1. Find the singular values of the matrices:
(i) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$, (iii) $\begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix}$, (iv) $\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$

2. Find the SVD of: (i) $\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$, (ii) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

3. Find the principal components of the data:
(i) $\begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 5 & 2 & 6 & 7 & 3 \\ 3 & 11 & 6 & 8 & 15 & 11 \end{bmatrix}$

4. Suppose three tests are administered to a random sample of college students. Let $X_1, ..., X_N$ be observation vectors in $\mathbb{R}^3$ that list the three scores of each student, and for $i = 1, 2, 3$, let x_j denote a student's score on the jth exam. Suppose the covariance matrix of the data is S. Let y be an index of student performance, with $y = c_1x_1 + c_2x_2 + c_3x_3$ and $c_1^2 + c_2^2 + c_3^2 = 1$. Choose c_1, c_2, c_3 so that the variance of y over the data set is as large as possible.
(i) $S = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$, (ii) $S = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 11 & 4 \\ 2 & 4 & 5 \end{bmatrix}$

1. Compute the quadratic form $x^T Ax$, when
(i) $\begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

2. Find the matrix of the quadratic form. Assume either x is in $\mathbb{R}^2$ or $\mathbb{R}^3$.
(i) $20x_1^2 - 15x_1x_2 - 10x_2^2$, (ii) $5x_1^2 + 3x_1x_2$, (iii) $4x_1x_2 + 6x_1x_3 - 8x_2x_3$, (iv) $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

3. Make a change of variable, $x = Py$, that transforms the given quadratic form into a quadratic form with no cross-product term. Give P and the new quadratic form.
(i) $x_1^2 + 10x_1x_2 + x_2^2$, (ii) $-5x_1^2 + 4x_1x_2 - 2x_2^2$, (iii) $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$, (iv) $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$

4. What is the largest value of the given quadratic form, if $x^T x = 1$?
(i) $5x_1^2 + x_2^2$, (ii) $5x_1^2 - 3x_2^2$, (iii) $5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$, (iv) $3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$.

5. Find (a) the maximum value of $Q(x)$ subject to the constraint $x^T x = 1$, (b) a unit vector u where this maximum is attained and (c) the maximum of $Q(x)$ subject to the constraints $x^T x = 1$ and $x^T u = 0$.
(i) $Q(x) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$, (ii) $Q(x) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$, (iii) $Q(x) = 5x_1^2 + 5x_2^2 - 4x_1x_2$, (iv) $7x_1^2 + 3x_2^2 + 3x_1x_2$.

Assignment 5

Diagonalize the following matrices:

1. $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}$

1) Find the singular values of the matrices,

i) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

$\rightarrow$ $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$$\boxed{\text{Characteristic Equation} = \lambda^2 - (\text{Trace of } A)\lambda + \text{Det of } A = 0}$$

$\lambda^2 - (1+9)\lambda + (9-0) = 0$

$\lambda^2 - 10\lambda + 9 = 0$

$\lambda = 9, \quad \lambda = 1$

$$\boxed{\sigma = \sqrt{\lambda}}$$

$\sigma_1 = \sqrt{9} = 3$

$\sigma_2 = \sqrt{1} = 1$

$\therefore$ The singular values of matrix is 3 and 1

ii) $\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow \quad AA^T = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$

$\lambda^2 - 25\lambda + 0 = 0$

$\lambda = 25 \qquad \lambda = 0$

$\sigma_1 = \sqrt{25} = 5$

$\sigma_2 = \sqrt{0} = 0$

iii) $\begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix}$

$\rightarrow \quad AA^T = \begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 6+0 & \sqrt{6}+\sqrt{6} \\ 0+0 & 1+6 \end{bmatrix} = \begin{bmatrix} 6 & 2\sqrt{6} \\ 0 & 7 \end{bmatrix}$

$AA^T = \begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 1 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 6+1 & 0+\sqrt{6} \\ 0\sqrt{6}+\sqrt{6} & 0+6 \end{bmatrix} = \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 6 \end{bmatrix}$

