COLUMN SPACE

In Linear Algebra the column Space $C(A)$ of a matrix [Sometimes called the range of a matrix] is the set of all possible linear combination of its column vectors. The Column space of a $m \times n$ matrix is a subspace of m-Dimensional Euclidean Space. The dimension of the column space is called the Rank of the matrix.

Definition:- Let A be a matrix of order $m \times n$, then the Column Space of A is denoted by $C(A)$ & defined by

$C(A) = \{$ set of all linear combinations of the columns of $A \}$

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then

$$C(A) = \left\{ \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots \alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \middle/ \alpha_i \in R \right\}$$

Ex:- $A = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 2 & 4 \end{bmatrix}_{3\times 2}$ then

$$C(A) = \left\{ x \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \middle/ x, y \in R \right\} \subseteq R^3$$

The set $C(A)$ is non-empty because if $x = 0$ & $y = 0$ then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in C(A)$

To prove that $C(A)$ is a subspace of R^m

Proof :- Let $\vec{b}$ & $\vec{b}' \in C(A)$

$\therefore \; A\vec{x} = \vec{b}$ & $A\vec{x'} = \vec{b'}$

clearly $\vec{b} + \vec{b}' = A\vec{x} + A\vec{x}'$

$= A(\vec{x} + \vec{x}')$

$\Rightarrow \vec{b} + \vec{b}' \in C(A)$

$\therefore$ $C(A)$ is closed under addition of R^m

(b) Let $\alpha \neq 0$, let $\vec{b} \in C(A)$

$\Rightarrow \vec{b} = A\vec{x}$

Consider $\alpha\vec{b} = \alpha(A\vec{x}) = A(\alpha\vec{x})$

$\therefore \; \alpha\vec{b} \in C(A)$

(c) Also $\vec{0}$ is attainable $A\vec{0} = \vec{0}$

$\vec{0} \in C(A)$

$\therefore$ $C(A)$ is a subspace of R^m

Ex:- Suppose $A = I_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$C(I) = \{$set of all Linear combinations of the columns of $I\}$

$= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Big/ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in R \}$

$= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} \Big/ \alpha_i \in R, \; i = 1, 2, \ldots 5 \right\}$

Note :- (1)

let A be a matrix such that $|A| \neq 0$
What is $C(A)$?

$\Rightarrow$ $b \in R^n$ is attainable $\Rightarrow$ $A\vec{x} = \vec{b}$

$$\vec{x} = A^{-1}\vec{b}$$

$$C(A) = R^n$$

[A is $n \times n$ matrix which is non singular [A^{-1} exists] Then the columns of A will be independent. In this case, the eq^n $A\vec{x} = \vec{b}$ is solvable for every $\vec{b}$ & Hence

$$C(A) = R^n \,].$$

(2) If $|A| = 0$ then, $\{\vec{0}\} \subset C(A) \subset R^n$

Does $A\vec{x} = \vec{b}$ have a solution for every RHS?

Consider 4 eq^{n}s 3 unknown

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ Then $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Also if $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ Then $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\boxed{A\vec{x} = \vec{b} \text{ can be solved if } \vec{b} \text{ is in } C(A)}$$

Null Space

Null space of a matrix A is denoted by N(A) & is defined as

$$N(A) = \{ \vec{x} \,/\, A\vec{x} = \vec{0} \}$$

To prove that N(A) is a Subspace

(i) Let $\vec{x} \in N(A) \Rightarrow A\vec{x} = 0$

$\vec{y} \in N(A) \Rightarrow A\vec{y} = 0$

Consider $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

$= 0 + 0 = 0$

$\therefore \vec{x} + \vec{y} \in N(A)$

(ii) Let $\vec{x} \in N(A) \Rightarrow A\vec{x} = 0$

Consider $\alpha \neq 0$, $\alpha A\vec{x} = \alpha \times 0 = 0$

$A(\alpha\vec{x}) = 0$

$\Rightarrow \alpha\vec{x} \in N(A)$

(iii) Clearly, $A\vec{0} = 0$

$\vec{0} \in N(A)$

$\therefore$ $\boxed{N(A) \text{ is a subspace of } R^n}$

If $A_{m \times n} \vec{x}_{n \times 1} = \vec{b}$

Row Space of A

The Row space of A is the column space of A^T denoted by $C(A^T)$ & is defined as the set of all linear combinations of the rows of A.

