1. **Orthogonal Vectors :-**

Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\vec{u}.\vec{v}=0$

ex $\vec{u}=(1,2)$ and $\vec{v}=(6,-3)$ are orthogonal in $\mathbb{R}^2$, as

$$\vec{u}.\vec{v} = (1,2).(6,-3) = 6-6=0$$

A set of vectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ are mutually orthogonal if every pair of vectors is orthogonal.

ie $\vec{v_i}.\vec{v_j}=0$, for all $i \neq j$.

The set of vectors $(1,0,-1)$, $(1,\sqrt{2},1)$, $(1,-\sqrt{2},1)$ are mutually orthogonal, since

$$(1,0,-1).(1,\sqrt{2},1) = 1+0-1=0$$
$$(1,0,-1),(1,-\sqrt{2},1) = 1+0-1=0$$
$$(1,\sqrt{2},1).(1,-\sqrt{2},1) = 1-2+1=0$$

3. **Orthogonal Subspaces**

Subspace S is orthogonal to subspace T means: every vector is S is orthogonal to every vector in T.

ex: In a plane, the space containing only the zero vector and any line through the origin are orthogonal subspaces.

A line through the origin and the whole plane are never orthogonal subspaces.

Two lines through the origin are orthogonal subspaces if they meet at right angles.

The row space of a matrix is orthogonal to the nullspace, because $Ax = 0$ means the dot product of x with each row of A is 0.
But then the product of x with any combination of rows of A must be 0.

The columnspace is orthogonal to the left nullspace of A because the rowspace of A^T is perpendicular to the nullspace of A^T, as $A^Ty = 0$.

$A_{m\times n}$

$N(A) \to \mathbb{R}^n$
$C(A) \to \mathbb{R}^m$
$R(A) \to \mathbb{R}^n$
$N(A^T) \to \mathbb{R}^m$

$A^T_{n\times m}$

ex: $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}_{2\times 3}$

$R_2 - 2R_1$ $\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Row space has dimension 1 with basis $\{(1, 2, 5)\}$

$Ax = 0 \Rightarrow$ $x_1 + 2x_2 + 5x_3 = 0 \Rightarrow x_1 = -2x_2 - 5x_3$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

Nullspace has dimension 2 with basis $\{(-2,1,0), (-5,0,1)\}$ which is orthogonal to the rowspace $\left\{ \begin{matrix} (-2,1,0)\cdot(1,2,5) = 0 \\ (-5,0,1)\cdot(1,2,5) = 0 \end{matrix} \right\}$

Not only is the nullspace orthogonal to the rowspace, their dimensions add up to the dimension of the whole space. The nullspace and the rowspace are orthogonal complements in $\mathbb{R}^n$.

Similarly the column space and the left nullspace are orthogonal complements in $\mathbb{R}^m$

Orthogonal complement :-

Let V be a subspace of $\mathbb{R}^n$.

The set $V^{\perp} = \{ \vec{w} \mid \vec{w} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V \}$

is called the orthogonal complement of V.

Note:-

* A vector $\vec{w}$ is in $V^{\perp}$ iff $\vec{w}$ is orthogonal to every vector in a set that spans V.
* $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

2. Orthogonal Sets :-

A set of vectors $\{u_1, \dots u_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $u_i \cdot u_j = 0$ whenever $i \neq j$.

ex $\{u_1, u_2, u_3\}$ such that $u_1 = (3,1,1)$, $u_2 = (-1,2,1)$,

$u_3 = (-\frac{1}{2}, -2, \frac{7}{2})$.

$$u_1 \cdot u_2 = (3,1,1) \cdot (-1,2,1) = -3 + 2 + 1 = 0$$

$$u_1 \cdot u_3 = (3,1,1) \cdot (-\tfrac{1}{2}, -2, \tfrac{7}{2}) = -\tfrac{3}{2} - 2 + \tfrac{7}{2} = 0$$

$$u_2 \cdot u_3 = (-1,2,1) \cdot (-\tfrac{1}{2}, -2, \tfrac{7}{2}) = \tfrac{1}{2} - 4 + \tfrac{7}{2} = 0$$

Each pair of distinct vectors is orthogonal, and so $\{u_1, u_2, u_3\}$ is an orthogonal set.

