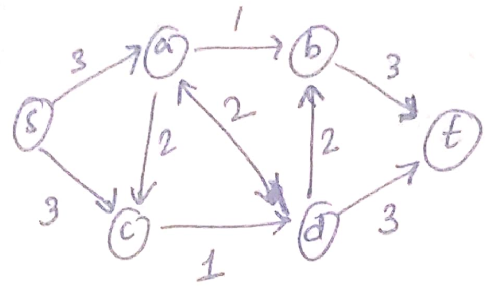
flow n/w

(I) ①

$G = (V, E)$
digraph
$(u,v) \in E$
$c(u,v) \geq 0$



If $f(s,a) = 1$
$f(a,s) = -1$

A flow in G is a real valued fn
$f : V \times V \rightarrow R$ such that

real value fn

① $f(u,v) \leq c(u,v)$: capacity rule

i.e flow on an edge cannot exceed capacity on the edge

② skew symmetry
$f(u,v) = -f(v,u)$

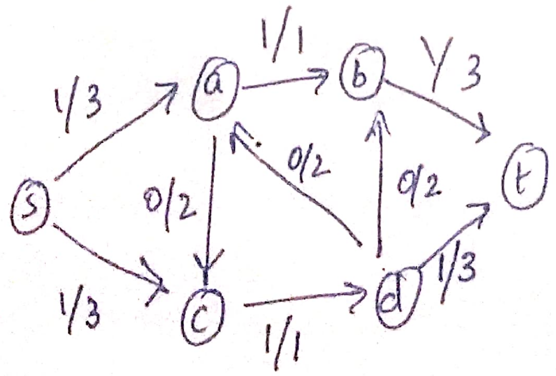
Eg: If $f(s,a) = 1$
then $f(a,s) = -1$

③ flow conservation
for every $v \notin \{s,t\}$

$$\sum_{u \in V} f(u,v) = \sum_{w \in V} f(v,w)$$

Amt of flow into a node = the amount of flow out of it.

f/c



Amt of flow into a = 0 + 1 = 1 ∴ Amt of flow ~~into a~~ out of 'a' = 1

b = 0 + 1 = 1 ∴ B = 0 + 1 = 1

c = 1 + 0 = 1 ∴ c = 2

2

- given a n/w ; what is the maximum amt of flow.

(Optimal path) from s to t.

Problem : given a flow n/w G, find maximum flow from 's' to 't'

Total Amt of flow from 's' to 't' is 2

Total outgoing flow from s is 2. (s to a is 1 + s to c is 1)

Total incoming flow to t is 2 (b to t is 1 + d to t is 1)

$\therefore$ The value of flow is $|f| = \sum_{u \in V} f(s,u) = \sum_{u \in V} f(u,t)$

$\therefore$ The total flow from the source is the same as the total flow into sink.

Residual N/W:

Consider an arbitrary flow in a n/w G. the residual n/w G_f has the same vertices as the original n/w and one or two edges for each edge (u,v) in the original n/w.

If f(u,v) is strictly less than capacity

1. if $f(u,v) < c(u,v)$ then there is a forward edge (u,v) with capacity $c_f(u,v) = c(u,v) - f(u,v)$

(2) If $f(u,v) > 0$ there is a backward edge (v,u) with capacity $c_f(v,u) = f(u,v)$

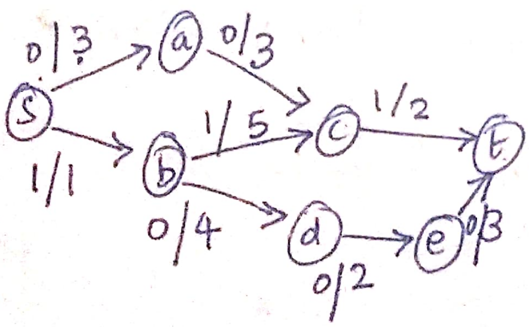
Consider an arbitrary flow f in a n/w, G

The residual n/w, G_f has the same vertices as the original n/w, and one or two edges for each edge (u,v) in the original n/w.

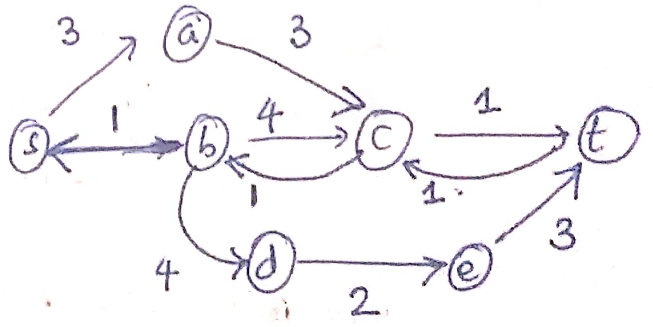
(1) If $f(u,v) < c(u,v)$ then there is a forward edge (u,v) with capacity $c_f(u,v) = c(u,v) - f(u,v)$

(2) If $f(u,v) > 0$, there is a backward edge (v,u) with capacity $c_f(v,u) = f(u,v)$.

f/c ; $f \leq c$



G



G_f

Augmenting path : s to t. $s \to a \to c \to t$

$P = \langle s, a, c, t \rangle$

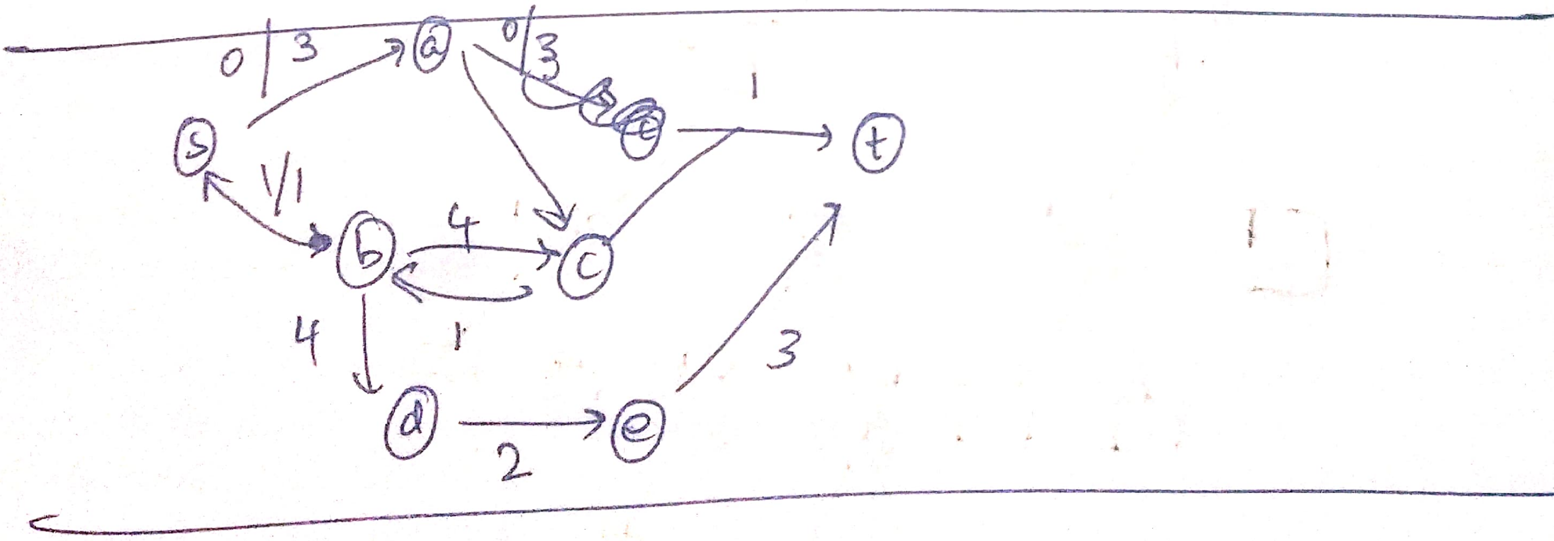
Path capacity $c_f(P)$ is the min capacity of an edge along that path

An augmenting path is simply a path from Source, s to sink, t in the Residual n/w, whose purpose is to increase the flow.

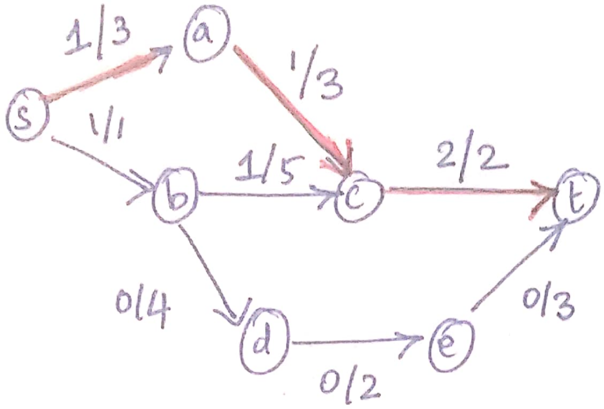
$$\delta(p) = \min\{\underset{3}{c(s,a)}, \underset{3}{c(a,c)}, \underset{1}{c(c,t)}\} = \min\{3,3,1\} = 1$$

path capacity is 1

- By considering the augmenting path $p = (s,a,c,t)$ we can increase the flow by $\delta(p) = 1$ unit as the path capacity.



