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Network Flows: The Max Flow/Min Cut Theorem

In this lecture, we prove optimality of the Ford-Fulkerson theorem, which is an immediate corollary of a well known theorem: The Max-Flow/Min-Cut theorem, which says:

The Max-Flow/Min-Cut Theorem:

Let (G, s, t, c) be a flow network and left f be a flow on the network. The following is equivalent:

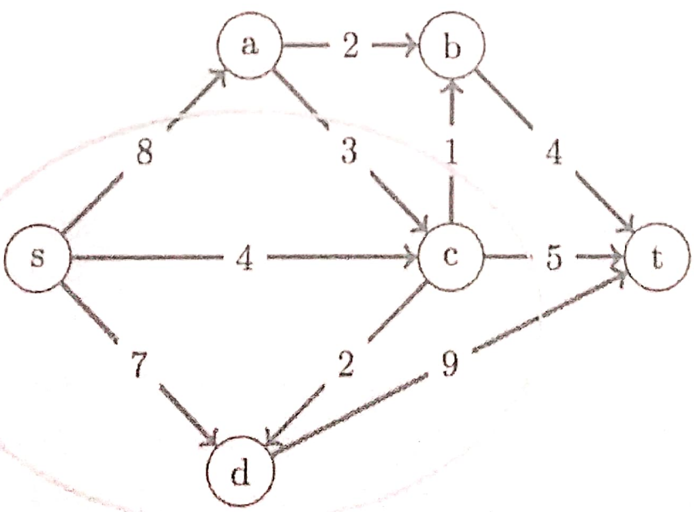
1. f is maximized.
2. G_f has no augmenting paths.
3. There exists a cut $C(S, T)$ such that $c(S, T) = val(f)$.

What is a cut?

Definition : Let (G, s, t, c) be a flow network, an **s-t cut** in G is a partition of V into two sets S and T such that:

1. $S \cup T = V$
2. $S \cap T = \emptyset$
3. $s \in S$ and $t \in T$

Below is an example of a cut $C(S, T)$ where $S = \{s, c, d\}$ and $T = \{a, b, t\}$.



Cap = {<s,a> + <c,t> + <d,t> <c,b>}

Cap = {8 + 5 + 9 + 1

= 23.

The **capacity** of a cut $C(S, T)$, denoted $c(S, T)$, is the sum of the capacities of the edges (u, v)[1] with $u \in S$

[1]Recall that edges are directed, and thus (u, v) means the edge from u to v

and $v \in T$. That is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v) \tag{1}$$

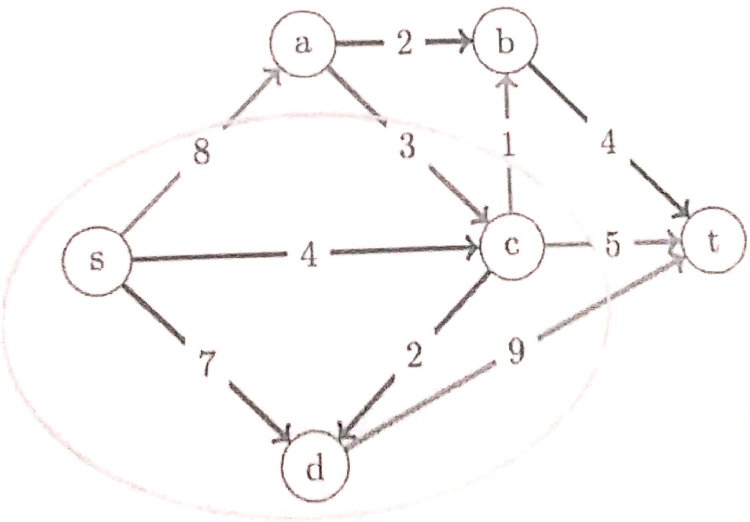
In the example above, $c(S,T) = 23$, we don't count the edge (a,c) since $a \in T, c \in S$.

