* Find the BASES and DIMENSION of 4 fundamental sub spaces

$$A = \begin{bmatrix} 1 & 2 & -2 & -3 \\ -1 & 2 & -1 & -2 \\ 2 & -4 & 2 & 4 \end{bmatrix}$$

Column Space

$$U = \begin{bmatrix} 1 & 2 & -2 & -3 \\ 0 & 4 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases = $\{(1, -1, 2), (2, 2, -4)\}$

Column Space = $C(A)$ = Span $\{(1,-1,2), (2,2,-4)\}$

Dimension = Rank = 2

Null Space

$$Ux = 0$$

$$\begin{array}{cccc} \left[\begin{matrix} 1 & 2 & -2 & -3 \\ 0 & 4 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \\ x_1 \quad x_2 \quad x_3 \quad x_4 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_2 - 3x_3 - 5x_4 = 0$$

WHEN $x_3 = 0$ $x_4 = 1$ $\Rightarrow$ $x_2 = 5/4$

WHEN $x_3 = 1$ $x_4 = 0$ $\Rightarrow$ $x_2 = 3/4$

$$1x_1 + 2x_2 - 2x_3 - 3x_4 = 0$$

WHEN $x_3 = 0$ $x_4 = 1$ $x_2 = 5/4 \Rightarrow x_1 = 1/2$

WHEN $x_3 = 1$ $x_4 = 0$ $x_2 = 3/4 \Rightarrow x_1 = 1/2$

Bases $= \{(1/2, 5/4, 0, 1), (1/2, 3/4, 1, 0)\}$

Null Space $= N(A) = \{ x_4(\ ) + x_3(\ ) \}$

Dimension $= n - r = 4 - 2 = 2$

Row Space

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -4 \\ -2 & -1 & 2 \\ -3 & -2 & 4 \end{bmatrix}$$

$$U^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Bases $= S(A^T) = \{(1, 2, -2, -3), (-1, 2, -1, -2)\}$

Row Space $= C(A^T) =$ Span of $\{(), ()\}$

Dimension $=$ Rank $= 2$

LEFT NULL SPACE

$$A^T y = 0$$

$$\begin{array}{c}\begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} y_1 & y_2 & y_3 \end{matrix}\end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$-4y_2 + 8y_3 = 0 \Rightarrow y_2 = 2y_3$$

$$y_1 - y_2 + 2y_3 = 0 \Rightarrow y_1 = 0$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Bases $= \{(0,2,1)\}$

Left Null Space $= N(A^T) =$ Span of $\{(0,2,1)\}$

Dimension $= m - n = 3 - 2 = 1$

Field $\rightarrow \mathbb{R}$
Vector $\rightarrow V$

$V = \{ (x, y) \mid x, y \in \mathbb{R} \}$

To show subset $W = \{ (x, y) \} \in V$

i) Closed under Vector Addition

$\alpha = (x_1, y_1)$
$\beta = (x_2, y_2)$

$$\alpha + \beta = (x_1, y_1) + (x_2, y_2)$$
$$= (x_1 + x_2, y_1 + y_2)$$
$$\in V$$

$\alpha \in V \qquad \beta \in V \qquad \alpha + \beta \in V$

ii) Closed under Scalar Multiplication

$$c(\alpha + \beta) = (cx, cy) = c\alpha + c\beta$$
$$= \in V$$

iii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
iv) $\alpha + 0 = \alpha$
v) $\alpha + \alpha^{-1} = 0$
vi) $c(\alpha + \beta) = c\alpha + c\beta$
vii) $(c + c')\alpha = c\alpha + c'\alpha$
viii) $(cc')\alpha = c(c'\alpha)$
ix) $\alpha \cdot 1 = \alpha$
x) $\alpha + \beta = \beta + \alpha$

\# $S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $a, b \in \mathbb{R}$

$\alpha = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$ $\beta = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ $\gamma = \begin{bmatrix} a_3 & 0 \\ 0 & b_3 \end{bmatrix}$

$\boxed{\text{i)} \quad \alpha + \beta \in V}$ Vector Addition

$\alpha + \beta = \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix}$ ✓

$\boxed{\text{ii)} \quad c \cdot \alpha \in V}$ Scalar Multiplication

$c \cdot \alpha = \begin{bmatrix} ca_1 & 0 \\ 0 & cb_1 \end{bmatrix}$ ✓

C iii) $\alpha + \beta = \beta + \alpha$

A iv) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$\begin{bmatrix} a_1 + (a_2 + a_3) & 0 \\ 0 & b_1 + (b_2 + b_3) \end{bmatrix} = \begin{bmatrix} (a_1 + a_2) + a_3 & 0 \\ 0 & (b_1 + b_2) + b_3 \end{bmatrix}$$

Id v) $\alpha + 0 = \alpha$ vi) $\alpha \cdot 1 = \alpha$

In vii) $\alpha + \alpha^{-1} = 0$

D viii) $(c + d)\alpha = c\alpha + d\alpha$

A ix) $(cd)\alpha = c(d\alpha)$

D x) $c(\alpha + \beta) = c\alpha + c\beta$

$2t^2 + t + 2$ ST Linearly Dependent

$t^2 - 2t$

$5t^2 - 5t + 2$

$-t^2 - 3t - 2$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 0 \\ 5 & -5 & 2 \\ -1 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Free Variables Exist

* Rank (2) < Unknowns (3)

* SOME Combination of equa^n is 0

$\Rightarrow$ Linearly Dependant

Linearly Independent Subset

$(2t^2 + t + 2)$ and $(t^2 - 2t)$

Basis $= \{(2, 1, 2), (1, -2, 0)\}$

Dimension = rank = 2

$T(x, y, z) = (x - z - y,\ x + y - z,\ x + y + z)$
ST T is a linear transformation

