Joint Probability Mass Function

Let S be a sample space and X and Y be two discrete random variables defined on S. Let $X = f(s)$, $Y = g(s)$, $s \in S$, where f and g are real-valued functions defined on S, and $f(S) = \{x_1, x_2, x_3, \ldots\}$ and $g(S) = \{y_1, y_2, y_3, \ldots\}$ be the image sets of S under f and g respectively, where $x_1, x_2, \ldots$ are the values of X and $y_1, y_2, \ldots$ are the values of Y.

The probability that $X = x_i$, denoted by $P(x_i)$, is defined by $P(X = x_i) = P(x_i) = P\{s \in S \mid f(s) = x_i\}$.

The probability that $Y = y_j$, denoted by $P(y_j)$, is defined by $P(Y = y_j) = P(y_j) = P\{s \in S \mid g(s) = y_j\}$.

Then their joint probability mass function is given by $p_{ij} = P(x_i, y_j) = P\{s \in S \mid f(s) = x_i \text{ and } g(s) = y_j\}$.

such that (i) $p_{ij} \geq 0$, for $i = 1, 2, 3 \ldots$, $j = 1, 2, 3 \ldots$

(ii) $\sum_i \sum_j p_{ij} = 1$

The values of X, Y and P(X, Y) can be displayed in the form of the following table, called the Probability Contingency Table or Joint Probability Table.

$X \backslash Y$	y_1	y_2	y_3	$\cdots$
x_1	P_{11}	P_{12}	P_{13}	$\cdots$
x_2	P_{21}	P_{22}	P_{23}	$\cdots$
x_3	P_{31}	P_{32}	P_{33}	$\cdots$
$\vdots$				

Marginal probability distributions

Let $P_i = P_{i1} + P_{i2} + P_{i3} + \cdots = \sum_j P_{ij}$ (fixed i)

$$Q_j = P_{1j} + P_{2j} + P_{3j} + \cdots = \sum_i P_{ij} \quad (\text{fixed } j)$$

The sets $\{P_i\}$, $i = 1, 2, 3 \ldots$, and $\{Q_j\}$, $j = 1, 2, 3, \ldots$ are called the marginal probability distributions of X and Y respectively. These P_i and Q_j are identical with the individual probability distributions of X and Y (respectively), so that

$$P_i = P(x_i) = P(X = x_i) \quad \text{and} \quad Q_j = P(y_j) = P(Y = y_j)$$

Stochastic Independence

The random variables X and Y, defined on the same sample space S, are said to be stochastically independent or statistically independent.

$$P_{ij} = P_i Q_j$$

Expectations

Consider a function $\phi(x, y)$ of X and Y.
The function $E[\phi(x, y)]$ defined by the formula

$$E[\phi(x,y)] = \sum_i \sum_j p_{ij}\, \phi(x_i, y_j)$$

is called the mathematical expectation (or expectation) of $\phi(x,y)$ in the joint distribution of X and Y.

Suppose $\phi(x,y) = (X-\mu_X)^r (Y-\mu_Y)^s$,

where $\mu_X = E(X)$ and $\mu_Y = E(Y)$ are the means of the distributions of X and Y respectively, r and s are non-negative integers. Then the above formula yields

$$E[(X-\mu_X)^r (Y-\mu_Y)^s] = \sum_i \sum_j p_{ij} (x_i-\mu_X)^r (y_j-\mu_Y)^s$$

The function $E[(X-\mu_X)^r (Y-\mu_Y)^s]$ is called the $(r,s)^{th}$ moment about the mean or the $(r,s)^{th}$ central moment of the joint distribution of X, Y and is denoted by μ_{rs}.

In particular, μ_{11} is called the covariance of X and Y and is denoted by $Cov(X,Y)$.

Thus, $Cov(X,Y) = \mu_{11} = E[(X-\mu_X)(Y-\mu_Y)]$

$$= \sum_i \sum_j p_{ij} (x_i-\mu_X)(y_j-\mu_Y)$$

Note: $Cov(X,Y)$ is the expectation of the function $\phi(x,y) = (X-\mu_X)(Y-\mu_Y)$

* If $Y = X$, then

$$Cov(X, X) = E\left[(X - \mu_X)^2\right] = Var(X)$$

* $$Cov(X, Y) = E[XY] - E[X]\,E[Y]$$

* let σ_X and σ_Y be the standard deviations of the distributions of X and Y respectively. Then $\rho(X, Y)$ is defined by

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y},$$

which is called the correlation coefficient of the joint distribution of X, Y.