$$C(A^T) = \{\alpha_1(\text{row}_1 \text{ of } A) + \alpha_2(\text{row}_2 \text{ of } A) + \cdots + \alpha_n(\text{row}_n \text{ of } A)\}$$

$$\boxed{\text{Row Space of } A = \text{Column Space of } A^T}$$

Left Null Space of A

Left Null Space of A is denoted by $N(A^T)$ & is defined as

$$N(A^T) = \{\vec{y} \,/\, A^T y = 0\}$$

The four Fundamental Subspaces of A are

(1) Column Space of A, $C(A)$

(2) Null Space of A, $N(A)$

(3) Row Space of A, $C(A^T)$

(4) Left Nullspace of A, $N(A^T)$

Ex (1) :- $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$. $\quad U = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row Space of A.

$$C(U^T) = \text{Span of } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} \right\}$$

$$C(A^T) = \text{Span of } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \\ 12 \end{bmatrix} \right\}$$

Linearly independent rows form the basis for $C(A^T)$

Left Null Space of A

$$N(A^T) = \{\vec{y} \,/\, A^T\vec{y} = \vec{0}\}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 8 & 7 \\ 5 & 12 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_4 - R_3 = R_4$

$y_3 \to$ Free Variable

$y_1, y_2 \to$ Pivotal Variables

$$UY = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 + 2y_2 + 3y_3 = 0$

$2y_2 - 2y_3 = 0 \Rightarrow y_2 = y_3$

$\Rightarrow y_1 = -3y_3 - 2y_3 = -5y_3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5y_3 \\ y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

$$N(A^T) = \text{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\}$$

dimension $N(A^T) = 1$

dimension $C(A^T) = 2$

$$\boxed{\dim C(A^T) + \dim N(A^T) = \dim R^3}$$

$$\boxed{\text{Rank} + \text{Nullity} = \dim \text{ of } R^{\text{no of Rows}}}$$

If the

$$C(A) = \text{Span}\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\8\\7\end{bmatrix} \right\} \rightarrow \dim = 2$$

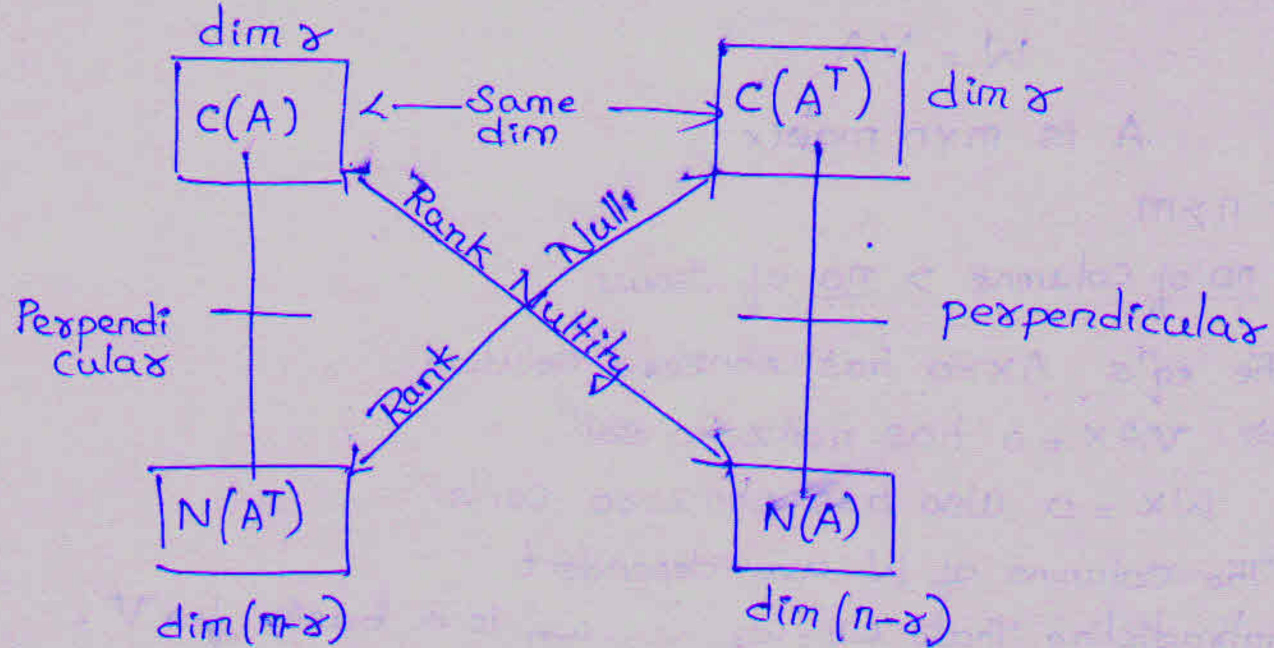
$$N(A) = \text{Span}\left\{ \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\end{bmatrix} \right\} \quad \dim = 2$$