$\rightarrow$

$$\alpha = (x, y, z)$$
$$\beta = (\bar{x}, \bar{y}, \bar{z}) \in V_3(R)$$

i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$

$$T(\alpha) = (x - z - y,\ x + y - z,\ x + y + z)$$
$$T(\beta) = (\bar{x} - \bar{z} - \bar{y},\ \bar{x} + \bar{y} - \bar{z},\ \bar{x} + \bar{y} + \bar{z})$$

$$T(\alpha + \beta) = T((x, y, z) + (\bar{x}, \bar{y}, \bar{z}))$$

$$= T(x + \bar{x},\ y + \bar{y},\ z + \bar{z})$$

$$= x + \bar{x} - (z + \bar{z}) - (y + \bar{y}),$$
$$x + \bar{x} + (y + \bar{y}) - (z + \bar{z}),$$
$$x + \bar{x} + (y + \bar{y}) + (z + \bar{z})$$

$$= x + \bar{x} - z - \bar{z} - y - \bar{y},$$
$$x + \bar{x} + y + \bar{y} - z - \bar{z},$$
$$x + \bar{x} + y + \bar{y} + z + \bar{z}$$

$$= (x - z - y,\ x + y - z,\ x + y + z) +$$
$$(\bar{x} - \bar{z} - \bar{y},\ \bar{x} + \bar{y} - \bar{z},\ \bar{x} + \bar{y} + \bar{z})$$

$$= T(\alpha) + T(\beta)$$

ii) $T(c \cdot \alpha) = c \cdot T(\alpha)$

$$T(c \cdot \alpha) = T(cx, cy, cz)$$

$$= (cx - cz - cy,$$
$$cx + cy - cz,$$
$$cx + cy + cz)$$

$$= c(x - z - y),$$
$$c(x + y - z),$$
$$c(x + y + z)$$

$$= c\left((x - z - y), (x + y - z), (x + y + z)\right)$$

$$= c \cdot T(\alpha)$$

Linear Transformation

$T(1,2,1) = (-3, 2, 5, -1)$
$T(2,1,1) = (0, 5, 5, 5)$
$T(1,1,2) = (-1, 1, 2, 0)$

A (preimage) = (Image)

$$A \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 5 & 1 \\ 5 & 5 & 2 \\ -1 & 5 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 5 & 1 \\ 5 & 5 & 2 \\ -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -1 \\ 2 & 2 & -1 \\ 4 & -2 & -1 \end{bmatrix}$$

$$T(x,y,z) = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T(x,y,z) = (x - 2y, \; 3x - z, \; 2x - 2y - z, \; 4x - 2y - z)$$

Range Space

$T(1,0,0) = (1,3,2,4)$
$T(0,1,0) = (-2,0,-2,-2)$
$T(0,0,1) = (0,-1,-1,-1)$

~~Basis of $R(T) = \{(1,3,2,4), (-2,0,-2,-2), (0,-1,-1,-1)\}$~~

~~Null Space~~

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -1 \\ 2 & -2 & -1 \\ 4 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of $R(T) = \{(1,3,2,4), (-2,0,-2,-2), (0,-1,-1,-1)\}$

Null Space

$Tx = 0$

$-2z = 0 \qquad z = 0$
$-6y + z = 0 \qquad y = 0$
$x - 2y = 0 \qquad x = 0$

$N(T) = \{(0,0,0)\}$

Rank Nullity theorem

Rank + Nullity = Dimension
3 + 0 = $\mathbb{R}3$

UNIT 2

* Orthogonal Vectors

$u \cdot v = 0$

$u = (1, 2)$
$v = (6, -3)$

$$\begin{aligned} u \cdot v &= (1, 2) \cdot (6, -3) \\ &= (1 \cdot 6) + (2 \cdot -3) \\ &= 6 + (-6) \\ &= 0 \end{aligned}$$

* Orthogonal Set

$(u, v, w, z) \in V$

$$\left.\begin{array}{lll} u \cdot v & u \cdot w & u \cdot z \\ v \cdot u & v \cdot w & v \cdot z \\ w \cdot u & w \cdot v & w \cdot z \\ z \cdot u & z \cdot v & z \cdot w \end{array}\right\} = 0$$

* ## Orthogonal Subspace

Subspace S orthogonal to Subspace T
$\Rightarrow$ Every vector in S og to Every vector in T

Rowspace og Nullspace
Columnspace og Left Nullspace

* ## Orthogonal Basis

Subspace W

* Basis of W
* Orthogonal Set

$\Big\}$ ob for W

$S = \{ u_1, u_2, u_3 \}$

$$\begin{array}{l} u_1 = \\ u_2 = \\ u_3 = \end{array} \begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ -\frac{1}{2} & -2 & \frac{7}{2} \end{vmatrix}$$

$3(7+2) - 1\left(-\frac{7}{2} + \frac{1}{2}\right) + 1(2+1)$
$27 + 3 + 3 = 33 \neq 0$

