

## LINEAR ALGEBRA II

### Linear Transformation:

Let  $U$  and  $V$  be two vector spaces over the same field  $F$ . The mapping  $T : U \longrightarrow V$  is said to be a linear transformation, if

$$(i) T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \quad \alpha, \beta \in U$$

$$(ii) T(c.\alpha) = c.T(\alpha) \quad \forall \quad c \in F, \alpha \in U.$$

A Linear transformation  $T : U \longrightarrow V$  is also called a linear map on  $U$ .

### Theorem:

A mapping  $T : U \longrightarrow V$  from the vector space  $U(F)$  into  $V(F)$  is a linear transformation iff  $T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta) \quad \forall \quad c_1, c_2 \in F \quad \text{and} \quad \alpha, \beta \in U$ .

### Properties of Linear Transformation

#### Theorem

If  $T : U \longrightarrow V$  is a linear transformation, then

$$(a) T(0) = 0', \text{ where } 0 \text{ and } 0' \text{ are zero vectors of } U \text{ and } V \text{ respectively.}$$

$$(b) T(-\alpha) = -T(\alpha) \quad \forall \quad \alpha \in U$$

$$(c) T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \dots + c_nT(\alpha_n)$$

### Theorem

If  $\beta_1, \beta_2, \dots, \beta_m$  be any basis of vector space  $V$  and  $\alpha_1, \alpha_2, \dots, \alpha_m$  be any  $m$  vectors of the vector space  $W$ , then there exists one and only one linear transformation  $T : V \longrightarrow W$  with  $T(\beta_i) = \alpha_i$  for  $i = 1, 2, \dots, m$ .

### Problems:

1. If  $T$  is a mapping from  $V_3(\mathbb{R})$  into  $V_3(\mathbb{R})$  defined by  $T(x_1, x_2, x_3) = (0, x_2, x_3)$ . Show that  $T$  is a linear transformation.

**Soln:** Let  $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in V_3(\mathbb{R})$

Consider,  $T(\alpha + \beta) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$

$$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$= T(0, x_2 + y_2, x_3 + y_3)$$

$$= (0, x_2, x_3) + (0, y_2, y_3)$$

$$= T(\alpha) + T(\beta)$$

Consider,  $T(c\alpha) = T(c(x_1, x_2, x_3))$

$$= T(cx_1, cx_2, cx_3)$$

$$= (0, cx_2, cx_3)$$

$$= c(0, x_2, x_3)$$

$$= cT(x_1, x_2, x_3)$$

$$= cT(\alpha)$$

$\therefore T$  is a linear transformation.

2. Find the linear transformation  $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$  such that  $T(1, 1) = (0, 1, 2), T(-1, 1) = (2, 1, 0)$ .

**Soln:**  $\{(1, 1), (-1, 1)\}$  forms a basis of  $\mathbb{R}^2$ .

Let  $\alpha = (x, y) \in \mathbb{R}^2$

$$(x, y) = c_1(1, 1) + c_2(-1, 1) \implies (x, y) = (c_1 - c_2, c_1 + c_2)$$

$$\implies x = c_1 - c_2, y = c_1 + c_2 \implies c_1 = \frac{x+y}{2}, c_2 = \frac{y-x}{2}$$

$$\therefore (x, y) = \frac{x+y}{2}(1, 1) + \frac{y-x}{2}(-1, 1)$$

$\therefore$  the required transformation is

$$T(x, y) = \frac{x+y}{2}(0, 1, 2) + \frac{y-x}{2}(2, 1, 0)$$

$$T(x, y) = (y-x, y, x+y)$$

3. Is there a linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  for which  $T(2, 2) = (4, -6)$  and  $T(5, 5) = (2, -3)$ ?

**Soln:** The vectors  $(2, 2)$  and  $(5, 5) \in \mathbb{R}^2$  are linearly dependent as  $(5, 5) = \frac{5}{2}(2, 2)$

$$\text{If } T \text{ is linear, } T(5, 5) = T\left(\frac{5}{2}(2, 2)\right) = \frac{5}{2}T(2, 2) = \frac{5}{2}(4, -6) = (10, -15)$$

$$\text{But } T(5, 5) = (2, -3) \neq (10, -15).$$

$\therefore$  a linear map with the given data doesn't exist.