$\dim R^4$

$$C(A^T) = \text{Span}\left\{ \begin{bmatrix}1\\2\\3\\5\end{bmatrix}, \begin{bmatrix}2\\4\\8\\12\end{bmatrix} \right\} \quad \dim = 2$$

$$N(A^T) = \text{Span}\left\{ \begin{bmatrix}-5\\1\\1\end{bmatrix} \right\} \quad \dim 1$$

$\rightarrow \dim R^3$



$$C(A^T) \perp N(A) \;\&\; C(A) \perp N(A^T)$$

If the dot product of 2 vectors are zero then vectors are $\perp^r$ to each other.

THEOREM

If $v_1, v_2, \dots v_m$ & $w_1, w_2, \dots w_n$ are two bases for a vector space V, then $m=n$. The no of vectors is the same

Proof :- Case I :- Suppose $n>m$ i.e, we have more w's than v's

Since $\{v_1, v_2, \dots v_m\}$ forms a basis of V,

each $w_1, w_2, \dots w_n$ can be expressed as a linear combinations of V's

$w_1 = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m$ → I column of a matrix multiplication.

$w_2 = a_{12}v_1 + a_{22}v_2 + \dots + a_{m2}v_m$

⋮

$w_m = a_{1n}v_1 + a_{2n}v_2 + \dots + a_{mn}v_m$

$$W = [w_1 \ w_2 \dots w_m] = [v_1 \ v_2 \dots v_m]\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = VA.$$

$$W = VA$$

A is $m \times n$ matrix

$\because n>m$

no of columns > no of rows

The eqⁿs $AX=0$ has nonzero solution

$\Rightarrow$ $VAX=0$ has nonzero solⁿ.

$\Rightarrow$ $WX=0$ also has nonzero solⁿs

$\Rightarrow$ The columns of W are dependent.

Contradicting that $w_1, w_2, \dots w_n$ is a basis for V.

Hence $n \not> m$. VA is always a square matrix.

III'ly $m \not> n$ we exchange v's & w's & repeat the same step.

$\therefore$ $m=n$.

EX:- $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

No of rows : No of equations

No of columns : No of variables

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$C(A) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} \;/\; \alpha_1, \alpha_2, \alpha_3 \in R \right\}$$

$$= \left\{ \begin{bmatrix} \alpha_1 + \alpha_3 \\ 5\alpha_1 + 4\alpha_2 + 9\alpha_3 \\ 2\alpha_1 + 4\alpha_2 + 6\alpha_3 \end{bmatrix} \;/\; \alpha_1, \alpha_2, \alpha_3 \in R \right\}$$

$$N(A) = \left\{ \vec{x} \;/\; A\vec{x} = 0 \right\}$$

$$\Rightarrow \quad u + w = 0$$
$$5u + 4v + 9w = 0$$
$$2u + 4v + 6w = 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 2$$

$$u + w = 0 \Rightarrow u = -w$$

$$4v + 4w = 0 \Rightarrow v = -w$$

$$\vec{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -w \\ -w \\ w \end{bmatrix} = w \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} / \alpha \in R \right\}$$

<u>Remark:</u> If $|A| \neq 0$, then $A\vec{x} = \vec{0}$ will have trivial solⁿ
i.e, $N(A) = \{\vec{0}\}$

(2) For what value of b is the vector ~~space~~ $\vec{B} = [1\ 2\ 3\ b]^T$ in the column space of the following matrix?

