Maximum Flow

Eg: The source produces the material at some steady rate and the sink consumes the material at the same rate.

- Flow n/w can model many problems, including liquid flowing thr' pipes, parts thr' assembly lines; current thr' electrical n/w & info thr' communication n/ws.

In maximum-flow problem, we wish to compute the greatest rate at which we can ship material from the source to the sink without violating any capacity constraints.

Flow N/ws: A flow n/w $G=(V,E)$ is a directed graph in which each edge $(u,v) \in E$ has a nonnegative capacity $c(u,v) \geq 0$.

If E contains an edge (u,v), then there is no edge (v,u) in the reverse direction.

- If $(u,v) \notin E$ then for convenience we define $c(u,v) = 0$ & disallow self-loops.

- Two vertices in a flow n/w : source, s and a sink, t

- ~~For~~ Each vertex $v \in V$, ~~the flow n/w~~ ~~lies on~~ lies on some path from the source to the sink.

i.e For each $v \in V$, the flow n/w contains a path from source to the sink.

$s \rightsquigarrow v \rightsquigarrow t$ $\quad$ $s \rightsquigarrow v \rightsquigarrow t$

- Let $G = (V, E)$ be a flow n/w with a capacity fn, 'c'

- Let 's' be the source of the n/w and let 't' be the sink

- A flow in G is a real-valued fn $f: V \times V \rightarrow R$ that satisfies the foll two properties.

Capacity constraint: For all $u, v \in V$, we require

$$0 \leq f(u,v) \leq c(u,v)$$

Flow conservation: For all $u \in V - \{s,t\}$, we require

$$\sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$$

i.e total flow out of v is equal to total flow into v.

When $(u,v) \notin E$, there can be no flow from u to v and $f(u,v) = 0$.

- $f(u,v)$ is non-negative and is the flow from vertex u to v.
- the value of flow is $|f|$.

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s).$$

i.e the total flow out of the source minus the flow into the source

$| \, |$ ⇒ denotes flow value & not absolute value or cardinality

(2) – Typically, a flow n/w will not have any edges into the source, and the flow into the source.

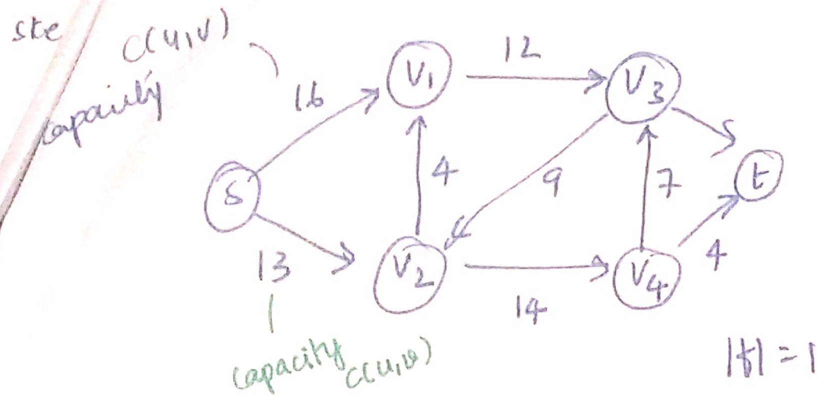
– $\sum_{v \in V} f(v, s) = 0.$

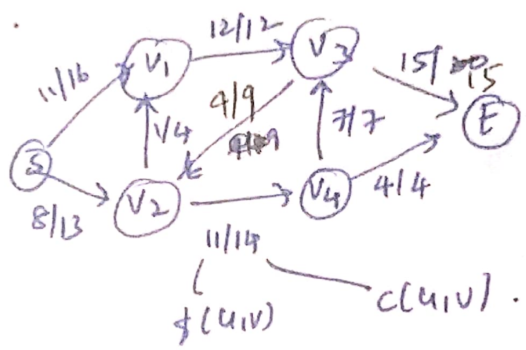
In max-flow prob, we are given a flow n/w G with source, s & sink, t. We wish to find a flow of maximum value.

Flow properties

- Flow from one vertex to another must be non-negative. & must not exceed the given capacity.
- Flow conservation property states that the total flow into a vertex other than the source or sink must be equal the total flow out of that vertex "flow in equals flow out"

(5)





$|f| = 19$

- Can ship items atmost $c(u,v)$ per day b/w u & v.
- Need to determine the largest no of 'p' of ~~crates~~ items per day that they can ship & then to produce this amt; since there is no point in producing more items than they can ship to their warehouse.

= Not interested in duration it takes for the items to reach;

= Interest is in 'p' items per day leave the factory & p crates per day arrive at the warehouse.

