Inner product:-

An inner product on a vector space V is a function that, to each pair of vectors u and v in V, associates a real number $\langle u,v\rangle$ and satisfies the following axioms for all u,v,w in V and all scalars c:

(i). $\langle u,v\rangle = \langle v,u\rangle$

(ii) $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

(iii) $\langle cu, v\rangle = c\langle u,v\rangle$

(iv) $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u=0$

A vector space with an inner product is called an inner product space.

ex 1:- Fix any two positive numbers - say, 4 and 5 - and for vectors $u=\langle u_1, u_2\rangle$ and $v=(v_1,v_2)$ in $\mathbb{R}^2$, set $\langle u,v\rangle = 4u_1v_1 + 5u_2v_2$. -(1). Show that (1) defines an inner product.

soln:- (i) $\langle u,v\rangle = 4u_1v_1 + 5u_2v_2 = 4v_1u_1 + 5v_2u_2 = \langle v,u\rangle$

(ii) $\langle u+v, w\rangle = 4(u_1+v_1)w_1 + 5(u_2+v_2)w_2$

$= 4u_1w_1 + 5u_2w_2 + 4v_1w_1 + 5v_2w_2$

$= \langle u,w\rangle + \langle v,w\rangle$

(iii) $\langle cu, v\rangle = 4(cu_1)v_1 + 5(cu_2)v_2$

$= c(4u_1v_1 + 5u_2v_2)$

$= c\langle u,v\rangle$

(iv) $\langle u,u\rangle = 4u_1^2 + 5u_2^2 \geq 0$.

& $\langle u,u\rangle = 0$ iff $u=(0,0)$.

$\therefore$ (1) defines an inner product.

ex 2: Let V be $\mathbb{P}_2$, with the inner product defined by $\langle p,q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2)$, where $t_0=0$, $t_1 = 1/2$, $t_2=1$. Let $p(t) = 12t^2$ and $q(t) = 2t-1$. Compute $\langle p,q\rangle$ and $\langle q,q\rangle$.

solⁿ: $\langle p,q\rangle = p(0)q(0) + p(\tfrac{1}{2})q(\tfrac{1}{2}) + p(1)q(1)$

$= 0\times(-1) + 3\times 0 + 12\times 1$

$= 12$

$\langle q,q\rangle = [q(0)]^2 + [q(\tfrac{1}{2})]^2 + [q(1)]^2$

$= (-1)^2 + 0^2 + 1^2$

$= 2$

Lengths, Distances and Orthogonality

Let V be an inner product space, with the inner product denoted by $\langle u,v\rangle$. The length or norm of a vector v to be the scalar $\|v\| = \sqrt{\langle v,v\rangle}$.

Equivalently, $\|v\|^2 = \langle v,v\rangle$.

A unit vector is one whose length is 1.

The distance between u and v is $\|u-v\|$.

Vectors u and v are orthogonal if $\langle u,v\rangle = 0$.

ex: Let $\mathbb{P}_2$ have the inner product $\langle p,q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2)$, where $t_0=0$, $t_1=1/2$, $t_2=1$. Compute the lengths of the vectors $p(t) = 12t^2$ and $q(t) = 2t-1$.

solⁿ: $\|p\|^2 = \langle p,p\rangle = [p(0)]^2 + [p(\tfrac{1}{2})]^2 + [p(1)]^2$

$= 0^2 + 3^2 + 12^2$

$= 153 \quad \Rightarrow \|p\| = \sqrt{153}$

$\|q\|^2 = \langle q,q\rangle = [q(0)]^2 + [q(\tfrac{1}{2})]^2 + [q(1)]^2 = (-1)^2 + 0^2 + 1^2 = 2 \quad \therefore \|q\| = \sqrt{2}$

The Gram-Schmidt Process

Suppose $\{u_1, u_2 \cdots u_p\}$ is a basis of an inner-product space W. Define $v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$\vdots$$

$$v_p = u_p - \frac{\langle u_p, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \cdots \frac{\langle u_p, v_{p-1}\rangle}{\langle v_{p-1}, v_{p-1}\rangle} v_{p-1}$$

Then $\{v_1, v_2 \cdots v_p\}$ is an orthogonal basis of W. The process of constructing an orthogonal basis is called Gram-Schmidt process.

ex Let V be $\mathbb{P}_4$ with the inner product defined by $\langle p, q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2) + p(t_3)q(t_3) + p(t_4)q(t_4)$ where $t_0 = -2$, $t_1 = -1$, $t_2 = 0$, $t_3 = 1$, $t_4 = 2$, and $\mathbb{P}_2$ be a subspace of V. Produce an orthogonal basis for $\mathbb{P}_2$ by applying Gram-Schmidt process to the polynomials $1, t, t^2$.