* S is orthogonal set
* S forms basis of $\mathbb{R}3$

$\Big\}$ S is ogb

* Orthonormal vector

- Unit Vector
- $u \cdot v = 0$ $u \cdot u = 1$

$v \cdot v = 1$

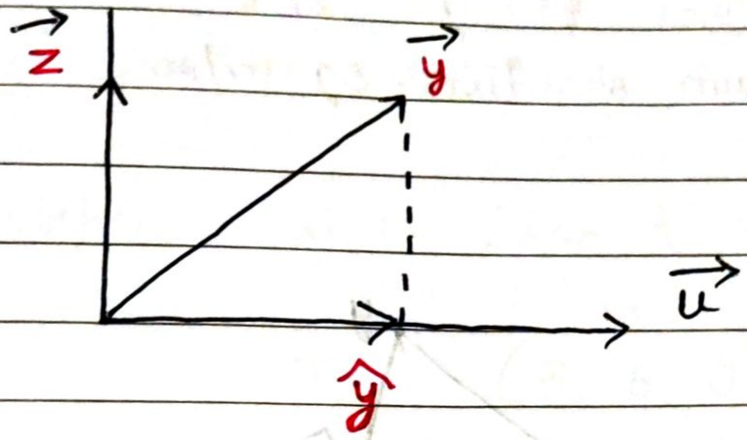
* Orthonormal Set

Orthogonal Set of Unit Vector

* Orthogonal Matrix

- Square Matrix A
- $A^{-1} = A^T$

* Orthogonal Projection



Vector $\vec{u}$
Decompose $\vec{y}$ into sum of two vectors

(i) $\hat{y} = \alpha \vec{u}$

(ii) $\vec{z} = \vec{y} - \hat{y}$

Then $\vec{z}$ is og $\vec{u}$
iff $\vec{z} \cdot \vec{u} = 0$

$$(\vec{y} - \alpha\vec{u}) \cdot (\vec{u}) = \vec{y} \cdot \vec{u} - \alpha \vec{u}\vec{u} = 0$$

$$\alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \qquad \hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$\hat{y}$ = og projection of $\vec{y}$ onto $\vec{u}$

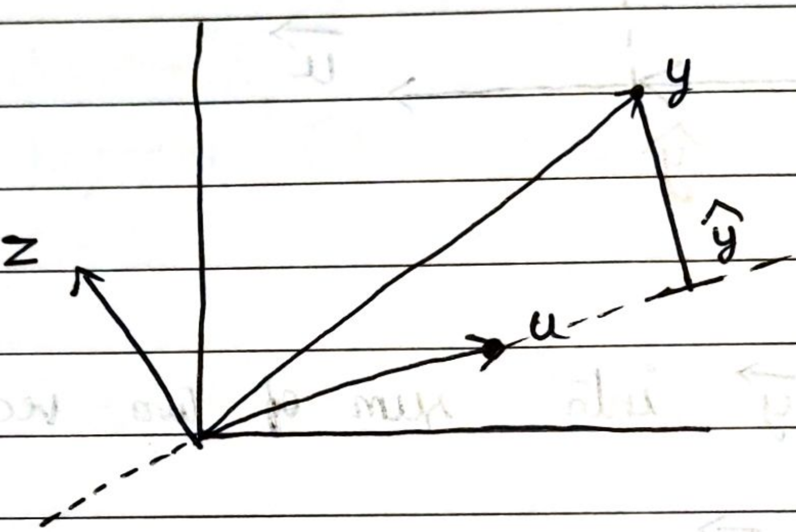
$\vec{z}$ = component of $\vec{y}$ og to $\vec{u}$

$\vec{y} = (7, 6)$ $\quad \vec{u} = (4, 2)$

i) – og projection of $\vec{y}$ onto $\vec{u}$

ii) – $\vec{y}$ as sum of two og vector

→



i) – $\hat{y} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \dfrac{(7,6) \cdot (4,2)}{(4,2) \cdot (4,2)} \vec{u}$

$\hat{y} = \dfrac{28 + 12}{16 + 4} \vec{u} = \dfrac{40}{20} (4, 2) = 2(4,2)$

$\hat{y} = (8, 2)$

ii) – $\vec{y} = \hat{y} + \vec{z}$

$\vec{z} = \vec{y} - \hat{y} = (7,6) - (8,2)$

$\vec{z} = (-1, 4)$

* Gram - Schmidt Process

- Producing Og (or) on basis for any nonzero subspace of $\mathbb{R}^n$

- Subspace $W = \text{Span}\{x_1, x_2\}$

$$x_1 = (3, 6, 0)$$
$$x_2 = (1, 2, 2)$$

- Orthogonal basis $\{v_1, v_2\}$ for W = ?

→ Let P be projection of x_2 onto x_1 ($\hat{y}$)

i) $v_1 = x_1$

ii) $v_2 = x_2 - \dfrac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$

iii) $v_3 = x_3 - \dfrac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \dfrac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$

* QR Factorization

- A is $m \times n$ matrix
  linearly independent columns

- $A = QR$

  Q = Columns form orthonormal basis
  R = Upper triangular Invertible

$A = QR \implies Q^T A = Q^T Q R$

$Q^T A = R$

$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$

→ i) Find orthogonal basis

$u_1 = (1 \ -1 \ -1 \ 1 \ 1)$
$u_2 = (2 \ 1 \ 4 \ -4 \ 2)$
$u_3 = (5 \ -4 \ -3 \ 7 \ 1)$

$v_1 = u_1 = (1 \ -1 \ -1 \ 1 \ 1)$

$v_2 = u_2 - \frac{u_2 v_1}{v_1 v_1} v_1 = (3 \ 0 \ 3 \ -3 \ 3)$

$v_3 = u_3 - \frac{u_3 v_1}{v_1 v_1} v_1 - \frac{u_3 v_2}{v_2 v_2} v_2 = (2 \ 0 \ 2 \ 2 \ -2)$

ii) Construct orthonormal vectors

$u_1 = \frac{1}{\sqrt{5}} \ \frac{-1}{\sqrt{5}} \ \frac{-1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}}$

