frac{\lambda}{\mu}\right)^2 P_0}{2!\left(1-\frac{\lambda}{\mu \times 2}\right)} = \dfrac{\left(\frac{1/6}{1/4}\right)^2 \times \frac{1}{2}}{2 \times \left(1 - \frac{1/6}{\frac{1}{4}\times 2}\right)}$

$= \frac{1}{6}$

$$P_0 = \frac{1}{\left[\left\{\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{2!\left(1-\frac{\lambda}{\mu \times 2}\right)}\left(\frac{\lambda}{\mu}\right)^2\right\}\right]}$$

$$= \frac{1}{\left\{1 + \frac{1/6}{1/4}\right\} + \left\{\frac{1}{2\left(1 - \frac{1/6}{\frac{1}{4}\times 2}\right)}\left(\frac{1/6}{1/4}\right)^2\right\}} = \frac{1}{2}$$

(b) Fraction of time when the girls are busy = traffic intensity $= \frac{\lambda}{\mu s} = \frac{1/6}{\frac{1}{4}\times 2} = \frac{1}{3}$

Fraction of time the girls are idle $= 1 - \frac{1}{3} = \frac{2}{3}$

expected percentage $\simeq 67\%$

(c) $E[W_q \mid W_s > 0] = \dfrac{1}{\mu \times 2 - \lambda} = \dfrac{1}{\frac{1}{4}\times 2 - \frac{1}{6}} = 3$ min

* A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min and cars arrive for service in a Poisson process at the rate of 30 cars per hour.

(a) What is the probability that an arrival would have to wait in line?

(b) Find the average waiting time, average time spent in the system and the average number of cars in the system

(c) For what percentage of time would a pump be idle on an average?

Solⁿ.

Soln. $S=4,\ \lambda = 30/hr,\ \mu = \frac{60}{6} = 10/hr$

(a) $P(w>0) = \dfrac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$

$P_0 = \dfrac{1}{\left\{\sum_{n=0}^{3} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{4!\left(1-\frac{\lambda}{\mu\times 4}\right)}\left(\frac{\lambda}{\mu}\right)^4\right\}}$

$= \dfrac{\left(\frac{30}{10}\right)^4 \times 0.0377}{4!\left(1-\frac{30}{10\times 4}\right)}$

$= \dfrac{1}{\left\{1+\frac{30}{10}+\frac{1}{2}\left(\frac{30}{10}\right)^2+\frac{1}{6}\left(\frac{30}{10}\right)^3\right\} + \left\{\frac{1}{24\left(1-\frac{30}{10\times 4}\right)}\left(\frac{30}{10}\right)^4\right\}}$

$= 0.5090$

$= 0.0377$

(b) $E[W_q] = \dfrac{1}{\mu} \times \dfrac{1}{S\times S!} \dfrac{\left(\frac{\lambda}{\mu}\right)^S}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$

$= \dfrac{1}{10} \times \dfrac{1}{4\times 4!} \dfrac{\left(\frac{30}{10}\right)^4}{\left(1-\frac{30}{10\times 4}\right)^2} \times 0.0377.$

$= 0.0509$ hrs

$E[W_s] = \dfrac{1}{\mu} + \dfrac{1}{\mu} \times \dfrac{1}{S\times S!} \dfrac{\left(\frac{\lambda}{\mu}\right)^S}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$

$= \dfrac{1}{10} + 0.0509$

$= 0.1509$ hrs

$E[N_s] = \dfrac{1}{S.\,S!} \dfrac{\left(\frac{\lambda}{\mu}\right)^{S+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 + \dfrac{\lambda}{\mu}$

$= \dfrac{1}{4\times 4!} \dfrac{(3)^5}{\left(1-\frac{3}{4}\right)^2} \times 0.0377 + 3$

$= 4.527.$

(c) fraction of time the pump is busy

$=$ traffic intensity

$= \dfrac{\lambda}{\mu s}$

$= \dfrac{30}{10\times 4}$

$= 0.75$

fraction of time the pump is idle

$= 1 - 0.75$

$= 0.25$

$= 25\%.$