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & -4 & -5 \\ 6 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & -4 & -5 & 2 \\ 6 & 3 & 0 & 3 \\ 1 & 1 & 3 & b \end{bmatrix}$$

Reduce to echelon form

$$= \Bigg[$$

$\rho(A) = 3$, $\rho(A|B) = 3$

$\therefore \boxed{b =\quad}$

$\therefore \vec{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ b \end{bmatrix}$ is Column space of A iff $b =$

$$\therefore \boxed{b \in cs(A) \Longleftrightarrow \text{Rank } A = \text{Rank}[A|B]}$$

(3) let $A = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 2 & 4 \end{bmatrix}$

The eq^n $A\vec{x} = 0$ gives

$x + 0 = 0$ $\quad\quad (x, y)^T = (0, 0)$

$5x + 0 = 0$

$2x + 4y = 0$

$\therefore (0,0)$ belongs to $N(A)$ & it is the only vector ~~for~~ in $N(A)$.

(4) $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 0 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$A\vec{x} = 0$

$x + z = 0$ $\quad\quad$ put $z = k$

$5x + 5z = 0$ $\quad\quad x = -k$

$2x + 4y + 6z = 0$ $\quad\quad 4y = -6k + 2k = -4k$

$y = -k$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ k \end{bmatrix}, \quad k \in R.$

$\therefore N(A) = \{(-k, -k, k) \mid k \in R\}$

$(0,0) \rightarrow$ Represents a point in 2D

$(-k,-k,k) \rightarrow$ lines passing through origin in 3D.

PROBLEMS

(1) Describe the four Subspaces in 3-Dimension Space associated with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Solⁿ :- Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(i) Column Space $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

dimension of $C(A) = 2$

(ii) Row Space $C(A^T) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

dimension of $C(A^T) = 2$

(iii) Null Space $N(A)$

consider $AX = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 = 0$ and $x_3 = 0$

let $x_1 = k$

$\therefore N(A) = \left\{ \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \text{ or } k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

dimension of $N(A) = n - r = 3 - 2 = 1$

(iv) Left Null Space $N(A^T)$

Consider $A^T y = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 = 0, \quad y_2 = 0, \quad y_3 = k$

$\therefore$

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(2) Find four Fundamental Subspaces for the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Sol^n:- Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \text{Span of } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$C(A^T) = \text{Span of } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$N(A) = \{x \mid Ax = 0\}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$Ux = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u + w = 0 \Rightarrow u = -w$$
$$v = 0$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -w \\ 0 \\ w \end{bmatrix} = w \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Null Space $N(A) =$ Span of $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Left Null Space: $N(A^T) = \{y \mid A^T y = 0\}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 + 2y_2 + 3y_3 = 0 \quad \text{---} (1)$$
$$y_2 + y_3 = 0 \quad \text{---} (2)$$

$y_2 = -y_3 \quad , \quad y_1 = -2(-y_3) - 3y_3 = -y_3$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -y_3 \\ -y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore N(A^T) =$ Span of $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ or Span of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(3) Find the bases for the column space, Row spaces, Null Spaces given that

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = r = 2$

$$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$C(U^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$C(A^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \checkmark$$

$N(A) = N(U) = \{ x / Ux = 0 \}$

$$Ux = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 \rightarrow$ Free Variable
$x_1, x_2 \rightarrow$ pivot el-

$x_1 + 3x_2 + 2x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3 \quad , \quad x_1 = x_3$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A) = N(U) = \text{Span} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\dim C(A) + \dim N(A) =$ No of Columns in A

$2 + 1 = 3$ [Rank-Nullity theorem]

$\dim C(A) = \rho(A)$

(4) If $A_{7\times 9}$ matrix with $\rho(A) = 5$

$5 + x = 9$

$\dim N(A) = 4$

$\dim C(A) = 5$

$\dim C(A^T) = 5$

$\dim N(A^T) = 7 - 5 = 2$

$= \text{Row} - \rho(A) = 2$

$$\boxed{\dim N(A^T) = \text{Rows of } A - \rho(A)}$$

(5) A & B is $n \times m$, If $AB = 0$; $\rho(A) + \rho(B) \leq n$ (True/False)

columns of B are in $N(A)$ True.