4. Let  $M(\mathbb{R})$  be the vector space of all  $2 \times 2$  matrices over  $\mathbb{R}$  and  $B$  be a fixed non-zero element of  $M(\mathbb{R})$ . Show that the mapping  $T : M(\mathbb{R}) \rightarrow M(\mathbb{R})$  defined by  $T(A) = AB - BA, \forall A \in M(\mathbb{R})$  is a linear map.

**Soln:** Let  $A$  and  $C \in M(\mathbb{R})$  be arbitrary.

$$\text{Consider, } T(A+C) = (A+C)B - B(A+C) = AB + CB - BA - BC = AB - BA + CB - BC = T(A) + T(C)$$

Let  $c \in \mathbb{R}$  be any scalar.

$$\text{Consider, } T(cA) = (cA)B - B(cA) = c(AB - BA) = cT(A)$$

$\therefore T$  is a linear transformation.

5. If  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation such that  $T(1, 0) = (1, 1)$  &  $T(0, 1) = (-1, 2)$ , show that  $T$  maps the square with vertices  $(0, 0), (1, 0), (1, 1), (0, 1)$  into a parallelogram.

**Soln:**  $\{(1, 0), (0, 1)\}$  forms a basis of  $\mathbb{R}^2$

$$(x, y) = x(1, 0) + y(0, 1)$$

$$\therefore T(x, y) = xT(1, 0) + yT(0, 1) = x(1, 1) + y(-1, 2) = (x - y, x + 2y)$$

$$\text{Now, } T(0, 0) = (0, 0) = A, T(1, 0) = (1, 1) = B, T(1, 1) = (0, 3) = C, T(0, 1) = (-1, 2) = D.$$

To show that  $A, B, C, D$  are vertices of a parallelogram, we shall show that the diagonals  $AC$  and  $BD$  bisect each other.

$$\text{Midpoint of } AC = \left(0, \frac{3}{2}\right), \text{ Midpoint of } BD = \left(0, \frac{3}{2}\right)$$

Diagonals bisect each other. Hence  $ABCD$  is a parallelogram.

6. If  $T : V_1(\mathbb{R}) \rightarrow V_3(\mathbb{R})$  is defined by  $T(x) = (x, x^2, x^3)$ , verify whether  $T$  is linear or not.

**Soln:** Let  $x, y \in V_1(\mathbb{R})$

$$\text{Consider, } T(x+y) = (x+y, (x+y)^2, (x+y)^3)$$

$$T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3) = (x+y, x^2+y^2, x^3+y^3)$$

$$\text{We can see that, } T(x+y) \neq T(x) + T(y)$$

$\therefore T$  is not a linear transformation.

7. Find the linear transformation  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $f(1, 0) = (1, 1)$  and  $f(0, 1) = (-1, 2)$ .

**Soln:** Let  $(x, y) \in \mathbb{R}^2$ . Then  $(x, y) = x(1, 0) + y(0, 1)$

$$\text{Define } f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ by } f(x, y) = xf(1, 0) + yf(0, 1) = x(1, 1) + y(-1, 2)$$

$$\text{Hence } f(x, y) = (x - y, x + 2y) \quad \forall (x, y) \in \mathbb{R}^2$$

8. Find the linear transformation  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $f(1, 1) = (0, 1)$  and  $f(-1, 1) = (3, 2)$ .

**Soln:**  $(1, 1), (-1, 1)$  of  $\mathbb{R}^2$  forms a basis of  $\mathbb{R}^2$

Let  $\alpha = (x, y) \in \mathbb{R}^2$  be arbitrary.

$$\text{Let } (x, y) = c_1(1, 1) + c_2(-1, 1) = (c_1 - c_2, c_1 + c_2)$$

$$\implies c_1 - c_2 = x, c_1 + c_2 = y \implies c_1 = \frac{x+y}{2}, c_2 = \frac{y-x}{2}$$

$$\therefore (x, y) = \frac{x+y}{2}(1, 1) + \frac{y-x}{2}(-1, 1)$$

$$\begin{aligned}
\text{Hence the required transformation is } f(x, y) &= \frac{x+y}{2}f(1, 1) + \frac{y-x}{2}f(-1, 1) \\
&= \frac{x+y}{2}(0, 1) + \frac{y-x}{2}(3, 2) \\
\therefore f(x, y) &= \left(\frac{3y-3x}{2}, \frac{3y-x}{2}\right).
\end{aligned}$$

**9.**  $f : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$  is defined by  $f(x, y, z) = (x + y, y + z)$ , show that  $f$  is a linear transformation.