Increase the flow along the path, P with one unit yield the following flow :



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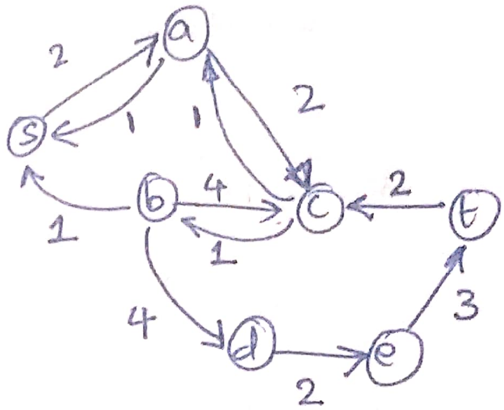


G_f

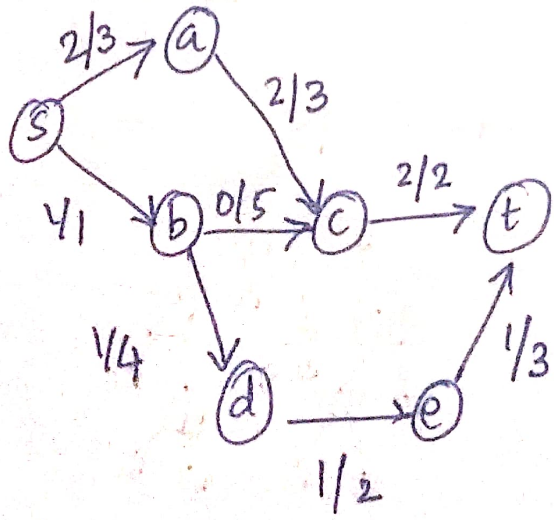
$|f| = 2$

FORD- FULKERSON ALGO

Augmenting Path : s-a, a-c, c-b, b-d, d-e, e-t.

$\delta(P) = \min\{2, 2, 1, 4, 2, 3\} = 1$ unit.

The flow after augmenting 1 unit of flow is :



Edge (c-b) is backward edge

∴ the flow will be cancelled.

(c-b) is used in order to cancel the flow (b,c)

$|f| = 3$

Compute the Residual N/w for the new flow

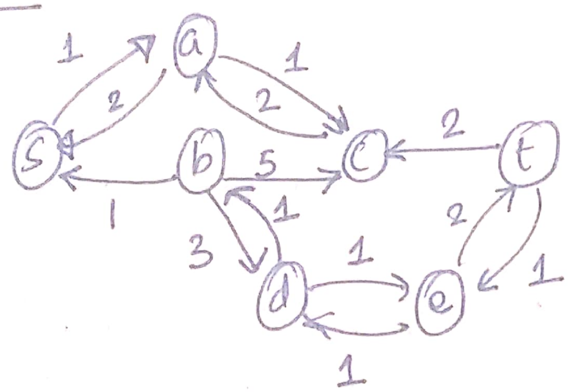
To check if this is the max flow;

find an augmenting path; if it exists continue

else stop. i.e the max flow.

Residual

G_f



Find an augmenting path from s to t

there is no augmenting path from s to t; does not exists in this residual n/w. STOP.

$\therefore$ the current flow $|f| = 3$ is the maximum.

Ford & Fulkerson, 1956 (Augmenting path algo)

Input: N/w G

o/p: Maximum flow, f.

$f = 0$

While (G_f contains an augmenting path, P in G_f)

{ Identify an augmenting path P in G_f;

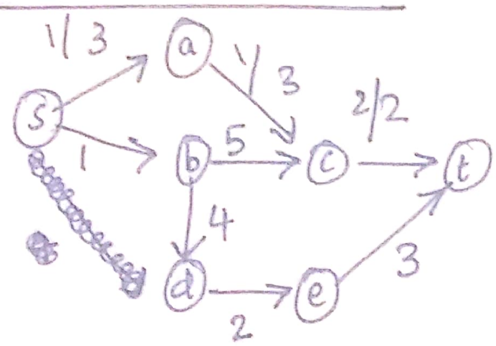
$\delta = \min\{ c_f(u,v) \mid (u,v) \in P \}$

augmenting δ unit flow along P & update G_f;

// (compute new G_f)

} Recover max flow from the final residual N/w.

without Residual N/W



$P_1 = \{s, b, c, t\}$; $\delta(P_1) = 1$

Another path, $P_2 = \{s, a, c, t\}$; $\delta(P_2) = 2$; we cannot send 2 ∵ max cap of this is 2

∴ we can send 1

$|f| = 2.$

If there is a wrong path; Res-N/w takes care of it.

Lemma 1: $|f|$ = flow across the cut

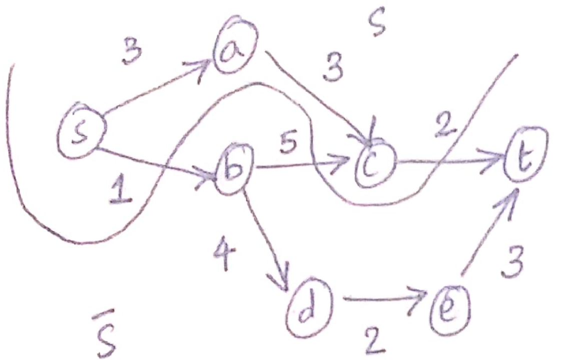
$|f| = f(S, \bar{S})$

Lemma 2: Let f be any flow and let $(S, \bar{S})$ be any s-t cut. Then the value of the flow

$|f| \leq c(S, \bar{S})$

Flow & min-cut

A cut $(S, \bar{S})$ partitions the vertex set V into two subset $S \subset V$ & $\bar{S} = V - S$ and it consists of edges with one endpoint in S and the other in $\bar{S}$.



$S = \{s, a, c\}$

$\bar{S} = \{b, d, e, t\}$

Capacity of cut $c(S, \bar{S}) = c(c, t) + c(s, b)$

$= 2 + 1 = \underline{\underline{3}}$

The cut $(S, \bar{S})$ is an <u>s-t cut</u> if $s \in S$ and $t \in \bar{S}$.

the capacity of the <u>s-t cut</u> is

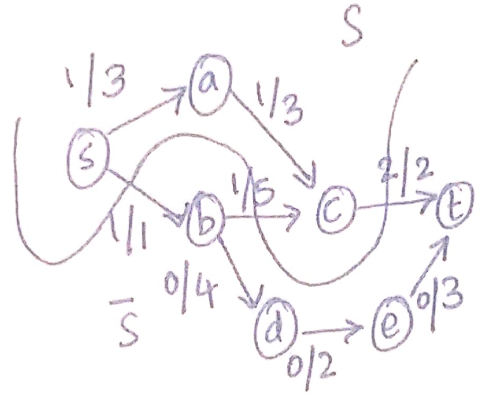
$$c(S, \bar{S}) = \sum_{u \in S \,\&\, v \in \bar{S}} c(u, v)$$

A min-cut is an st cut having minimum capacity.

$(S, \bar{S}) = \{(s, b), (c, t), (b, c)\}$ b is in $\bar{S}$, c is in S.

If f is a flow in G, then the net flow across the cut $S, \bar{S}$ is defined to be

$$f(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$



$S = \{s, a, c\}$

$\bar{S} = \{b, d, e, t\}$

$f(S, \bar{S}) = f(s, b) + f(c, t) - f(b, c)$

$f(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } \bar{S}} (f(e))$

$f(S, \bar{S}) = 1 + 2 - 1 = 2$

<u>Lemma 1</u> Let f be a flow in a flow n/w, G with source, s & sink t. and let $(S, \bar{S})$ be a s-t cut of G then the net flow across $(S, \bar{S})$ is $|f| = f(S, \bar{S})$

<u>Proof</u>: $|f| = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ into } s} f(e) = \sum_{v \in S} \left[\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right] = f(S, \bar{S})$

$\boxed{S = \{a, c, s\} \quad v = a, c, s}$

<u>Illustration of Lemma 1</u>

$|f| = \sum_{e \text{ out of } s} f(e) = f(s, a) + f(s, b)$

$= [f(\cancel{s}, a) + f(s, b)] + [f(\cancel{a}, c) - f(\cancel{s}, a)] + [f(c, t) - f(\cancel{a}, c) - f(b, c)]$

$= f(s, b) + f(c, t) - f(b, c)$

The flow across this cut S, S complement is equal to the flow value

Theorem: (Max-flow, Min-cut Theorem)

- If f is a flow in a flow n/w $G = (V, E)$ with source s and sink, t, then the foll cond are equivalent.