$u_2 = \frac{1}{2} \ 0 \ \frac{1}{2} \ \frac{-1}{2} \ \frac{1}{2}$

$u_3 = \frac{1}{2} \ 0 \ \frac{1}{2} \ \frac{1}{2} \ \frac{-1}{2}$

$u_2 = \sqrt{\Sigma v_2^2} = 3^2 + 0 + 3^2 + (-3)^2 + 3^2 = \sqrt{36} = 6$

iii) $Q = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$

iv) $R = Q^T A$

$$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

* Least Square

$A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \\ 3 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 7 \\ 4 \\ -3 \\ 1 \end{bmatrix}$

$$\boxed{A^T A x = A^T b}$$

i) $A^T A = \begin{bmatrix} 1 & -2 & -5 & 3 \\ 2 & 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \\ 3 & -1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 39 & -41 \\ -41 & 66 \end{bmatrix}$

ii) $A^T b = \begin{bmatrix} 1 & -2 & -5 & 3 \\ 2 & 5 & 6 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ -3 \\ 1 \end{bmatrix}$

$A^T b = \begin{bmatrix} 17 \\ 15 \end{bmatrix}$

$$A^TA\,x = A^Tb$$

$$\begin{bmatrix} 39 & -41 \\ -41 & 66 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 \\ 15 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 39 & -41 & 17 \\ -41 & 66 & 15 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 39 & -41 & 17 \\ 0 & 893 & 1282 \end{array}\right] \quad 41R_1 + 39R_2$$

$$893\,x_2 = 1282$$

$$x_2 = \frac{1282}{893} = 1.4356$$

$$39x_1 - 41\left(\frac{1282}{893}\right) = 17$$

$$x_1 = \frac{\cancel{36596}}{\cancel{39}} \quad \frac{75.86}{39} = 1.9451$$

$$x = (A^TA)^{-1}A^Tb \quad \text{(Same Answer)}$$

G-S process. Orthonormalise

$B = \{ (1, 2, -1) \quad (1, 2, 1) \quad (1, 1, 1) \}$

$\qquad\quad x_1 \qquad\qquad x_2 \qquad\qquad x_3$

$V_1 = x_1 = (1, 2, -1)$

$V_2 = x_2 - \dfrac{x_2 \cdot V_1}{V_1 \cdot V_1} V_1$

$= (1, 2, 1) - \dfrac{(1, 2, 1)(1, 2, -1)}{(1, 2, -1)(1, 2, -1)} (1, 2, -1)$

$V_2 = \langle 1, 2, 5 \rangle$

$V_3 = x_3 - \dfrac{x_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \dfrac{x_3 V_2}{V_2 \cdot V_2} V_2$

$V_3 = (1, 1, 1) - \dfrac{(1, 1, 1) \cdot (1, 2, -1)}{(1, 2, -1)(1, 2, -1)} (1, 2, -1)$

$- \dfrac{(1, 1, 1)(1, 2, 5)}{(1, 2, 5)(1, 2, 5)} (1, 2, 5)$

$V_3 = (1, 1, 1) - (1, 2, -1) - (4, 8, 20)$

$V_3 = (-4, -9, -18)$

$V_3 = (1, 3, 2)$

$\text{o.n} \Rightarrow \left(\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}} \right) \left(\dfrac{1}{\sqrt{30}}, \dfrac{2}{\sqrt{30}}, \dfrac{5}{\sqrt{30}} \right) \left(\quad \right)$

* Least Square - Application to linear models

$y = \beta_0 + \beta_1 x + \beta_2 x^2$

$\{(1,4)\ (2,4)\ (3,6)\ (4,7)\ (5,8)\}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$A^T y = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{(A^T A)^{-1}} \cdot \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{A^T y}$$

* INNER PRODUCT

It is a function that associates a real number $\langle u, v \rangle$ to u, v vector in V, such that

i) $\langle u, v \rangle = \langle v, u \rangle$

ii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

iii) $\langle cu, v \rangle = c \langle u, v \rangle$

iv) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

Inner Product Space :
Vector Product with inner product

$u = (u_1, u_2)$ $\quad$ $v = (v_1, v_2)$

$\langle u, v \rangle = 4 u_1 v_1 + 5 u_2 v_2$

ST it is an inner product

$<p, q> = p(t_0)\,q(t_0) + p(t_1)\,q(t_1) + p(t_2)\,q(t_2)$

$t_0 = 0 \qquad t_1 = 1/2 \qquad t_2 = 1$

$P(t) = 12t^2$

$q(t) = 2t - 1$

Compute $<p, q>$ and $<q, q>$

* $V \rightarrow$ Inner Product Space with $<u, v>$

* Length $= \|v\| = \sqrt{<v, v>}$

* Distance $= \|u - v\| = \sqrt{<u - v,\ u - v>}$

* u and v are orthogonal if $<u, v> = 0$

Find for the above problem,

- length of p and q
- distance b/w p and q
- verify if p and q are orthogonal

UNIT 3

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* Diagonalization

$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$

→ $\boxed{A = P D P^{-1}}$

i) Find Eigen values

$$\lambda^3 - 11\lambda^2 = 0$$

$$\lambda = 11, 0, 0$$

ii) Find Eigen vectors

For $\lambda = 11$, $A - 11I$

$$A - 11I = \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix}$$

$$\frac{x_1}{11} = \frac{-x_2}{-33} = \frac{x_3}{44} \Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{4}$$

$$V_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

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For $\lambda = 0$, $A - \lambda I$

$$A - 0I = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$$

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0$$

$x_3 = 0$, $x_2 = 1$ $\Rightarrow$ $x_1 = -2(1) - 0 = -2$
$x_3 = 1$, $x_2 = 0$ $\Rightarrow$ $x_1 = 0 - 1 = -1$

$$V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

iii) All eigen vectors are linearly independent
$\Rightarrow$ P exists

iv) $A = PDP^{-1}$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P^{-1}$$

* Quadratic Forms

Quadratic form on $\mathbb{R}^n$ is a function Q whose value at vector x is given by,

$$\boxed{Q(x) = x^T A x}$$

$A \rightarrow$ symmetric matrix (Matrix of Quadratic f)