Sol^n The polynomials $1, t, t^2$ evaluated at $-2, -1, 0, 1, 2$ are given as below

$$\underset{P_0}{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}, \underset{P_1}{\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}}, \underset{P_2}{\begin{bmatrix}4\\1\\0\\1\\4\end{bmatrix}}$$

The polynomial vector t is orthogonal to the polynomial vector 1. Therefore let $P_0(t) = 1$, $P_1(t) = t$.

Now, $$P_2(t) = t^2 - \frac{\langle t^2, P_0\rangle}{\langle P_0, P_0\rangle} P_0(t) - \frac{\langle t^2, P_1\rangle}{\langle P_1, P_1\rangle} P_1(t)$$

$$= t^2 - \frac{10}{5} \times 1 - \frac{0}{10} \times t$$

$$P_2(t) = t^2 - 2$$

$\therefore \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\\-2\\-1\\2\end{bmatrix}$ is the orthogonal basis of $\mathbb{P}_2$ of V

Best approximation in Inner Product Spaces

Suppose a function f in V has to be approximated by a function g from a specified subspace W of V. The closeness of the approximation of f depends on the way $\|f-g\|$ is defined. We will consider only the case in which the distance between f and g is determined by an innerproduct. In this case, the best approximation to f by functions in W is the orthogonal projection of f onto the subspace W.

ex let V be $\mathbb{P}_4$ with the inner product

$$\langle p,q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2) + p(t_3)q(t_3) + p(t_4)q(t_4),$$

where $t_0=-2,\ t_1=-1,\ t_2=0,\ t_3=1,\ t_4=2$. let P_0, P_1 and P_2 be the orthogonal basis, where $P_0=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, $P_1\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$, $P_2\begin{bmatrix}2\\-1\\-2\\-1\\2\end{bmatrix}$ for the subspace $\mathbb{P}_2$.

Find the best approximation to $p(t)=5-\frac{1}{2}t^4$ by polynomials in $\mathbb{P}_2$.

Soln: $p(t)=5-\frac{1}{2}t^4$ evaluated at $-2,-1,0,1,2$ is $p=\begin{bmatrix}-3\\9/2\\5\\9/2\\-3\end{bmatrix}$

The best approximation in V to p by polynomials in $\mathbb{P}_2$ is

$$\hat{p} = \text{proj}_{\mathbb{P}_2} p = \frac{\langle p,P_0\rangle}{\langle P_0,P_0\rangle}P_0 + \frac{\langle p,P_1\rangle}{\langle P_1,P_1\rangle}P_1 + \frac{\langle p,P_2\rangle}{\langle P_2,P_2\rangle}P_2$$

$$= \frac{8}{5}\times 1 + \frac{(-31)}{14}\times(t^2-2)$$

$$\hat{P} = \frac{211}{35} - \frac{31t^2}{14}$$ is the polynomial closest to p to all polynomials in $\mathbb{P}_2$.

An inner product for $C[a,b]$

$C[a,b]$ the set of all continuous functions on an interval $a \le t \le b$ is a vector space.

For f, g in $C[a,b]$, $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$

defines an innerproduct on $C[a,b]$, since

(i) $\langle f, g\rangle = \int_a^b f(t)\,g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f\rangle$

(ii) $\langle f+g, h\rangle = \int_a^b (f(t)+g(t))\,h(t)\,dt = \int_a^b f(t)\,h(t)\,dt + \int_a^b g(t)\,h(t)\,dt$

$= \langle f, h\rangle + \langle g, h\rangle$

(iii) $\langle cf, g\rangle = \int_a^b (c f(t))\,g(t)\,dt = c\int_a^b f(t)\,g(t)\,dt = c\langle f, g\rangle$

(iv) $\langle f, f\rangle = \int_a^b f(t)\,\cancel{f}(t)\,dt = \int_a^b [f(t)]^2\,dt \ge 0$ $\left[\because [f(t)]^2\right.$ is continuous and nonnegative on $\left.[a,b]\right]$

if $\langle f, f\rangle = 0$ then $\int_a^b [f(t)]^2\,dt = 0 \Rightarrow f(t)$ is a zero function.

if $f(t)$ is a zero function, then $\int_a^b [f(t)]^2\,dt = 0 \Rightarrow \langle f, f\rangle = 0$.

ex. Let V be the space $C[0,1]$ with the innerproduct $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, and let W be the subspace spanned by the polynomials $p_1(t) = 1$, $p_2(t) = 2t-1$, $p_3(t) = 12t^2$. Use the Gram-Schmidt process to find an orthogonal basis for W.