**Soln:** Let  $\alpha = (x_1, y_1, z_1)$  and  $\beta = (x_2, y_2, z_2)$  be any two elements of  $V_3(\mathbb{R})$ .

$$\begin{aligned}
\text{Consider, } f(\alpha + \beta) &= f(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\
&= (x_1 + x_2 + y_1 + y_2, y_1 + y_2 + z_1 + z_2) = ((x_1 + y_1) + (x_2 + y_2), (y_1 + z_1) + (y_2 + z_2)) \\
&= (x_1 + y_1, y_1 + z_1) + (x_2 + y_2, y_2 + z_2) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2) \\
\therefore f(\alpha + \beta) &= f(\alpha) + f(\beta).
\end{aligned}$$

$$\begin{aligned}
\text{Consider, } f(c\alpha) &= f(cx_1, cy_1, cz_1) \\
&= (cx_1 + cy_1, cy_1 + cz_1) = c(x_1 + y_1, y_1 + z_1) = cf(x_1, y_1, z_1) \\
\therefore f(c\alpha) &= cf(\alpha)
\end{aligned}$$

Hence  $f$  is a linear transformation.

**10.** If  $T$  is a mapping from  $V_2(\mathbb{R})$  into  $V_2(\mathbb{R})$  defined by  $T(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$ , show that  $T$  is a linear transformation.

**Soln:** Let  $\alpha = (x_1, x_2), \beta = (y_1, y_2) \in V_2(\mathbb{R})$

$$\begin{aligned}
\text{Consider, } T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2) \\
&= ((x_1 + y_1) \cos \theta - (x_2 + y_2) \sin \theta, (x_1 + y_1) \sin \theta + (x_2 + y_2) \cos \theta) \\
&= ((x_1 \cos \theta - x_2 \sin \theta) + (y_1 \cos \theta - y_2 \sin \theta), (x_1 \sin \theta + x_2 \cos \theta) + (y_1 \sin \theta + y_2 \cos \theta)) \\
&= (x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta) + (y_1 \cos \theta - y_2 \sin \theta, y_1 \sin \theta + y_2 \cos \theta) \\
&= T(x_1, x_2) + T(y_1, y_2) \therefore T(\alpha + \beta) = T(\alpha) + T(\beta)
\end{aligned}$$

$$\begin{aligned}
\text{Consider, } T(c\alpha) &= T(cx_1, cx_2) \\
&= (cx_1 \cos \theta - cx_2 \sin \theta, cx_1 \sin \theta + cx_2 \cos \theta) = c(x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta) \\
&= cT(x_1, x_2) \\
\therefore T(c\alpha) &= cT(\alpha)
\end{aligned}$$

Hence  $T$  is a linear transformation.

### Ordered Basis:

Let  $V$  be a vector space over a field  $F$  and  $d[V] = n > 0$ .

Choose a basis  $\alpha_1, \alpha_2, \dots, \alpha_n$  of  $V$  written in this order. This basis  $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is called an ordered basis of  $V$ .

### Coordinate Vector:

Let  $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be an ordered basis of the vector space. Then  $\alpha \in V$  can be uniquely written as  $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n; c_i \in F$ . The scalars  $c_1, c_2, \dots, c_n$  are called the coordinates of the vector  $\alpha$  w.r.t the ordered basis  $B$ . The n-tuple  $(c_1, c_2, \dots, c_n)$  is called the coordinate vector of  $V$  relative to the basis  $B$ .

### Problems

**1.** Find the co-ordinate vector of  $(3, -2, 1)$  relative to

(i) ordered standard basis

(ii) the ordered basis  $\{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$  of  $\mathbb{R}^3$ .

$$\begin{aligned}
\text{Soln: (i) Let } (3, -2, 1) &= c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (c_1, c_2, c_3) \\
\implies c_1 &= 3, c_2 = -2, c_3 = 1
\end{aligned}$$

$\therefore$  co-ordinates of  $(3, -2, 1)$  relative to standard basis is  $(3, -2, 1)$ .

$$\text{(ii) Let } (3, -2, 1) = c_1(1, 1, 1) + c_2(1, 0, 0) + c_3(1, 1, 0) = (c_1 + c_2 + c_3, c_1 + c_3, c_1)$$



Hence  $A_T = \begin{pmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{pmatrix}$

3. Let  $R_3[x]$  denote the vector space of all real polynomials. Define  $T : R_3[x] \rightarrow R_3[x]$  by  $T(p) = p'$  denotes the differential coefficient of  $p$ . Find the matrix of the linear transformation  $T$  relative to the basis  $\{1, x, x^2, x^3\}$ .

