RV COLLEGE OF ENGINEERING®

(An Autonomous Institution affiliated to VTU, Belagavi)

I Semester- Master of Technology

Common to MCE / MCN

LINEAR ALGEBRA, PROBABILTY AND QUEUEING THEORY

Time: 03 Hours *Maximum Marks: 100*

Instructions to candidates:

1. Answer FIVE full questions selecting one from each unit.
2. Each unit consisting of two questions of 20 marks each.

UNIT-1

1	a	If $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$ is defined by $T(x,y,z) = (x-y-z, x+y-z, x+y+z)$, show that T is a linear transformation.	04
	b	If $\mathbb{R}$ is the field of real numbers and V is the set of vectors in a plane, i.e., $V = \{(x,y) \mid x, y \in \mathbb{R}\}$, which is closed under vector adition and scalar multiplication, prove that V is a vector space over the field $\mathbb{R}$.	06
	c	Find the bases and dimension of the four fundamental sub spaces of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & -3 \\ -1 & 2 & -1 & -2 \\ 2 & -4 & 2 & 4 \end{bmatrix}$.	10
		OR	
2	a	Show that the set $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times2}$ the set of all 2×2 matrices.	04
	b	Show that the vectors $\{2t^2+t+2,\ t^2-2t,\ 5t^2-5t+2,\ -t^2-3t-2\}$ are linearly dependent in $\mathbb{P}_2$. Extract a linearly independent subset. Also find the basis and dimension of the subspace spanned by them.	06
	c	Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$, such that $T(1,2,1) = (-3,2,5,-1)$, $T(2,1,1) = (0,5,5,5)$, $T(1,1,2) = (-1,1,2,0)$. Also find the range space and null space of the Linear transformation.	10

UNIT-2

3	a	Suppose $\mathbb{P}_2$ is a vector space having the inner product defined by $< p,q > = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2)$, where $t_0 = -1$, $t_1 = 0$, $t_2 = 1$. Compute the lengths of the vectors $p(t) = 3t - t^2$ and $q(t) = 3 + 2t^2$.	04
	b	Find a least-squares solution of the inconsistent system $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \\ 3 & -1 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 4 \\ -3 \\ 1 \end{bmatrix}$.	06
	c	Obtain the QR factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.	10
		OR	
4	a	Let V b the space $C[0,1]$, with the inner product $< f,g > = \int_0^1 f(t)g(t)dt$, and let $f(t) = e^t$, $g(t) = 2t - 1$, then find $< f,g >$, $< f,f >$ and $< g,g >$.	04
	b	Using Gram-Schmidt process orthonormalize the basis of $\mathbb{R}^3$ $B = \{(1,2,-1), (1,2,1), (1,1,1)\}$	06

c A simple curve that often makes a good model for the variable costs a company, as a function of the sales level x, has the form $y = \beta_0 + \beta_1 x + \beta_2 x^2$. Find the least-squares curve of the form above to fit the data $\{(1,4),(2,4),(3,6),(4,7),(5,8)\}$. Also predict the y values for $x = 8$ and $x = 10$. **10**

UNIT-3

5 a Suppose the quadratic form is given by $Q(x) = 11x_1^2 + 9x_2^2 + 7x_3^2 + 8x_1x_2 - 8x_2x_3$. Find:

- i) The maximum value of $Q(x)$ subject to the constraint $x^T x = 1$
- ii) A unit vector u where the maximum in i) is attained
- iii) The maximum of $Q(x)$ subject to the constraints $x^T x = 1$ and $x^T u = 0$
- iv) A unit vector v where the maximum in iii) is attained
- v) The new quadratic form $Q(y)$ after the change of variable, $x = Py$ is applied. **10**

b Obtain a Singular Value Decomposition (SVD) of the matrix $A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$. **10**

OR

6 a Decompose the matrix $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ as $A = PDP^{-1}$. **10**

10 b Given the matrix of observations as: $\begin{bmatrix} 12 & 6 & 9 & 15 & 13 & 5 \\ 19 & 22 & 6 & 3 & 2 & 20 \end{bmatrix}$, convert the matrix to mean deviation form, construct the covariance matrix and hence find its principal components. Also determine what percentage of the information is retained from the first principal component. **10**

UNIT-4

7 a Consider an experiment that consists of 2 throws of a fair die. Let X be the number of 4s and Y be the number of 5s obtained in the two throws. Find:

