1. The response time is the speed of page downloads, and it is critical for a mobile website. Let X denote the number of bars of service, and let Y denote the response time (to the nearest second) for a particular user and site.

$Y \downarrow X \rightarrow$	1	2	3
4	0.15	0.1	0.05
3	0.02	0.1	0.05
2	0.02	0.03	0.2
1	0.01	0.02	0.25

Find (i) the marginal probability distributions of X and Y (ii) the probability that the number of bars of the service is at least 2, (iii) the response time is less than 2 seconds, (iv) both the number of bars of the service and the response time is greater than 2.

2. Determine the value of c that makes the function $f(x, y) = c(x + y)$ a joint probability mass function over the nine points with $x = 1, 2, 3$ and $y = 1, 2, 3$. Also find the expected values of X and Y.

3. The joint probability distribution of two discrete random variables X and Y is given as below. Find (i) $P(Y|X = 2)$

$Y \downarrow X \rightarrow$	1	2	3
1	0.05	0.05	0.10
3	0.05	0.10	0.35
5	0.00	0.20	0.10

4. A ballpoint pen is selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, construct the joint probability distribution table.

5. Determine the value of c that makes the function $f(x, y) = c(x + y)$ a joint probability mass function over the nine points with $x = 1, 2, 3$ and $y = 1, 2, 3$. Determine (i) the marginal distributions of X and Y, (ii)$E(X), E(Y), E(XY), Cov(X, Y)$.

6. Let X and Y be the number of hardware failures in two computer labs in a given month. The joint distribution of X and Y is given by $P(X, Y)$, where $P(0, 0) = 0.52, P(0, 1) = 0.20, P(0,2) = 0.04, P(1,0) = 0.14, P(1,1) = 0.02, P(1,2) = 0.01, P(2,0) = 0.06, P(2,1) = 0.01, P(2, 2) = 0$. Compute (i) the marginal probability distributions of X and Y, (ii) the conditional probability of Y, given $X = 1$, (iii) the conditional probability of X, given $Y = 2$.

7. For two flips of a fair coin, let X equal the total number of tails and let Y equal the number of heads on the last flip. Find (i) the marginal distribution of X, (ii) marginal distribution of Y, (iii) the joint PMF $P(X, Y)$, (iv) $E(X), E(Y), E(XY), Cov(X, Y)$.

8. Random variables X and Y have the joint PMF

$$f(x, y) = \begin{cases} c|x + y|, & x = -2, 0, 2;\ -1, 0, 1 \\ 0, & otherwise \end{cases}.$$

Find (i) the constant c, (ii) $P(Y < X)$, (iii) $P(Y > X)$, (iv) $P(X = Y)$, (v) $P(X < 1)$.

9. A program consists of two modules. The number of errors, X, in the first module and the number of errors, Y , in the second module have the joint distribution, $P(0,0) = P(0,1) = P(1,0) = 0.2, P(1,1) = P(1,2) = P(1,3) = 0.1, P(0,2) = P(0,3) = 0.05$. Find (i) the marginal distributions of X and Y, (ii) the probability of no errors in the first module, and (iii) the distribution of the total number of errors in the program. Also, (iv) find out if errors in the two modules occur independently.

10. An internet service provider charges its customers for the time of the internet use rounding it up to the nearest hour. The joint distribution of the used time(X, hours) and the charge per hour(Y, cents) is given in the table below:

$Y \downarrow X \rightarrow$	1	2	3	4
1	0	0.06	0.06	0.10
2	0.10	0.10	0.04	0.04
3	0.40	0.10	0	0

Each customer is charged $Z = X \cdot Y$ cents, which is the number of hours multiplied by the price of each hour. Find the distribution of Z.

1. The function $f(x,y) = \frac{2}{3}(x + 2y)$ is a joint probability density function over the range $0 \le x \le 1, 0 \le y \le 1$. Determine (i) $E[X]$, (ii) $E[Y]$, (iii) $Cov[X,Y]$.

