MIT - Linear Algebra
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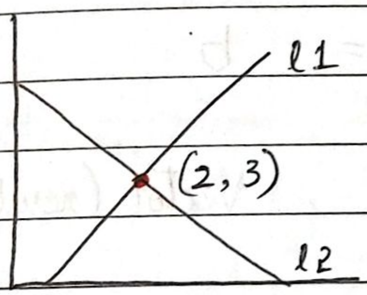
## * Geometry of Linear Equations

$$2x - y = 1 \quad (l1)$$
$$x + y = 5 \quad (l2)$$

- Row picture

concentrates on rows (separate equations)



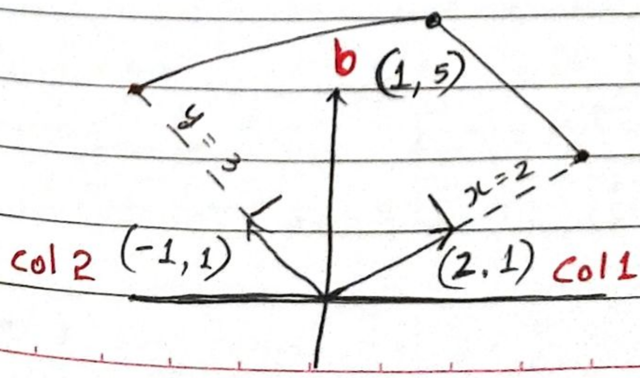

Intersection Point can be found by Elimination

- Column picture

two separate equations are really one vector equa^n

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

By deduction, when $x = 2$ and $y = 3$, it holds
Vectors become $(4, 2)$ and $(-3, 3)$



linear combination

to find the right amount of $x$ and $y$

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The goal is to find $x$ and $y$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$A \qquad x \qquad = \qquad b$

Matrix (of coefficients) — Vector (to find) — Vector (result)

↓

this is to be multiplied with column vectors, that gives result

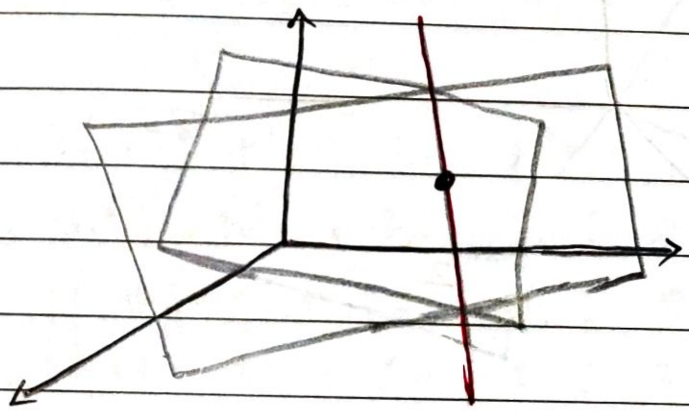
$$\boxed{Ax = b}$$

* $2x - y = 0$
  $-x + 2y - z = -1$
  $-3y + 4z = 4$

$3 \times 3$

* $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

* Each equation describes a plane



planes meet (at a line)

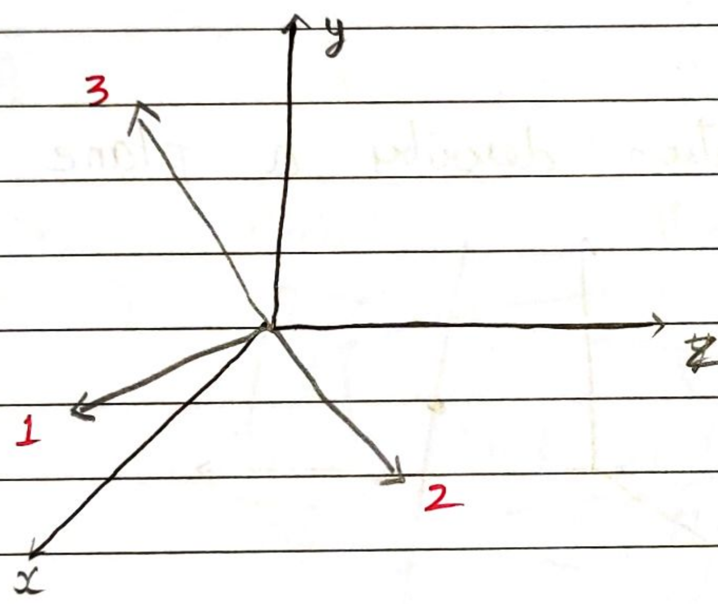
2 planes meet at a line
3 planes meet at a point
That point is the solution
(row picture of $3 \times 3$)

