

(7)

Joint probability density function

Let x and y be two continuous random variables. Suppose that there exists a real-valued function $p(x, y)$ of x and y such that the following conditions hold:

- (i) $p(x, y) \geq 0$ for all x, y
- (ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) dx dy$ exists and is equal to 1.

Then $p(x, y)$ is called the joint probability density function of x and y .

If $[a, b]$ and $[c, d]$ are two intervals, then the probability that $x \in [a, b]$ and $y \in [c, d]$, denoted by $P(a \leq x \leq b, c \leq y \leq d)$, is defined by the formula

$$P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d p(x, y) dy dx$$

Further, the function

$$P_1(x) = \int_{-\infty}^{\infty} p(x, y) dy$$

is called the marginal density function of x , and the function

$$P_2(y) = \int_{-\infty}^{\infty} p(x, y) dx$$

is called the marginal density function of y .

Further $p_1(x)$ is the density function of x and $p_2(y)$ is the density function of y .

Consequently,

$$P(a \leq x \leq b) = \int_a^b p_1(x) dx = \int_{x=a}^b \int_{y=-\infty}^{\infty} p(x, y) dy dx$$

$$P(c \leq y \leq d) = \int_c^d p_2(y) dy = \int_{y=c}^d \int_{x=-\infty}^{\infty} p(x, y) dx dy$$

The variables x and y are said to be stochastically independent (or just independent) if $p_1(x)p_2(y) = p(x, y)$

If $\phi(x, y)$ is a function of x and y , then the expectation of $\phi(x, y)$ is defined by

$$E[\phi(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y) p(x, y) dx dy$$

As particular cases, we get

$$E[x] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x p(x, y) dy dx = \int_{-\infty}^{\infty} x p_1(x) dx$$

$$E[y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y p(x, y) dy dx = \int_{-\infty}^{\infty} y p_2(y) dy$$

$$E[xy] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy p(x, y) dy dx$$

The covariance between x and y is $Cov(x, y) = E[xy] - E[x]E[y]$

* Find the constant k so that

$$p(x,y) = \begin{cases} k(x+1)e^{-y}, & 0 < x < 1, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a joint probability density function.
Are x and y independent?

Sol: The function $p(x,y) = k(x+1)e^{-y} \geq 0 \quad \forall x, y,$
if $k \geq 0.$

Further, $\int_{y=0}^{\infty} \int_{x=0}^1 k(x+1)e^{-y} dx dy = 1$

$$\Rightarrow \int_{y=0}^{\infty} \left[k e^{-y} \left(\frac{x^2}{2} + x \right) \right]_0^1 dy = 1$$

$$\Rightarrow \int_{y=0}^{\infty} k e^{-y} \cdot \frac{3}{2} dy = 1$$

$$\Rightarrow \left[\frac{3k}{2} \cdot \frac{e^{-y}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow 0 + \frac{3k}{2} = 1$$

$$\Rightarrow k = \frac{2}{3}$$



with $k = \frac{2}{3}$, the marginal density functions

are, $p_1(x) = \int_{-\infty}^{\infty} p(x,y) dy$

$$= \frac{2}{3}(x+1) \int_0^{\infty} e^{-y} dy$$

$$= \frac{2}{3}(x+1) \left[\frac{e^{-y}}{-1} \right]_0^{\infty}$$

$$= \frac{2}{3}(x+1), \quad 0 < x < 1$$

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y) dx$$

$$= \frac{2}{3} e^{-y} \int_0^{\infty} (x+1) dx$$

$$= \frac{2}{3} e^{-y} \left[\frac{x^2}{2} + x \right]_0^1$$

$$= 2e^{-y}, \quad y > 0.$$

$$p_1(x) \cdot p_2(y) = p(x,y)$$

$\therefore x$ and y are stochastically independent

* The joint probability density function of two random variables x and y is given by (9)

$$p(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal density functions of x and y .
Also, find the covariance between x and y .