- i) The joint distribution of X and Y
- ii) The marginal distributions of X and Y
- iii) The expected values of X, Y and XY
- iv) $Cov(X,Y)$. **10**

b Random variables X and Y have the joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{24}xy, & 1 < x < 3,\ 2 < y < 4 \\ 0, & otherwise \end{cases}$$

Find:

- i) The conditional probability density function $f_{Y|X}(y|x)$
- ii) The conditional probability density function $f_{X|Y}(x|y)$ **06**

c If X and Y are independent random variables, each having probability density function, $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$ and $U = X + Y$ and $V = X - Y$, find the joint probability density function of U and V, using transformation of random variables. **04**

OR

8 a If the joint probability density function for (x, y) is

$$f(x,y) = \begin{cases} c(4-x-y), & 0 \le x \le 2,\ 0 \le y \le 2,\ c > 0 \\ 0, & otherwise \end{cases}$$, determine

i) The value of c

ii) $P(x < 1, y > 1)$

iii) $P(1/2 < y < 3/2)$

iv) $P(y < x)$ — **10**

b Find the covariance matrix for the two random variables X_1 and X_2 whose joint probability is represented as follows:

X_1 \ X_2	0	1	2
0	0.12	0.12	0.08
1	0.13	0.21	0.23
2	0.07	0.02	0.02

06

c The random variables X_1 and X_2 denote the length and width, respectively of a manufactured part. X_1 and X_2 are independent normal variates with $\mu_1 = 2cms$, $\mu_2 = 5cms$, $\sigma_1 = 0.1cms$, $\sigma_2 = 0.2cms$. Find the probability that the perimeter $Y = 2X_1 + 2X_2$ exceeds $14.5cms$. — **04**

UNIT-5

9 a A T.V. repairman finds that the time spent on his job has an exponential distribution with a mean of 30 minutes. The repair sets in the order in which they came in and if the arrival of sets is approximately Poisson with an average of 48minutes,

i) What is the average number of T.V.s waiting in the system to be repaired?

ii) What is the average waiting time of the T.V. to be repaired in the queue? $\frac{\lambda}{\mu-\lambda}$

iii) What is the proabability that the number of T.V.s to be repaired in the system exceeds 5?

iv) What is the repairman's expected idle time? — **10**

b A super market has 2 girls attending to sales at the counters. If the service time for each customer is exponential with mean 5 minutes and if people arrive in Poission fashion at the rate of 10 per hour,

i) What is the probability that a customer has to wait for service?

ii) What is the expected percentage of idle time for each girl?

iii) If the customer has to wait in the queue, what is the expected length of his waiting time?

iv) What is the average number of customers in the system? — **10**

OR

10 a Customers arrive at a one window drive-in bank according to Poisson distribution with mean 10per hour. Service time per customer is exponential with a mean of 5 minutes. The space in front of the window, including that for the serviced car can accommodate a maximum of 3 cars. Other cars can wait outside this space.

i) What is the probability that an arriving customer can drive directly to the space in front of the window?

ii) What is the probability that an arriving customer will have to wait outside the indicated space?

iii) How long is an arriving customer expected to wait before being serviced? — **10**

Linear Algebra, Probability and Queueing Theory 22MATIIB
Scheme and Solution

1a $V = \{(x,y) \mid x,y \in \mathbb{R}\}$.
let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, $\gamma = (x_3, y_3) \in V$ and $c, c' \in \mathbb{R}$. 1

(i) $\alpha + \beta = (x_1+x_2, y_1+y_2) = \beta + \alpha$ 1

(ii) $(\alpha+\beta)+\gamma = ((x_1+x_2)+x_3, (y_1+y_2)+y_3) = \alpha + (\beta+\gamma)$ 1

(iii) $\alpha + 0 = (x_1, y_1) + (0,0) = 0 + \alpha$ ∴ $0 = (0,0)$ is the zero vector 1

(iv) $\alpha + \alpha^{-1} = (x_1, y_1) + (-x_1, -y_1) = 0 = \alpha^{-1} + \alpha$ ∴ $\alpha^{-1} = (-x_1, -y_1)$ is the inverse element 1

(v) $c.(\alpha+\beta) = (cx_1 + cx_2, cy_1 + cy_2) = c.\alpha + c.\beta$ 1

(vi) $(c+c').\alpha = ((c+c')x_1, (c+c')y_1) = c.\alpha + c'.\alpha$

(vii) $(c.c').\alpha = (c.c'x_1, c.c'y_1) = c.(c'.\alpha)$ 1

(viii) $1.\alpha = (1.x_1, 1.y_1) = \alpha$ where 1 is the unit element

∴ From (i) to (viii). V is a vector space

1b $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$, $T(x,y,z) = (x-y-z, x+y-z, x+y+z)$.
let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in V_3(\mathbb{R})$, $c \in \mathbb{R}$.