This definition of capacity of a cut is very natural, and it suggests we can define the flow of a cut in a similar manner. That is, given a cut $C(S,T)$ with capacity $c(S,T)$, and a flow f, how much of f crosses from S to T? Intuitively this would be the flow going from S to T minus whatever flow was sent back. And this is exactly it.

The flow of a cut $C(S,T)$, denoted $f(C,T)$ is defined as:

$$f(S,T) = \left(\sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) \right) \tag{2}$$

Again, if we consider the previous example, redrawn below: $f(S,T)$ would equal the sum of the flow sent across the blue edges minus whatever flow sent down the red edges.



By the capacity constraint, we know that $f(u,v) \leq c(u,v), \forall (u,v) \in E$. Using this fact and (1) and (2) above, we get the following lemma:

Lemma 1. *Let (G, s, t, c) be a flow network, $C(S,T)$ an $s-t$ cut and f a flow, then:*

$$f(S,T) \leq c(S,T)$$

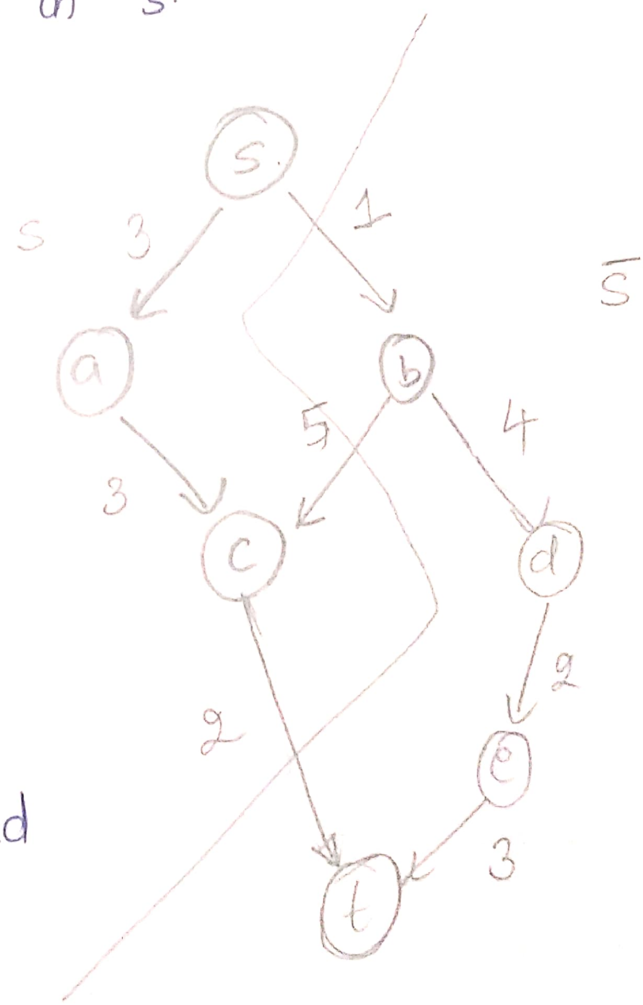
Proof. For every (u,v) in the network, we know that

$$f(u,v) \leq c(u,v)$$

MAX FLOW and Minimum Cut Theorem

A Cut $(S, \bar{S})$ partitions the vertex set V into two subsets $S \subset V$ and $\bar{S} = V - S$.

and it consists of edges with one endpoint in S and the other in $\bar{S}$.