Sol^n. let $q_1 = P_1$,

$$q_2 = P_2 - \frac{\langle P_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1 = 2t-1 - \frac{\int_0^1 (2t-1)\cdot 1\,dt}{\int_0^1 1\cdot 1\,dt} = 2t-1-\frac{(t^2-t)]_0^1}{t]_0^1} = 2t-1$$

$$q_3 = P_3 - \frac{\langle P_3, q_1\rangle}{\langle q_1, q_1\rangle} q_1 - \frac{\langle P_3, q_2\rangle}{\langle q_2, q_2\rangle} q_2 = 12t^2 - \frac{\int_0^1 12t^2\,dt}{\int_0^1 1\,dt}(1) - \frac{\int_0^1 (24t^3-12t^2)\,dt}{\int_0^1 (4t^2-4t+1)\,dt}(12t-1) = 12t^2 - \frac{4}{1}(1) - \frac{2}{(1/3)}(12t-1)$$

$$= 12t^2 - 12t + 2.$$

Created With Tiny Scanner

Applications of inner product spaces to Fourier Series

Continuous functions are often approximated by linear combinations of sine and cosine functions.
Consider functions on $0 \le t \le 2\pi$.
Any function in $C[0, 2\pi]$ can be approximated as closely as desired by a function of the form

$$\frac{a_0}{2} + a_1 \cos t + \dots + a_n \cos nt + b_1 \sin t + \dots + b_n \sin nt \quad -(1)$$

for a sufficiently large value of n.
The function (1) is called a trignometric polynomial.
If a_n and b_n are not both zero, the polynomial is said to be of order n.
The set $[1, \cos t, \cos 2t, \dots, \cos nt, \sin t, \sin 2t, \dots, \sin nt]$ $-(2)$
is orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(t) g(t)\, dt \quad -(3)$$

Let W be the subspace of $C[0, 2\pi]$ spanned by the functions in (2). Given f in $C[0, 2\pi]$, the best approximation to f by functions in W is called the n^{th} order Fourier approximation to f on $[0, 2\pi]$. Since the functions in (2) are orthogonal, the best approximation is given by the orthogonal projection onto W. In this case, the coefficients a_k and b_k in (1) are called the Fourier Coefficients of f. The standard formula for an orthogonal projection shows that

$$a_k = \frac{\langle f, \cos kt\rangle}{\langle \cos kt, \cos kt\rangle}, \quad b_k = \frac{\langle f, \sin kt\rangle}{\langle \sin kt, \sin kt\rangle}, \quad k \geq 1$$

And since $\langle \cos kt, \cos kt\rangle = \pi$, $\langle \sin kt, \sin k\pi\rangle = \pi$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\, dt, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\, dt,$$

$$\cancel{\frac{a_0}{2} = \frac{1}{\pi}\int_0^{2\pi} f(t).1\,dt} \text{ as } \frac{a_0}{2} = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi}\int_0^{2\pi} f(t).1\, dt$$

~~ex~~ Find the n^{th}-order Fourier approximation to the function $f(t) = t$ on the interval $[0, 2\pi]$.

Sol^n $\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} t\, dt = \frac{1}{2\pi}\cdot\left[\frac{t^2}{2}\right]_0^{2\pi} = \pi$

for $k > 0$, $a_k = \frac{1}{\pi}\int_0^{2\pi} t\cos kt\, dt = \frac{1}{\pi}\left[t \times \frac{\sin kt}{k} + \frac{\cos kt}{k^2}\right]_0^{2\pi}$

$$= \frac{1}{\pi}\left[\left(2\pi \times \frac{0}{k} + \frac{1}{k^2}\right) - \left(0 + \frac{1}{k^2}\right)\right] = 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} t\sin kt\, dt = \frac{1}{\pi}\left[t \times \left(-\frac{\cos kt}{k}\right) + \frac{\sin kt}{k^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{2\pi \cdot 1}{k} + \frac{0}{k^2}\right) - (-0 + 0)\right]$$

$$= -\frac{2}{k}$$

$\therefore$ the n^{th}-order Fourier approximation of $f(t) = t$ is

$$t = \pi - 2\sin t - \sin 2t - \frac{2}{3}\sin 3t - \cdots - \frac{2}{n}\sin nt.$$

* The expression $f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m\cos mt + b_m \sin mt)$ is called the Fourier series for f on $[0, 2\pi]$.

The term $a_m \cos mt$ is the projection of f onto the one-dimensional subspace spanned by $\cos mt$.

The term $b_m \sin mt$ is the projection of f onto the one-dimensional subspace spanned by $\sin mt$.