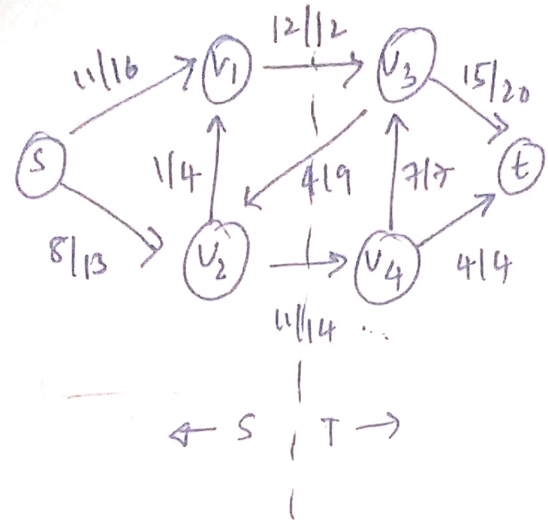
1. f is a maximum flow in G
2. The residual n/w G_f contains no augmenting paths
3. $|f| = c(S, T)$ for some cut (S, T) of G.

The running time of FORD-FULKERSON Algo is $O(E|f^*|)$

$|f^*| \Rightarrow$ max flow.

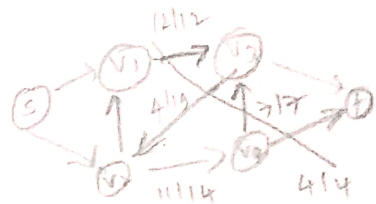
The value of the max-flow is equal to the capacity of the min cut.

Cuts of flow n/ws :-



$S = \{s, v_1, v_2\}$

$T = \{v_3, v_4, t\}$



$|f| = 12+7+4 = 23$
-4
$= 19$

Netflow across this cut is : $|f| = \sum\limits_{e \text{ out of } s} f(e) - \sum\limits_{e \text{ into } s} f(e)$

$f(S, \bar{S})$

$c(S,\bar{S}) = 12+7+4 = 23$

$= \sum\limits_{v \in S} \left[\sum\limits_{e \text{ out of } v} f(e) - \sum\limits_{e \text{ into } v} f(e) \right]$.

$\sum f(e) = f(v_1, v_3) + f(v_2, v_4) - f(v_3, v_2)$

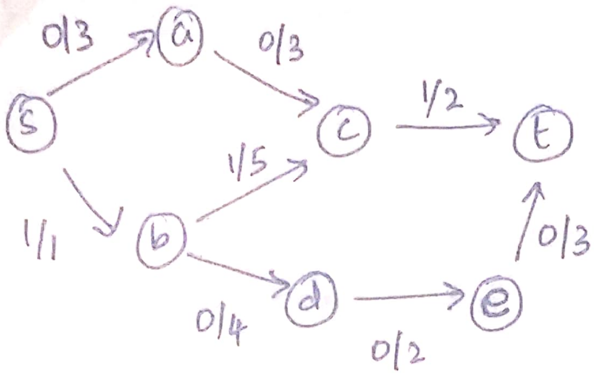
$|f| = 12 + 11 - 4 = \underline{\underline{19}}$.

capacity of this cut is : $c(S, \bar{S}) = \sum\limits_{u \in S \,\&\, v \in \bar{S}} c(u,v)$.

$= 12+14 = \underline{\underline{26}}$.

Lemma: the Netflow across any cut is same and it equals $|f|$, the value of the flow.

↳ Let f be a flow in a flow N/w G, with source s and sink t, let (S,T) be any cut of G. Then the net flow across (S,T) is $f(S,T) = |f|$.



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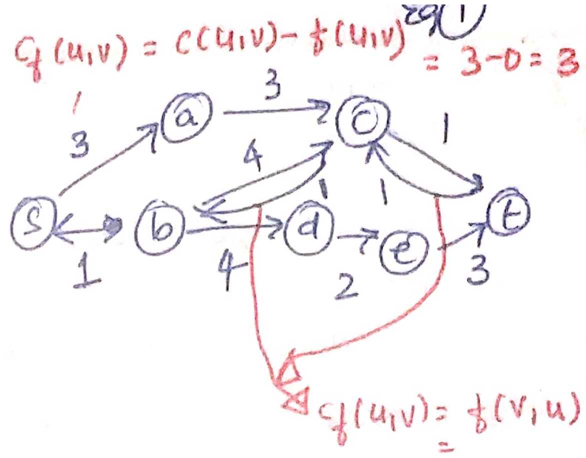
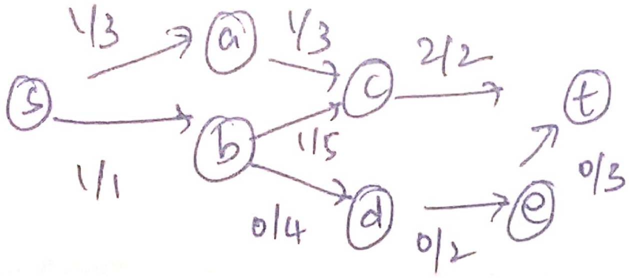


G_f

Augmenting path in G_f from s to t : $s \to a \to c \to t$

$\delta(p) = \min\{3, 3, 1\} = 1$; flow = flow+1.



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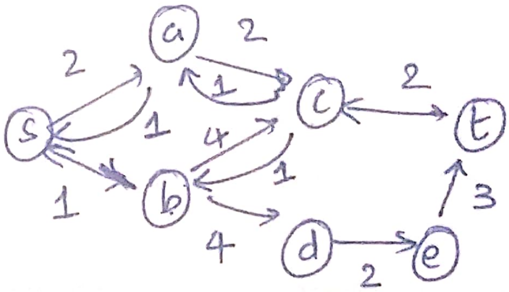
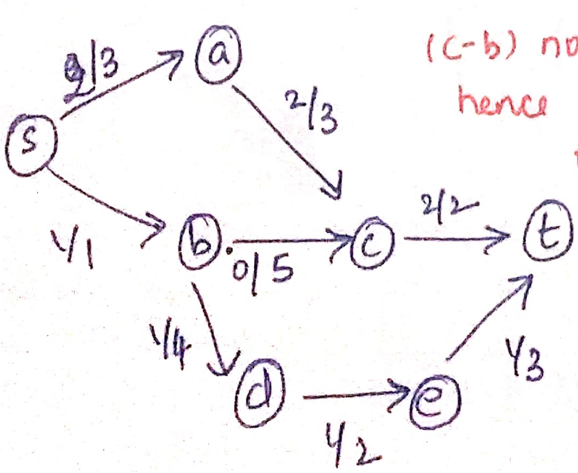


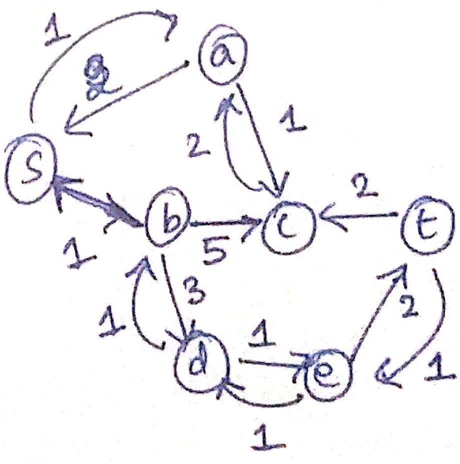
G_f

Augmenting path in G_f from s to t : ⟨s-a-c-b-d-e-t⟩

$\delta(p) = \min\{2, 2, 1, 4, 2, 3\} = 1$. ; flow = flow+1



(c-b) no edge in G; hence subtract the flow by (1); instead of adding.

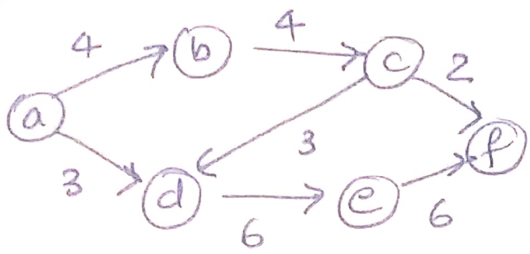


G

G_f

Since there is no augmenting path in G_f ; stop.

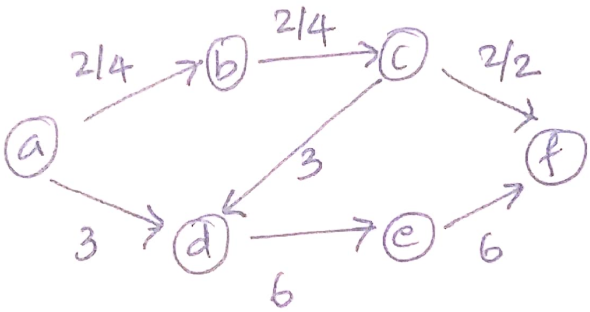
Max flow $|f| = 3$.