- $|f| = 19 \quad \because \quad \Sigma f(s,v)$ = total flow out of the source
$= 11 + 8 = 19$

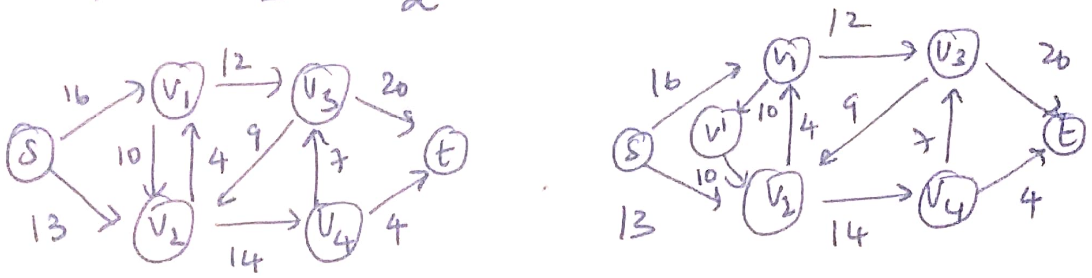
$\Sigma f(v,s)$ = flow into the source $= 0$.

$\Sigma f(v,s) = 0$

$|f| = 19 - 0 = 19.$

An eg of flow: Modeling problems with antiparallel edges

- Suppose there is an option to increase the no of items from v_1 to v_2



- It violates our assumption if edge $(v_1 v_2) \in E$ then $(v_2, v_1) \notin E$
- Two edges (v_1, v_2) and (v_2, v_1) antiparallel.
- ∴ Transform into an equivalent one containing no antiparallel edges. (v_1, v') and (v', v_2)
- ∴ satisfies the property that if an edge is in the n/w, the ~~reverse~~ edge is not. reverse

N/w with multiple sources & sink.

$S = \{s_1, s_2, \dots s_m\}$ $T = \{t_1, t_2, \dots t_n\}$

- Convert the n/w from (a) to ordinary flow n/w with only a single source & a single sink.
- We add supersource s and add a directed edge (s, s_i) with capacity $c(s, s_i) = \infty$ for each $i = 1, 2, \dots m$
- Create a supersink, t $c(t_i, t) = \infty$ $i = 1, 2, \dots n$

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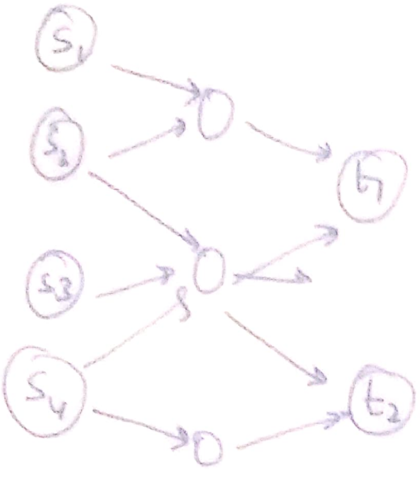
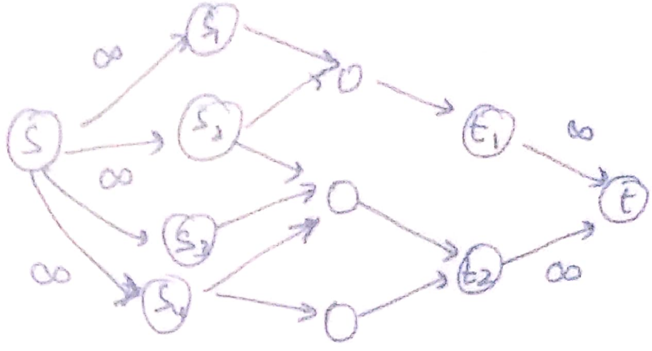


(a)



(b)

FORD-FULKERSON METHOD : (1) Iteratively increases the value of the flow.

(2) Start with $f(u,v) = 0$ for all $u,v \in V$.

(3) Increase the flow by finding an augmenting path in an associated "residual n/w", G_f.

(4) Once we know the edges of an augmenting path in G_f, we can identify specific edges in G for which we can change the flow so that we increase the value of flow.

(5) Repeatedly augment the flow until the Residual n/w has no more augmenting paths.

(6) Max-flow min cut theorem upon termination, ~~this~~ ~~process~~ yields a max. flow.

FORD-FULKERSON-METHOD (G,s,t)

(3) initialize flow f to 0

while there exists an augmenting path p in the residual n/w G_f

augment flow f along p

return f.

Residual N/ws:

given G, flow f, G_f - Residual n/w

G_f - consists of edges with capacities that repts how we can change the flow on edges of G.

$$c_f(u,v) = c(u,v) - f(u,v)$$

flow on the edge

residual capacity (+ve value)

edge's capacity

- the only edges of G that are in G_f are those that can admit more flow

- If flow equals capacity for edges (u,v) then $c_f(u,v) = 0$.

they are not in G_f.

- G_f may also contain edges that are not in G.

- In order to represent a +ve decrease of a +ve flow $f(u,v)$ on an edge in G, we place an edge (v,u) into G_f with residual capacity $c_f(v,u) = f(u,v)$.

- i.e an edge that can admit flow in opp direction to (u,v)

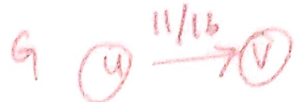
- we have a flow n/w $G = (V,E)$ with source s and sink, t.