* $$Cov(X, Y) = \sum_i \sum_j P_{ij} (x_i - \mu_X)(y_i - \mu_Y)$$

$$E[XY] = \sum_i \sum_j P_{ij}\, x_i\, x_j$$

$$\mu_X = E[X] = \sum_i x_i P_i = \sum_i \sum_j P_{ij}\, x_i$$

$$\mu_Y = E[Y] = \sum_j y_j Q_j = \sum_i \sum_j P_{ij}\, y_j$$

Also $\sum_i \sum_j P_{ij} = 1$.

$$\therefore Cov(X, Y) = \sum_i \sum_j P_{ij} \{x_i y_j - \mu_Y \sum_i \sum_j P_{ij} x_i - \mu_X \sum_i \sum_j P_{ij} y_j + \mu_X \mu_Y \sum_i \sum_j P_{ij}$$

$$= E[XY] - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X \mu_Y \cdot 1$$

$$= E[XY] - E[X]\,E[Y]$$

The joint probability distribution of two random variables X and Y is given by the following table: (3)

X \ Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine the individual (marginal) distributions of X and Y. Also, verify that X and Y are stochastically independent.

Sol

X \ Y	2	3	4	
1	0.06	0.15	0.09	$P_1 = 0.3$
2	0.14	0.35	0.21	$P_2 = 0.7$
	$Q_1 = 0.2$	$Q_2 = 0.5$	$Q_3 = 0.3$	

$\therefore$

X	1	2
P(X)	0.3	0.7

Y	2	3	4
P(Y)	0.2	0.5	0.3

$P_1 Q_1 = 0.06 = P_{11}$, $P_1 Q_2 = 0.15 = P_{12}$, $P_1 Q_3 = 0.09 = P_{13}$

$P_2 Q_1 = 0.14 = P_{21}$, $P_2 Q_2 = 0.35 = P_{22}$, $P_2 Q_3 = 0.21 = P_{23}$

Thus $P_i Q_j = P_{ij}$ for all allowable values of i and j. Hence X and Y are stochastically independent.

* A joint distribution of two random variables X and Y is given by the following table:

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

. Are X and Y independent?

Determine (i) The marginal distributions of X and Y.
(ii) $E(X)$ and $E(Y)$, (iii) $E(XY)$

Sol^n From the given table, we have

(i)

X	1	5
P(X)	1/2	1/2

,

Y	-4	2	7
P(Y)	3/8	3/8	2/8

$P_1 Q_1 = \frac{3}{16} \neq P_{11}$ Thus $P_i Q_j \neq P_{ij}$ for $i=1, j=1$,
Therefore, X and Y are not independent.

(ii) $E(X) = \mu_X = x_1 P_1 + x_2 P_2 = 1\times\frac{1}{2} + 5\times\frac{1}{2} = 3$

$E(Y) = \mu_Y = y_1 Q_1 + y_2 Q_2 + y_3 Q_3 = (-4)\times\frac{3}{8} + 2\times\frac{3}{8} + 7\times\frac{2}{8} = 1$

(iii) $E(XY) = \sum_i \sum_j P_{ij} x_i y_j$

$= (P_{11} x_1 y_1 + P_{12} x_1 y_2 + P_{13} x_1 y_3) + (P_{21} x_2 y_1 + P_{22} x_2 y_2 + P_{23} x_2 y_3)$

$= \frac{1}{8}\times 1\times(-4) + \frac{1}{4}\times 1\times 2 + \frac{1}{8}\times 1\times 7 + \frac{1}{4}\times 5\times(-4) + \frac{1}{8}\times 5\times 2 + \frac{1}{8}\times 5\times 7$

$= \cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}} + \frac{7}{8} - 5 + \frac{5}{4} + \frac{35}{8}$

$= \frac{3}{2}$

* The joint distribution of two random variables X and Y is given by the following table: (4)

$X \backslash Y$	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find the marginal distribution of X and Y, and evaluate Cov(X, Y) and $\rho(X, Y)$.

