Linear Algebra Probability and Queuing Theory (MMA202T) CIE 2.

Quiz

1. $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$, $u = (3,2)$, $v = (2,1)$

$u - v = (1,1)$ (1) $\|u-v\| = \sqrt{2\times1\times1 + 3\times1\times1} = \sqrt{5}$ (1)

2. $Q(x) = 10x_1x_2$ $A = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix} \Rightarrow \lambda^2 - 0\lambda + (-25) = 0 \Rightarrow \lambda^2 = 25 \Rightarrow \lambda = \pm 5$ (1)

$\therefore$ max val' is 5 (1)

3. $A = \begin{bmatrix} a & 2 \\ 0 & b \end{bmatrix}$ $\left.\begin{matrix} a+b = 7 \\ ab = 12 \end{matrix}\right\} \Rightarrow a = 3, b = 4$ (1) $\therefore A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$ (1)

4. $A = [1 \ -2 \ 2]$ $AA^T = [9]$ (1) $\therefore$ singular value is $\sqrt{9} = 3$ (1)

5. $\lambda = 84.36$ $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$ (1) $y_1 = 0.45x_1 + 0.89x_2$ (1)

Test

1a. $\langle p, q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2) + p(t_3)q(t_3)$, $t_0 = -2, t_1 = -1, t_2 = 1, t_3 = 2$

$p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$ (1)

$P_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $P_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$; $P_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 4 \end{bmatrix}$ (1) $P_0(t) = 1$ (1); $P_1(t) = t - \frac{\langle t, P_0\rangle}{\langle P_0, P_0\rangle} P_0(t)$

$P_1(t) = t - \frac{0}{4} \times 1 = t$ (1)

$P_2(t) = t^2 - \frac{\langle t^2, P_0\rangle}{\langle P_0, P_0\rangle}\times 1 - \frac{\langle t^2, P_1\rangle}{\langle P_1, P_1\rangle}\times t = t^2 - \frac{10}{4}\times 1 - \frac{0}{14}\times t = t^2 - \frac{5}{2}$ (2)

1b. $f(t) = 2$, $[0, 2\pi]$ $\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} 2\,dt = \frac{1}{2\pi}\left[2t\right]_0^{2\pi} = 2$ (1)

$a_n = \frac{1}{2\pi}\int_0^{2\pi} 2\cos kt\,dt = \frac{1}{\pi}\left[\frac{\sin kt}{k}\right]_0^{2\pi} = \frac{1}{\pi}\times(0-0) = 0$ (1)

$b_n = \frac{1}{2\pi}\int_0^{2\pi} 2\sin kt\,dt = \frac{1}{\pi}\left[\frac{-\cos kt}{k}\right]_0^{2\pi} = \frac{1}{k\pi}\left[-\cos 2\pi + \cos 0\right] = 0$ (1)

$\therefore f(t) = 2$ (1)

2a. $Q(x) = 5x_1^2 + 4x_2^2 + 4x_3^2 + x_1x_2 - x_1x_3 - 4x_2x_3$.

$A = \begin{bmatrix} 5 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 4 & -2 \\ -\frac{1}{2} & -2 & 4 \end{bmatrix} \Rightarrow \lambda^3 - 13\lambda^2 + 50\lambda - 56 = 0$ $\Rightarrow \lambda = 7, 4, 2$ (2)

(i) max val' is 7. s.t $x^Tx = 1$. (1)

(ii) $A - 7I = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}$ $\Rightarrow u = \begin{bmatrix} 5 \\ 5 \\ -5 \end{bmatrix} \sim \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$ is unit vector (2)

(iii) max val' is 4 s.t $x^Tx = 1$ & $x^Tu = 0$. (1)

2b. $Q(x) = 6x_1^2 + 4x_1x_2 + 3x_2^2$ $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow \lambda^2 - 9\lambda + 14 = 0 \Rightarrow \lambda = 7, 2$. (2)

$\lambda = 7$ $A - 7I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\lambda = 2$ $A - 2I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\therefore P = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$ (2)

$Q(y) = 7y_1^2 + 2y_2^2$ (1)