2. The random variable X denotes the time until a computer server connects to your machine (in milliseconds), and Y denotes the time until the server authorizes you as a valid user (in milliseconds). Each of these random variables measures the wait from a common starting time and $X < Y$. The joint probability density function for X and Y is given as $f(x,y) = k\, e^{-0.001x-0.002y}$ for $0 < x < y < \infty$. Determine the constant k and the probability that $x < 1000$.

3. The joint density for the random variables (X,Y) is given as:

$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0, & elsewhere \end{cases}$. Find the conditional density $f(x|y)$ and $P\left(x > \frac{1}{2} \middle| y = 0.5\right)$.

4. The time, in minutes, it takes to reboot a certain system is a continuous variable with the density function $f(x) = c(10 - x)^2$, if $0 < x < 10$ and $f(x) = 0$, otherwise. Compute the constant c.

5. Let the random variable X denote the time until a computer server connects to your machine (in milliseconds), and let Y denote the time until the server authorizes you as a valid user (in milliseconds). Each of these random variables measures the wait from a common starting time and $X < Y$. Assume that the joint probability density function for X and Y is $f(x,y) = 2e^{-x-y}$ for $x < y$. (i) Verify that $f(x,y)$ is a valid joint density function, (ii) compute the probability that Y exceeds 2 milliseconds, the marginal density function of X, the conditional probability density function for Y, given that $X = x$.

6. Determine the value of c such that the function $f(x,y) = cxy$ for $0 < x < 3$ and $0 < y < 3$ satisfies the properties of a joint probability density function. Determine (i) $E(X), E(Y), E(XY)$.

7. A popular clothing manufacturer receives Internet orders via two different routing systems. The time between orders for each routing system in a typical day is known to be exponentially distributed with a mean of 3.2 minutes. Both systems operate independently. (i) What is the probability that no orders will be received in a 5-minute period? In a 10-minute period? (ii) What is the probability that both systems receive two orders between 10 and 15 minutes after the site is officially open for business?

8. Determine the value of c that makes the function $f(x, y) = ce^{-2x-3y}$, a joint probability density function over the range $0 < x$ and $x < y$. Determine the following: (i) $P(X < 1, Y < 2)$, (ii) $P(1 < X < 2)$, (iii) $P(Y > 3)$, (iv) $P(X < 2, Y < 2)$, (v) $E(X)$, (vi) $E(Y)$, (vii) Marginal probability distribution of X, (viii) Conditional probability distribution of Y given $X = 1$, (ix) $P(Y < 2 \mid X = 1)$, (x) Conditional probability distribution of X given $Y = 2$.

9. The random variables X and Y have joint density function $f(x, y) = 12xy(1 - x)$, $0 < x < 1, 0 < y < 1$ and equal to 0 otherwise. (i) Are X and Y independent? Find (i) $E[X]$, (ii) $E[Y]$, (iii) $Var(X)$, (iv) $Var(Y)$.

10. X and Y are random variables with the joint PDF $f(x, y) = \begin{cases} 2, & x + y \leq 1, x \geq 0, y \geq 0 \\ 0, & otherwise \end{cases}$. Determine (i) $P_1(X)$, (ii) $P_2(Y)$, (iii) $P(Y|X = x)$.

1. Find the covariance matrix for the two random variables X_1 and X_2 whose joint probability is represented as follows:

$X_1 \downarrow X_2 \rightarrow$	2	4
1	0.24	0.06
3	0.56	0.14

2. Find the covariance matrix for the two random variables X_1 and X_2 whose joint probability is represented as follows: $f(x_1, x_2) = e^{-x-y}$ over $x > 0, y > 0$.

3. Let X_1 and X_2 be independent discrete random variables each having the probability distribution $\binom{2}{x_1, x_2, 2 - x_1 - x_2}\left(\frac{1}{4}\right)^{x_1}\left(\frac{1}{3}\right)^{x_2}\left(\frac{5}{12}\right)^{2-x_1-x_2}$, where $x = 0, 1, 2; x_2 = 0, 1, 2; x_1 + x_2 \leq 2$ and *zero* otherwise. Let $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_1+X_2}$. Find the joint pdf of Y_1 and Y_2. Use $\left\{\binom{a}{b, c, d} = \frac{a!}{b!c!d!}\right\}$.