(i) $T(\alpha+\beta) = T(x_1+x_2, y_1+y_2, z_1+z_2) = (x_1+x_2-y_1-y_2-z_1-z_2,\ x_1+x_2+y_1+y_2-z_1-z_2,\ x_1+x_2+y_1+y_2+z_1+z_2)$ 2

(ii) $T(c\alpha) = T(cx_1, cy_1, cz_1) = cx_1 - cy_1 - cz_1,\ cx_1 + cy_1 - cz_1,\ cx_1 + cy_1 + cz_1$ $= T(\alpha)+T(\beta)$, $= c.T(\alpha)$ 2

∴ From (i) & (ii) T is a linear transformation.

1c. $$A = \begin{bmatrix} 1 & 2 & -2 & -3 \\ -1 & 2 & -1 & -2 \\ 2 & -4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & -3 \\ 0 & 4 & -3 & -5 \\ 0 & -8 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & -3 \\ 0 & 4 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ 2

Basis of row space $= \{(1,2,-2,-3), (-1,2,-1,-2)\}$ 1

Basis of column space $= \{(1,-1,2), (2,2,-4)\}$ 1

$Ax = 0 \to$ $\left.\begin{aligned} x_1 + 2x_2 - 2x_3 - 3x_4 &= 0 \\ 4x_2 - 3x_3 - 5x_4 &= 0 \end{aligned}\right\}$ let x_3 & x_4 be f.v $\boxed{x_2 = \frac{3}{4}x_3 + \frac{5}{4}x_4}$

$x_1 = -2\left(\frac{3}{4}x_3 + \frac{5}{4}x_4\right) + 2x_3 + 3x_4$ 2

$\boxed{x_1 = \frac{2}{4}x_3 + \frac{2}{4}x_4}$

Basis of nullspace $= \left\{\left(\frac{2}{4}, \frac{3}{4}, 1, 0\right), \left(\frac{2}{4}, \frac{5}{4}, 0, 1\right)\right\}$ or $\{(2,3,4,0), (2,5,0,4)\}$ 1

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -4 \\ -2 & -1 & 2 \\ -3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -8 \\ 0 & -3 & 6 \\ 0 & -5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ 2

$A^TY = 0 \Rightarrow$ $\left.\begin{aligned} y_1 - y_2 + 2y_3 &= 0 \\ 4y_2 - 8y_3 &= 0 \end{aligned}\right\}$ let y_3 be f.v. $\boxed{y_2 = 2y_3}$ $y_1 = y_2 - 2y_3 = 2y_3 - 2y_3$ 1

$y_1 = 0$

Basis of left nullspace $= \{(0, 2, 1)\}$

2a $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

Let $\alpha = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$, $\beta = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \in S$ & $c \in \mathbb{R}$.

(i) $\alpha + \beta = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{bmatrix} \in S$ $\therefore$ From (i) & (ii) S is a subspace of $M_{2\times 2}$ 2

(ii) $c.\alpha = \begin{bmatrix} ca_1 & 0 \\ 0 & cb_1 \end{bmatrix} \in S$ 2

2b $2t^2+t+2,\ t^2-2t,\ 5t^2-5t+2,\ -t^2-3t-2$.

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \\ 5 & -5 & 2 \\ -1 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & -5 & -2 \\ 0 & -15 & -6 \\ 0 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ 2

$\therefore$ The given vectors are linearly dependent. 1

$\{2t^2+t+2,\ t^2-2t\}$ is a linearly independent subset. 1
and it forms a basis of a subspace of P_2 with dimension 2. 2

2c. $T(1,2,1) = (-3,2,5,-1)$, $T(2,1,1) = (0,5,5,5)$, $T(1,1,2) = (-1,1,2,0)$.

$$A \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 5 & 1 \\ 5 & 5 & 2 \\ -1 & 5 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 5 & 1 \\ 5 & 5 & 2 \\ -1 & 5 & 0 \end{bmatrix} \times \frac{1}{-4} \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$ 1+2