$S = \{s, a, c\}$

$\bar{S} = \{b, d, e, t\}$

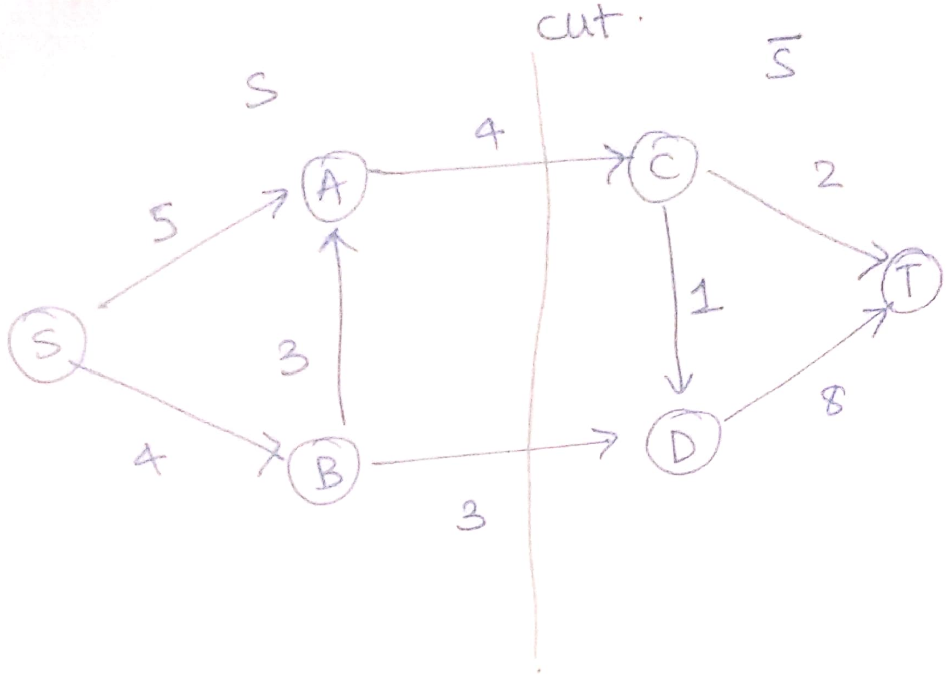
The cut $(S, \bar{S})$ is an S-T cut if $s \in S$ and $t \in \bar{S}$.

Capacity of the S-T cut is

$$c(S, \bar{S}) = \sum_{u \in S \,\&\, v \in \bar{S}} c(u, v)$$

Capacity of the cut $c(S, \bar{S}) = c(s, b) + c(c, t)$

$= 1 + 2 = \underline{\underline{3}}$.

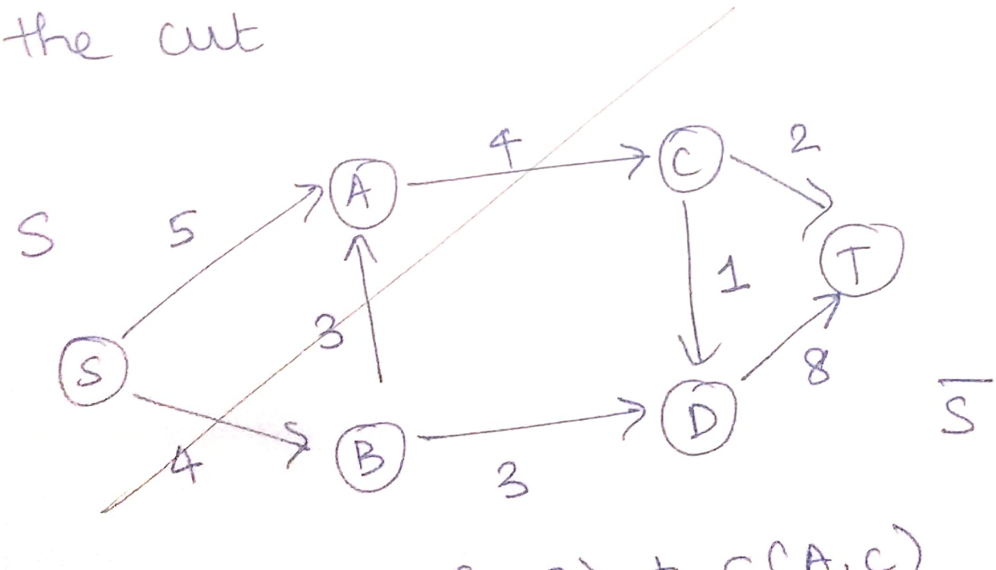


$S = \{S, A, B\}$

$\bar{S} = \{C, D, T\}$

$c(S, \bar{S}) = c(A, C) + c(B, D)$

Capacity of the cut $= 4 + 3 = 7$.