4. Let X_1 and X_2 be independent random variables each having the probability distribution $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & elsewhere \end{cases}$. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint pdf of Y_1 and Y_2.

5. Suppose X_1 and X_2 have a bivariate normal distribution with $\sigma_{X_1} = 4, \sigma_{X_2} = 1, \mu_{X_1} = 2, \mu_{X_2} = 1, \rho = 0.8$. Find the mean and variance of the random variable $Y = X_1 - X_2$, which follows a normal distribution.

6. Obtain the axes of constant probability density contours for a bivariate normal distribution with the covariance matrix $\Sigma = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}$.

1) The number of bars of service (X) and response time (Y) for mobile is given below

Y \ X	1	2	3
4	0.15	0.1	0.05
3	0.02	0.1	0.05
2	0.02	0.03	0.2
1	0.01	0.02	0.25

→ i) Marginal Probability of X and Y

X	1	2	3
P(X)	0.2	0.25	0.55

ii) $P(X \geq 2) = P(X=2) + P(X=3)$

$= 0.25 + 0.55 = 0.8$

iii) $P(Y < 2) = P(Y=1) = 0.28$

iv) $P(X > 2, Y > 2) = P(X=3, Y=3) + P(X=3, Y=4)$

$= 0.05 + 0.05 = 0.1$

2) Determine value of c that makes function

$f(x, y) = c(x+y)$ with $x = 1, 2, 3$ and $y = 1, 2, 3$

→

X \ Y	1	2	3
1	2c	3c	4c
2	3c	4c	5c
3	4c	5c	6c

$\sum \sum P_{ij} = 1$

As $f(x, y)$ is a joint probability mass function

$\sum \sum P_{ij} = 36c = 1$

$\Rightarrow \boxed{c = \frac{1}{36}}$

X	1	2	3
P(X)	9/36	12/36	15/36

Y	1	2	3
P(Y)	9/36	12/36	15/36

Expected Value of X = $E[X] = \sum x_i P_i = 1 \times \frac{1}{4} + 2 \times \frac{1}{3} + 3 \times \frac{5}{12}$

$E[X] = 2.1667$

Expected Value of Y = $E[Y] = \sum y_i Q_i = 1 \times \frac{1}{4} + 2 \times \frac{1}{3} + 3 \times \frac{5}{12}$

$E[Y] = 2.1667$

3) The joint probability distribution of two discrete fn random variables X and Y is given below

Y \ X	1	2	3
1	0.05	0.05	0.10
3	0.05	0.10	0.35
5	0.00	0.20	0.10

→ $P(Y|X=2) = ?$

Y	1	3	5
$P(Y\|X=2)$	0.05	0.10	0.20

$$P(Y|X=2) = 0.05 + 0.10 + 0.2$$

$$= 0.35$$

4) A pen is selected at random from box that has 3 blue pens, 2 red, 3 green. If X is no of blue and Y is red, construct joint probability distribution table

→ $X = \{0, 1\}$ Blue pens

$Y = \{0, 1\}$ Red pens

Total Pens $= 3B + 2R + 3G$

$= 8$

$P_{ij} \Rightarrow P_{00} =$ 0R and 0B $= \frac{3}{8}$

$P_{01} =$ 0B and 1R $= \frac{2}{8}$

$P_{10} =$ 1B and 0R $= \frac{3}{8}$

$P_{11} =$ 1B and 1R $= \frac{0}{8}$

X \ Y	0	1
0	3/8	2/8
1	3/8	0

5) Determine value of c that make fn
$f(x,y) = c(x+y)$ with $X = 1,2,3$ and $Y = 1,2,3$