Compute quadratic form $x^T A x$ when
$A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Q(x) = x^T A x$$
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5x_1 + 1/3\, x_2 \\ 1/3\, x_1 + x_2 \end{bmatrix}$$

$$= x_1 (5x_1 + 1/3\, x_2) + x_2 (1/3\, x_1 + x_2)$$

$$= 5x_1^2 + \frac{1}{3} x_1 x_2 + \frac{1}{3} x_1 x_2 + x_2^2$$

$$Q(x) = 5x_1^2 + \frac{2}{3} x_1 x_2 + x_2^2$$

Find the Matrix of Quadratic Form

1) $Q(x) = 20x_1^2 - 15x_1x_2 - 10x_2^2$

$$A = \begin{bmatrix} x_1 & (12) \\ (12) & x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 20 & -15/2 \\ -15/2 & -10 \end{bmatrix}$$

2) $Q(x) = 5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

$$A = \begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$$

- "Change of Variable" in quadratic form

If x is variable vector,
then change of vector

$$x = Py$$

$$x^T A x = (Py)^T A (Py)$$
$$x^T A x = y^T (P^T A P) y$$

Make a change of variable $x = Py$ that transforms qf into qf without cross product

$x_1^2 + x_1 x_2 (10) + x_2^2$

→ $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

→ Eigen Value : $\lambda^2 - 2\lambda - 24 = 0$

$\lambda = 6, -4$

→ Eigen Vector :

$A - 6I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \Rightarrow \frac{x_1}{-5} = \frac{-x_2}{5} \Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A + 4I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \Rightarrow \frac{x_1}{5} = \frac{-x_2}{5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

→ P and D

$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$

→ $Q(x) = x^T A x$

$Q(x) = x^T (P D P^T) x$ $\because A = PDP^T$

$x = Py$

$Q(x) = (Py)^T (P D P^T) Py$

$Q(x) = y^T \underline{P^T P} D \underline{P^T P} y$

$Q(x) = y^T D y$

$Q(x) = Q(y)$

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→ Quadratic form with no cross product = $Q(y)$

$$Q(y) = y^T D y$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 6y_1^2 - 4y_2^2$$

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* Constrained Optimization

- To find min and max value of $Q(x)$
- Problem can be arranged so that x varies over set of unit vectors

- $||x|| = 1$

$x^T x = 1$

$x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 1$

What is the largest value of quadratic form if $x^T x = 1$

1) $-5x_1^2 + 4x_1x_2 - 2x_2^2$

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$\lambda^2 + 7\lambda + 6 = 0$

$\lambda = -1, -6$

Max value $= -1$

2) $5x_1 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix}$$

$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$\lambda = 9, 6, 3$

Largest value of $Q(x)$ such that $x^T x = 1$ is 9

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\# Find

(a) max value of $Q(x)$, constraints, $x^T x = 1$

(b) unit vector u where this max is obtained

(c) max value of $Q(x)$, constraints,
$x^T x = 1$ and $x^T u = 0$

$$Q(x) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$$

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\lambda^3 + 3\lambda^2 + (-6)\lambda - 8 = 0$$
$$\lambda = 2, -1, -4$$

(a) max value of $Q(x)$ is 2
[max value of $Q(x)$ subject to constraint $x^T x = 1$ is the greatest eigen value]

(b) unit vector u where max value of $Q(x)$ occurs corresponds to greatest eigen value.

For $\lambda = 2$ $\quad A - 2I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}$

$$X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \rightsquigarrow u_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

at u_1, max value is attained

(c) max value of $Q(x)$ subject to constraints $x^T x = 1$ and $x^T u = 0$ occurs at second highest eigen value's unit vector

For $\lambda = -1$

$$A + 1I = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \rightsquigarrow u_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

at u_2, max value is attained subjected to constraints

Singular Values of Matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

i) Find AA^T

ii) Eigen values of AA^T

iii) Singular Values, $\sigma = \sqrt{\lambda}$

Singular Value Decomposition

$$A = U \Sigma V^T$$

i) $A^T A \rightarrow$ Eigen Value $\rightarrow$ Eigen Vector $\rightarrow V$

ii) $\Sigma = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}$ bcos $\sigma = \sqrt{\lambda}$

iii) $AA^T \rightarrow$ Eigen value $\rightarrow$ Eigen vector $\rightarrow U$

* Eigen Vector $\rightarrow$ unit vector

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* Mean, Covariance, Principal Component Analysis

- Matrix of Observations

- Mean Vector

$$M = \frac{X_1 + \cdots + X_N}{N}$$

- Mean Deviation form

$$B = X - M$$

- Covariance Matrix

$$S = \frac{1}{N-1} B B^T$$

Student	1	2	3	4	5
Weight	120	125	125	135	145
Height	61	60	64	68	72

- Find covariance matrix for data
- Make a Principal Component Analysis of the data to find a single size index that explains most of the variation in data

→ $X = \begin{bmatrix} 120 & 125 & 125 & 135 & 145 \\ 61 & 60 & 64 & 68 & 72 \end{bmatrix}$

Mean Vector

$M = \begin{bmatrix} 130 \\ 65 \end{bmatrix}$

Mean Deviation form $(B = X - N)$

$B = \begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}$

Covariance Matrix

$$S = \frac{1}{N-1} B B^T$$

$$S = \frac{1}{5-1} \begin{bmatrix} 400 & 190 \\ 190 & 100 \end{bmatrix}$$

$u_1 \rightarrow$ Principal Component

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$$S = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