- Let f be flow in G, and consider a pair of vertices $u,v \in V$. we define residual capacity $c_f(u,v)$ by

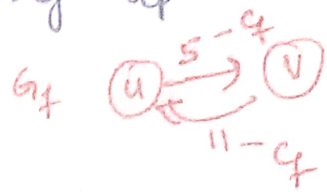
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

- ∵ of our assumption that $(u,v) \in E$ implies $(v,u) \notin E$; exactly one case above applies to each ordered pair of vertices

If $c(u,v) = 16$ & $f(u,v) = 11$



then we can increase $f(u,v)$ by upto to



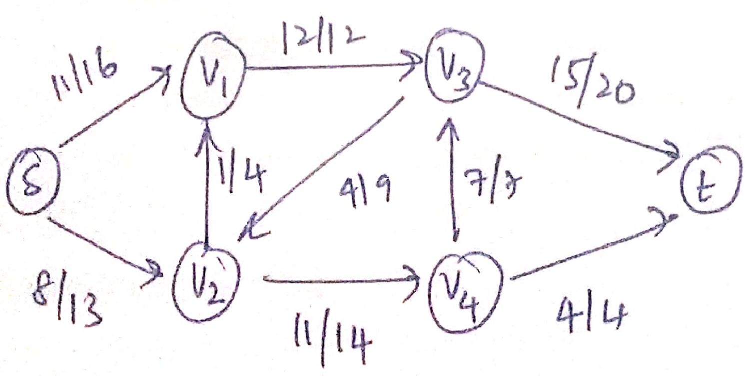
$c_f(u,v) = 5$ units

- we also wish to allow an algo to return upto 11 units of flow from v to u, & hence $c_f(v,u) = 11$.

- Given a flow n/w $G = (V, E)$ and a flow, f the residual n/w of G induced by f is $G_f = (V, E_f)$ where

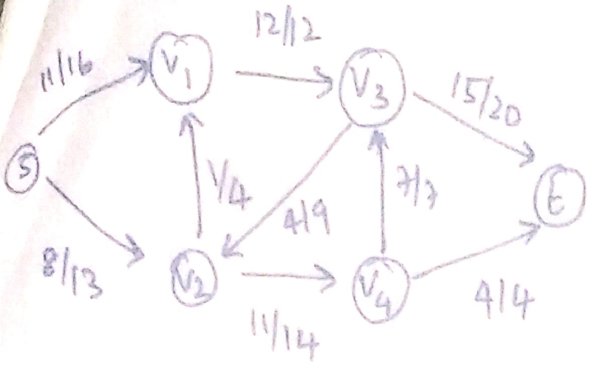
$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

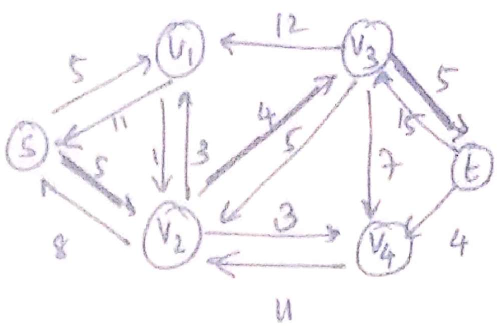
- Each edge on residual n/w can admit a flow that is greater than 0.



flow n/w. G ; flow f

Residual n/w G_f
augmenting path shaded.





$|E_f| \leq 2|E|$

- The residual n/w G_f is $\parallel^{lar}$ to flow n/w with cap, c_f.

- It does not satisfy our definition of a flow n/w $\because$ it may contain both an edge (u,v) and its reversal (v,u).

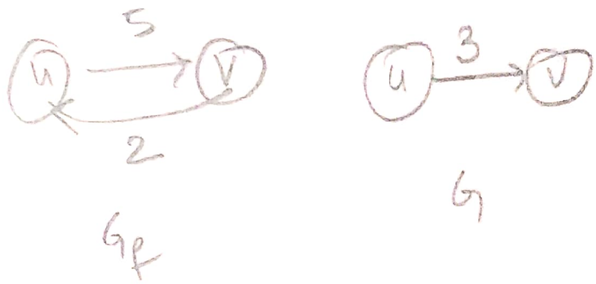
- If f is a flow in G and f' is a flow in corresponding residual n/w G_f, we define $f \uparrow f'$, the augmentation of flow f by f' to be a fn from $V \times V$ to R defined by,

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

- We increase the flow on (u,v) by $f'(u,v)$ but decrease it by $f'(v,u)$ $\because$ pushing flow on the reverse edge in the residual n/w signifies decreasing the flow in the original n/w.

- Pushing flow on the reverse edge in the residual n/w is also known as <u>cancellation</u>.

Eg: If we send 5 items from u to v and send 2 from v to u; it is as good as sending just 3 items from u to v & none from v to u.



<u>Augmenting paths:</u>

- given $G = (V,E)$
- flow f, an augmenting path p is a simple path from s to t in G_f.
- we can increase flow on edge (u,v) of an augmenting path upto $c_f(u,v)$ without violating capacity constraint on whichever of (u,v) & (v,u) is in original flow n/w G.
- Eg: we can increase by 4 units $c_f(v_2,v_3) = 4$.

$$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$$