If $S=\{u_1,\dots,u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Proof: If $c_1u_1+c_2u_2+\dots+c_pu_p=0$, for scalars $c_1,\dots,c_p$,

then $(c_1u_1+c_2u_2+\dots+c_pu_p)\cdot u_1=0\cdot u_1$

$\Rightarrow\quad (c_1u_1)\cdot u_1+(c_2u_2)\cdot u_1+\dots+(c_pu_p)\cdot u_1=0\cdot u_1$

$\Rightarrow\quad c_1(u_1\cdot u_1)+c_2(u_2\cdot u_1)+\dots+c_p(u_p\cdot u_1)=0\cdot u_1$

$\Rightarrow\quad c_1(u_1\cdot u_1)=0$ $\left[\because u_2\cdot u_1=\dots=u_p\cdot u_1=0 \text{ as } \{u_1,\dots u_p\} \text{ is an orthogonal set}\right]$.

$\Rightarrow\quad c_1=0$

Similarly $c_2=\dots=c_p=0$

$\therefore$ S is linearly independent.

4. <u>Orthogonal</u> <u>basis</u>:-

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

<u>ex</u>: $S=\{u_1,u_2,u_3\}$, $u_1=(3,1,1)$, $u_2=(-1,2,1)$, $u_3=(-\frac{1}{2},-2,\frac{7}{2})$ is an orthogonal basis for $\mathbb{R}^3$ as (i) S is an orthogonal set and (ii) S forms a basis of $\mathbb{R}^3$.

$$\begin{vmatrix}3&1&1\\-1&2&1\\-\frac{1}{2}&-2&\frac{7}{2}\end{vmatrix}=3(7+2)-1(-\tfrac{7}{2}+\tfrac{1}{2})+1(2+1)=27+3+3=33\neq0$$

3. Orthonormal Sets

A set $\{u_1, \dots, u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.

If W is the subspace spanned by such a set, then $\{u_1, \dots, u_p\}$ is an orthonormal basis for W, since the set is automatically linearly independent.

ex $\{e_1, \dots, e_n\}$, the standard basis for $\mathbb{R}^n$, is an orthonormal set.

Any nonempty subset of $\{e_1, \dots, e_n\}$ is orthonormal, too.

5. example: Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where $v_1 = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$, $v_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $v_3 = \left(-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right)$

$$v_1 \cdot v_2 = -\frac{3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$$

$$v_1 \cdot v_3 = -\frac{3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$$

$$v_2 \cdot v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal set.

$$v_1 \cdot v_2 = \frac{9}{\sqrt{11}} + \frac{1}{11} + \frac{1}{11} = 1$$

$$v_2 \cdot v_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

$$v_3 \cdot v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$$

which shows that v_1, v_2 and v_3 are unit vectors.

Thus $\{v_1, v_2, v_3\}$ is an orthonormal set.

Since the set is linearly independent, its three vectors form a basis for $\mathbb{R}^3$.

ex: Show that $\{u_1, u_2\}$, where $u_1 = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$u_2 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ is an orthonormal basis for $\mathbb{R}^2$.

6. Orthogonal matrix

A ~~square~~ matrix A with real entries and satisfying the condition $A^{-1} = A^T$ is called an orthogonal matrix.

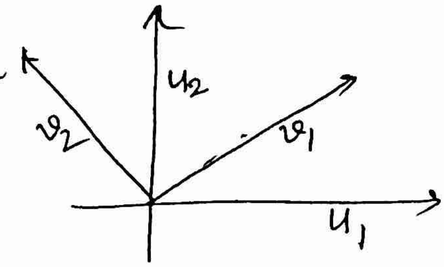
The vectors $u_1 = (1,0)$ and $u_2 = (0,1)$ form an orthonormal basis $B = \{u_1, u_2\}$.

Rotating the vectors u_1 and u_2 anticlockwise by an ange θ, we obtain $R_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

$v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$.

Then $C = \{v_1, v_2\}$ is also an orthonormal basis

The transition matrix from the basis C to the basis B is given by



$$P_{B\leftarrow C}\quad \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{array}\right]$$

$$P_{B\leftarrow C} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

~~clearly~~ $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\quad P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Clearly $P^{-1} = P^T$.

$\therefore$ P is an orthogonal matrix.

* Suppose that $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ are two orthonormal bases of a vectorspace V. Then the transition matrix P from the basis C to the basis B is an orthogonal matrix.

example:
The matrix $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal,

Since $A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The row vector of A, namely $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$ and $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ are orthonormal.
So are the column vectors of A.