$c(S, \bar{S}) = c(S, B) + c(A, C)$

$4 + 4 = 8$.

Edge/capacity of (B, A) not included $\because$

Edge is from $\bar{S}$ to S and not from S to $\bar{S}$.

A min-cut is an s-t cut having minimum capacity.

MAX FLOW ≤ Capacity of the min-cut of the Network.

Also, MAX FLOW ≤ Capacity of any Cut.

If max. flow ≤ Capacity of any cut, then

maximum flow ≤ capacity of the minimum cut.

There exists a cut C, such that:

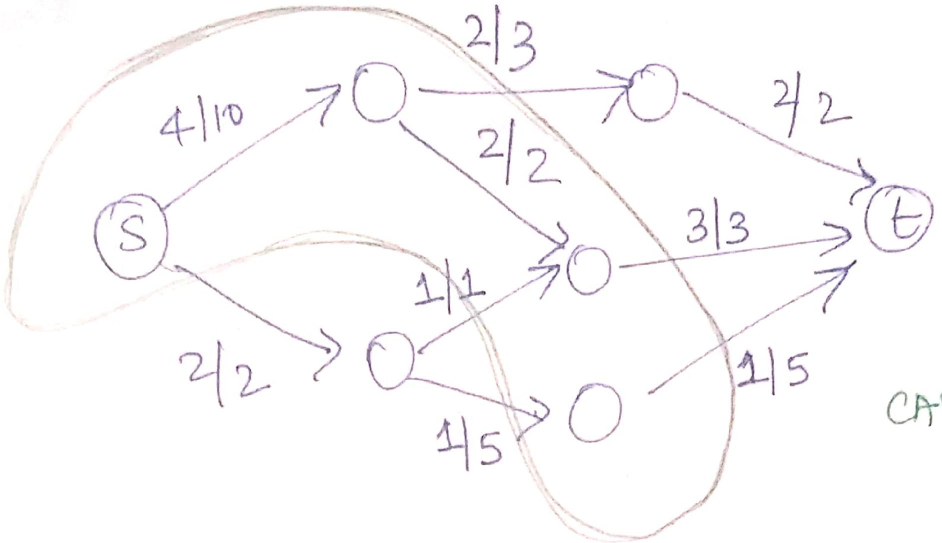
Capacity of cut, C = max. flow.

If f is a flow in G, then the net flow across the cut $(S, \bar{S})$ is defined to be

$$f(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$

$$f(S,T) \text{ across } cut(S,T) = \sum_{u \text{ in } S} \sum_{v \text{ in } T} f(u,v) - \sum_{u \text{ in } S} \sum_{v \text{ in } T} f(v,u)$$

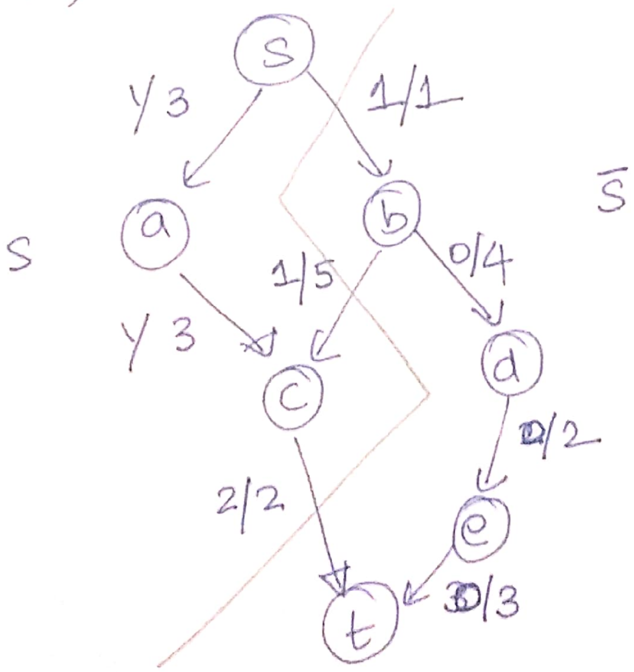
$$= (2+2+3+1) - (1+1) = \underline{6}.$$