3. $A=\begin{bmatrix}2 & -1 & -1\\ 1 & 4 & 1\\ -1 & -1 & 2\end{bmatrix} \Rightarrow \lambda^3-8\lambda^2+21\lambda-18=0$ (1)

$\Rightarrow \lambda = 2, 3, 3$ (2)

$\lambda=2 \quad A-2I=\begin{bmatrix}0 & -1 & -1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix} \Rightarrow x=\begin{bmatrix}1\\-1\\1\end{bmatrix}$ (1), $\lambda=3 \quad A-3I \begin{bmatrix}-1 & -1 & -1\\ 1 & 1 & 1\\ -1 & -1 & -1\end{bmatrix} \Rightarrow \begin{matrix} x_1+x_2+x_3=0 \\ x_1=-x_2-x_3 \end{matrix} \quad x=\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ (1) (1)

$\therefore P=\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$ (1), $D=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$ (1), $P^{-1}=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$ (1)

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4. $A=\begin{bmatrix}2 & -1\\ 2 & 2\end{bmatrix}$

$A^TA=\begin{bmatrix}8 & 2\\ 2 & 5\end{bmatrix}$ (1)

$AA^T=\begin{bmatrix}5 & 2\\ 2 & 8\end{bmatrix}$ (1)

$\Rightarrow \lambda = 9, 4$

$\lambda^2-13\lambda+36=0$ (1)

$\lambda = 9, 4$

$\lambda=9 \quad A^TA-9I=\begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix} \Rightarrow x=\begin{bmatrix}-4\\-2\end{bmatrix} \sim \begin{bmatrix}2\\1\end{bmatrix}$ (1)

$\lambda=9 \quad AA^T-9I=\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}$ (1)

$\lambda=4 \quad A^TA-4I=\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix} \Rightarrow x=\begin{bmatrix}1\\-2\end{bmatrix}$ (1)

$\Rightarrow x=\begin{bmatrix}-1\\-2\end{bmatrix} \sim \begin{bmatrix}1\\2\end{bmatrix}$

$V=\begin{bmatrix}2/\sqrt5 & 1/\sqrt5\\ 1/\sqrt5 & -2/\sqrt5\end{bmatrix}$ (1)

$\lambda=4 \quad AA^T-4I=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ (1)

$\Rightarrow x=\begin{bmatrix}4\\-2\end{bmatrix} \sim \begin{bmatrix}2\\-1\end{bmatrix}$ (1)

$U=\begin{bmatrix}1/\sqrt5 & 2/\sqrt5\\ 2/\sqrt5 & -1/\sqrt5\end{bmatrix}$ (1)

$\Sigma=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$ (1)

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5. $X=\begin{bmatrix}20 & 16 & 14 & 18 & 15 & 19\\ 9 & 7 & 8 & 6 & 5 & 7\end{bmatrix}$ $\quad M=\begin{bmatrix}17\\7\end{bmatrix}$ (2) $\quad B=\begin{bmatrix}3 & -1 & -3 & 1 & -2 & 2\\ 2 & 0 & 1 & -1 & -2 & 0\end{bmatrix}$ (2)

$S=\frac{1}{5}\begin{bmatrix}28 & 6\\ 6 & 10\end{bmatrix}=\begin{bmatrix}5.6 & 1.2\\ 1.2 & 2\end{bmatrix}$ (2) $\Rightarrow \lambda^2-7.6\lambda+9.76=0$ (1) $\Rightarrow \lambda = \begin{matrix}5.9633\\ 1.6367\end{matrix}$ (1)

$\lambda=5.9633 \quad S-5.96I=\begin{bmatrix}-0.36 & 1.2\\ 1.2 & -3.96\end{bmatrix} \Rightarrow x=\begin{bmatrix}-3.96\\-1.2\end{bmatrix}$ (1) $\sim \begin{bmatrix}3.96/\sqrt{17.1216}\\ 1.2/\sqrt{17.1216}\end{bmatrix} \sim \begin{bmatrix}0.9565\\ 0.2899\end{bmatrix}$ (1) first principal component

$\frac{5.9633}{7.6}=78.46\%$ of information is retrieved. (1)