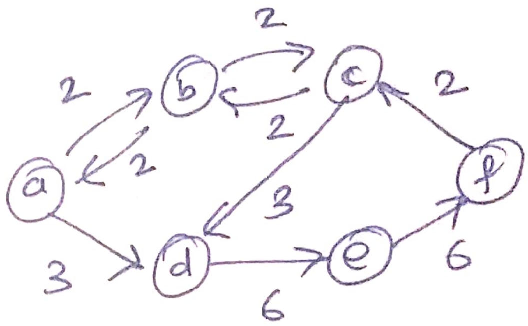


$|f| = 7$

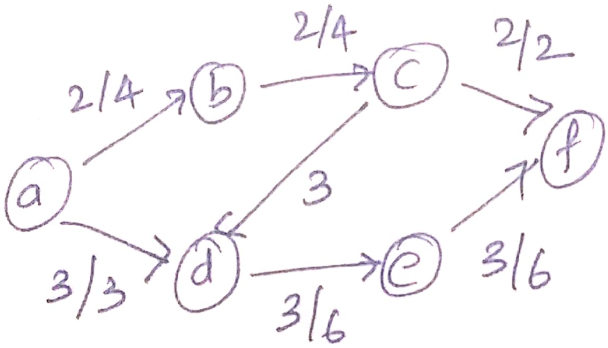
$\Rightarrow$

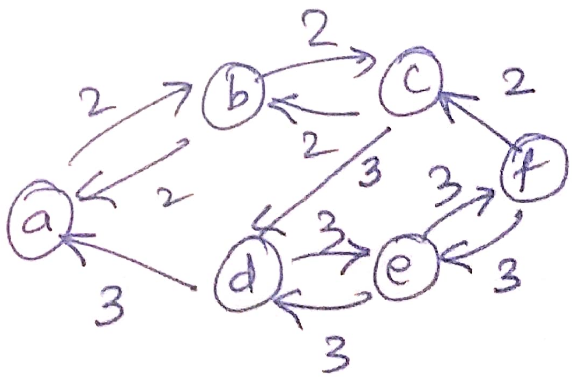
a-b-c-f : min{2}

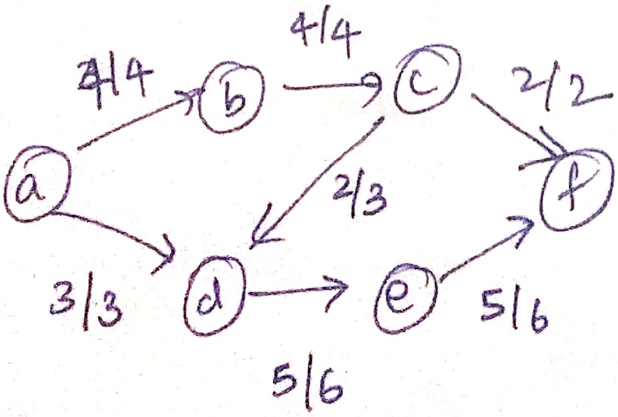




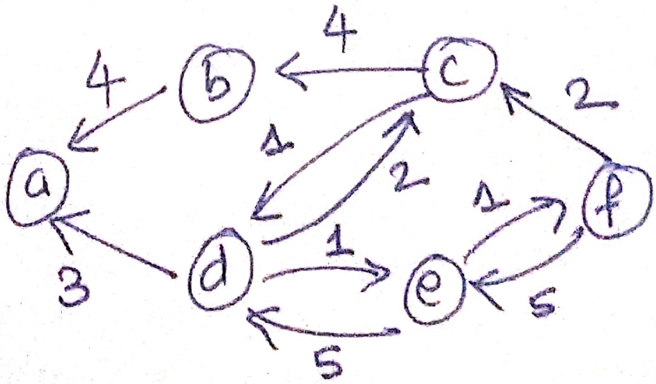
a, d, e, f
min{3, 6, 6} = 3







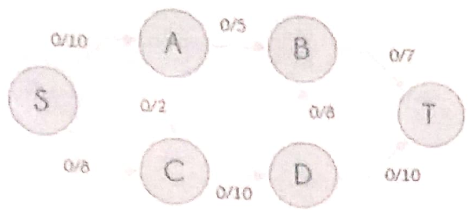
a, b, c, d, e, f
Min = 2

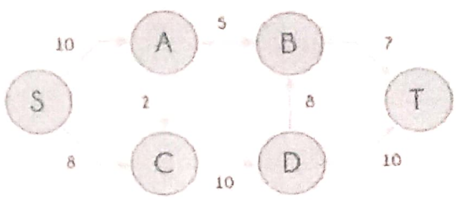


$|f| = 7$

Network (G)

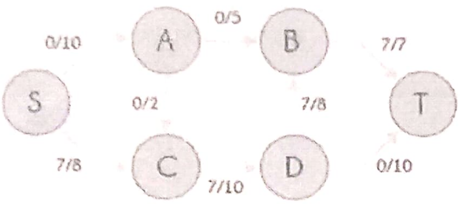
Residual Graph (G_R)

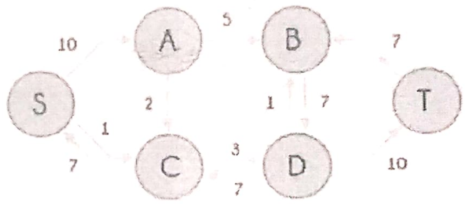




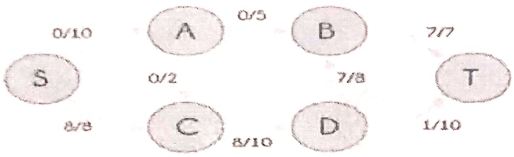
Flow = 0

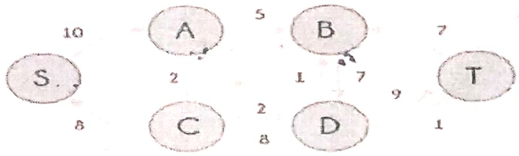
Path 1: S - C - D - B - T → Flow = Flow + 7





Path 2: S - C - D - T → Flow = Flow + 1

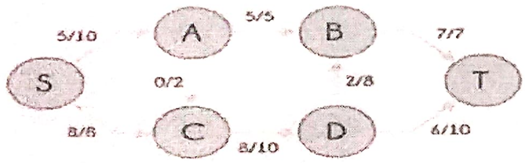


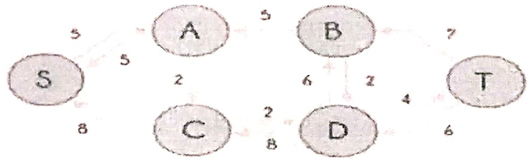


S-A-B-D-T

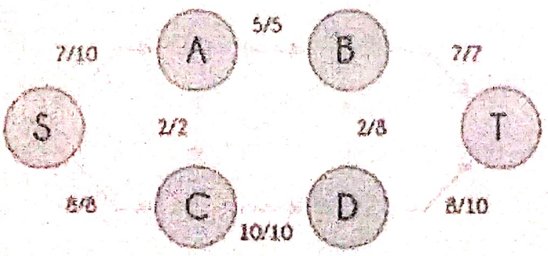
Path 3: S - A - B - T → Flow = Flow + 5

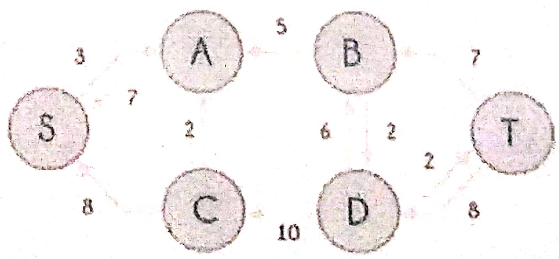
^





Path 4: S - A - C - D - T → Flow = Flow + 2



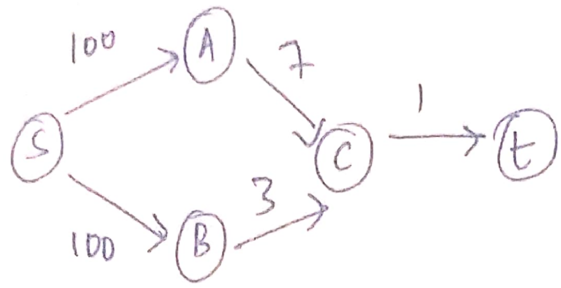


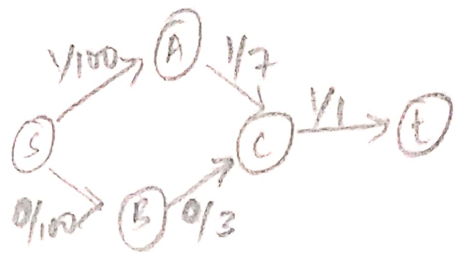
|f| = 15

No More Paths Left

Max Flow = 15

Flow Algo

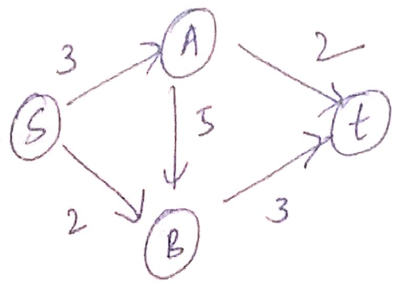




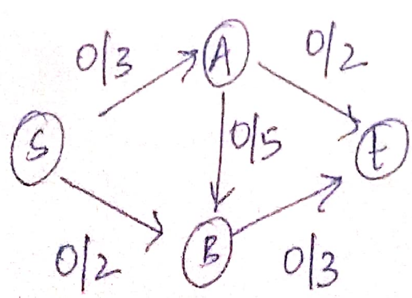
S is water treatment facility & t is our home. We are interested in finding out the amount of water that can flow through to our literal bathroom sink.