$\lambda^2 - 13\lambda + (42 - 6) = 0$

$\lambda^2 - 13\lambda + 36 = 0$

$\lambda = 9 \qquad \lambda = 4$

$\sigma_1 = \sqrt{9} = 3$

$\sigma_2 = \sqrt{4} = 2$

iv) $\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$

$\rightarrow$ $$A A^T = \begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 2 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 3+4 & 0+2\sqrt{3} \\ 0+2\sqrt{3} & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2\sqrt{3} \\ 2\sqrt{3} & 3 \end{bmatrix}$$

$$\lambda^2 - (7+3)\lambda + (21 - (2\sqrt{3}\,2\sqrt{3})) = 0$$

$$\lambda^2 - 10\lambda + (21 - 12) = 0$$

$$\lambda^2 - 10\lambda + 9 = 0$$

$$\lambda = 9 \qquad \lambda = 1$$

$$\boxed{\begin{aligned} \sigma_1 &= \sqrt{9} = 3 \\ \sigma_2 &= \sqrt{1} = 1 \end{aligned}}$$

PTO

2) Find the SVD of

i) $\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

→ Singular Value Decomposition

Every $m \times n$ matrix factors into

$$A = U \Sigma V^T$$

$U \rightarrow$ Columns of U are eigen vector of AA^T

$V \rightarrow$ Columns of V are eigen vector of $A^T A$

$\Sigma \rightarrow$ Diagonal of Σ are singular values

We will use $A^T A = V \Sigma^T \Sigma V^T$ (or) $AA^T = U \Sigma^T \Sigma U^T$

$A V \Sigma^{-1} = U$ (or) $A U \Sigma^{-1} = V$

* $A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

Eigen Values : $\lambda^2 - 90\lambda + 0 = 0$

$\lambda = 90$ and $\lambda = 0$

Eigen Vector : $A^T A - \lambda I$

For $\lambda = 90$, $A^T A - 90 I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} - \begin{bmatrix} 90 & 0 \\ 0 & 90 \end{bmatrix} = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$

For $\lambda = 90$, $-27x_1 = 81x_2 \Rightarrow \dfrac{x_1}{-81} = \dfrac{x_2}{27} \Rightarrow \dfrac{x_1}{-3} = \dfrac{x_2}{1}$

$$X = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

For $\lambda = 0$, $A^T A - 0I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

$$-27x_1 + 9x_2 = 0 \Rightarrow \frac{-x_1}{9} = \frac{-x_2}{27} \Rightarrow \frac{x_1}{1} = \frac{x_2}{3}$$

$$X = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$V = \begin{bmatrix} -3\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = V^T$$

Σ has order of A, which is 3×2

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \sigma = \sqrt{\lambda}$$

PTO

* To find U

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

Eigen Values : $\lambda^3 - 90\lambda^2 + 0\lambda - 0 = 0$

$$\lambda^3 - 90\lambda^2 = 0$$

$$\lambda = 90, \quad \lambda = 0$$

Eigen Vectors :

For $\lambda = 90$, $AA^T - 90I = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} - \begin{bmatrix} 90 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 90 \end{bmatrix}$

$$= \begin{bmatrix} -80 & -20 & -20 \\ -20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix}$$

$$\frac{x_1}{900} = \frac{-x_2}{1800} = \frac{x_3}{-1800} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-2}$$

$$X = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

PTO

For $\lambda = 0$, $AA^T - 0I = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$

$$\begin{bmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = \underline{1}$

free variables $= n - r = 2$

We evaluate X by taking $x_2 = 1$ and $x_3 = 0$
$x_2 = 0$ and $x_3 = 1$

$10x_1 - 20x_2 - 20x_3 = 0$

① $\Rightarrow$ $10x_1 - 20 - 0 = 0$

$10x_1 = 20$

$x_1 = 2$

$X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

② $\Rightarrow$ $10x_1 - 0 - 20 = 0$

$10x_1 = 20$

$x_1 = 2$

$X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

$$U = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

ii) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

→ Find U:

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Eigen Values : $\lambda^2 - 34\lambda + 225 = 0$

$\lambda = 25 \qquad \lambda = 9$

For $\lambda = 25$, $AA^T - 25I = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}$

$$\frac{x_1}{-8} = \frac{-x_2}{8} \Rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

For $\lambda = 9$, $AA^T - 9I = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

$$\frac{x_1}{8} = \frac{-x_2}{8} \Rightarrow X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Find V:

$$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Eigen Values: $\lambda^3 - 34\lambda^2 + 225\lambda + \cancel{}00 = 0$

$\lambda = 25 \quad \lambda = 9 \quad \lambda = 0$

Eigen Vectors, $A^TA - 25I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \Rightarrow \dfrac{x_1}{200} = \dfrac{-x_2}{-200} = \dfrac{x_3}{0}$