$T(x,y,z) = (x-2y,\ 3x-z,\ 2x+2y-z,\ 4x-2y-z)$ $\quad A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & -1 \\ 4 & -2 & -1 \end{bmatrix}$ 1+1

$$\begin{bmatrix} -3 & 2 & 5 & -1 \\ 0 & 5 & 5 & 5 \\ -1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 2 & 5 & -1 \\ 0 & 5 & 5 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 2 & 5 & -1 \\ 0 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ 1

$\therefore$ range space $= \{c_1(-3,2,5,-1) + c_2(0,5,5,5)\}$ 1

$$T(x,y,z) = (0,0,0,0) \Rightarrow \begin{array}{ll} x-2y = 0 \Rightarrow x = 2y \\ 3x - z = 0 & z = 3x = 6y \\ 2x+2y-z = 0 & 4y+2y-z = 0 \Rightarrow z = 6y \\ 4x-2y-z = 0 & y=1 \Rightarrow x=2,\ z=6 \end{array}$$ 2

$\therefore$ null space $= \{c_1(2,1,6)\}$ 1

3a $\langle p,q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2)$, $t_0=-1, t_1=0, t_2=1$

$p(t) = 3t - t^2$, $q(t) = 3 + 2t^2$.

$p(t_0) = -4$, $p(t_1) = 0$, $p(t_2) = 2$, $q(t_0) = 5$, $q(t_1) = 3$, $q(t_2) = 5$ 1+1

$\langle p,p\rangle = 20$, $\langle q,q\rangle = 59$, length of $p(t)$ is $\sqrt{20}$, length of $q(t)$ is $\sqrt{59}$ 1+1

3b. $A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \\ 3 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 7 \\ 4 \\ -3 \\ 1 \end{bmatrix}$ $A^TAx = A^Tb \Rightarrow \begin{bmatrix} 39 & -41 \\ -41 & 66 \end{bmatrix} x = \begin{bmatrix} 17 \\ 15 \end{bmatrix}$ 2+1

$x = \frac{1}{893}\begin{bmatrix} 66 & 41 \\ 41 & 39 \end{bmatrix}\begin{bmatrix} 17 \\ 15 \end{bmatrix} = \begin{bmatrix} 1.9451 \\ 1.4356 \end{bmatrix}$ is the solution. 2+1

3c $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $x_1 = (1,2,-1,0)$, $x_2 = (2,2,0,1)$, $x_3 = (1,1,1,0)$ 1

$u_1 = x_1 = (1,2,-1,0)$ 2

$u_2 = x_2 = (2,2,0,1) - \frac{6}{6}(1,2,-1,0) = (1,0,1,1)$ 3

$u_3 = (1,1,1,0) - \frac{2}{6}(1,2,-1,0) - \frac{2}{3}(1,0,1,1) = (0,1,2,-2)$ 2

$Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 0 \\ 2/\sqrt{6} & 0 & 1/3 \\ -1/\sqrt{6} & 1/\sqrt{3} & 2/3 \\ 0 & 1/\sqrt{3} & -2/3 \end{bmatrix}$ $R = Q^TA = \begin{bmatrix} \sqrt{6} & \sqrt{6} & 2/\sqrt{6} \\ 0 & \sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$ 2

4a $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt \qquad f(t)=e^t,\ g(t)=2t-1.$

$\langle f,g\rangle = \int_0^1 e^t(2t-1)\,dt = \left[(2t-1)e^t - 2e^t\right]_0^1 = e - 2e + 1 + 2 = 3-e$ 2

$\langle f,f\rangle = \int_0^1 e^t\cdot e^t\,dt = \int_0^1 e^{2t}\,dt = \left[\frac{e^{2t}}{2}\right]_0^1 = \frac{e^2-1}{2}$ 1

$\langle g,g\rangle = \int_0^1 (2t-1)(2t-1)\,dt = \int_0^1 (4t^2-4t+1)\,dt = \left[\frac{4t^3}{3} - \frac{4t^2}{2} + t\right]_0^1 = \frac{1}{3}$ 1

4b $B = \{(1,2,-1), \cancel{(4,3,2)}, (1,2,1), (1,1,1)\}$ 1

$u_1 = (1,2,-1),$

$\cancel{u_2 = (4,3,2) - \frac{18}{63}(1,2,-1) = \left(\frac{8}{3}, \frac{1}{3}, \frac{10}{3}\right) \equiv (8,1,10)}$