→ We have calculate the table

X \ Y	1	2	3
1	2/36	3/36	4/36
2	3/36	4/36	5/36
3	4/36	5/36	6/36

$$C = \frac{1}{36}$$

$$E[X] = 2.1667$$

$$E[Y] = 2.1667$$

$$E[XY] = \sum\sum x_i y_j P_{ij}$$

$$= 1\times1\times\frac{2}{36} + 1\times2\times\frac{3}{36} + 1\times3\times\frac{4}{36} +$$

$$2\times1\times\frac{3}{36} + 2\times2\times\frac{4}{36} + 2\times3\times\frac{5}{36} +$$

$$3\times1\times\frac{4}{36} + 3\times2\times\frac{5}{36} + 3\times3\times\frac{6}{36}$$

$$= \frac{14}{3} = 4.6667$$

$$Cov(X, Y) = E[XY] - E[X] \cdot E[Y]$$

$$= 4.6667 - (2.1667)(2.1667)$$

$$= -0.0279$$

6)

X \ Y	0	1	2
0	0.52	0.2	0.04
1	0.14	0.02	0.01
2	0.06	0.01	0

i)

X	0	1	2
P(X)	0.76	0.17	0.07

Y	0	1	2
P(Y)	0.72	0.23	0.05

ii) $P(Y \mid X = 1)$

Y	0	1	2
$P(Y \mid X = 1)$	0.14	0.02	0.01

iii)

X	0	1	2
$P(X \mid Y = 2)$	0.04	0.01	0

7) For two flips of a coin, let X be no of tails and Y be no of heads on last flip. Find

→ $S = \{TT, HT, TH, HH\}$

$X = \{0, 1, 2\}$ No of tails

$Y = \{0, 1\}$ No of heads on last flip

i) $P_i(X) \Rightarrow$

X	0	1	2
$P(X)$	$1/4$	$2/4$	$1/4$

ii) $Q_j(Y) \Rightarrow$

Y	0	1
$P(Y)$	$2/4$	$2/4$

iii) Probability Mass Function, $P(X, Y) = ?$

X \ Y	0	1
0	0	$1/4$
1	$1/4$	$1/4$
2	$1/4$	0

$$E[X] = \sum x_i P_i = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = 1$$

$$E[Y] = \sum y_j P_j = 0 \times \frac{2}{4} + 1 \times \frac{2}{4} = \frac{1}{2}$$

$$E[XY] = \sum \sum x_i y_j P_{ij}$$

$$= 0 \times 0 \times 0 + 0 \times 1 \times \frac{1}{4} + 1 \times 0 \times \frac{1}{4} +$$

$$1 \times 1 \times \frac{1}{4} + 2 \times 0 \times \frac{1}{4} + 2 \times 1 \times 0$$

$$E[XY] = \frac{1}{4}$$

$$Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$$

$$= \frac{1}{4} - 1 \times \frac{1}{2} = -\frac{1}{4}$$

$$Cov(X,Y) = -0.25$$

8) Random variables X and Y have joint PMF

$$f(x,y) = \begin{cases} c\,|x+y|, & x = -2, 0, 2;\ \ -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Find i) constant c

ii) $P(Y < X)$

iii) $P(Y > X)$

iv) $P(X = Y)$

v) $P(X < 1)$

8) →

$X \backslash Y$	-1	0	1
-2	3c	2c	c
0	c	0	c
2	c	2c	3c

i) $\sum \sum P_{ij} = 1$

$14c = 1$

$$\boxed{c = \frac{1}{14}}$$

ii) $P(y < x) = \sum \sum P_{ij}$ for $y < x$

$= \frac{1}{14} + \frac{1}{14} + 2 \times \frac{1}{14} + 3 \times \frac{1}{14}$

$= \frac{7}{14} = 0.5$

iii) $P(x > y) = \sum \sum P_{ij}$ for $y > x$

$= 3 \times \frac{1}{14} + 2 \times \frac{1}{14} + \frac{1}{14} + \frac{1}{14}$

$= 0.5$

iv) $P(x = y) = 0$

v) $P(X < 1) = c + c + 3c + 2c + c = 8c = \frac{8}{14} = 0.5714$

PROBABILITY DENSITY FUNCTION

1) The fn $f(x,y) = \frac{2}{3}(x+2y)$ is a joint PDF over $0 \le x \le 1$ and $0 \le y \le 1$.