<u>Sol^n</u> From the given table

X	2	4	6
P(X)	1/4	2/4	1/4

Y	1	3	9
P(Y)	3/6	2/6	1/6

$$E[X] = \sum x_i P_i = 2\times\frac{1}{4} + 4\times\frac{2}{4} + 6\times\frac{1}{4} = 4$$

$$E[Y] = \sum y_j Q_j = 1\times\frac{3}{6} + 3\times\frac{2}{6} + 9\times\frac{1}{6} = 3$$

$$E[XY] = \sum_i \sum_j P_{ij}\, x_i y_j = (P_{11}x_1y_1 + P_{12}x_1y_2 + P_{13}x_1y_3)$$
$$+ (P_{21}x_2y_1 + P_{22}x_2y_2 + P_{23}x_2y_3)$$
$$+ (P_{31}x_3y_1 + P_{32}x_3y_2 + P_{33}x_3y_3)$$

$$= \left(\cancel{2\times}\frac{1}{8}\times 2\times 1 + \frac{1}{24}\times 2\times 3 + \frac{1}{12}\times 2\times 9\right) + \left(\frac{1}{4}\times 4\times 1 + \frac{1}{4}\times 4\times 3 + 0\times 4\times 9\right)$$
$$+ \left(\frac{1}{8}\times 6\times 1 + \frac{1}{24}\times 6\times 3 + \frac{1}{12}\times 6\times 9\right)$$

$$E[XY] = 12$$

$$Cov(X, Y) = E[XY] - E[X]E[Y] = 12 - 4\times 3 = 0$$

As $$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0$$

* The distributions of two stochastically independent random variables X and Y defined on the same sample space are given by the following tables:

X	0	1
P(X)	0.2	0.8

Y	1	2	3
P(Y)	0.1	0.4	0.5

Find the joint distribution of X and Y. Also evaluate Cov(X, Y).

Sol

Sol^n Using $P_{ij} = P_i Q_j \Rightarrow$

X \ Y	1	2	3
0	0.02	0.08	0.10
1	0.08	0.32	0.40

$P_i = P(x_i)$

$Q_j = P(y_j)$

$E[X] = \Sigma x_i P_i = 0 \times 0.2 + 1 \times 0.8 = 0.8$

$E[Y] = \Sigma y_j Q_j = 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.5 = 2.4$

$E[XY] = \sum_i \sum_j P_{ij} x_i x_j = 0.02 \times 0 \times 1 + 0.08 \times 0 \times 2 + 0.10 \times 0 \times 3$
$+ 0.08 \times 1 \times 1 + 0.32 \times 1 \times 2 + 0.40 \times 1 \times 3$

$= 1.92$.

$\therefore Cov(X, Y) = E[XY] - E[X].E[Y]$

$= 1.92 - 0.8 \times 2.4$

$= 0$

* The joint probability function for two discrete (5) random variables X and Y is given by $f(x,y) = c(2x+y)$ where x and y can assume all integral values such that $0 \le x \le 2$, $0 \le y \le 3$, and $f(x,y) = 0$ otherwise. Find (i) the value of the constant c, (ii) $P(X=2, Y=1)$, (iii) $P(X \ge 1, Y \le 2)$, (iv) $P(X+Y \le 1)$ (v) $P(X+Y>1)$.

Solⁿ

X \ Y	0	1	2	3
0	0	c	2c	3c
1	2c	3c	4c	5c
2	4c	5c	6c	7c

(i) $\sum_i \sum_j P_{ij} = 1$

$\Rightarrow (0+c+2c+3c) + (2c+3c+4c+5c) + (4c+5c+6c+7c) = 1$

$\Rightarrow 42c = 1 \Rightarrow c = 1/42$.

(ii) $P(X=2, Y=1) = P(x_3, y_2)\ P_{32} = 5c = 5/42$

(iii) $P(X \ge 1, Y \le 2) = P(X=1, Y \le 2) + P(X=2, Y \le 2)$

$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$

$+ P(X=2, Y=0) + P(X=2, Y=1), P(X=2, Y=2)$

$= P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33}$

$= (2c+3c+4c+5c) + (4c+5c+6c+7c)$

$= 24c = 24/42 = 4/7$

(iv) $P(X+Y \le 1) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=1, Y=0)$

$= P_{11} + P_{12} + P_{21}$

$= 0 + c + 2c$

$= 3c = 3/42 = 1/14$

(v) $P(X+Y>1) = 1 - P(X+Y \le 1)$

$= 1 - \frac{1}{14}$

$= 13/14$

* A coin is tossed three times. Let X be equal to 0 or 1 according as a head or a tail occurs on the first toss. Let Y be equal to the total number of heads which occur. Determine (i) the marginal distributions of X and Y, and (ii) the joint distribution of X and Y, (iii) expected values of X, Y, X+Y and XY, (iv) σ_X and σ_Y (v) Cov(X, Y) and $\rho(X, Y)$.