Soln: $p_1(x) = \int_{-\infty}^{\infty} p(x,y) dy = \begin{cases} \int_x^1 2 dy, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

$$= \begin{cases} [2x]_x^1, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2 - 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$p_2(y) = \int_{-\infty}^{\infty} p(x,y) dx = \begin{cases} \int_0^y 2 dx, & 0 < y < 1 \\ \int_0^1 0 dx, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} [2x]_0^y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$= \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[x] = \int_{-\infty}^{\infty} x p_1(x) dx$$

$$= \int_0^1 x [2(1-x)] dx$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

$$E[y] = \int_{-\infty}^{\infty} y p_2(y) dy$$

$$= \int_0^1 y \times 2y dy$$

$$= 2 \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{2}{3}$$

$$E[xy] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy p(x,y) dx dy$$

$$= \int_0^1 \int_0^y xy \times 2 dx dy$$

$$= \int_0^1 2y \times \frac{x^2}{2} \Big|_0^y dy$$

$$= \int_0^1 y^3 dy$$

$$= \frac{y^4}{4} \Big|_0^1 = \frac{1}{4}$$

$$\text{1. } \text{Cov}(x,y)$$

$$= E[xy] - E[x]E[y]$$

$$= \frac{1}{4} - \frac{1}{3} \cdot \frac{2}{3}$$

$$= \frac{1}{36}$$

* If the joint probability function for (x,y) is ⑩

$$f(x,y) = \begin{cases} C(x^2+y^2), & 0 \leq x \leq 1, 0 \leq y \leq 1, C \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- determine ① the value of the constant C ,
 ② marginal density functions of x and y ,
 ③ $P(x < 1/2, y > 1/2)$, ④ $P(1/4 < x < 3/4)$, ⑤ $P(y < 1/2)$.

① $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = \int_{x=0}^1 \int_{y=0}^1 C(x^2+y^2) dy dx$

$$\int_{x=0}^1 C \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^1 dx = \int_{x=0}^1 C \left(x^2 + \frac{1}{3} \right) dx$$

$$1 = C \left[\frac{x^3}{3} + \frac{x}{3} \right] \Big|_0^1 \Rightarrow 1 = \frac{2C}{3} \Rightarrow C = \frac{3}{2} .$$

② marginal density function of x is

$$p_1(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_0^1 \frac{3}{2}(x^2+y^2) dy = \frac{3}{2} \left[x^2 y + \frac{y^3}{3} \right] \Big|_0^1 = \frac{1}{2} (3x^2 + 1)$$

marginal density function of y is

$$p_2(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_0^1 \frac{3}{2}(x^2+y^2) dx = \frac{3}{2} \left[\frac{x^3}{3} + \frac{y^2 x}{2} \right] \Big|_0^1 = \frac{1}{2} (3y^2 + 1)$$



$$\textcircled{iii} \quad P(x < \frac{1}{2}, y > \frac{1}{2}) = \int_{x=0}^{1/2} \int_{y=1/2}^1 \frac{3}{2}(x^2 + y^2) dy dx$$

$$= \frac{3}{2} \int_{x=0}^{1/2} \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^{1/2} dx = \frac{3}{2} \int_{x=0}^{1/2} \left(\frac{x^2}{2} + \frac{1}{24} \right) dx$$

$$= \frac{3}{2} \left[\frac{x^3}{6} + \frac{1}{24} x \right]_0^{1/2} = \frac{3}{2} \left[\frac{1}{48} + \frac{1}{48} \right] = \frac{1}{4}$$

$$\textcircled{iv} \quad P\left(\frac{1}{4} < x < \frac{3}{4}\right) = \int_{x=1/4}^{3/4} \int_{y=0}^1 \frac{3}{2}(x^2 + y^2) dy dx$$

$$= \frac{3}{2} \int_{x=1/4}^{3/4} \left(x^2 y + \frac{y^3}{3} \right) \Big|_0^{1/2} dx = \frac{3}{2} \int_{x=1/4}^{3/4} \left(x^2 + \frac{1}{3} \right) dx$$

$$= \frac{3}{2} \left[\frac{x^3}{3} + \frac{x}{3} \right]_{1/4}^{3/4} = \frac{1}{2} \left[\frac{27}{64} - \frac{1}{64} + \frac{3}{48} - \frac{1}{48} \right] = \frac{29}{64}$$

$$\textcircled{v} \quad P(y < \frac{1}{2}) = \int_{x=0}^1 \int_{y=0}^{1/2} \frac{3}{2}(x^2 + y^2) dy dx$$

$$= \frac{3}{2} \left[\int_{x=0}^1 x^2 y + \frac{y^3}{3} \right]_0^{1/2} dx = \frac{3}{2} \left[\int_{x=0}^1 \left(\frac{x^2}{2} + \frac{1}{24} \right) dx \right]$$

$$= \frac{3}{2} \left[\frac{x^3}{6} + \frac{x}{24} \right]_0^1 = \frac{3}{2} \left[\frac{1}{6} + \frac{1}{24} \right] = \frac{5}{16}$$

* Verify that

$$f(x, y) = \begin{cases} e^{-(x+y)} & , x \geq 0, y \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

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is a density function of a joint probability distribution. Then evaluate the following:

- (i) $P(1/2 < x < 2, 0 < y < 4)$ (ii) $P(x < 1)$, (iii) $P(x \leq y)$
 (iv) $P(x > y)$ (v) $P(x + y \leq 1)$.