$u_2 = (1,2,1) - \frac{4}{6}(1,2,-1) \cancel{- \frac{20}{165}(8,1,10)} \equiv \left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right) \equiv (1,2,5)$ 2

$u_3 = (1,1,1) - \frac{21}{63}(1,2,-1) - \frac{84}{\frac{30}{15}}(1,2,5) = \left(\frac{6}{15}, -\frac{3}{15}, 0\right) \equiv (2,-1,0)$ 2

orthonormal basis = $\left(\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{-1}{\sqrt6}\right), \left(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right), \left(\frac{2}{\sqrt5}, \frac{-1}{\sqrt5}, 0\right)$ 1

4c. $y = \beta_0 + \beta_1 x + \beta_2 x^2$

$(1,4), (2,4), (3,6), (4,7), (5,8)$ 1

$A^T A \beta = A^T b$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 7 \\ 8 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 29 \\ 98 \\ 386 \end{bmatrix}$$ 2+1

$$\Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 4.6 & -3.3 & 0.5 \\ -3.3 & 2.6714 & -0.4286 \\ 0.5 & -0.4286 & 0.0714 \end{bmatrix} \begin{bmatrix} 29 \\ 98 \\ 386 \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 0.6714 \\ 0.0714 \end{bmatrix}$$ 2+1

$\therefore y = 3 + 0.6714x + 0.0714x^2$ 1

at $x=8$ $y = 12.9408$ 1

at $x=10$ $y = 16.8540$ 1

$\underline{5a}$. $Q(x) = 11x_1^2 + 9x_2^2 + 7x_3^2 + 8x_1x_2 - 8x_2x_3$.

$$A = \begin{bmatrix} 11 & 4 & 0 \\ 4 & 9 & -4 \\ 0 & -4 & 7 \end{bmatrix} \Rightarrow \lambda^3 - 27\lambda^2 + 207\lambda - 405 = 0 \Rightarrow 15, 9, 3.$$ 1+2

(i) max' $\cancel{at}$ = 15 at $x^Tx = 1$ 1

(ii) $A - 15I = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -6 & -4 \\ 0 & -4 & -8 \end{bmatrix} \Rightarrow \frac{x_1}{32} = \frac{-x_2}{-32} = \frac{x_3}{-16} \Rightarrow x = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ 2

$u = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$.

(iii) max' $\cancel{at}$ = 9 at $x^Tx = 1$ and $x^Tu = 0$. 1

(iv) $A - 9I = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 0 & -4 \\ 0 & -4 & -2 \end{bmatrix} \Rightarrow \frac{x_1}{-16} = \frac{-x_2}{-8} = \frac{x_3}{-16} \Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ $v = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$ 2

(v) $Q(y) = 15y_1^2 + 9y_2^2 + 3y_3^2$ 1

$\underline{5b}$. $A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$.

$AA^T = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} \Rightarrow \lambda^2 - 13\lambda + 36 = 0 \Rightarrow \lambda = 9, 4.$ 2, 1

$AA^T - 9I = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$AA^T - 4I = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 1

$\therefore U = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. 1

$A^TA = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix} \Rightarrow \lambda^2 - 13\lambda + 36 = 0$, $\lambda = 9, 4$ 1

$A^TA - 9I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ 1

$A^TA - 4I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 1

$\therefore V = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. 1

$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. 1

6a. $A = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix} \Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$ 2

$\Rightarrow \lambda = 1, 1, 0$

$A - I = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{bmatrix} \Rightarrow 2x_1 - x_2 - 2x_3 = 0 \quad x_1 = \frac{1}{2}[x_2 + 2x_3]$

$\therefore x = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ 3

$A - 0I = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \Rightarrow \frac{x_1}{-2} = \frac{-x_2}{2} = \frac{x_3}{-2} \quad x = \begin{bmatrix} +1 \\ +1 \\ 1 \end{bmatrix}$ 1

$\therefore P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ 1+1+2

1

6b. $X = \begin{bmatrix} 12 & 6 & 9 & 15 & 13 & 5 \\ 19 & 22 & 6 & 3 & 2 & 20 \end{bmatrix} \quad M = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$ 1+1