$\dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{0}$

$$X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$A^TA - 9I = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \quad \dfrac{x_1}{-8} = \dfrac{-x_2}{-8} = \dfrac{x_3}{-32} \Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{1} = \dfrac{x_3}{-4}$

$$X = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{bmatrix}$$

$A^TA - 0I = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \quad \dfrac{x_1}{100} = \dfrac{-x_2}{100} = \dfrac{x_3}{-50}$

$\dfrac{x_1}{-2} = \dfrac{x_2}{+2} = \dfrac{x_3}{+1}$

$$X = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & 2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{bmatrix} \qquad V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & -1/3\sqrt{2} & 4/3\sqrt{2} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$\lambda = 25, 9, 0$

$\sigma = 5, 3, 0$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$$

We've found U, V^T and Σ which for SVD of $A = U\Sigma V^T$

3) Find the principal components of the data

i) $\begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$

$\rightarrow$ Mean, $M = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$ $\qquad B = X - M = \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix}$

Covariance, $S = \dfrac{1}{n-1} BB^T$ $\qquad S = \dfrac{1}{5} \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 10 & -4 \\ -6 & -1 \\ -9 & 5 \\ -10 & 3 \\ 8 & -5 \end{bmatrix}$

$$S = \frac{1}{5} \begin{bmatrix} 430 & -135 \\ -135 & 80 \end{bmatrix}$$

$$S = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$$

Eigen Value: $\lambda^2 - 102\lambda + 647 = 0$

$\lambda = 95.204 \quad \lambda = 6.796$

Eigen Vector: $S - 95.204\,I = \begin{bmatrix} -9.204 & -27 \\ -27 & -79.204 \end{bmatrix}$

$$\frac{x_1}{-79.204} = \frac{-x_2}{-27} \Rightarrow X = \begin{bmatrix} -79.204 \\ -27 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -79.204/83.679 \\ -27/83.679 \end{bmatrix} = \begin{bmatrix} -0.9465 \\ 0.3226 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -0.9465 \\ 0.3226 \end{bmatrix}$$

For $\lambda = 6.796$, $S - 6.796I = \begin{bmatrix} 79.2041 & -27 \\ -27 & 9.2041 \end{bmatrix}$

$$\frac{x_1}{9.2041} = \frac{-x_2}{-27} \Rightarrow X = \begin{bmatrix} 9.2041 \\ -27 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 9.2041/28.5256 \\ 27/28.5256 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.3226 \\ 0.9465 \end{bmatrix}$$

$\therefore$ u_1 and u_2 are the principal components

ii) $\begin{bmatrix} 1 & 5 & 2 & 6 & 7 & 3 \\ 3 & 11 & 6 & 8 & 15 & 11 \end{bmatrix}$

$\rightarrow$ $M = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$ $\quad B = X - M = \begin{bmatrix} -3 & 1 & -2 & 2 & 3 & -1 \\ -6 & 2 & -3 & -1 & 6 & 2 \end{bmatrix}$

$$S = \frac{1}{n-1} BB^T = \frac{1}{5}\begin{bmatrix} 28 & 40 \\ 40 & 90 \end{bmatrix} = \begin{bmatrix} 5.6 & 8 \\ 8 & 18 \end{bmatrix}$$

Eigen Values: $\lambda^2 - 23.6\lambda + 36.8 = 0 \Rightarrow \lambda = 21.921, 1.679$

Eigen Vectors: $\lambda - 21.921 I = \begin{bmatrix} -16.321 & 8 \\ 8 & -3.921 \end{bmatrix}$

$$\frac{x_1}{-3.921} = \frac{-x_2}{8} \Rightarrow X = \begin{bmatrix} +3.921/8.909 \\ +8/8.909 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0.4401 \\ 0.8982 \end{bmatrix}$$

$$\lambda - 1.679I = \begin{bmatrix} 3.921 & 8 \\ 8 & 16.321 \end{bmatrix} \quad \frac{x_1}{16.321} = \frac{-x_2}{8} \Rightarrow X = \begin{bmatrix} 16.321/18.176 \\ -8/18.176 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 0.8980 \\ -0.4401 \end{bmatrix}$$

$\therefore$ u_1 and u_2 are principal components

4) i) $S = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

$\rightarrow$ Eigen Values: $\lambda^3 - 18\lambda^2 + (38+35+26)\lambda - 162 = 0$