100 → Variance of Weight

25 → Variance of Height

* Corelation $= \dfrac{47.5}{\sqrt{100}\sqrt{25}} = 0.95$

* Eigen Value : $\lambda^2 - 125\lambda + 243.75 = 0$

$\lambda = 123.02 , 1.98$

* Eigen Vector :

$$S - 123.02\, I = \begin{bmatrix} -23.02 & 47.5 \\ 47.5 & -98.02 \end{bmatrix}$$

$$\frac{x_1}{-98.02} = \frac{-x_2}{47.5} \Rightarrow X = \begin{bmatrix} 98.02 \\ 47.5 \end{bmatrix} \rightarrow u_1 = \begin{bmatrix} 0.90 \\ 0.44 \end{bmatrix}$$

~~$S - 1.98 I$~~

$y = c_1 x_1 + c_2 x_2$

$y = 0.90\, w + 0.44\, h$

$$u_1 = \begin{array}{cc} \begin{bmatrix} 0.90 \\ 0.44 \end{bmatrix} & \begin{array}{l} \text{weight} \\ \text{height} \end{array} \end{array}$$

Variance of this index over the ~~index~~ dataset is 123.02

And it accounts for $\dfrac{123.02}{123.02 + 1.98} = 0.984$

98.4 % of the variance of the data

UNIT 4

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S - Sample Space

$X = \{x_1 \quad x_2 \quad x_n\}$

$Y = \{y_1 \quad y_2 \quad y_3\}$

$P_i = P(x_i) = P(X = x_i) = P(s \in S \mid f(s) = x_i$

$P_j = P(y_j) = P(Y = y_j) = P(s \in S \mid g(s) = y_j$

* JOINT PROBABILITY MASS

→ $P_{ij} = P(x_i, y_j) \geq 0$

→ $\sum_i \sum_j P_{ij} = 1$

* JOINT Distribution Table

X \ Y	y_1	y_2	y_3	
x_1	P_{11}	P_{12}	P_{13}	P_1
x_2	P_{21}	P_{22}	P_{23}	P_2
	Q_1	Q_2	Q_3	

* Marginal Distribution

$P_i = P_{i1} + P_{i2} + P_{i3}$

$Q_j = P_{1j} + P_{2j} + P_{3j}$

Stoc Inde

$P_{ij} = P_i Q_j$

* Expectations

$$E(X) = \sum x_i \, P(x_i)$$

$$E(Y) = \sum y_j \, P_j$$

$$E(XY) = \sum \sum x_i \, y_j \, P_{ij}$$

$$E(X+Y) = \sum \sum (x_i + y_j) \, P_{ij}$$

$$E(X^2) = \sum (x_i)^2 \, P_i$$

* Variance

$$V(X) = \sigma_x^2 = E(X^2) - [E(X)]^2$$

* Covariance

$$Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$$

* $P(X, Y) = \dfrac{Cov(X, Y)}{\sigma_x \, \sigma_y}$

$$R_X = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] \\ E[X_1 X_2] & E[X_2]^2 \end{bmatrix}$$

$$C_X = R_X - E[X_1] \cdot E[X]^T$$

Y \ X	-1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

Marginal Distribution

$Q(X=-1) = Q_1 = 1+3+2 = 6/15$
$Q(X=0) = Q_2 = 2+2+1 = 5/15$
$Q(X=1) = Q_3 = 1+1+1 = 3/15$

$P(Y=0) = P_1 = 1+2+1 = 4/15$
$P(Y=1) = P_2 = 3+2+1 = 6/15$
$P(Y=2) = P_3 = 2+1+2 = 5/15$

X	-1	0	1
Q(X)	6/15	5/15	3/15

Y	0	1	2
P(Y)	4/15	6/15	5/15

Conditional Distribution

$P(X \mid Y=2)$

$\dfrac{P(X=-1 \cap Y=2)}{P(Y=2)} = \cancel{6/15}\ \dfrac{2/15}{5/15}$

$P(X=0 \cap Y=2)$
$P(X=1 \cap Y=2)$

$P(X \leq 1, Y = 2) =$

$P(X = -1, Y = 2) + P(X = 0, Y = 2) + P(X = 1, Y = 2)$

$$\frac{2}{15} + \frac{1}{15} + \frac{2}{15} = \frac{1}{3}$$

COIN tossed 3 times

$X = \{0, 1\}$ H & T on 1st toss

$Y =$ Total no of heads occuring

i) Joint Probability Distribution

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

$X = \{\overset{H}{0}, \overset{T}{1}\}$

$Y = \{0, 1, 2, 3\}$

X \ Y	0	1	2	3
0	0	1/8	2/8	1/8
1	1/8	2/8	1/8	0

$P_{22} = P(X = 0, Y = 2) = P(X = \text{Head}, Y = 2 \text{ Heads})$

$= P\{HTH, HHT\}$

$$= \frac{2}{8}$$

ii) ~~Joint Probability~~ Marginal Distribution

$$P_1 = P(X=0) = 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = 1/2$$

$$P_2 = P(X=1) = \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 = 1/2$$

$$Q_1 = Q(Y=0) = 0 + \frac{1}{8} = 1/8$$

$$Q_2 = Q(Y=1) = 1/8 + \frac{2}{8} = 3/8$$

$$Q_3 = Q(Y=2) = 2/8 + \frac{1}{8} = 3/8$$

$$Q_4 = Q(Y=3) = 1/8 + 0 = 1/8$$

iii) Expectations

$$E(X) = \sum x_i \, P(x_i)$$

$$= 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)$$

$$= 1/2 = 0.5$$

$$E(Y) = \sum y_j \, Q(y_j)$$

$$= 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= 12/8 = 3/2 = 1.5$$

$$= \cancel{1.625}$$

$$E(XY) = \sum \sum x_i \, y_j \, P_{ij}$$

$$= (0 \times 0 \times 0) + \left(0 \times 1 \times \frac{1}{8}\right) + \left(0 \times 2 \times \frac{2}{8}\right) + 0$$

$$(1 \times 0 \times \ ) + \left(1 \times 1 \times \frac{2}{8}\right) + \left(1 \times 2 \times \frac{1}{8}\right) + (1 \times 3 \times 0)$$

$$= \frac{2}{8} + \frac{2}{8} = 1/2$$

$$E(X+Y) = \frac{1}{8} + \frac{4}{8} + \frac{3}{8} + \frac{1}{8} + \frac{4}{8} + \frac{3}{8} + 0 = \frac{16}{8} = 2$$