$B = \begin{bmatrix} 2 & -4 & -1 & 5 & 3 & -5 \\ 7 & 10 & -6 & -9 & -10 & 8 \end{bmatrix} \quad S = \frac{1}{5} BB^T = \begin{bmatrix} 16 & -27 \\ -27 & 86 \end{bmatrix}$

$\Rightarrow \lambda^2 - 102\lambda + 647 = 0$ 2

$\Rightarrow \quad \Rightarrow \lambda = 95.2041, \; 6.7959.$

$S - \cancel{95.20}\,6.80\, I = \begin{bmatrix} 9.2 & -27 \\ -27 & 79.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 79.2 \\ 27 \end{bmatrix} = \begin{bmatrix} 0.9465 \\ 0.3227 \end{bmatrix}$ ~~first~~ second principal component 2

$S - \cancel{6.80}\,95.20\, I = \begin{bmatrix} -79.2 & -27 \\ -27 & -9.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -9.2 \\ 27 \end{bmatrix} = \begin{bmatrix} -0.3225 \\ 0.9466 \end{bmatrix}$ ~~second~~ first principal component. 2

$\frac{95.20}{(95.20 + 6.80)} = 0.9333 \quad 93.33\%$ 1

7a. $S\{(1,1),(1,2),\ldots (6,1),\ldots (6,6)\}$

$X = \underline{no}$ of $4^s = \{0,1,2\}$ $\quad Y = \underline{no}$ of $5^s = \{0,1,2\}$

(i)

$X \backslash Y$	0	1	2
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

(ii)

X	0	1	2
P(X)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Y	0	1	2
P(Y)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

2 + 1 + 1

(iii) $E(X) = \frac{12}{36} = \frac{1}{3}$ 1.

$E(Y) = \frac{12}{36} = \frac{1}{3}$ 1

$E(XY) = \frac{2}{36} = \frac{1}{18}$ 2.

(iv) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{1}{18} - \frac{1}{9}$ 2

$= -\frac{1}{18}.$

7b $f_{X,Y}(x,y) = \begin{cases} \frac{1}{24}xy, & 1<x<3,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$

$f_X(x) = \int_2^4 \frac{1}{24}xy\,dy = \left.\frac{xy^2}{48}\right]_2^4 = \frac{x}{4}$ 2

(i) conditional $f_{Y|X}(y|x) = \begin{cases} \frac{y}{6}, & 1<x<3, 2<y<4 \\ 0, & \text{otherwise} \end{cases}$ 1

$f_Y(y) = \int_1^3 \frac{1}{24}xy\,dx = \left.\frac{x^2y}{48}\right]_1^3 = \frac{y}{6}.$ 2

(ii) conditional. $f_{X|Y}(x|y) = \begin{cases} \frac{x}{4}, & 1<x<3, 2<y<4 \\ 0, & \text{otherwise} \end{cases}$ 1

7c $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$, $f(y) = \begin{cases} \lambda e^{-\lambda y}, & y>0 \\ 0, & \text{otherwise} \end{cases}$

$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)}, & x,y>0 \\ 0, & \text{otherwise} \end{cases}$ 1

$\begin{matrix} U = X+Y \\ V = X-Y \end{matrix} \Rightarrow \begin{matrix} X = \frac{U+V}{2} \\ Y = \frac{U-V}{2} \end{matrix}$ Jacobian $= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$ $\quad f_{U,V}(u,v) = \begin{cases} \frac{1}{2}\lambda^2 e^{-\lambda u}, & u+v>0,\ u-v>0 \\ 0, & \text{otherwise} \end{cases}$

2 + 1

8a. $f(x,y)=\begin{cases} c(4-x-y), & 0\le x\le 2,\ 0\le y\le 2,\ c>0 \\ 0, & \text{otherwise} \end{cases}$

(i) $\int_{x=0}^{2}\int_{y=0}^{2} c(4-x-y)\,dy\,dx = \int_{x=0}^{2} c\left[4y-xy-\frac{y^2}{2}\right]_0^2 dx = \int_{x=0}^{2} c[6-2x]\,dx$ 3

$= c\left[6x-\cancel{2}\frac{x^2}{\cancel{2}}\right]_0^2 = 8c \Rightarrow c=\frac{1}{8}$

(ii) $P(x<1, y>1) = \int_{x=0}^{1}\int_{y=1}^{2} \frac{1}{8}[4-x-y]\,dy\,dx = \int_{x=0}^{1}\frac{1}{8}\left[4y-xy-\frac{y^2}{2}\right]_1^2 dx$ 2