Sol? $f(x, y) \geq 0$ for all x and y .

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^{\infty} \int_0^{\infty} e^{-(x+y)} dx dy = \int_0^{\infty} e^{-y} \cdot \frac{e^{-x}}{-1} dy$$

$$= \int_0^{\infty} e^{-y} \cdot 1 dy = \left[\frac{e^{-y}}{-1} \right]_0^{\infty} = 1$$

$\therefore f(x, y)$ is a density function.

$$(i) P(1/2 < x < 2, 0 < y < 4) = \int_{x=1/2}^2 \int_{y=0}^4 e^{-x} \cdot e^{-y} dy dx$$

$$= \int_{x=1/2}^2 \left[e^{-x} \cdot \frac{e^{-y}}{-1} \right]_0^4 dx = \int_{x=1/2}^2 e^{-x} \left[1 - \frac{1}{e^4} \right] dx = \left(1 - \frac{1}{e^4} \right) \left[\frac{e^{-x}}{-1} \right]_{1/2}^2$$

$$= \left(1 - \frac{1}{e^4} \right) \left(e^{-1/2} - e^{-2} \right)$$

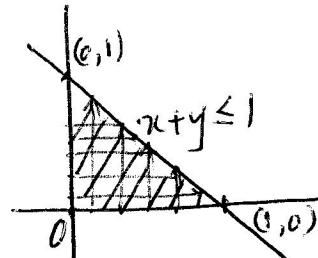
$$(ii) P(x < 1) = \int_{x=0}^1 \int_{y=0}^{\infty} e^{-x} \cdot e^{-y} dy dx = \int_{x=0}^1 e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^{\infty} dx$$

$$= \int_{x=0}^1 e^{-x} (1) dx = \left[\frac{e^{-x}}{-1} \right]_0^1 = 1 - e^{-1}$$

$$\begin{aligned}
 \text{iii) } P(x \leq y) &= \int_{y=0}^{\infty} \int_{x=0}^{y} e^{-x} e^{-y} dx dy \\
 &= \int_{y=0}^{\infty} e^{-y} \left[\frac{e^{-x}}{-1} \right]_0^y dy = \int_{y=0}^{\infty} e^{-y} (1 - e^{-y}) dy \\
 &= \left[\frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2} \right]_0^{\infty} = 1 - \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

$$\text{iv) } P(x > y) = 1 - P(x \leq y) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\begin{aligned}
 \text{v) } P(x+y \leq 1) &= \iint_A e^{-x} e^{-y} dy dx \\
 &= \int_{x=0}^{1-x} \int_{y=0}^{1-x} e^{-x} e^{-y} dy dx \\
 &= \int_{x=0}^{1-x} e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^{1-x} dy dx = \int_{x=0}^{1-x} e^{-x} (1 - e^{-1+x}) dx \\
 &= \int_{x=0}^{1-x} (e^{-x} - e^{-1}) dx = \left[\frac{e^{-x}}{-1} - \frac{e^{-1+x}}{-1} \right]_0^{1-x} = -e^{-1} - e^{-1+1} \\
 &= 1 - 2e^{-1}
 \end{aligned}$$



* For the distribution given by the density function (12)
 $f(x,y) = \begin{cases} \frac{1}{96}xy, & 0 < x < 4, 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$

evaluate (i) $P(1 < x < 2, 2 < y < 3)$ (ii) $P(x \geq 3, y \leq 2)$

(iii) $P(y \leq x)$, (iv) $P(y > x)$ (v) $P(xy \leq 3)$ (vi) $P(x+y > 3)$

$$\text{(i)} P(1 < x < 2, 2 < y < 3) = \int_{x=1}^2 \int_{y=2}^3 \frac{1}{96}xy dy dx$$

$$= \frac{1}{96} \int_{x=1}^2 x \left[\frac{y^2}{2} \right]_2^3 dx = \frac{1}{192} \int_{x=1}^2 x \cdot \frac{5}{2} dx = \frac{5}{192 \times 2} \left[\frac{x^2}{2} \right]_1^2$$