**Soln:**

$$T(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$$

$$T(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$$

$$T(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$$

$$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$$

Hence the matrix of  $T$  is  $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. Given the matrix  $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ . Determine the linear transformation  $T : V_3(\mathbb{R}) \rightarrow$

$V_2(\mathbb{R})$  relative to bases  $B_1$  and  $B_2$  given by

(i)  $B_1$  and  $B_2$  are standard bases of  $V_3(\mathbb{R})$  and  $V_2(\mathbb{R})$  respectively.

(ii)  $B_1 = \{(1, 1, 1), (1, 2, 3), (1, 0, 0)\}$ ,  $B_2 = \{(1, 1), (1, -1)\}$

**Soln:** (i)  $B_1$  and  $B_2$  are standard bases.

Define  $T : V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$  by

$$T(1, 0, 0) = 1(1, 0) + 3(0, 1) = (1, 3)$$

$$T(0, 1, 0) = -1(1, 0) + 1(0, 1) = (-1, 1)$$

$$T(0, 0, 1) = 2(1, 0) + 0(0, 1) = (2, 0)$$

$$\text{Now, } (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\text{Then } T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) = x(1, 3) + y(-1, 1) + z(2, 0)$$

$$\therefore T(x, y, z) = (x - y + 2z, 3x + y) \text{ is the required transformation.}$$

(ii)  $B_1 = \{(1, 1, 1), (1, 2, 3), (1, 0, 0)\}$ ,  $B_2 = \{(1, 1), (1, -1)\}$

Define  $T : V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$  by

$$T(1, 1, 1) = 1(1, 1) + 3(1, -1) = (4, -2)$$

$$T(1, 2, 3) = -1(1, 1) + 1(1, -1) = (0, -1)$$

$$T(1, 0, 0) = 2(1, 1) + 0(1, -1) = (2, 2)$$

$$\text{Now, } (x, y, z) = c_1(1, 1, 1) + c_2(1, 2, 3) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + 2c_2, c_1 + 3c_2)$$

$$\implies c_1 + c_2 + c_3 = x, c_1 + 2c_2 = y, c_1 + 3c_2 = z \implies c_1 = 3y - 2z, c_2 = z - y, c_3 = x - 2y + z$$

$$\therefore (x, y, z) = (3y - 2z)(1, 1, 1) + (z - y)(1, 2, 3) + (x - 2y + z)(1, 0, 0)$$

$$T(x, y, z) = (3y - 2z)T(1, 1, 1) + (z - y)T(1, 2, 3) + (x - 2y + z)T(1, 0, 0)$$

$$= (3y - 2z)(4, -2) + (z - y)(0, -1) + (x - 2y + z)(2, 2)$$

$$\text{Hence } T(x, y, z) = (2x + 8y - 6z, 2x - 8y + 4z) \text{ is the required transformation.}$$

5. For the matrix  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ , find the corresponding linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  w.r.t

the basis  $\{(1, 0), (1, 1)\}$

**Soln:**  $B_1 = \{(1, 0), (1, 1)\}$ ,  $B_2 = \{(1, 0), (1, 1)\}$

Define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T(1, 0) = 1(1, 0) + 3(1, 1) = (4, 3)$$

$$T(1, 1) = 2(1, 0) + 4(1, 1) = (6, 4).$$

$$\text{Now, } (x, y) = c_1(1, 0) + c_2(1, 1) = (c_1 + c_2, c_2)$$

$$\implies c_1 + c_2 = x, c_2 = y \implies c_1 = x - y, c_2 = y$$

$$\therefore (x, y) = (x - y)(1, 0) + y(1, 1)$$

$$\text{Then } T(x, y) = (x - y)T(1, 0) + yT(1, 1) = (x - y)(4, 3) + y(6, 4) = (4x + 2y, 3x + y)$$

Hence  $T(x, y) = (4x + 2y, 3x + y)$  is the required transformation.