$= \int_{x=0}^{1}\frac{1}{8}\left[\frac{5}{2}-x\right]dx = \frac{1}{8}\left[\frac{5x}{2}-\frac{x^2}{2}\right]_0^1 = \frac{1}{4}.$

(iii) $P\left(\frac{1}{2}<y<\frac{3}{2}\right) = \int_{x=0}^{2}\int_{y=\frac{1}{2}}^{\frac{3}{2}} \frac{1}{8}[4-x-y]\,dy\,dx = \int_{x=0}^{2}\frac{1}{8}\left[4y-xy-\frac{y^2}{2}\right]_{y=\frac{1}{2}}^{3/2} dx$ 2

$= \int_{x=0}^{2}\frac{1}{8}[3-x]\,dx = \frac{1}{8}\left[3x-\frac{x^2}{2}\right]_0^2 = \frac{1}{2}$

(iv) $P(y<x) = \int_{x=0}^{2}\int_{y=0}^{x} \frac{1}{8}[4-x-y]\,dy\,dx = \int_{x=0}^{2}\frac{1}{8}\left[4y-xy-\frac{y^2}{2}\right]_0^x dx$ 3

$= \int_{x=0}^{2}\frac{1}{8}\left[4x-\frac{3x^2}{2}\right]dx = \frac{1}{8}\left[\frac{4x^2}{2}-\frac{\cancel{3}x^3}{2\times\cancel{3}}\right]_0^2 = \frac{1}{2}$

8b.

x_1 \ x_2	0	1	2
0	0.12	0.12	0.08
1	0.13	0.21	0.23
2	0.07	0.02	0.02

x_1	0	1	2
$P(x_1)$	0.32	0.57	0.11

$E(x_1) = 0.79$ $\mu_x = \begin{bmatrix} 0.79 \\ 1.01 \end{bmatrix}$ 1

x_2	0	1	2
$P(x_2)$	0.32	0.35	0.33

$E(x_2) = 1.01$

$E(x_1^2) = \cancel{0.3491}$ 1.01, $E(x_2^2) = \cancel{1.67}$ 1.67 $E(x_1x_2) = 0.79$. $R_x = \begin{bmatrix} 1.01 & 0.79 \\ 0.79 & 1.67 \end{bmatrix}$ 1+1+1

$C_x = \begin{bmatrix} 0.3859 & -0.0079 \\ -0.0079 & 0.6499 \end{bmatrix}$ +1

$\begin{bmatrix} 0.39 & 0 \\ 0 & 0.65 \end{bmatrix}$

8c $\mu_1=2$, $\mu_2=5$, $\sigma_1=0.1$, $\sigma_2=0.2$

$P(Y=2X_1+2X_2>14.5)$ $Y=2X_1+2X_2 \Rightarrow E[Y]=2\times2+2\times5=14$ 1

$Var[Y] = 4\times0.1^2+4\times0.2^2 = 0.2.$ 1

$= P\left[\frac{Y-\mu_Y}{\sigma_Y} > \frac{14.5-14}{\sqrt{0.2}}\right] = P(Z>1.12) = 0.13$ 2

9a. $\mu = \frac{1}{30}$ per min or $\frac{60}{30} = 2$ per hour.

$\lambda = \frac{1}{48}$ per min or $\frac{60}{48} = \frac{5}{4} = 1.25$ per hour 2

(i) $E[N_s] = \frac{\lambda}{\mu-\lambda} = \frac{5/4}{2-\frac{5}{4}} = \frac{5}{3} = 1.67$ | $\mu = \frac{1}{30} = 0.0333$, $\lambda = \frac{1}{48} = 0.0208$; $\frac{0.0208}{(0.0333-0.0208)} = 1.664$ 2

(ii) $E[W_q] = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5/4}{2\left(2-\frac{5}{4}\right)} = \frac{5}{6} = 0.83$ | $\frac{0.0208}{0.0333(0.0333-0.0208)} = 50.05$ 2

(iii) $P[N_s > 5] = \left(\frac{5/4}{2}\right)^{5+1} = \left(\frac{5}{8}\right)^6 = \boxed{0.0596}$ $\left(\frac{0.0208}{0.0333}\right)^6 = 0.0594$ 2

(iv) $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{5/4}{2} = \frac{3}{8} = 0.375$ | $1 - \frac{0.0208}{0.0333} = 0.3754$ 2