Sol^n: $S = \begin{Bmatrix} HHH, & HHT, & HTH, & HTT, \\ THH, & THT, & TTH, & TTT \end{Bmatrix}$

(i)

X	0	1
P(x)	4/8	4/8

*

Y	0	1	2	3
P(y)	1/8	3/8	3/8	1/8

(ii)

X \ Y	0	1	2	3
0	0	1/8	2/8	1/8
1	1/8	2/8	1/8	0

(iii) $E[X] = \mu_X = \sum x_i P(x_i)$

$= 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$

$E[Y] = \mu_Y = \sum y_j P(y_j)$

$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{12}{8}$

$E[X+Y] = \sum_i \sum_j P_{ij}(x_i + y_j)$

$= P_{11}(x_1 + y_1) + P_{12}(x_1 + y_2) + P_{13}(x_1 + y_3) + P_{14}(x_1 + y_4) + P_{21}(x_2 + y_1) + P_{22}(x_2 + y_2) + P_{23}(x_2 + y_3) + P_{24}(x_2 + y_4)$

$= 0(0+0) + \frac{1}{8}(0+1) + \frac{2}{8}(0+2) + \frac{1}{8}(0+3) + \frac{1}{8}(1+0) + \frac{2}{8}(1+1) + \frac{1}{8}(1+2) + 0(1+3)$

$= \frac{16}{8} = 2$

$E[XY] = \sum_i \sum_j P_{ij} x_i y_j$

$= P_{11} x_1 y_1 + P_{12} x_1 y_2 + P_{13} x_1 y_3 + P_{14} x_1 y_4 + P_{21} x_2 y_1 + P_{21} x_2 y_2 + P_{23} x_2 y_3 + P_{24} x_2 y_4$

$= 0 + 0 + 0 + 0 + 0 + \frac{2}{8} + \frac{2}{8} + 0$

$= \frac{1}{2}$

(iv) $\sigma_X^2 = E(X^2) - \mu_X^2$

$= \sum x_i^2 P(x_i) - [E(X)]^2$

$= 0^2 \times \frac{4}{8} + 1^2 \times \frac{4}{8} - \left(\frac{4}{8}\right)^2 = \frac{1}{4}$

$\sigma_Y^2 = E(Y^2) - \mu_Y^2$

$= \sum y_j^2 P(y_j) - [E(Y)]^2$

$= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} - \left(\frac{12}{8}\right)^2 = \frac{3}{4}$

$Cov(X, Y) = E(XY) - \mu_X \mu_Y$

$= \frac{1}{2} - \frac{1}{2} \times \frac{3}{2} = -\frac{1}{4}$

$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{-1/4}{(1/2)(\sqrt{3}/2)} = -\frac{1}{\sqrt{3}}$

1. The joint probability distribution of two random variables X and Y is defined by the function $P(X, Y) = \frac{1}{27}(2X + Y)$ where X and Y assume the integer values 0, 1, 2. Find the marginal distributions of X and Y. Are X and Y independent?

Ans:

X	0	1	2
P(x)	1/9	1/3	5/27

Y	0	1	2
P(Y)	2/9	1/3	4/9

X and Y are not independent

2. Consider an experiment that consists of 2 throws of a fair die. Let X be the number of 4s and Y be the number of 5s obtained in the two throws. Find the joint probability distribution of X and Y. Also evaluate $P(2X + Y < 3)$.

Ans:

X \ Y	0	1	2
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

$P(2X + Y < 3) = 11/12$.

3. The joint distribution of two random variables X and Y ~~as~~ as given below:

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Find the marginal distributions of X and Y. Also determine μ_X, μ_Y and the covariance of X and Y.

Ans.

X	1	2
P(X)	0.6	0.4

Y	-2	-1	4	5
P(Y)	0.3	0.3	0.1	0.3

$\mu_X = 1.4$

$\mu_Y = 1.0$

$Cov(X, Y) = -0.5$