Here the flow can clearly be the atmost the capacity of our smallest edge leading into 't'.

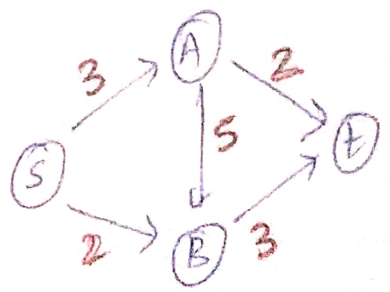
Consider the following N/W:



1. Initially set the flow along every edge to '0'



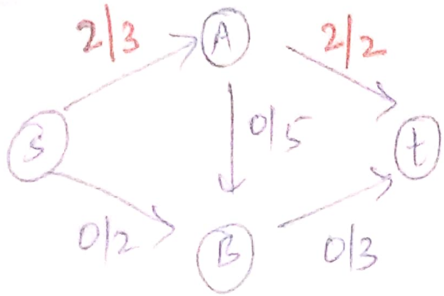
G



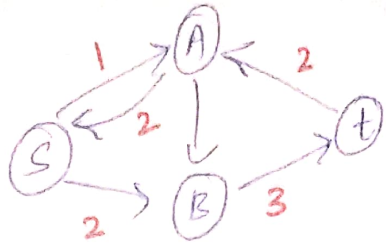
G_f (Residual graph)

Augmenting Path: $\min\{(S,A),(A,t)\}$
$= \min\{3, 2\} = 2$

Path $(S,A),(A,t)$ $\Rightarrow$ Update/Increase flow of 2 in G



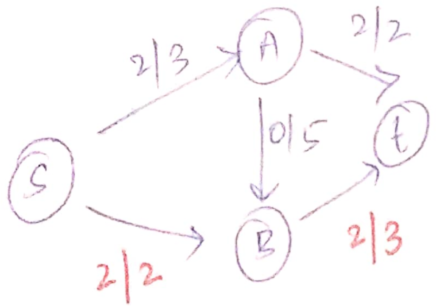
G



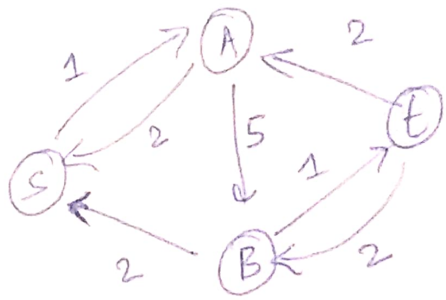
G_f

Aug path = min{(S,B), (B,t)}
= min{2,3} = 2

Update/Increase flow of 2 along the path (S,B), (B,t) in G



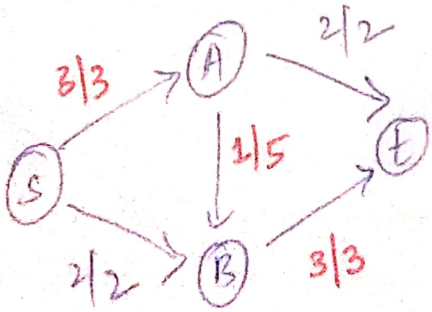
G



G_f

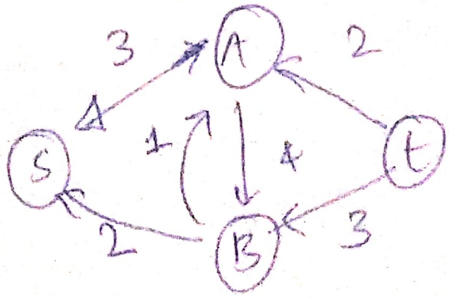
Aug path = min{(S,A), (A,B), (B,T)}
= min{1, 5, 1} = 1

Increase the flow of 1 along the path S → A → B → T



|f| = 5

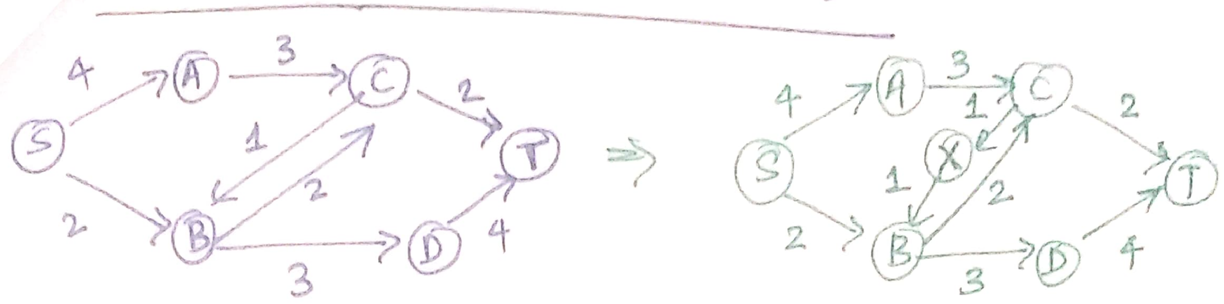
G



G_f

No more Augmenting path exists; hence exit ∴ |f| = 5

GRAPH WITH ANTI-PARALLEL EDGES



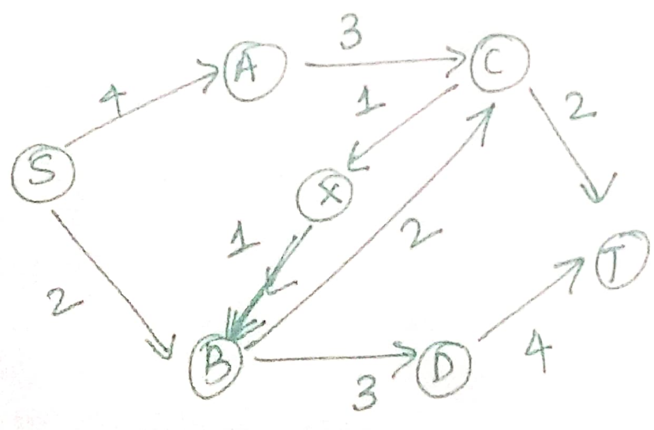
It violates the assumption that if edge $(v_1, v_2) \in E$

then $(v_2, v_1) \notin E$.

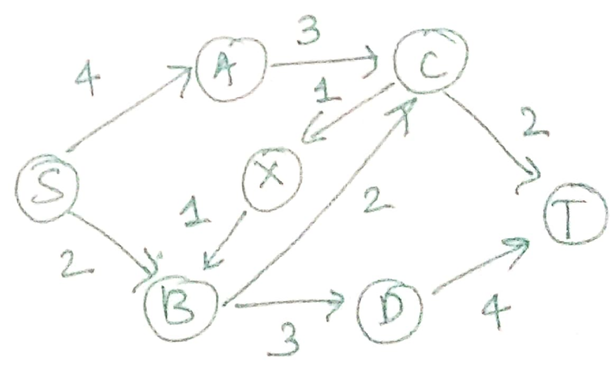
Edges (B, C) & (B, C) are anti-parallel.

$\therefore$ Transform into an equivalent one containing no anti-parallel edges (B, x) & (x, c).

$\therefore$ It satisfies the property that if an edge is in the n/w; the reverse edge is not present.



G



Gf

Increase the flow by 2 in G;

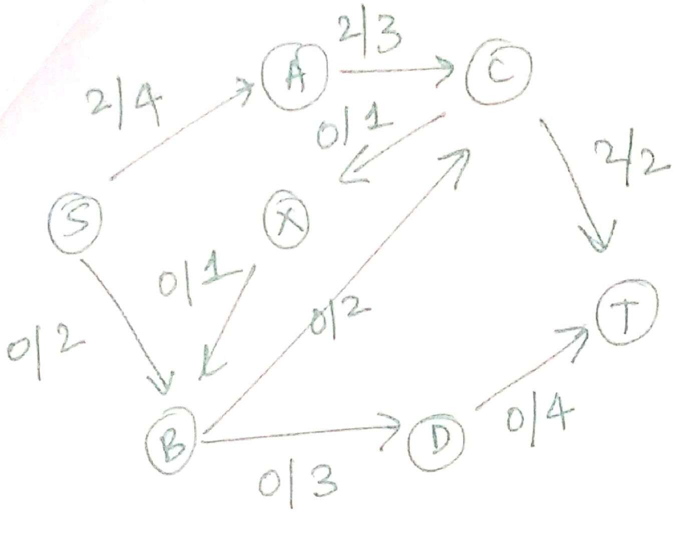
along the path

S – A – C – T

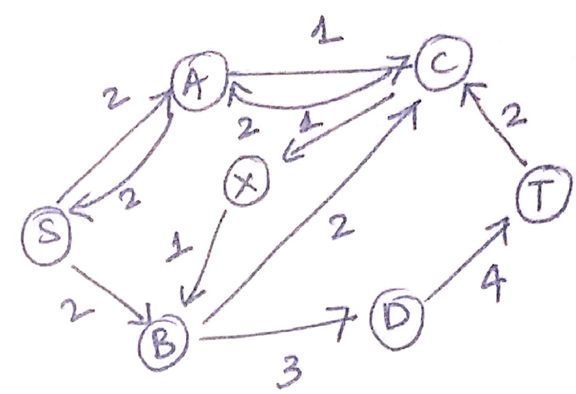
Augmenting path: $\min\{(S-A), (A-C), (C-T)\}$

$= \min\{4, 3, 2\}$

$= \underline{2}$.