9b. $s = 2$, $\mu = \frac{1}{5}$ per min or $\frac{60}{5} = 12$ per hour 2

$\lambda = 10$ per hour

$$P_0 = \frac{1}{\left\{\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \left\{\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right\}}$$

$$= \frac{1}{1 + \frac{5}{6} + \frac{1}{2\times\left(\frac{7}{12}\right)}\times\left(\frac{5}{6}\right)^2}$$

$= 0.4118$. 2+2

(i) $P(N \geq 2) = \dfrac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$

$= \dfrac{\left(\frac{5}{6}\right)^2 \times 0.4118}{2\left(1-\frac{5}{12}\right)}$

$= 0.2451$

(ii) Fraction of time when the girls are busy $= \frac{\lambda}{\mu s} = \frac{5}{12}$

fraction of time when the girls are idle $= \frac{7}{12}$ % $= 58\%$ 1

(iii) $E[W_q | W_s > 0] = \frac{1}{\mu s - \lambda} = \frac{1}{24-10} = 0.0714$ 1

(iv) $E[N_s] = \dfrac{1}{s.s!}\dfrac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 + \dfrac{\lambda}{\mu}$ 2

$= \dfrac{1}{2\times 2} \times \dfrac{\left(\frac{5}{6}\right)^3}{\left(1-\frac{5}{12}\right)^2} \times 0.4118 + \dfrac{5}{6}$

$= 1.0084$

10a $\lambda = 10$ per hr $\quad \mu = \frac{1}{5}$ per min $= \frac{60}{5} = 12$ per hr. $k = 3$ $\quad \lambda \neq \mu$. 2

(i) $P_0 = \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1-\frac{5}{6}}{1-\left(\frac{5}{6}\right)^4} = 0.3219.$ 2

(ii) $P_3 = \left(\frac{\lambda}{\mu}\right)^3 \times \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^4} = \frac{5}{6} \times \dfrac{1-\frac{5}{6}}{1-\left(\frac{5}{6}\right)^4} = 0.5787 \times 0.3219 = 0.1863$ 2

(iii) $E[W_q] = \frac{1}{\lambda'} E[N_q]$

$$= \frac{1}{\mu(1-P_0)} \times \left[E[N] - \frac{\lambda'}{\mu} \right]$$

$$= \frac{1}{\mu(1-P_0)} \times \left\{ \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} - \frac{\mu(1-P_0)}{\mu} \right\}$$ 4

$$= \frac{1}{12(1-0.3219)} \times \left\{ \frac{10}{12-10} - \frac{4\left(\frac{5}{6}\right)^4}{1-\left(\frac{5}{6}\right)^4} - (1-0.3219) \right\}$$

$$= 0.0733$$

10b $\lambda = \frac{1}{30}$/min $= \frac{60}{30} = 2$ per hour, $\quad s = 2,\ k = 6$ 2

$\mu = \frac{1}{20}$/min $= \frac{60}{20} = 3$ per hour,

(i) $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{\mu s}\right)^{n-s} \right]^{-1} = \left[1 + \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right)^2 \left\{1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4\right\}\right]^{-1}$ 2

$= 0.5003.$

(ii) $E[N_q] = P_0 \times \left(\frac{\lambda}{\mu}\right)^s \times \dfrac{\rho}{s!(1-\rho)^2}\left[1 - \rho^{k-s} - (k-s)(1-\rho)\,\rho^{k-s}\right],\ \rho = \frac{\lambda}{\mu s}$

$= 0.5003 + \left(\frac{2}{3}\right)^2 + \dfrac{(1/3)}{2\left(\frac{2}{3}\right)^2}\left[1 - \left(\frac{1}{3}\right)^4 - 4 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^4\right] = 0.0796.$ 3

(iii) $E[W_q] = \frac{1}{\lambda'} E[N_q]$, $\quad \lambda' = \mu\left[s - \sum_{n=0}^{s-1} (s-n)P_n\right]$

$= 3\left[2 - 2P_0 - P_1\right]$ 1+2

$= \frac{1}{1.9976} \times 0.0796$ $\quad = 3\left[2 - 2 \times 0.5003 - \frac{2}{3} \times 0.5003\right]$

$= 0.0398$ $\quad = 1.9976$

Jayalatha. G

PROFESSOR
Department of Mathematics
RVCE, Bangalore-560 059