$$= \frac{5 \times 3}{96 \times 4} = \frac{5}{128}$$

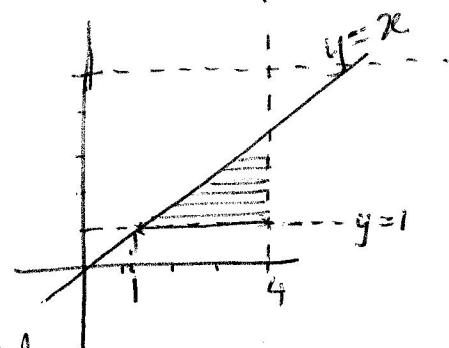
$$\text{(ii)} P(x \geq 3, y \leq 2) = \int_{x=3}^4 \int_{y=1}^2 \frac{1}{96}xy dy dx$$

$$= \frac{1}{96} \left[\frac{x^2}{2} \right]_3^4 \times \left[\frac{y^2}{2} \right]_1^2 = \frac{1}{96} \times \frac{7}{2} \times \frac{3}{2} = \frac{7}{128}$$

$$\text{(iii)} P(y \leq x) = \int_{x=1}^4 \int_{y=1}^x \frac{1}{96}xy dy dx$$

$$= \frac{1}{96} \int_{x=1}^4 \left[\frac{xy^2}{2} \right]_1^x dx = \frac{1}{192} \int_{x=1}^4 (x^3 - x) dx$$

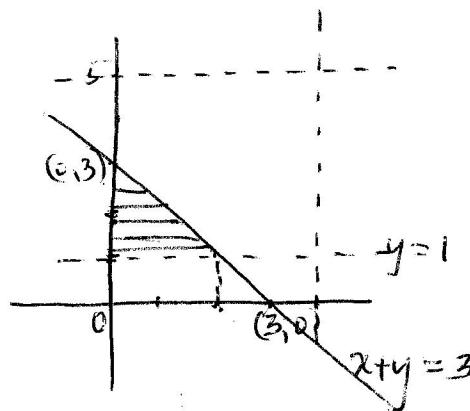
$$= \frac{1}{192} \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^4 = \frac{1}{192} \left[\frac{256}{4} - \frac{1}{4} - \frac{16}{2} + \frac{1}{2} \right] = \frac{225}{768}$$



$$\textcircled{iiv} \quad P(y > x) = 1 - P(y \leq x) = 1 - \frac{225}{768} = \frac{543}{768}$$

$$\textcircled{iv} \quad P(x+y \leq 3)$$

$$= \int_{x=0}^2 \int_{y=1}^{3-x} \frac{1}{96} xy dy dx$$



$$\begin{aligned} &= \frac{1}{96} \int_{x=0}^2 \left[xy \right]_1^{3-x} dx = \frac{1}{96 \times 2} \int_0^2 [x(3-x)^2 - x] dx \\ &= \frac{1}{192} \int_0^2 (x^3 - 6x^2 + 8x) dx = \frac{1}{192} \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right]_0^2 \\ &= \frac{1}{192} \left[\frac{16}{4} - \frac{48}{3} + \frac{32}{2} \right] = \frac{1}{48} \end{aligned}$$

$$\textcircled{v} \quad P(x+y > 3) = 1 - P(x+y \leq 3)$$

$$= 1 - \frac{1}{48}$$

$$= \frac{47}{48}$$

1. The joint probability density function of two random variables x and y is given by (13)

$$p(x,y) = \begin{cases} (8/9)xy, & 0 \leq x \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal density functions of x and y .

Ans: $p_1(x) = \frac{4}{9}x(4-x^2), 1 \leq x \leq 2, p_2(y) = \begin{cases} \frac{4}{9}y(y^2-1), & 1 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$

2. Find the constant k so that

$$f(x,y) = \begin{cases} kxy, & 0 \leq y \leq 4, 0 \leq x \leq y, \\ 0, & \text{otherwise} \end{cases}$$

is a joint probability density function. Then check whether x and y are stochastically independent or not.

Ans: $k = \frac{1}{32}$; not independent.

3. For the distribution defined by the density

function $f(x,y) = \begin{cases} 3xy(x+y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

find the covariance between x and y .

Ans: $\frac{1}{2} - \left(\frac{17}{24}\right)^2$

4. For the distribution with density function

$$p(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0 < x < 2, 2 < y < 4 \\ 0, & \text{otherwise}, \end{cases}$$
 evaluate

(i) $P(x < 1, y < 3)$
(ii) $P(x+y < 3)$

Ans: (i) $\frac{3}{8}$, (ii) $\frac{5}{24}$