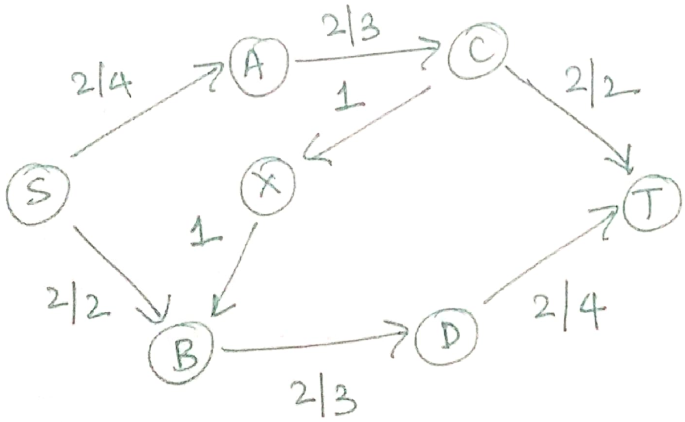
G



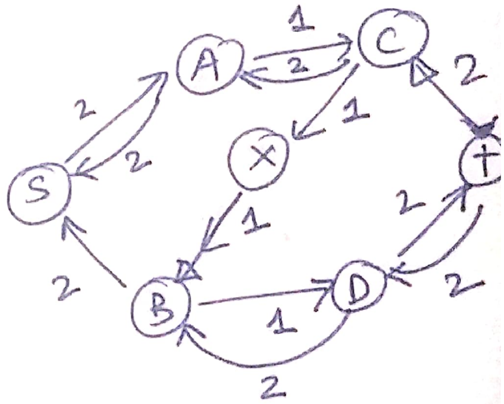
Aug path: min$\left\{ \begin{matrix} S-B, & B-D, & D-T \\ 2, & 3, & 4 \end{matrix} \right\} = 2.$

Gf

Increase the flow by 2 along the path S-B-D-T in G.



G



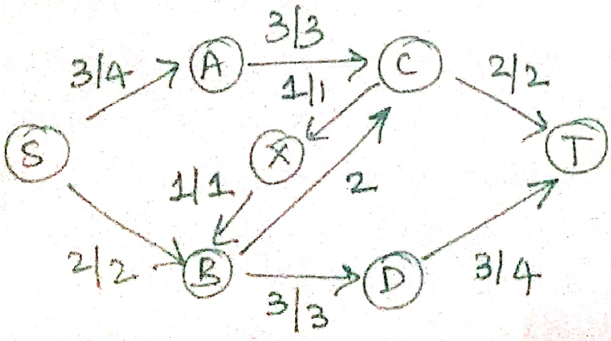
Gf

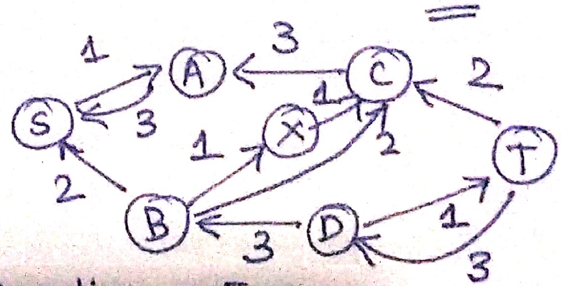
Aug path: min$\{$(S-A), (A-C), (C-X), (X-B), (B-D), (D-T)$\}$.

: min$\{2, 1, 1, 1, 1, 2\}$

$= \underline{\underline{1}}.$

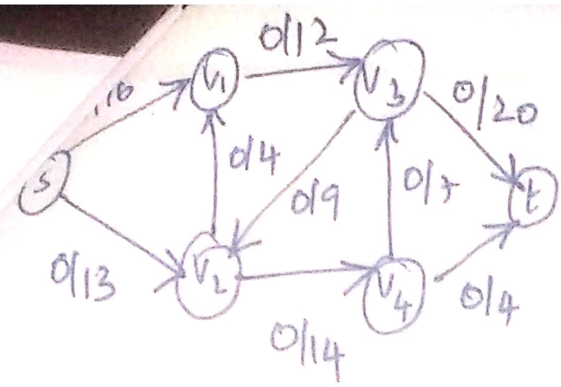
Increase flow by 1 along S-A-C-X-B-D-T in G.



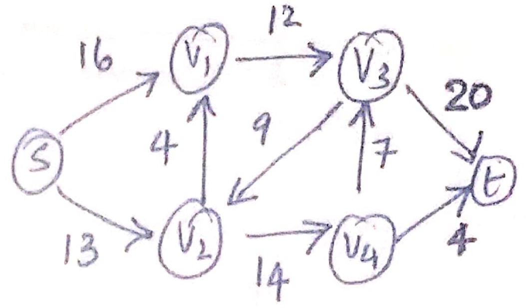


MAX-FLOW $|f| = 5$.

No more Augmenting path ; STOP.



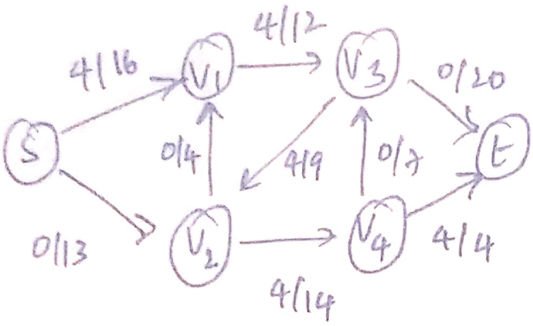
G



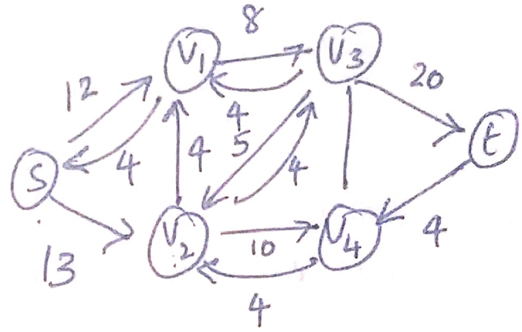
G_f

Augmenting path in G_f : $s - v_1 - v_3 - v_2 - v_4 - t$.

$\delta(p) = \min\{16, 12, 9, 14, 4\} = 4$ flow = flow + 4 in G



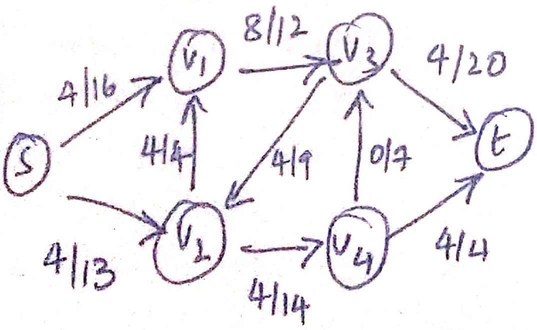
G



G_f

Augmenting path in G_f : $s - v_2 - v_1 - v_3 - t$. flow = flow + 4 in G

$\delta(p) = \min\{13, 4, 8, 20\} = 4$



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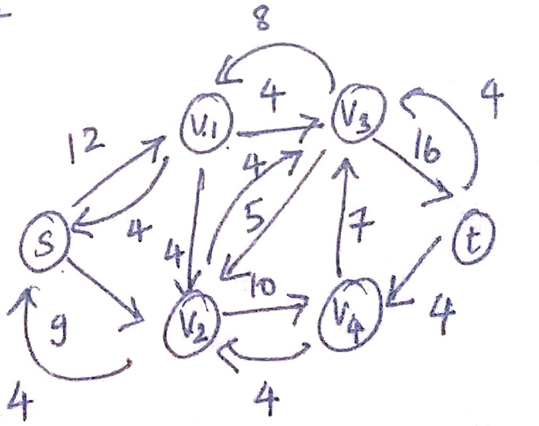
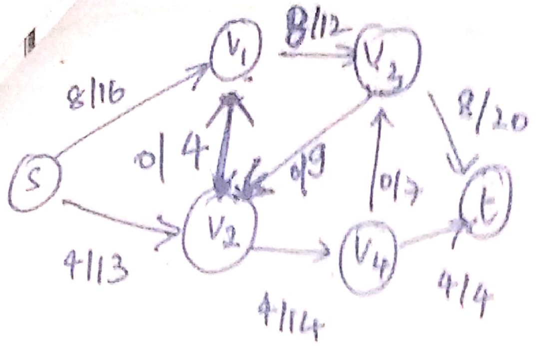