* Suppose that A is an $n \times n$ matrix with real entries.
Then (a) A is orthogonal iff the row vectors of A form an orthonormal basis of $\mathbb{R}^n$. ~~bao~~
(b) A is orthogonal iff the column vectors of A form an orthonormal basis of $\mathbb{R}^n$.

ex:- Show that the matrix $U = \begin{bmatrix} \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -\frac{4}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} \end{bmatrix}$ is an orthogonal matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

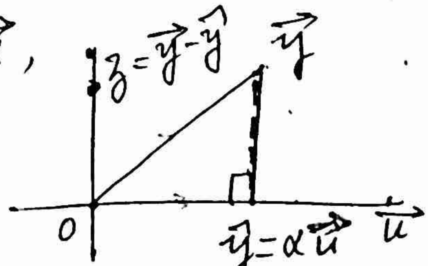
7. Orthogonal Projections:

Given a nonzero vector $\vec{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\vec{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\vec{u}$ and the other orthogonal to $\vec{u}$. We wish to write

$$\vec{y} = \overset{\alpha\vec{u}}{\hat{y}} + \vec{z} \quad (1)$$

where $\hat{y} = \alpha\vec{u}$ for some scalar α and $\vec{z}$ is some vector orthogonal to $\vec{u}$.

Given any scalar α, let $\vec{z} = \vec{y} - \alpha\vec{u}$, so that (1) is satisfied.



Then $\vec{y} - \overset{\vec{z}}{\hat{y}}$ is orthogonal to $\vec{u}$ iff

Finding α to make $y - \hat{y}$ orthogonal to $\vec{u}$.

$$0 = (\vec{y} - \alpha\vec{u})\cdot\vec{u} = \vec{y}\cdot\vec{u} - (\alpha\vec{u})\cdot\vec{u}$$

$$= \vec{y}\cdot\vec{u} - \alpha(\vec{u}\cdot\vec{u})$$

That is, (1) is satisfied with $\vec{z}$ orthogonal to $\vec{u}$ iff $\alpha = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$ and $\overset{\alpha\vec{u}}{\hat{y}} = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$.

The vector $\alpha\vec{u} = \hat{y}$ (denoted as $\hat{y}$) is called the orthogonal projection of $\vec{y}$ onto $\vec{u}$, and the vector $\vec{z}$ is called the component of $\vec{y}$ orthogonal to $\vec{u}$.

ex:- Let $\vec{y} = (7,6)$ and $\vec{u} = (4,2)$.

Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$.

Then write $\vec{y}$ as the sum of two orthogonal vectors, one in Span$\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

Sol^n:

$$\vec{y}\cdot\vec{u} = (7,6)\cdot(4,2) = 28+12 = 40$$

$$\vec{u}\cdot\vec{u} = (4,2)\cdot(4,2) = 16+4 = 20.$$

The orthogonal projection of $\vec{y}$ onto $\vec{u}$ is

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = 2(4,2) = (8,4).$$

The component of $\vec{y}$ orthogonal to $\vec{u}$ is

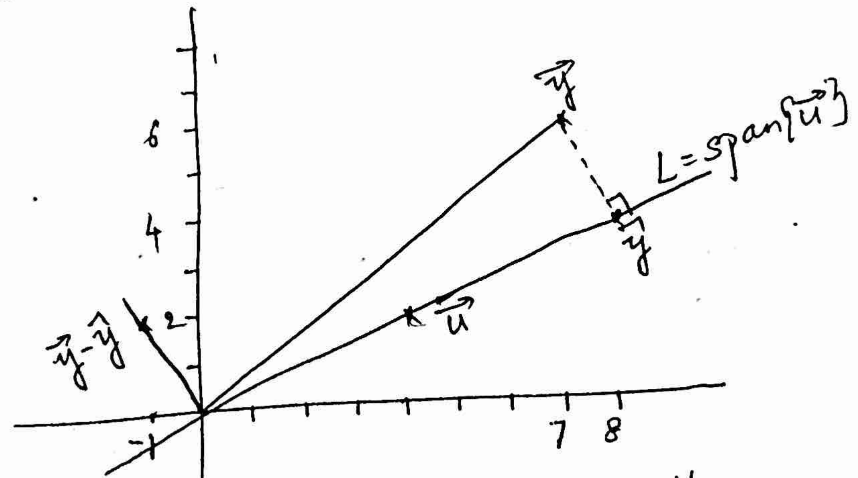
$$\vec{y}-\hat{y} = (7,6)-(8,4) = (-1,2)$$

The component of $\vec{y}$ in span$\{\vec{u}\}$ is

$$\alpha\vec{u} = 2(4,2) = (8,4).$$

$$\therefore\ \vec{y} = \alpha\vec{u} + (\vec{y}-\hat{y})$$

$$= (8,4)+(-1,2)$$



ex: Let $\vec{y} = (2,3)$ and $\vec{u} = (4,-7)$. Write $\vec{y}$ as the sum of two orthogonal vectors, one in span$\{\vec{u}\}$ and a vector orthogonal to $\vec{u}$.