$\Rightarrow \dim C(B) \leq \dim N(A)$

$\rho(B) \leq n - \rho(A)$

6. choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \& \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$$

$$C(U) = \text{Span}\{col_1, col_2, col_4\}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}$$

$$\rho(A) = 3$$

$$\dim c(A) = 3$$

$c(U)$ & $c(A)$ $\rightarrow$ are the bases for this which is found.

7. find the Largest possible no of independent Vectors among

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad V_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad V_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

This no is the _____ of the Space spanned by the v's.

sol^n:-

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$R_4 \to R_4 + R_3 \sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3$

$C(A) = \text{Span}\{\text{Col}1, \text{Col}2, \text{Col}3\}$

$= \text{Span}\{v_1, v_2, v_3\}$

(8) Obtain the four fundamental Subspaces associated with the following matrices. Establish a relation b/w $N(A)$ & $C(A^T)$ and $N(A^T)$ & $C(A)$ so obtained.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} \; R_3 \to R_3 + \tfrac{1}{2}R_2 \sim \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

$C(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\dim C(A) = 2$.

$\dim N(A) = n - r = 2 - 2 = 0$.

$AX = 0 \Rightarrow UX = 0$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0$

$x_1 + x_2 = 0 \Rightarrow x_1 = 0$

$N(A) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r.$

$\dim [N(A^T)] = 3 - 2 = 1$

$\dim C(A^T) = 2$

Basis $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0 .$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & +2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2y_2 - y_3 = 0$ let $y_3 = k$.

$y_1 - y_2 + 2y_3 = 0$ $y_2 = k/2$.

$$y_1 = -2k + \frac{k}{2} = -3/2\,k$$

$$\therefore k \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix} \in N(A^T)$$

$C(A)$ is $\perp^r$ to $N(A^T)$

$N(A)$ is $\perp^r$ to $C(A^T)$

(9) $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\rho(A) = 1$

$\dim C(A) = 1$ $\dim N(A) = n - r = 2 - 1 = 1$

$\dim C(A^T) = 1$ $\dim N(A^T) = m - r = 2 - 1 = 1$

$C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

<u>$N(A)$</u> :- $UX = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $x_1 - x_2 = 0$, $x_1 = x_2$

$$N(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$C(A^T) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$N(A^T)$:- $A^T Y = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $y_1 = -2y_2$, $N(A^T) = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

10. $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 6 \\ 5 & 15 & 25 \\ -3 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \rho(A) = 2$

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 15 \\ -2 \end{bmatrix} \right\}$$

$\dim C(A) = 2$

$\dim C(A^T) = 2$

$\dim N(A) = n - r = 3 - 2 = 1$

$\dim N(A^T) = m - r = 4 - 2 = 2$

$$Ux = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -7x_2 - 9x_3 = 0$ $\qquad x_1 + 3x_2 + 5x_3 = 0$

$\boxed{x_3 = k}$ $\quad \boxed{x_2 = -9/7\, k}$. $\qquad \boxed{x_1 = k}$

$$\therefore \quad N(A) = \begin{bmatrix} 1 \\ -9/7 \\ 1 \end{bmatrix}$$

$$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 5 & -3 \\ 3 & 2 & 15 & -2 \\ 5 & 6 & 25 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -3 \\ 0 & -7 & 0 & -7 \\ 0 & 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$$

$\boxed{y_4 = 0}$

$\Rightarrow y_2 + y_4 = 0$.

$\boxed{y_2 = 0}$.

$y_1 + 3y_2 + 5y_3 - 3y_4 = 0$

$\cancel{y_3 = k}$ $\quad \cancel{y_1 = -5k}$. $\quad y_1 + 5y_3 = 0$

$$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\}$$

?

$$\dim C(A^T) = 2$$

$$N(A^T) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\dim N(A^T) = 4 - 2 = 2 .$$