CAPACITY C(S,T) of CUT(S,T) = MAX FLOW ACROSS THE CUT

$$\sum_{u \in S} \sum_{v \in T} c(u,v) = 2+3+3+5 = \underline{\underline{13}}.$$

f(s,t) across cut(s,t) $(2+2+3+1) - (1+1) = \underline{\underline{6}}.$



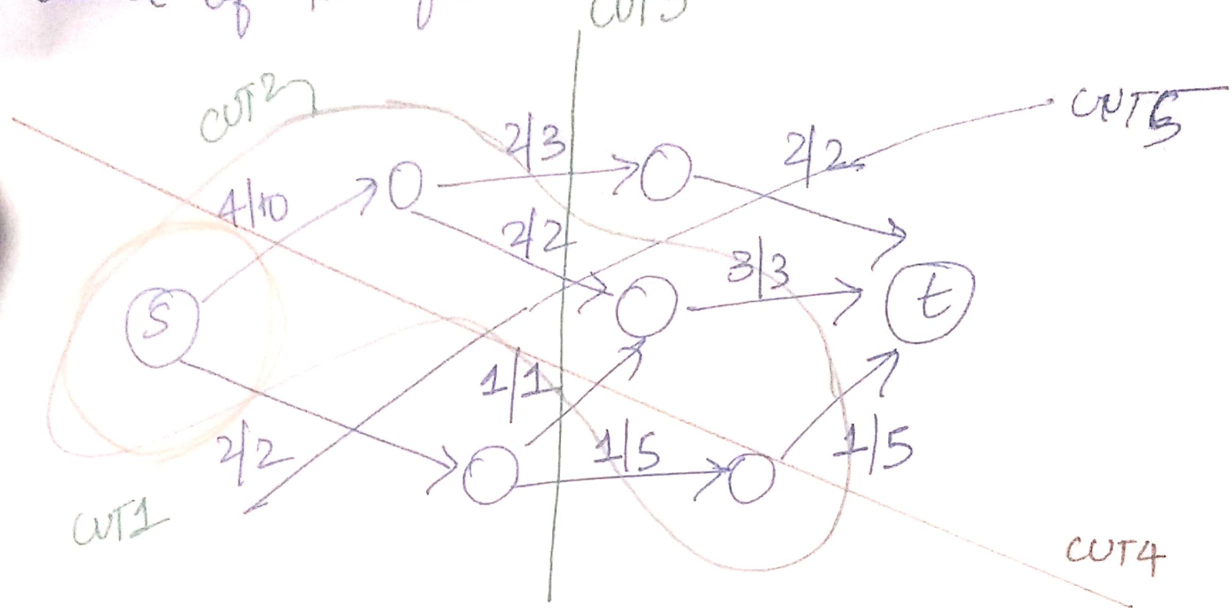
$$\text{flow}(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } \bar{S}} f(e)$$

$$= (1+2) - 1 = \underline{\underline{2}}.$$

$$\text{capacity}(S, \bar{S}) = 1+2 = \underline{\underline{3}}.$$

LEMMA

- Flow across any cut is the same, and equals the value of the flow



flow across cut1 = 6 = 4+2 = 6 ‖ capacity c1 = 10+2 = 12

— " — CUT2 = (2+2+3+1) − (1+1) = 6.

cap c2 = 3+3+ … = 11

flow across CUT3 = (2+2+1+1) = 6.

Cap C3 = 3+2+ … = 11

flow across CUT4 = (4+1+1) = 6.