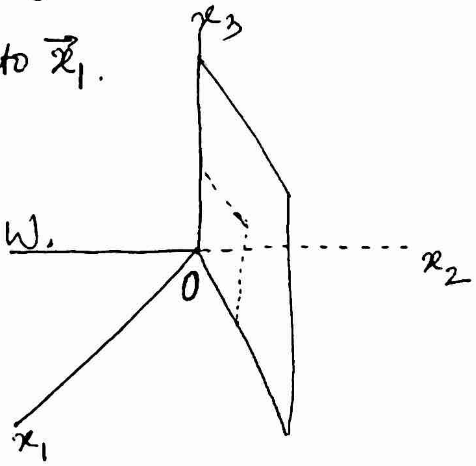
Gram-Schmidt Orthogonalization.

The Gram-Schmidt process is a single algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$.

ex! Let $W = \text{Span}\{x_1, x_2\}$ where $x_1 = (3,6,0)$ and $x_2 = (1,2,2)$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.

Let $\vec{p}$ be the projection of $\vec{x}_2$ onto $\vec{x}_1$.

The component of $\vec{x}_2$ orthogonal to $\vec{x}_1$ is $\vec{x}_2 - \vec{p}$, which is in W.



Let $\vec{v}_1 = \vec{x}_1$ and

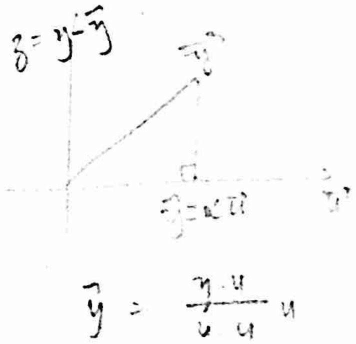
$$\vec{v}_2 = \vec{x}_2 - \vec{p}$$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1$$

$$= (1,2,2) - \frac{(1,2,2)\cdot(3,6,0)}{(3,6,0)\cdot(3,6,0)}(3,6,0)$$

$$\vec{v}_2 = (0,0,2).$$

Then $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set of nonzero vectors in W. Since $\dim W = 2$, the set $\{\vec{v}_1, \vec{v}_2\}$ is a basis in W.



ex: Let $W = \text{Span}\{v_1, v_2\}$ where $v_1 = (1,1)$ & $v_2 = (2,-1)$.
Construct an orthogonal basis $\{u_1, u_2\}$ for W.

Sol^n: $u_1 = v_1$

Set $u_1 = (1,1)$

and
$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$
$$= (2,-1) - \frac{(2,-1)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1)$$
$$= \left(\frac{3}{2}, -\frac{3}{2}\right)$$

ex: Let $W = \text{Span}\{v_1, v_2, v_3\}$, where $v_1 = (0,1,2)$, $v_2 = (1,1,2)$, $v_3 = (1,0,1)$. Construct an orthogonal basis $\{u_1, u_2, u_3\}$ for W.

Sol^n: $u_1 = v_1$

Set $u_1 = (0,1,2)$

and
$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$
$$= (1,1,2) - \frac{(1,1,2)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2)$$
$$u_2 = (1,0,0)$$

and
$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 \cancel{+} - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$
$$= (1,0,1) - \frac{(1,0,1)\cdot(0,1,2)}{(0,1,2)\cdot(0,1,2)}(0,1,2) - \frac{(1,0,1)\cdot(1,0,0)}{(1,0,0)\cdot(1,0,0)}(1,0,0)$$
$$= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0)$$
$$u_3 = \left(0, -\frac{2}{5}, \frac{1}{5}\right)$$

The Gram-Schmidt Process [gram-shmit]

Given a basis $\{x_1, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

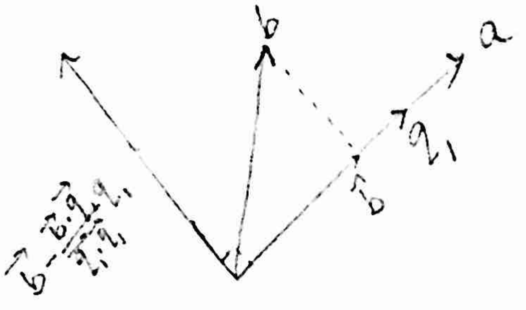
$$\vdots$$

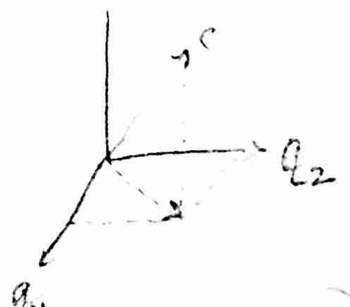
$$v_p = v_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W.