$\lambda = 9, 6, 3$

Eigen Vector:

$$\lambda - 9I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \Rightarrow \frac{x_1}{2} = \frac{-x_2}{-4} = \frac{x_3}{4} \Rightarrow X = \begin{bmatrix} 1 \\ +2 \\ 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/3 \\ +2/3 \\ 2/3 \end{bmatrix}$$

$u_1 = \begin{bmatrix} 1/3 \\ +2/3 \\ 2/3 \end{bmatrix}$ $\therefore c_1 = 1/3$ $\quad c_2 = +2/3$ $\quad c_3 = 2/3$

$$c_1^2 + c_2^2 + c_3^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$\therefore$ Variance of y is max

$$y = \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3$$

ii) $S = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 11 & 4 \\ 2 & 4 & 5 \end{bmatrix}$

→ Eigen Values: $\lambda^3 - 21\lambda^2 + 99\lambda - 135 = 0 \Rightarrow \lambda = 15, 3, 3$

Eigen Vector:

$$S - 15I = \begin{bmatrix} -10 & 4 & 2 \\ 4 & -4 & 4 \\ 2 & 4 & -10 \end{bmatrix} \Rightarrow \frac{x_1}{24} = \frac{-x_2}{-48} = \frac{x_3}{24} \Rightarrow X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad \therefore \; c_1 = 1/\sqrt{6} \quad c_2 = 2/\sqrt{6} \quad 1/\sqrt{6}$$

$$c_1^2 + c_2^2 + c_3^2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$$

Variance of $y = \frac{1}{\sqrt{6}}x_1 + \frac{2}{\sqrt{6}}x_2 + \frac{1}{\sqrt{6}}x_3$ is max

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5) Compute quadratic form $x^T A x$, when

i) $\begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ $Q(x) = x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Q(x) = \begin{bmatrix} 5x_1 + \frac{1}{3}x_2 & \frac{1}{3}x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Q(x) = 5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$$

ii) $\begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

→ From $Q(x) = 5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$

$$= 5(6)^2 + \frac{2}{3}(6 \times 1) + (1)^2 = 180 + 4 + 1$$

$$= 185$$ //

iii) $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\rightarrow$ $Q(x) = x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Q(x) = \begin{bmatrix} 4x_1^2 + 3x_2 & 3x_1 + 2x_2 + x_3 & x_2 + x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Q(x) = 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$

iv) $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$\rightarrow$ $Q(x) = 4(2)^2 + 2(-1)^2 + (5)^2 + 6(2)(-1) + 2(-1)(5)$

$Q(x) = 21$

6) Find the matrix of quadratic form. Assume x in $\mathbb{R}^2$ or $\mathbb{R}^3$

i) $20x_1^2 - 15x_1x_2 - 10x_2^2$

$\rightarrow$ $A = \begin{bmatrix} 20 & -15/2 \\ -15/2 & -10 \end{bmatrix}$

ii) $5x_1^2 + 3x_1x_2$

$\rightarrow$ $A = \begin{bmatrix} 5 & 3/2 \\ 3/2 & 0 \end{bmatrix}$

iii) $4x_1x_2 + 6x_1x_3 - 8x_2x_3$

$\rightarrow$ $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{bmatrix}$

iv) $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

$\rightarrow$ $A = \begin{bmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{bmatrix}$

7) Make a change of variable $x = Py$ that transforms given quadratic form into quadratic form with no cross product term

i) $x_1^2 + 10x_1x_2 + x_2^2$

$\rightarrow$ $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \Rightarrow \lambda^2 - 2\lambda - 24 = 0$

$\lambda = 6, -4$

$A - 6I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \Rightarrow \dfrac{x_1}{-5} = \dfrac{-x_2}{5} \Rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A - (-4)I = \begin{bmatrix} \cancel{4}5 & 5 \\ 5 & \cancel{4}5 \end{bmatrix} \Rightarrow \dfrac{x_1}{5} = \dfrac{-x_2}{5} \Rightarrow X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$

$Q(x) = \cancel{PDP}\ x^T A x$ and $A = PDP^T$

$$
\begin{aligned}
Q(x) &= x^T (PDP^T) x \\
&= (Py)^T PDP^T (Py) \\
&= y^T \underline{P^T P}\, D\, \underline{P^T P} y \\
&= y^T D y = Q(y)
\end{aligned}
$$

$\therefore$ Quadratic form with no cross-product $= y^T D y$

$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$= 6y_1^2 - 4y_2^2$

ii) $-5x_1^2 + 4x_1x_2 - 2x_2^2$

$\rightarrow \quad A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow \lambda^2 - (-7)\lambda + 6 = 0$

$\lambda = -1, -6$

$A - (-1)I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow \frac{x_1}{-1} = \frac{-x_2}{2} \Rightarrow X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A - (-6)I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \frac{x_1}{4} = \frac{-x_2}{2} \Rightarrow X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$.