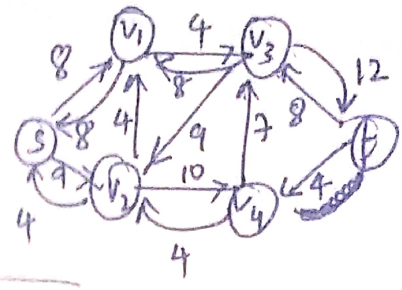
G_f

Augmenting path in G_f : $s - v_1 - v_2 - v_3 - t$. $\delta(p) = \min\{12, 4, 4, 16\} = 4$

flow = flow + 4 in G



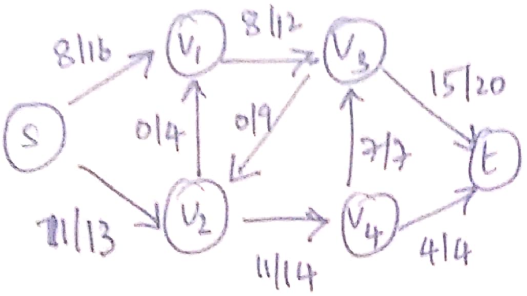
G



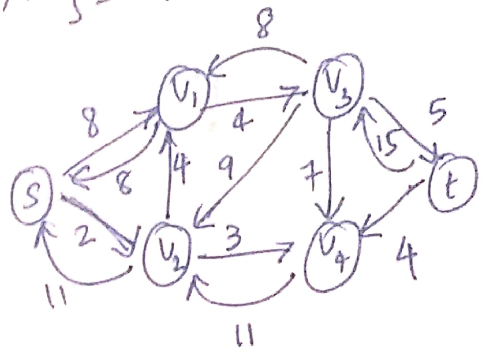
Gf

Augmenting path in Gf : s - v_2 - v_4 - v_3 - t.

$\delta(p) = \min\{9, 10, 7, 12\} = 7$.



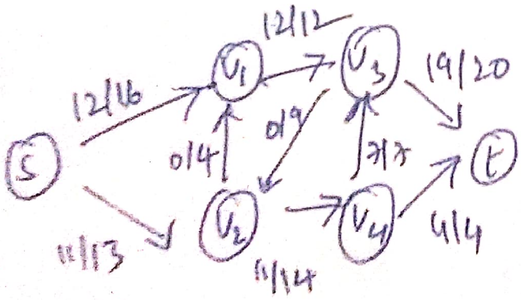
G



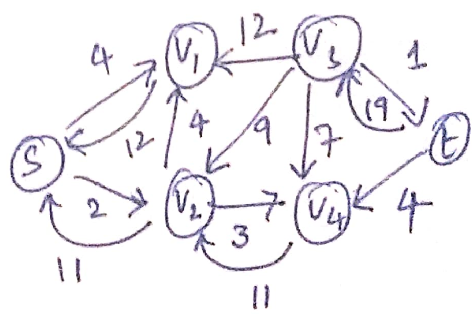
Gf

Augmenting path in Gf : s - v_1 - v_3 - t.

$\delta(p) = \min\{8, 4, 5\} = 4$ flow = flow + 4 in G



G

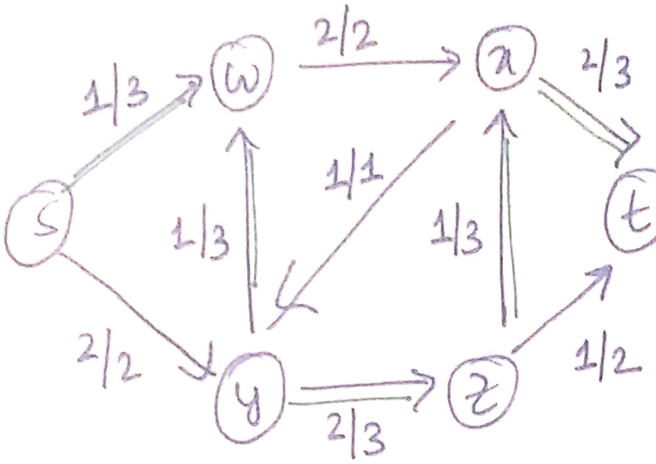


Gf

In Gf there are no more augmenting path and hence max flow has been reached.

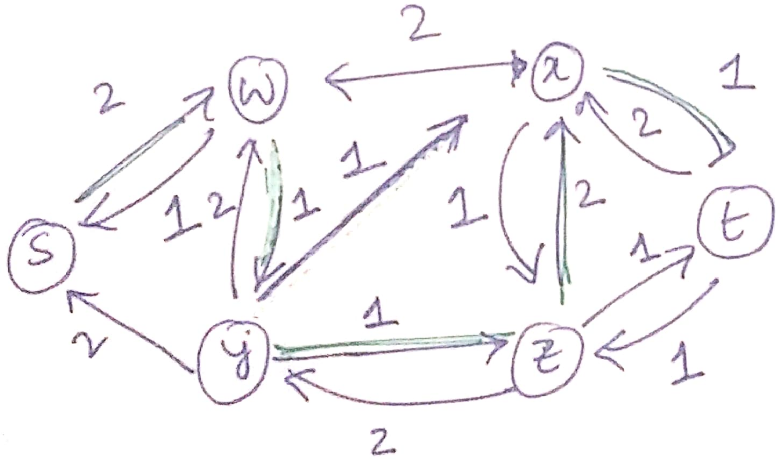
max flow = 19 + 4 ⇒ 23 from G.

$|f| = 23$

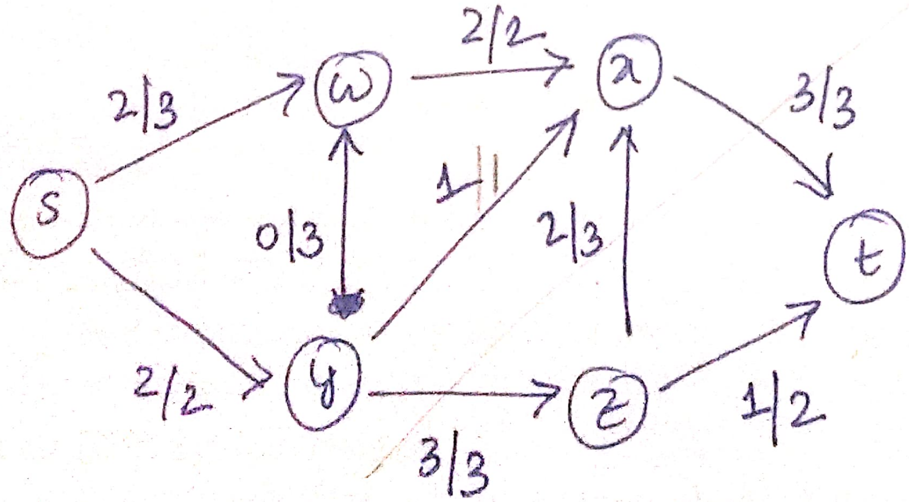


Augmenting path $p = \langle s, w, y, z, x, t \rangle$ in G_f.

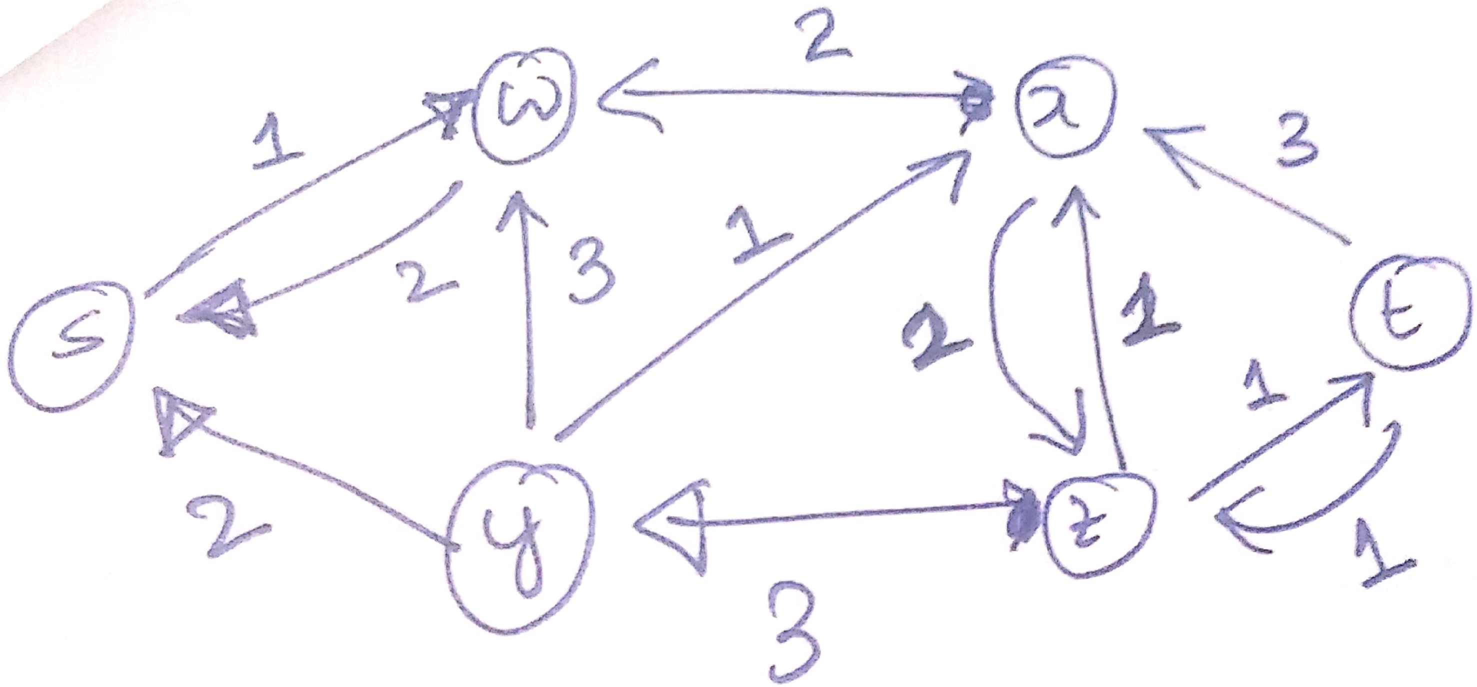
What is the minimum residual capacity of this path.