### Range of a Linear transformation:

Let  $T : V \longrightarrow W$  be a linear transformation. The range of  $T$  is the set  $R(T) = \{T(\alpha) / \alpha \in V\}$

### Kernel(or null space) of a Linear transformation:

Let  $T : V \longrightarrow W$  be a linear transformation. The kernel(or null space) of  $T$  is the set  $N(T) = \{\alpha \in V / T(\alpha) = 0\}$ , where 0 is the zero vector of  $W$ .

### Note:

- (i) For the identity map  $I : V \longrightarrow V$  the range is the entire space  $V$  and the kernel is the zero subspace.
- (ii) For the zero linear map  $T : V \longrightarrow W$  defined by  $T(\alpha) = 0 \forall \alpha \in V$ , the range  $R(T) = \{0\} =$  zero space of  $W$  and the null space  $N(T) = V$ .

### Theorem

Let  $T : V \longrightarrow W$  be a linear transformation. Then

- (a)  $R(T)$  is a subspace of  $W$ .
- (b)  $N(T)$  is a subspace of  $V$ .
- (c)  $T$  is one-one iff  $N(T) = \{0\}$ , where 0 is the zero vector of  $W$ .

### Theorem

Let  $T : V \longrightarrow W$  be a linear transformation. If the vectors  $\alpha_1, \alpha_2, \dots, \alpha_n$  generate  $V$ , then the vectors  $T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)$  generates  $R(T)$ .

### Rank and Nullity of a Linear Transformation:

Let  $T : V \longrightarrow W$  be a linear transformation. The dimension of the range space  $R(T)$  is called the rank of the linear transformation  $T$  and is denoted by  $r(T)$ . The dimension of the null space  $N(T)$  is called the nullity of the linear transformation  $T$  and is denoted by  $n(T)$ .

### Rank-Nullity theorem

Let  $T : V \longrightarrow W$  be a linear transformation and  $V$  be a finite dimensional vector space. Then  $r(T) + n(T) = d[V] (d[R(T)] + d[N(T)] = d[V])$ .

### Problems:

1. Let  $T : V \longrightarrow W$  be a linear transformation defined by  $T(x, y, z) = (x + y, x - y, 2x + z)$ . Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.

**Soln:**  $T(e_1) = T(1, 0, 0) = (1, 1, 2) = \alpha_1$

$T(e_2) = T(0, 1, 0) = (1, -1, 0) = \alpha_2$

$T(e_3) = T(0, 0, 1) = (0, 0, 1) = \alpha_3$

$\{\alpha_1, \alpha_2, \alpha_3\}$  generates  $R(T)$

Consider  $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$   $|A| = -2 \neq 0$

$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$  is linearly independent. Thus it is a basis of  $R(T)$ .

$d[R(T)] = 3$ .

Let  $\alpha \in R(T)$

Then  $\alpha = c_1(\alpha_1) + c_2(\alpha_2) + c_3(\alpha_3) = c_1(1, 1, 2) + c_2(1, -1, 0) + c_3(0, 0, 1) = (c_1 + c_2, c_1 - c_2, 2c_1 + c_3)$

$\therefore R(T) = \{(c_1 + c_2, c_1 - c_2, 2c_1 + c_3) / c_1, c_2, c_3 \in \mathbb{R}\}$

Suppose  $T(x, y, z) = (0, 0, 0)$

$\implies (x + y, x - y, 2x + z) = (0, 0, 0) \implies x + y = 0, x - y = 0, 2x + z = 0 \implies x = 0, y = 0, z = 0$

$$\begin{aligned}\therefore N(T) &= \{(0, 0, 0)\} \\ d[N(T)] &= 0 \\ \text{rank} + \text{nullity} &= 3 + 0 = 3 = d[V_3(\mathbb{R})].\end{aligned}$$

**2.** Find the range, null space, rank, nullity of the linear transformation  $T : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$  defined by  $T(x, y, z) = (y - x, y - z)$  and hence verify the rank-nullity theorem.