Cap C4 = 10+1+ … = 16

flow across CUT5 = (2+2+2) = 6

Cap C5 = 2+2+ … = 6

NOTE: CAPACITY OF CUTS IS NOT NECESSARILY THE SAME

minimum cut = cut whose capacity is minimum

MAX FLOW cannot be more than capacity of minimum cut.

MAX FLOW = CAPACITY OF MINIMUM CUT

Corollary: Maximum flow cannot be more than capacity of minimum cut.

Max-flow min-cut Theorem: Let f be a flow in a flow network G. Then the following conditions are equivalent:

(i) f is a maximum flow in G

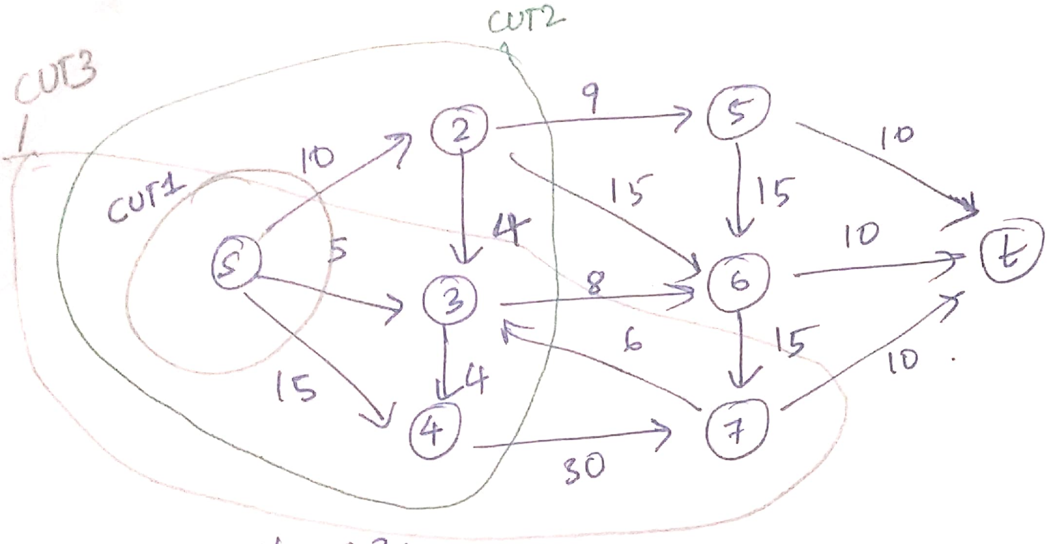
(ii) residual network G_f contains no augmenting path

(iii) there is a cut (S,T) with $|f| = c(S,T)$

Combine (iii) with corollary: maximum flow = capacity of minimum cut

3.

MAX-FLOW; MIN-CUT



Capacity of CUT1 : 30

Capacity of CUT2 : 62 = 9+15+8+30=62

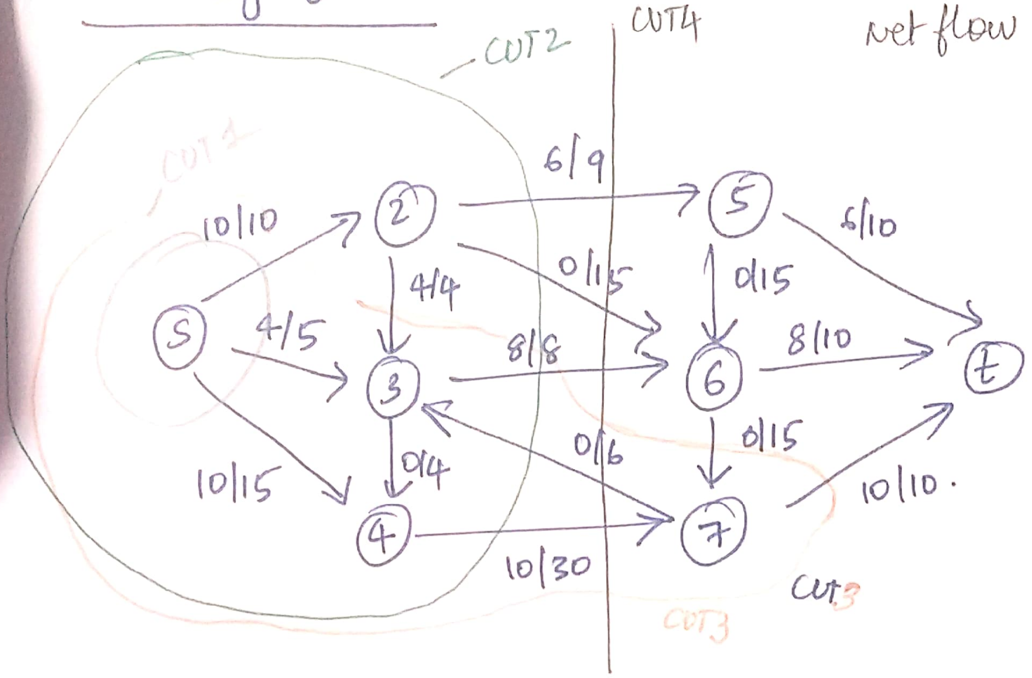
(Do not consider reverse edge from 7 to 3 node