* We stop the focus on row picture.
Column picture, combination of columns,
is better to solve as dimensions increase

That's our true goal, to look beyond,
into $n$ dimensions. With $n$ equations and $n$ unknowns

* column picture ( of 3x3 )

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\3\end{bmatrix} + z\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$$




$x = 0$ $y = 0$ $z = 1$

We will actually arrive at $(x, y, z)$ through systematic Elimination

* Can I solve $Ax = b$ for every $b$ ?
=> Do the linear combination of columns fill 3 D space ?

If yes, then linear independence

* Solve using ELIMINATION method

$$x - y - z + u = 0$$
$$2x + 2z = 8$$
$$-y - 2z = -8$$
$$3x - 3y - 2z + 4u = 7$$

P1) $$A : b = \left[\begin{array}{cccc:c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right]$$

P2) $$\begin{array}{c} \\ -2R_1 \\ \\ -3R_1 \end{array} \left[\begin{array}{cccc:c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

P3) $$\begin{array}{c} \\ \\ 2R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{cccc:c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & 0 & 0 & -2 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

ROW) $R_3 \leftrightarrow R_4$

P3) $$\left[\begin{array}{cccc:c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right]$$

$$A : b = \begin{array}{c} \begin{array}{cccccc} x & y & z & u & & \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 & : & 0 \\ 0 & 2 & 4 & -2 & : & 8 \\ 0 & 0 & 1 & 1 & : & 7 \\ 0 & 0 & 0 & -2 & : & -8 \end{bmatrix} \end{array}$$

Backward Substitution

$-2u = -8 \Rightarrow \boxed{u = 4}$

$z + u = 7 \Rightarrow z - 4 = 7$

$\boxed{z = 11}$

$2y + 4z + (-2)u = 8 \Rightarrow 2y + 44 - 8 = 8$

$2y = 28$

$\boxed{y = 14}$

$x - y - z + u = 0 \Rightarrow x - 14 - 11 + 4 = 0$

$\boxed{x = -21}$

From $Ax = b$

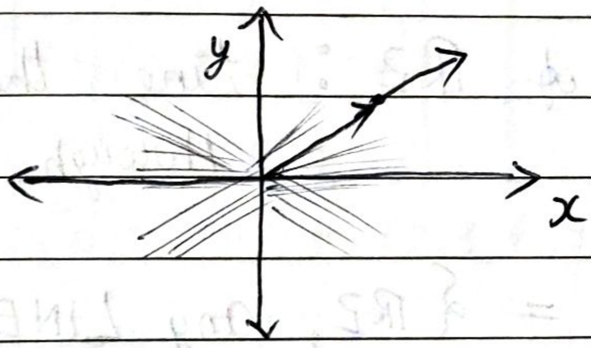
$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

## * SPACE

When we say "space",
we are putting a lot of things into
one set

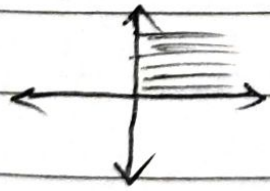
## * Vecton Space

$\mathbb{R}2$ $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$, ...