$x^T A x = y^T D y \Rightarrow$

$Q(y) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$Q(y) = -y_1^2 - 6y_2^2$

iii) $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2$

$\rightarrow \quad A = \begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix} \Rightarrow \lambda^3 - 27\lambda^2 + (207)\lambda - 405 = 0$

$\lambda = 15, 9, 3$

$A - 15I = \begin{bmatrix} -6 & -4 & 4 \\ -4 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix} \Rightarrow \frac{x_1}{32} = \frac{-x_2}{16} = \frac{x_3}{32} \Rightarrow X = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$A - 9I = \begin{bmatrix} 0 & -4 & 4 \\ -4 & -2 & 0 \\ 4 & 0 & 2 \end{bmatrix} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{8} = \frac{x_3}{8} \Rightarrow X = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$$A - 3I = \begin{bmatrix} 6 & -4 & 4 \\ -4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} \Rightarrow \frac{x_1}{32} = \frac{+x_2}{+32} = \frac{x_3}{-16} \Rightarrow X = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \qquad D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$x^T A x = y^T D y$$

$$Q(y) = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q(y) = 15y_1^2 + 9y_2^2 + 3y_3^2$$

iv) $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$

$$\rightarrow \quad A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \Rightarrow \lambda^3 - 12\lambda^2 + 21\lambda + 98 = 0$$

$$\lambda = 7, 7, -2$$

$$A - 7I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow -4x_1 - 2x_2 + 4x_3 = 0$$

$$x_1 = -\frac{1}{2}x_2 + x_3$$

$$X = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \Rightarrow \frac{x_1}{36} = \frac{x_2}{18} = \frac{x_3}{-36} \Rightarrow X = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{5} & 1/\sqrt{2} & 2/3 \\ 2/\sqrt{5} & 0 & 1/3 \\ 0 & 1/\sqrt{2} & -2/3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$x^T A x = y^T D y$

$$Q(y) = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q(y) = 7y_1^2 + 7y_2^2 - 2y_3^2$$

8) What is the largest value of given quadratic form, $x^T x = 1$

i) $5x_1^2 + x_2^2$

$\rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda^2 - 6\lambda + 5 = 0$

$\lambda = 5, 1$

Maximum value of $Q(x)$ is 5

ii) $5x_1^2 - 3x_2^2$

$\rightarrow \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow \lambda^2 - 2\lambda - 15 = 0$

$\lambda = 5, -3$

Maximum is 5

iii) $5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_3 - 4x_2x_3$

$\rightarrow \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix} \Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$\lambda = 9, 6, 3$

Maximum is 9

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iv) $3x_1^2 + 2x_2^3 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$

$\rightarrow$ $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

$\lambda^3 - 7\lambda^2 + (10)\lambda + 0 = 0$

$\lambda = 5, 2, 0$

Maximum value is 5

9) Find (a) max value of $Q(x)$ to constraint $x^T x = 1$

(b) unit vector u where maximum is attained

(c) max of $Q(x)$ subject to $x^T x = 1$ & $x^T u = 0$

i) $Q(x) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$

$\rightarrow$ $A = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$ $\lambda^3 + 3\lambda^2 - 6\lambda - 8 = 0$

$\lambda = 2, -1, -4$

(a) Maximum value of $Q(x)$ is $\lambda = 2$

(b) For $\lambda = 2$, $A - 2I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 3 & -2 \end{bmatrix}$

$\frac{x_1}{2} = \frac{-x_2}{-4} = \frac{x_3}{4}$ $\Rightarrow$ $X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$U_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\therefore$ at u_1, max value is attained

(c) $A + 1\lambda = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$\frac{x_1}{-4} = \frac{-x_2}{2} = \frac{x_3}{4} \Rightarrow X = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$u_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ $\therefore$ at u_2, max value is attained contrained to $U_2^T u_2 = 1$ and $u_2^T u_1 = 0$

ii) $Q(x) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$

$\rightarrow$ $A = \begin{bmatrix} 7 & -4 & -2 \\ -4 & 1 & -4 \\ -2 & -4 & 7 \end{bmatrix}$ $\lambda^3 - 15\lambda^2 + 27\lambda + 243 = 0$

$\lambda = 9, 9, -3$

(a) Max value of $Q(x)$ is 9

(b) For $\lambda = 9$, $A - 9I = \begin{bmatrix} -2 & -4 & -2 \\ -4 & -8 & -4 \\ -2 & -4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$-x_1 - 2x_2 - 2x_3 = 0$

$x_1 = -2x_2 - x_3$ $\Rightarrow$ $X = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

at $u_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ max value is attained.