17
15
32
30
62

Capacity of CUT3 : 10 + 8 + 10 = 28.

OBSERVATION 1

Let f be a flow and let (S,T) be any S-T cut. Then, the net flow across the cut is equal to the amount reaching t.



net flow = 6+0+8+10
=24

Capacity =
$C_1 = 10+5+15 = 30$
$C_2 = 9+15+8+30 = 62$
$C_3 = 10+8+10 = 28$

Net flow across CUT 1 = amt reaching t.

10 + 4 + 10 = 6+8+10

24 = 24

Net flow across CUT2 = amt reaching t.

6+8+10 = 6+8+10

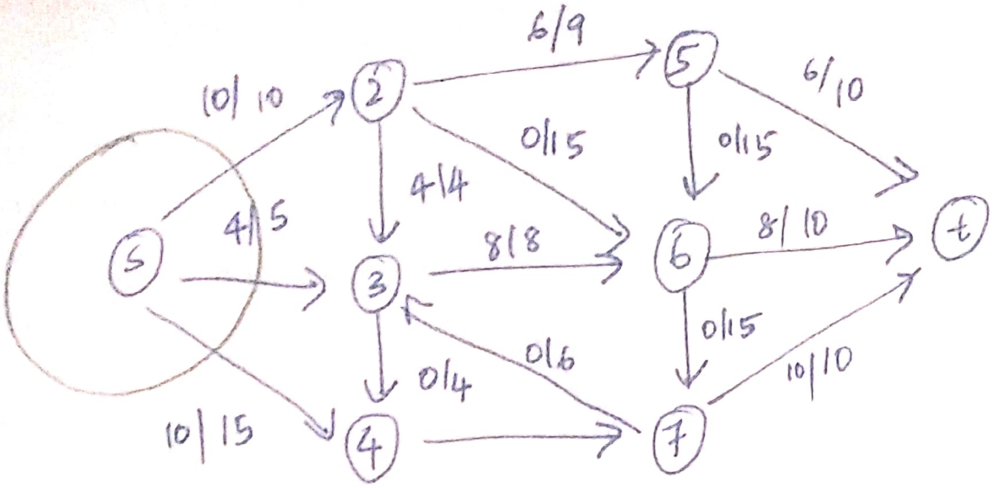
24 = 24

Net flow across CUT3 = amt reaching t

10+8+10−4 = 6+8+10

24 = 24

vation2: Let f be a flow, and let (S,T) be any (S,T) cut. Then, the value of the flow is atmost the capacity of the cut.



cut capacity = 30 => flow value ≤ 30.

24 ≤ 30

Observation 3: MAX-FLOW & MIN CUT

Let f be a flow, and let (S,T) be any S-T cut whose capacity equals the value of f. Then f is a max flow and (S,T) is a min-cut.

cut capacity = 28 ; max flow = 28.