$s-w-y-z-x-t$: $\min\{2, 1, 1, 2, 1\} = 1.$

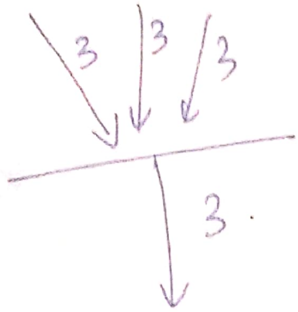


$|f| = 4$



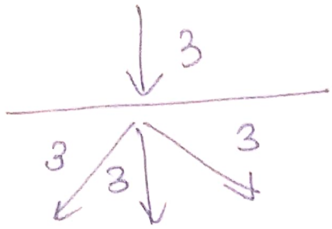
∴ MAX FLOW $|f| = 4$

①



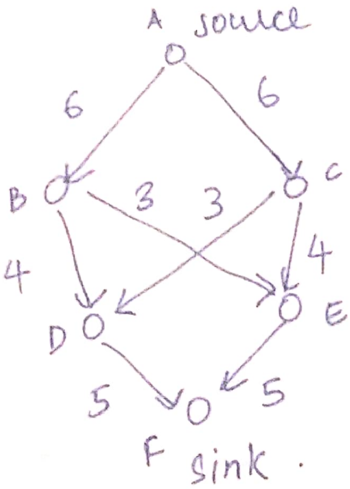
$|f| = 3$

②

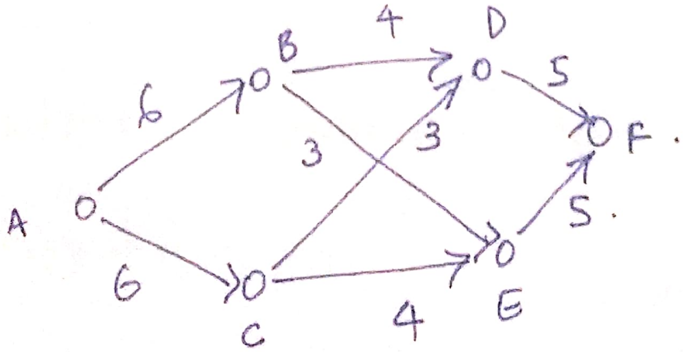


$|f| = 3$

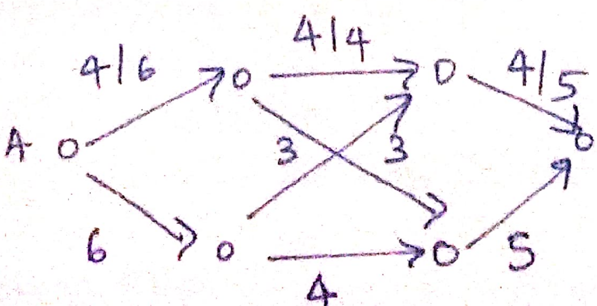
③

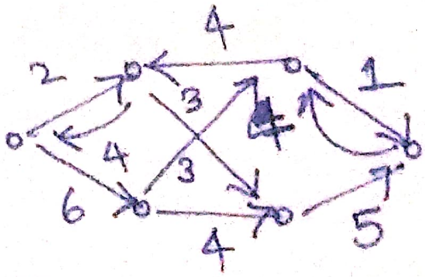


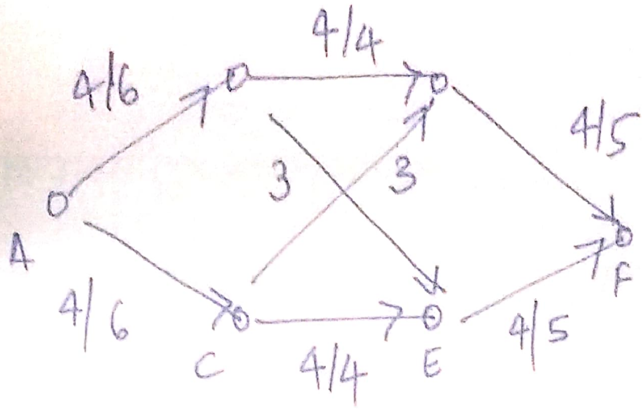
$|f| = 10$



A - B - D - F : min{4}

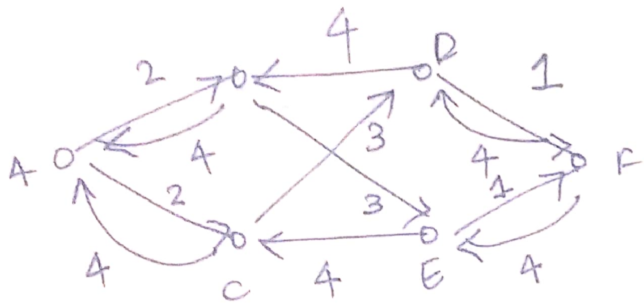






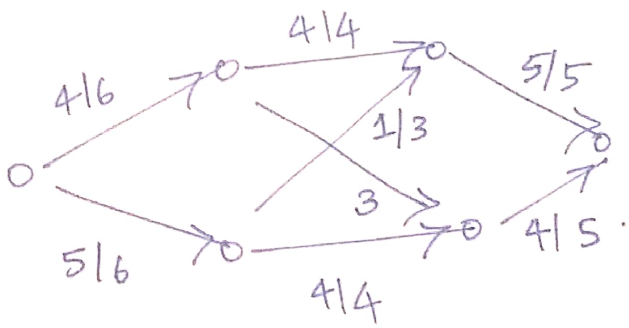
A-C-E-F

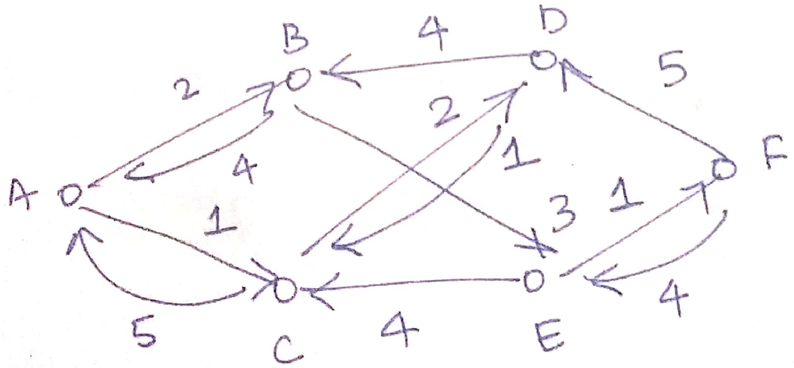
min{4}.

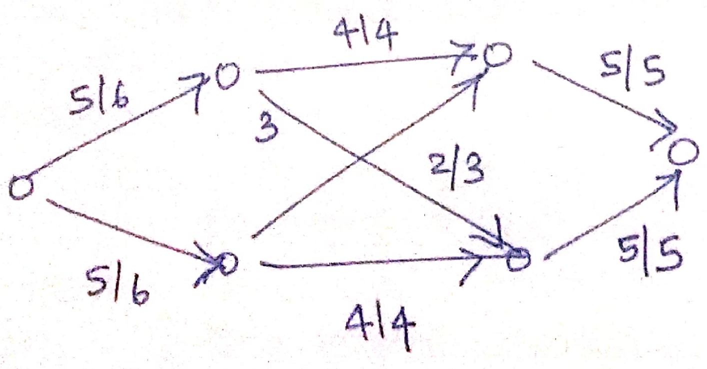


A-C-D-F.

{2, 3, 1} : 1







A-B-E-F

{2, 2, 1}: 1

|f| = 10