→ $P_1(x) = \frac{2}{3}(x+1)$

$$P_2(y) = \int_0^1 \frac{2}{3}(x+2y)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + 2xy\right]_0^1$$

$$= \frac{2}{3}\left(\frac{1}{2} + 2y\right)$$

i) $E(X) = \int_0^1 x\, P_1(x)\,dx = \int_0^1 x\,\frac{2}{3}(x+1)\,dx$

$$= \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{2}{3}\left(\frac{5}{6}\right) = \frac{5}{9}$$

ii) $E(Y) = \int_0^1 y\, P_2(y)\,dy = \int_0^1 y\,\frac{2}{3}\left(\frac{1}{2} + 2y\right)dy$

$$= \frac{2}{3}\left[\frac{y^2}{4} + \frac{2y^3}{3}\right]_0^1 = \frac{2}{3}\left[\frac{1}{4} + \frac{2}{3}\right]$$

$$= \frac{11}{18}$$

$$E(X,Y) = \int_0^1\int_0^1 xy\, f(x,y)\, dy\, dx$$

$$= \int_0^1\int_0^1 xy\, \frac{2}{3}(x+2y)\, dy\, dx$$

$$= \frac{2}{3}\int_0^1 x\left(\frac{xy^2}{2} + \frac{2y^3}{3}\right]_0^1 dx$$

$$= \frac{2}{3}\int_0^1 x\left[\frac{x}{2} + \frac{2}{3}\right]_0^1 dx = \frac{2}{3}\left[\frac{x^3}{6} + \frac{x^2}{2}\cdot\frac{2}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\frac{1}{6} + \frac{1}{3}\right]_0^1 = \frac{2}{3}\left(\frac{3}{6}\right) = \frac{1}{3}$$

iii> $\mathrm{Cov}(X,Y) = E(X,Y) - E(X)\cdot E(Y)$

$$= \frac{1}{3} - \frac{5}{9} \times \frac{11}{18}$$

$$= 0.006172$$

2) $f(x,y) = k\, e^{-0.001x - 0.002y}$

where $0 < x < y < \infty$

$$\int_0^\infty \int_x^\infty f(x,y)\, dy\, dx = 1$$

$$= \int_0^\infty \int_x^\infty k e^{-0.001x - 0.002y}\, dy\, dx = 1$$

$$= \int_0^\infty \left[\frac{k e^{-0.001x - 0.002y}}{-0.002} \right]_x^\infty dx$$

$$= \frac{-k}{0.002} \int_0^\infty \left(e^{-0.001x - 0.002\cancel{x}} \right) dx$$

$$= \frac{-k}{0.002} \int_0^\infty e^{-0.003x}\, dx = \frac{-k}{0.002} \left[\frac{e^{-0.003x}}{-0.003} \right]_0^\infty$$

$$\frac{-k}{0.002(-0.003)} \left[e^{-0.003x} \right]_0^\infty = \frac{k}{0.002 \times 0.003} \left(e^{-\infty} - e^{-0} \right) \xrightarrow{} (-1)$$

$$\frac{k}{0.002 \times 0.003} (-1) = 1 \Rightarrow k = 0.002 \times 0.003 \times -1$$

$$\boxed{k = -6 \times 10^{-6}}$$

3)
$$f(x,y) = \begin{cases} 10xy^2 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$\rightarrow$
$$P(x,y) = \int_0^y \int_x^1 f(x,y)\, dy\, dx$$

$$= \int_0^y \int_x^1 10xy^2\, dy\, dx$$

$$= \int_0^y 10x \left[\frac{y^3}{3}\right]_x^1 dx$$

$$= \int_0^y 10x \left[\frac{1}{3} - \frac{x^3}{3}\right] dx$$

$$= 10 \left[\frac{x^2}{6} - \frac{x^5}{15}\right]_0^y = 10\left(\frac{5y^2 - 2y^5}{30}\right)$$

$$= \frac{5}{3}y^2 - \frac{2}{3}y^5$$

$$P\left(X > \frac{1}{2} \,\middle|\, y\right) = 0.5$$

$$\int_{1/2}^1 \int_{0.5}^{0,5} 10xy^2\, dy\, dx = 0$$