

LINEAR ALGEBRA III

QR Factorization:

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{col } A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

ex.1. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Soln:

Construction an orthonormal basis for Col A

The columns of A are the vectors $\{x_1, x_2, x_3\}$

Let $v_1 = x_1 = (1, 1, 1, 1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (0, 1, 1, 1) - \frac{(0, 1, 1, 1) \cdot (1, 1, 1, 1)}{(1, 1, 1, 1) \cdot (1, 1, 1, 1)} (1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - \frac{3}{4} (1, 1, 1, 1) = (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (0, 0, 1, 1) - \frac{(0, 0, 1, 1) \cdot (1, 1, 1, 1)}{(1, 1, 1, 1) \cdot (1, 1, 1, 1)} (1, 1, 1, 1) - \frac{(0, 0, 1, 1) \cdot (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})}{(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \cdot (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})} (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

$$= (0, 0, 1, 1) - \frac{2}{4} (1, 1, 1, 1) - \frac{2}{3} (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = (0, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$$

$\therefore \{v_1, v_2, v_3\}$ forms an orthogonal basis of Col A .

$\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}), (0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}$ forms an orthonormal basis of Col A.

$$\therefore Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

To construct an upper triangular invertible matrix

We have $A = QR \implies Q^T A = Q^T Q R \implies Q^T A = IR \implies Q^T A = R$ i.e., $R = Q^T A$.

$$\therefore R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$$

ex.2. Find a QR factorization of $A =$

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

Soln:

$\{x_1, x_2, x_3\}$ are the columns of the matrix A .

Let $v_1 = x_1 = (1, -1, -1, 1, 1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (2, 1, 4, -4, 2) - \frac{(2, 1, 4, -4, 2) \cdot (1, -1, -1, 1, 1)}{(1, -1, -1, 1, 1) \cdot (1, -1, -1, 1, 1)} (1, -1, -1, 1, 1)$$

$$= (2, 1, 4, -4, 2) - \frac{-5}{5} (1, -1, -1, 1, 1) = (3, 0, 3, -3, 3)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (5, -4, -3, 7, 1) - \frac{(5, -4, -3, 7, 1) \cdot (1, -1, -1, 1, 1)}{(1, -1, -1, 1, 1) \cdot (1, -1, -1, 1, 1)} (1, -1, -1, 1, 1) - \frac{(5, -4, -3, 7, 1) \cdot (3, 0, 3, -3, 3)}{(3, 0, 3, -3, 3) \cdot (3, 0, 3, -3, 3)} (3, 0, 3, -3, 3)$$

$$= (5, -4, -3, 7, 1) - \frac{20}{5}(1, -1, -1, 1, 1) - \frac{-12}{36}(3, 0, 3, -3, 3) = (2, 0, 2, 2, -2).$$

$\therefore \{(1, -1, -1, 1, 1), (3, 0, 3, -3, 3), (2, 0, 2, 2, -2)\}$ forms an orthogonal basis of Col A.

$\{(1/\sqrt{5}, -1/\sqrt{5}, -1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}), (1/2, 0, 1/2, -1/2, 1/2), (1/2, 0, 1/2, 1/2, -1/2)\}$ forms an orthonormal basis of Col A.

$$\therefore Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

ex.3. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

Soln: The columns of A are the vectors $\{x_1, x_2, x_3\}$

where $x_1 = (3, 1, -1, 3), x_2 = (-5, 1, 5, -7), x_3 = (1, 1, -2, 8)$.

Let $v_1 = (3, 1, -1, 3)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (-5, 1, 5, -7) - \frac{(-5, 1, 5, -7) \cdot (3, 1, -1, 3)}{(3, 1, -1, 3) \cdot (3, 1, -1, 3)} (3, 1, -1, 3)$$

$$= (-5, 1, 5, -7) - \frac{-40}{20} (3, 1, -1, 3) = (1, 3, 3, -1)$$

$$v_3 = \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (1, 1, -2, 8) - \frac{(1, 1, -2, 8) \cdot (3, 1, -1, 3)}{(3, 1, -1, 3) \cdot (3, 1, -1, 3)} (3, 1, -1, 3) - \frac{(1, 1, -2, 8) \cdot (1, 3, 3, -1)}{(1, 3, 3, -1) \cdot (1, 3, 3, -1)} (1, 3, 3, -1)$$

$$= (1, 1, -2, 8) - \frac{30}{20} (3, 1, -1, 3) - \frac{-10}{20} (1, 3, 3, -1) = (-3, 1, 1, 3)$$

$\{(3, 1, -1, 3), (1, 3, 3, -1), (-3, 1, 1, 3)\}$ is an orthogonal basis for the column space of the given matrix.

ex.4. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

Soln: The columns of A are the vectors $\{x_1, x_2, x_3\}$

where $x_1 = (-1, 3, 1, 1), x_2 = (6, -8, -2, -4), x_3 = (6, 3, 6, -3)$.

Let $v_1 = (-1, 3, 1, 1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (6, -8, -2, -4) - \frac{(6, -8, -2, -4) \cdot (-1, 3, 1, 1)}{(-1, 3, 1, 1) \cdot (-1, 3, 1, 1)} (-1, 3, 1, 1)$$

$$= (6, -8, -2, -4) - \frac{-36}{12} (-1, 3, 1, 1) = (3, 1, 1, -1)$$

$$v_3 = \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (6, 3, 6, -3) - \frac{(6, 3, 6, -3) \cdot (-1, 3, 1, 1)}{(-1, 3, 1, 1) \cdot (-1, 3, 1, 1)} (-1, 3, 1, 1) - \frac{(6, 3, 6, -3) \cdot (3, 1, 1, -1)}{(3, 1, 1, -1) \cdot (3, 1, 1, -1)} (3, 1, 1, -1)$$

$$= (6, 3, 6, -3) - \frac{6}{12} (-1, 3, 1, 1) - \frac{30}{12} (3, 1, 1, -1) = (-1, -1, 3, -1)$$

$\{(-1, 3, 1, 1), (3, 1, 1, -1), (-1, -1, 3, -1)\}$ is an orthogonal basis for the column space of the given matrix.