In addition, $\operatorname{Span}\{v_1, \ldots v_k\} = \operatorname{Span}\{x_1, \ldots x_k\}$ for $1 \le k \le p$.

The construction, which converts a skewed set of axes into a perpendicular set, is known as Gram-Schmidt Orthogonalization.





$$c - \left(\frac{c \cdot q_1}{q_1 \cdot q_1} q_1 + \frac{c \cdot q_2}{q_2 \cdot q_2} q_2 \right)$$

example:

Find an orthogonal basis for the column space of the matrix $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

Sol^n: The columns of A are the vectors $\{x_1, x_2, x_3\}$

$x_1 = (3,1,-1,3)$, $x_2 = (-5,1,5,-7)$

$x_3 = (1,1,-2,8)$

Let $v_1 = (3,1,-1,3)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (-5,1,5,-7) - \frac{(-5,1,5,-7)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3)$$

$$= (-5,1,5,-7) - \frac{(-40)}{(20)}(3,1,-1,3)$$

$-5+6 \mid 1+2$

$5-2 \mid -7+6$

$v_2 = (1,3,3,-1)$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1}(v_1) \mp \frac{x_3 \cdot v_2}{v_2 \cdot v_2}(v_2)$$

$$= (1,1,-2,8) - \frac{(1,1,-2,8)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)}(3,1,-1,3) - \frac{(1,1,-2,8)\cdot(1,3,3,-1)}{(1,3,3,-1)\cdot(1,3,3,-1)}(1,3,3,-1)$$

$$= (1,1,-2,8) - \frac{30}{20}(3,1,-1,3) - \frac{(-10)}{20}(1,3,3,-1)$$

$1 - \frac{9}{2} + \frac{1}{2} = -\frac{6}{2}$

$1 - \frac{3}{2} + \frac{3}{2} = 1$

$-2 + \frac{3}{2} + \frac{3}{2} = \frac{4}{2}$

$8 - \frac{9}{2} - \frac{1}{2} = \frac{6}{2}$

$v_3 = (-3,1,1,3)$

$\{(3,1,-1,3), (1,3,3,-1), (-3,1,1,3)\}$ is an orthogonal basis for the column space of the given matrix.

example

Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

The columns of A are $\{x_1, x_2, x_3\}$,

where $x_1 = (-1,3,1,1)$, $x_2 = (6,-8,-2,-4)$, $x_3 = (6,3,6,-3)$

let $v_1 = (-1,3,1,1)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (6,-8,-2,-4) - \frac{(6,-8,-2,-4)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1)$$

$$= (6,-8,-2,-4) - \frac{(\cancel{-36})^{3}}{\cancel{12}}(-1,3,1,1)$$

$-6-24-2-4$
$1+9+1+1$
$6-3, -8+9$
$-2+3, -4+3$

$$v_2 = (3,1,1,-1)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2,$$

$$= (6,3,6,-3) - \frac{(6,3,6,-3)\cdot(-1,3,1,1)}{(-1,3,1,1)\cdot(-1,3,1,1)}(-1,3,1,1) - \frac{(6,3,6,-3)\cdot(3,1,1,-1)}{(3,1,1,-1)(3,1,1,-1)}(3,1,1,-1)$$

$$= (6,3,6,-3) - \frac{\cancel{6}}{\cancel{12}_2}(-1,3,1,1) - \frac{\cancel{30}^{5}}{\cancel{12}_2}(3,1,1,-1)$$

$$= (6,3,6,-3) - \left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{15}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2}\right)$$

$$v_3 = \left(\cancel{-\frac{5}{2}}, \cancel{-\frac{3}{2}}, \cancel{\frac{5}{2}}, \cancel{\frac{1}{2}}\right) \quad (-1,-1,3,-1)$$

$-6+9+6-3$
$18+3+6+3$
$9+1+1-1$
$6+\frac{1}{2}-\frac{15}{2} = \cancel{\frac{12+1-15}{2}}$
$3-\frac{3}{2}-\frac{5}{2}$
$6-\frac{1}{2}-\frac{5}{2} = \frac{12-1-6}{2}$
$-3-\frac{1}{2}+\frac{5}{2} = \cancel{-6+1+6}$

$\{(-1,3,1,1), (3,1,1,-1), (-1,-1,3,-1)\}$ is an orthogonal basis for the column space of the given matrix.