$\mathbb{R}2$, set of all $x$ and all $y$,
is closed under addition & multiplication

Not a VS in $\mathbb{R}2$ = Positive Vectors



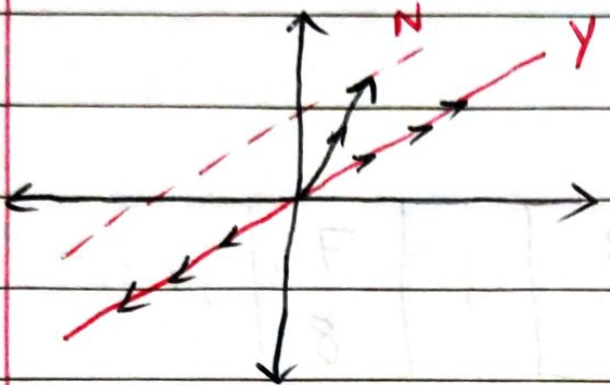
$\begin{bmatrix} x \\ y \end{bmatrix} + (-99) \begin{bmatrix} x \\ y \end{bmatrix} < 0$

not closed under scalar multiplication

||| $\mathbb{R}3$, $\mathbb{R}4$, ... $\mathbb{R}n$
(n - dimensions)

* Subspace

Subset of VS that satisfies all the properties of VS




Subspace of $\mathbb{R}2$ : Line that pass through zero vector

$\mathbb{R}2$ SS = { $\mathbb{R}2$, any LINE through (0,0), Zero vector alone }

$\mathbb{R}3$ SS = { $\mathbb{R}3$, any PLANE through (0,0,0) any LINE through (0,0) zero vector alone }

* $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

## * Column Space

If $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

Column Space contains ALL linear combinations of the columns of A

$C(A) = \{ (1, 2, 3, 4), (1, 1, 1, 1),$

$7 \times (1, 2, 3, 4)$ multiplication

$(1, 2, 3, 4) + (2, 3, 4, 5)$ addition

Q) When is $Ax = b$ solvable?

→ When vector b can be expressed as linear combination of columns of A

i.e b is in the column space

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

## * Null Space

$Ax = 0$

For what $\{x\}$, we get 0

Contains all vectors x such that $Ax = 0$

$C(A)$ subspace of $\mathbb{R}^4$ $\boxed{A}$

$N(A)$ subspace of $\mathbb{R}^3$ $\boxed{x}$

# Solving $Ax = 0$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Elimination on $A$
(but it doesn't change $x$ or RHS $= 0$)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Echelon Form (staircase)

→ can't exchange rows
⇒ C2 is dependent on C1

RANK = Number of Pivots = 2

$Ux = 0$

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ w & x & y & z \end{array}$$

↑ Pivot Columns ($r$)
↑ Free Columns ($n-r$)

Assign any value to variable $(x, z)$
and solve for $(w, y)$

Let's set $x = 1$ and $z = 0$

$2y + 4z = 0 \Rightarrow y = 0$

$w + 2x + 2y + 2z = 0$

$w + 2 + 0 + 0 = 0 \Rightarrow w = -2$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

If we take $x = 0$ and $z = 1$

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{SO} \Rightarrow \underbrace{x}_{\{\text{nullspace}\}} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

now reduced Echelon Form → Zero above below Pivot

→ Pivot = 1

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$0 = I$ $\quad 0 = F$ $\quad Rx = 0$ $\quad x = \begin{bmatrix} -F \\ I \end{bmatrix} c$

# Solving $Ax = b$

Augmented Matrix $= [A : b]$

$$\left[\begin{array}{cccc:c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

$$0 = b_3 - b_2 - b_1$$

$Ax = b$ is solvable
if a combination of rows of $A$ gives zero row, then the same combination of entries of $b$ must give $0$

- To find complete solution to $Ax = b$

i) $x_{particular}$ → Set free variables $(x_2, x_4) = 0$
→ Solve for pivot variables

$$b_3 - b_2 - b_1 = 0 \Rightarrow b = \begin{array}{ccc} b1 & b2 & b3 \\ \hline \end{array}$$
$$b = \begin{bmatrix} 1 & 5 & 6 \end{bmatrix}$$

$2x_3 = 5 - 2(1) \Rightarrow x_3 = 3/2$

$x_1 + 2x_3 = 1 \Rightarrow x_1 = -2$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

ii) $x_{\text{nullspace}}$

$$x = x_p + x_n$$

$$\begin{aligned} Ax_p &= b \\ Ax_n &= 0 \\ \hline A(x_p + x_n) &= b \end{aligned}$$

$$x = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

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$A_{m \times n}$ of rank $r$