Covariance

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$
$$= \frac{1}{2} - \frac{1}{2}\,\frac{13}{8}$$
$$= -5/16 = -0.3125$$
$$= \frac{1}{2} - \frac{1}{2}\times\frac{3}{2} = \frac{1}{2} - \frac{3}{4}$$
$$= -1/4 = -0.25$$

Variance

$$E(X^2) = 1/2$$
$$E(Y^2) = 3$$

$$[E(X)]^2 = 1/4 \qquad [E(Y)]^2 = 9/4$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$
$$= 1/2 - 1/4$$
$$= 1/4$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2$$
$$= 3 - 9/4$$
$$= 3/4$$

$P(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{-1/4}{\sqrt{\frac{1}{4}}\sqrt{\frac{3}{4}}} = \frac{-1}{4}\times\frac{16}{3} = \frac{-4}{3}$

$$= \frac{-1}{4}\times\frac{2}{1}\times\frac{2}{1}\times\frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

Die 2 throws

X = No of 4s
Y = No of 5s

→ S = { 11, 12, 13, 14, 15, 16,
21, 22, 23, 24, 25, 26,
31 32 33 34 35 36
41 42 43 44 45 46
51 52 53 54 55 56
61, 62, 63, 64, 65, 66 }

X = {0, 1, 2} 4s
Y = {0, 1, 2} 5s

i) Joint Distribution Table

X \ Y	0	1	2
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

ii) Joint Distribution Table and Marginal Distribution

X \ Y	0	1	2	P(X)
0	4/9	2/9	1/36	25/36
1	2/9	1/18	0	10/36
2	1/36	0	0	1/36
Q(Y)	25/36	10/36	1/36	1

X	0	1	2
P(X)	25/36	10/36	1/36

Y	0	1	2
Q(Y)	25/36	10/36	1/36

iii) $E(X) = \sum x \, P(x)$

$= 0 + 10/36 + 2/36$

$= 12/36 = 1/3$

$E(Y) = 0 + 10/36 + 2/36$

$= 12/36 = 1/3$

$E(XY) = \sum \sum x_i \, y_j \, P_{ij}$

$= (0 \times 0 \times \frac{16}{36}) + (0) + (0) +$

$(1 \times 0) + (1 \times 1 \times \frac{2}{36}) + 0 +$

$0 + 0 + 0$

$= 2/36 = 1/18$

iv) $Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$

$= \frac{1}{18} - \frac{12}{36} \cdot \frac{12}{36}$

$= -2591 / 18 = -143.94$

$= -1/18$

Integration

* $\int (1) = [x]$

* $\int x = \left[\frac{x^2}{2}\right]$

* $\int x^3 = \left[\frac{x^4}{4}\right]$

* $\int (x^2+y^2)\,dx = \left[\frac{x^3}{3} + y^2 x\right]$

* $\int (x^2+y^2)\,dy = \left[x^2 y + \frac{y^3}{3}\right]$

* $\int x^2 y^2 \,dy = x^2 \left[\frac{y^3}{3}\right]$

* $\int e^{ax}\,dx = \left[\frac{e^{ax}}{+a}\right]$

* JOINT PROBABILITY DENSITY Function

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y) \, dx \, dy = 1$$

* Marginal Distribution

X $\quad P_{i\cdot} = P_1(x) = \int_{-\infty}^{\infty} P(x, y) \, dy$

Y $\quad P_{\cdot j} = P_2(y) = \int_{-\infty}^{\infty} P(x, y) \, dx$

* Stochastic Independence

$$P_1(x) \cdot P_2(y) = P(x, y)$$

* Expectation

$$E[X] = \int x \cdot P_1(x) \, dx$$

$$E[Y] = \int y \, P_2(y) \, dy$$

$$E[X+Y] = \iint (x+y) \, P(x, y) \, dy \, dx$$

$f_{X,Y}(x,y) = \begin{cases} \frac{1}{24}xy & 1<x<3 \quad 2<y<4 \\ 0 & \text{otherwise} \end{cases}$

$\rightarrow P(x,y) = \begin{cases} \frac{1}{24}xy \\ 0 \end{cases}$

* $P_1(x) = \int_2^4 P(x,y)\,dy$

$= \int_2^4 \frac{xy}{24}\,dy$

$= \frac{x}{24}\left[\frac{y^2}{2}\right]_2^4 = \frac{x}{24}\left[\frac{16}{2} - \frac{4}{2}\right]$