(c) For $\lambda = -3$, $A + 3I = \begin{bmatrix} 10 & -4 & -2 \\ -4 & 4 & -4 \\ -2 & -4 & 10 \end{bmatrix}$

$$\frac{x_1}{24} = \frac{+x_2}{+48} = \frac{x_3}{24} \Rightarrow X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$ is max value with constraints

iii) $Q(x) = 5x_1^2 + 5x_2^2 - 4x_1 x_2$

$$\rightarrow \quad A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\lambda^2 - 10\lambda + 21 = 0$$

$$\lambda = 7, 3$$

(a) Max value of $Q(x)$ is $\lambda = 7$

(b) For $\lambda = 7$, $A - 7I = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$

$$\frac{x_1}{-2} = \frac{-x_2}{-2} \Rightarrow X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(c) For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\frac{x_1}{2} = \frac{+x_2}{+2} \Rightarrow X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$ is where max value of $Q(x)$ subject to constraints $x^T x = 1$ & $x^T x = 0$ occurs (at second largest λ)

iv) $7x_1^2 + 3x_2^2 + 3x_1x_2$

$\rightarrow$ $$A = \begin{bmatrix} 7 & 3/2 \\ 3/2 & 3 \end{bmatrix}$$

$$\lambda^2 - 10\lambda + \frac{75}{4} = 0$$

$$\lambda = \frac{15}{2}, \frac{5}{2}$$

(a) Max value of $Q(x)$ is greatest eigen value $\lambda_1 = \frac{15}{2}$

(b) Max value of $Q(x)$ subject to constraints $x^Tx = 1$ occurs at unit vector u_1

$$A - \frac{15}{2}I = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -9/2 \end{bmatrix}$$

$$\frac{x_1}{-9/2} = \frac{-x_2}{3/2} \Rightarrow X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

(c) Max value of $Q(x)$ to constraints $x^Tx = 1$ and $x^Tu = 0$ occurs at unit vector u_2, corresponding to second large eigen

$$A - \frac{5}{2}I = \begin{bmatrix} 9/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$$

$$\frac{x_1}{1/2} = \frac{-x_2}{3/2} \Rightarrow X = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

10) Diagonalize the following matrices:

i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$

$\rightarrow$ $\lambda^3 - 11\lambda^2 = 0$

$\lambda = 11, 0, 0$

$A - 11I = \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix}$

$\frac{x_1}{11} = \frac{-x_2}{-33} = \frac{x_3}{44} \Rightarrow X = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$A - 0I = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + 2x_2 + x_3 = 0$

$x_1 = -2x_2 - x_3$

$X = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1/11 & 2/11 & 1/11 \\ -3/11 & 5/11 & -3/11 \\ -4/11 & -8/11 & 7/11 \end{bmatrix}$

ii) $\begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix}$

$\rightarrow \quad \lambda^3 - 27\lambda^2 + 207\lambda - 405 = 0$

$\lambda = 15, 9, 3$

For $\lambda = 15$

$$A - 15I = \begin{bmatrix} -6 & -4 & 4 \\ -4 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

$$\frac{x_1}{32} = \frac{-x_2}{16} = \frac{x_3}{32} \Rightarrow X = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 9$, $A - 9I = \begin{bmatrix} 0 & -4 & 4 \\ -4 & -2 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

$$\frac{x_1}{-4} = \frac{-x_2}{-8} = \frac{x_3}{8} \Rightarrow X = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$, $A - 3I = \begin{bmatrix} 6 & -4 & 4 \\ -4 & 4 & 0 \\ -4 & 0 & 8 \end{bmatrix}$

$$\frac{x_1}{32} = \frac{-x_2}{-32} = \frac{x_3}{-16} \Rightarrow X = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$P = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 2/9 & -1/9 & 2/9 \\ -1/9 & 2/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$

$A = PDP^{-1}$