$r \leq m$ also $r \leq n$

* Full Column Rank $(r = n)$

There's a pivot in every column

$r$ pivot variables

$0$ free variables

$N(A) = \{\text{Zero Vector}\}$

$Ax = b \Rightarrow x = x_p$

{Unique solution if it exists}

{i.e 0 or 1 solution}

Eg: $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

* Full Row Rank $(r = m)$

m pivots

n - m free variables

{ $Ax = b$ is solvable for every b }

Eg : $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 & \_ & \_ \\ 0 & 1 & \_ & \_ \end{bmatrix}$

* $\boxed{r = m = n}$ invertible, square matrix

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$R = I$

$Ax = b$ $N(A) = \{ \text{Zero Vector} \}$

Every b exists

Summary

Date / /

* $r = m = n$

$R = I$

(( 1 solution ))

$r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

(( 0 or 1 solution ))

* $r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$

(( $\infty$ solution ))

$r < m, \quad r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(( 0 or $\infty$ solution ))

# Find all solutions

$$x - 2y - 2z = b_1$$
$$2x - 5y - 4z = b_2$$
$$4x - 9y - 8z = b_3$$

$$A = \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right]$$

$$U = \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

IF $b_3 - b_2 - 2b_1 \neq 0$ THEN no solutions

IF $b_3 - b_2 - 2b_1 = 0$ THEN

$$R = \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & +1 & 0 & +2b_1 - b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

* $x_p$

$Ax_p = b$ free variable, $z = 0$

$$x_p = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$$

→ Directly $\{b\}$ from R

OR

→ $kx + ky + kz = b$ (Solve) U

→ $-y = b_2 - 2b_1$

→ $x - 2y - 2z = b_1$

$x_n$

* $Ax_n = 0$ each free variable $= 1$

$z = 1$, others $= 0$

Consider $R$, but neglect $b$ $\therefore$ $b = 0$

$1y = 0 \Rightarrow y = 0$

$1x - 2z = 0 \Rightarrow x = 2$

$$x_s = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

* $x = x_p + c \cdot x_s$

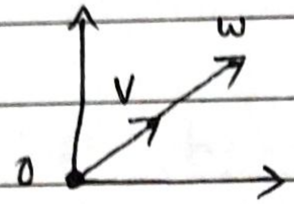
# * Independence

Vectors $x_1, x_2, x_3, \ldots x_n$

These vectors are independent if no combination gives zero vector

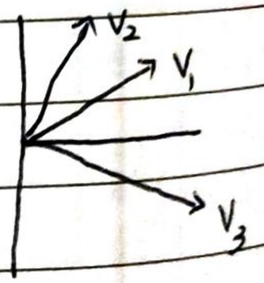
$$c x_1 + d x_2 + \cdots + e x_n \neq 0$$

{except $c, d, e$ all $= 0$}




$$\left.\begin{aligned} \underline{2}\, v + \underline{(-1)}\, w &= 0 \\ \underline{0}\, v + \underline{5}\; 0 &= 0 \end{aligned}\right\} \text{Dependent}$$

$$A = \begin{array}{ccc} \left[\begin{matrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{matrix}\right] \\ v_1 \quad v_2 \quad v_3 \end{array} \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$




$$\left.\begin{array}{l} m < x \\ \text{equations} < \text{unknowns} \\ \Rightarrow \text{there's solution} \end{array}\right\} \begin{array}{l} r < n \\ \text{Free variables} \checkmark \\ \text{dependent} \end{array}$$

$N(A) = \{\text{Zero vector}\}$ independent

$N(A) = \{c\}$ dependent

- Span

Vectors $V_1, V_2, \ldots V_l$ span a space means the space consists of all combinations of those vectors

Combinations of columns form Column Space

* Basis

Basis for a space is a sequence of vectors $V_1, V_2, \ldots, V_d$ such that

→ they are independent
→ they span the space

Space : $\mathbb{R}3$
Bases : $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{matrix} \right\}, \ldots$

* Given a space ($\mathbb{R}_n$), every basis for the space has same number of vectors

* Dimension

Dimension of $C(A)$ = Rank of A
Dimension of $N(A)$ = $n - r$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$
$\uparrow \quad \uparrow$

Dimension = 2
Basis = Pivot Columns