$= \frac{x}{24}(6) = \frac{x}{4}$

* Conditional Probability

$P(y|x) = \frac{P(x,y)}{P(x)} = \frac{xy}{24} \times \frac{4}{x} = \frac{y}{6}$

$= \begin{cases} y/6 & 1<x<3 \quad 2<y<4 \\ 0 & \text{otherwise} \end{cases}$

$$* \quad P_2(y) = \int_1^3 P(x,y)\, dx$$

$$= \int_1^3 \frac{xy}{24}\, dx$$

$$= \frac{y}{24}\left[\frac{x^2}{2}\right]_1^3 = \frac{y}{24}\left(\frac{9}{2} - \frac{1}{2}\right)$$

$$= \frac{y}{24}(4) = \frac{y}{6}$$

* Condiditional Probabrlity

$$P(x|y) = \frac{P(x,y)}{P(y)} = \frac{xy}{24} \times \frac{6}{y} = \frac{x}{4}$$

$$= \begin{cases} x/4 \\ 0 \end{cases}$$

UNIT 5

$(a \mid b \mid c) : (d \mid e)$

$(M \mid M \mid 1) : (\infty \mid FIFO)$

$(M \mid M \mid S) : (\infty \mid FIFO)$

Arrival

Service

No of Servers

Capacity

Queve Discipline

$\mu \rightarrow$ Service Rate

$\lambda \rightarrow$ Arrival Rate

Arrivals, Poisson, avg time 12 min

Length of phone call, Exponentially, with mean 4 min

$\lambda = \frac{1}{12}$ $\mu = \frac{1}{4}$

i) Average no of person waiting in system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = \frac{3}{8} = \frac{1}{2}$$

ii) Probability person arriving has to wait

$$P(N > 0) = \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

iii) Probability it takes more than 10 min wait

$$P(W > k) = e^{-(\mu - \lambda)k}$$

$$P(W > 10) = e^{-\left(\frac{1}{4} - \frac{1}{12}\right)10}$$

$$= e^{-\frac{10}{6}}$$

$$= 0.18887$$

iv) Average waiting time of each person

$$E[W] = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{4} - \frac{1}{12}} = 6$$

iv) Idle time

$$P_0 = 1 - \frac{\lambda}{\mu}$$

v) Active time

$$P_a = 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

* Average NUMBER waiting (L_s)

$$\frac{\lambda}{\mu-\lambda}$$

* Average TIME waiting $E(W)$

$$\frac{1}{\mu-\lambda}$$

---

* Probability Person has to WAIT

$$P(N>0) = \frac{\lambda}{\mu}$$

* Probability WAIT time > 10 mins

$$P(W>10) = e^{-(\mu-\lambda)10}$$

* Probability QUEUE time > 10 mins

$$P(N>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

* Probability NUMBER is > k

$$P(N>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

* Idle TIME

$P_0 = 1 - \frac{\lambda}{\mu}$

* Active TIME

$P_a = 1 - P_0$

* Average TIME in Queue

$\frac{\lambda}{\mu(\mu - \lambda)}$

* $S = 2$ $\quad \mu = \frac{1}{5}$ $\quad \lambda = 12$

→ Fraction of time BUSY $= \frac{\lambda}{\mu S}$

→ Fraction of time IDLE $= 1 -$ above

$\mu = \frac{1}{10}$ $\quad \lambda = \frac{1}{12}$

$\Rightarrow$ $\cancel{E(s)} = \cancel{\frac{\lambda}{\mu}}$

a) Expected Number in ~~Queue~~ System

$$\frac{\lambda}{\mu - \lambda} = \frac{1}{12} \times \frac{60}{1} = 5$$

Expected Number in ~~System~~ Queue

$$\frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\left(\frac{1}{12}\right)^2}{\left(\frac{1}{10}\right)\left(\frac{1}{60}\right)} = \frac{25}{6} = 4.17$$

b) Idle time $= P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{6}$

c) Average WAIT time

$$\frac{1}{\mu - \lambda} = \frac{1}{1/60} = 60 \text{ mins}$$

d) Wait time $\Rightarrow$ 75 mins

$$\frac{1}{\mu - \lambda_{new}} > 75$$

$$\lambda_{new} > \mu - \frac{1}{75} \Rightarrow \lambda > \frac{13}{150}$$

$$\lambda \uparrow \text{ by } \frac{13}{150} - \frac{1}{12} = \frac{1}{300}$$

e) Avg time in Queue

$$\frac{\lambda}{\mu(\mu-\lambda)} = 50$$

f) $P(W > 30) = e^{-(\mu-\lambda)30}$

$$= 0.6065$$

g) People who have to WAIT percentage

$$P[W > 0] = 1 - P[W=0]$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = 83.3\%$$

h) More than 3 unit in Queue

$$P[N > 3] = \left(\frac{\lambda}{\mu}\right)^4 = 0.4823$$

Unit I

- Four Fundamental
- Row and Column picture
- Linear Transform
- Subspace and vector space

Unit II

- QR Factorization
- Least square

Unit III

- SVD
- Diagonalization
- Quadratic form
- Mean and Covariance

Unit IV

- Joint probability
- Gaussian Distribution

Unit V

- Compute the unique final probability vector of the stochastic Matrix $A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$. Also show that A is regular

NOTES

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 + 10 \\ 4 \quad 3 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 7 \end{bmatrix}$$

* Inverse of a Matrix

* A is invertible if it is
→ non singular
→ det $|A| \neq 0$

* $Ax \neq 0$ for any x

* $AA^{-1} = I = A^{-1}A$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

- Gauss - Jordan

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right]$$

A^{-1}

* Characteristic Equation

$$\lambda^2 - (\text{Trace of } A)\lambda + (\text{Det of } A) = 0$$

$$\lambda^3 - (\text{Trace of } A)\lambda^2 + \begin{pmatrix}\text{Sum of Minor} \\ \text{along diagonal}\end{pmatrix}\lambda - (\text{Det of } A) = 0$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\text{Trace of } A = 1 + 2 + 3 = 6$$

Sum of Minors along diagonal }

$$M_{11} = (3 \times 2) - (1 \times 0) = 6$$
$$M_{22} = (3 \times 1) - (2 \times 2) = -1$$
$$M_{33} = (2 \times 1) - (0 \times 0) = 2$$

$$6 - 1 + 2 = 7$$

$$\text{Det of } A = 1\,(3 \times 2 - 1 \times 0) - 0\,(\quad) + 2\,(0 - 4)$$