**Soln:**

$$T(1, 0, 0) = (-1, 0) = \alpha_1$$

$$T(0, 1, 0) = (1, 1) = \alpha_2$$

$$T(0, 0, 1) = (0, -1) = \alpha_3.$$

$$R(T) = L\{\alpha_1, \alpha_2, \alpha_3\}$$

$$\text{Consider } \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}, R_1 = -R_1 \implies \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}, R_2 = R_2 - R_1 \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix},$$

$$R_3 = R_3 + R_2 \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \therefore d[R(T)] = 2 \text{ and basis of } R(T) = \{(-1, 0), (1, 1)\}.$$

Let  $\alpha \in R(T)$

$$\text{Then } \alpha = c_1\alpha_1 + c_2\alpha_2 = c_1(-1, 0) + c_2(1, 1) = (-c_1 + c_2, c_2)$$

$$\therefore R(T) = \{(-c_1 + c_2, c_2) / c_1, c_2 \in \mathbb{R}\}.$$

$$\text{Suppose } T(x, y, z) = (0, 0) \implies (y - x, y - z) = (0, 0) \implies y = x, y = z \implies x = y = z$$

$$\therefore N(T) = \{(a, a, a) / a \in \mathbb{R}\} \text{ and basis of } N(T) = \{(1, 1, 1)\}. \therefore d[N(T)] = 1.$$

$$\text{Hence rank} + \text{nullity} = 2 + 1 = 3 = d[V_3(\mathbb{R})].$$

**3.** Find the linear transformation  $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$  whose range space is spanned by  $(1, 0, -1)$  and  $(1, 2, 2)$ .

**Soln:** Define  $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$  such that

$$f(1, 0, 0) = (1, 0, -1)$$

$$f(0, 1, 0) = (1, 2, 2)$$

$$f(0, 0, 1) = (0, 0, 0)$$

Since  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  generates  $\mathbb{R}^3$ , the image vectors  $\{(1, 0, -1), (1, 2, 2)\}$  generate  $R(f)$ .

Let  $(x, y, z) \in \mathbb{R}^3$ .

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\therefore f(x, y, z) = xf(1, 0, 0) + yf(0, 1, 0) + zf(0, 0, 1) = x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0)$$

$$= (x + y, 2y, -x + 2y)$$

Hence  $f(x, y, z) = (x + y, 2y, -x + 2y)$  is the required transformation.

**4.** Find a linear transformation  $T : V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$  whose kernel is spanned by  $(1, 1, -1)$  and  $(1, 2, 2)$ .

**Soln:**  $(1, 1, -1)$  and  $(1, 2, 2)$  are L.I.

Let  $S$  be the subspace spanned by the vectors  $\alpha_1 = (1, 1, -1)$  and  $\alpha_2 = (1, 2, 2)$

$$\therefore S = \{c_1\alpha_1 + c_2\alpha_2 / c_1, c_2 \in \mathbb{R}\}$$

$$\text{Now } c_1\alpha_1 + c_2\alpha_2 = c_1(1, 1, -1) + c_2(1, 2, 2) = (c_1 + c_2, c_1 + 2c_2, 2c_2 - c_1)$$

$$\therefore S = \{(c_1 + c_2, c_1 + 2c_2, 2c_2 - c_1) / c_1, c_2 \in \mathbb{R}\}.$$

Choose a vector of  $V_3(\mathbb{R})$  outside this set  $S$ .

$(1, 0, 0)$  is not in  $S$ .

Hence the set  $B = \{(1, 1, -1), (1, 2, 2), (1, 0, 0)\}$  is a basis of  $V_3(\mathbb{R})$ .

Now define  $T$  by

$$T(1, 1, -1) = (0, 0, 0)$$

$$T(1, 2, 2) = (0, 0, 0)$$

$$T(1, 0, 0) = (0, 0, 1).$$

Let  $\alpha \in V_3(\mathbb{R})$

$$\alpha = (x_1, x_2, x_3) = c_1(1, 1, -1) + c_2(1, 2, 2) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + 2c_2, -c_1 + 2c_2)$$

$$\implies c_1 = \frac{x_2 - x_3}{2}, c_2 = \frac{x_2 + x_3}{4}, c_3 = \frac{4x_1 - 3x_2 + x_3}{4}$$

$$\therefore (x_1, x_2, x_3) = \frac{x_2 - x_3}{2}(1, 1, -1) + \frac{x_2 + x_3}{4}(1, 2, 2) + \frac{4x_1 - 3x_2 + x_3}{4}(1, 0, 0)$$

Hence  $T(x_1, x_2, x_3) = (0, 0, \frac{4x_1 - 3x_2 + x_3}{4})$  is the required transformation.