PROBABILITÁ E STATISTICA

COUNTING

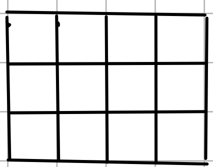
- How many possible outcomes satisfy an event?
→ This is counting

Ex:
- Roll a dice, possible outcomes: $\{1, 2, 3, 4, 5, 6\}$
- Even outcomes? $= \{2, 4, 6\}$
- if i roll 2 dices? $= \{(1,1), (1,2), (1,3) \dots\}$
→ The total amount of possibilities is 36 possible tuples.

Product Rule of counting

If 2 events are independent one from the other and the possible outcomes from the first is $|m|$ and the possible outcomes from the second is $|n|$, the total amount of outcomes is $\boxed{\text{Total} = |m||n|}$
[independent means that the choice from the first set doesn't affect the choice of the second set]

ex:



- 12 pixels, 17 mln colours x pixel
- How many different images?

$\rightarrow \{(17 \cdot 10^6, \text{pixel}2, \text{pixel}3 \dots)\} = \boxed{(17 \cdot 10^6)^{12}} = 5{,}88 \cdot 10^{77}$

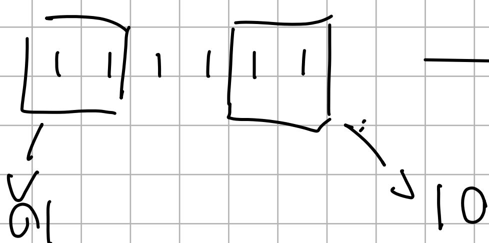
$\text{pixel}2 \rightarrow 17 \cdot 10^6$, $\text{pixel}3 \rightarrow 17 \cdot 10^6$

→ The colour of one pixel doesn't change the colour of other pixels

Sum rule of counting

There's the mutual exclusive case when 2 sets are empty
In any other case i just sum the n° of elements in the sets

ex



→ 2^4 possibilities for every combination but i also have repetitions.

In the sum rule of counting there must be the rule $|A \text{ AND } B| = \emptyset$ so the 2 sets must be mutually exclusive which mean that there shouldn't be the same element in both the sets.
So the rule is:

$$\# = |A| + |B| - |A \text{ AND } B| = 16 + 16 - 4$$

ex:

$$\boxed{0\ 1}\ 1\ 1\ \boxed{1\ 0}$$

Fixed $\quad 2^2 \quad$ Fixed

→ In the end i'll have 4 combinations in both the sets.

Or rule of counting (Exclusion/Inclusion)

if $A \cap B$ may not be empty the total amount of outcomes could be:

$$\boxed{|A| + |B| - |A \cap B|}$$

ex:
How many ways to order BOBA?

B
Fixed

BOBA	1
BOAB	2
BBOA	3
BBAO	4
BABO	5
BAOB	6

- change the 1st letter 4 time
- change the 2nd letter 3 times fixing the 1st
- change the 3rd letter 2 times fixing the 1st and 2nd

Result = $4 \times 3 \times 2$

→ This if the letters are not duplicated

if the letters are duplicated i need to take into consideration that I have less possibilities to choose from.

ex:

ABCD → result = $4 \times 3 \times 2 = 24$ possibilities

Resolution :

$\underset{\downarrow}{\underline{4}}\ \underline{3}\ \underline{2}\ \underline{1}$

Here i can choose between 4 letters , given that i choose A then in the second part i can choose between 3 remains letters. and so on

Since the choice of the first is independent from the next choices i can multiply the possibilities.
No matter which letter i choose there will be always 3 other choices left
The choice in the first step change which letter after but not how many letters.

Permutations

An ordered arrangements of objects.
The n° of unique ordering of n distinct objects : $\boxed{n!}$

ex:
I have 9 possible digits and I need to build a 6 digit code

$n-k \longrightarrow$ n° of free slots

$$\prod_{n}^{n-k} x = \frac{n!}{k!} = \frac{9!}{3!}$$

$n \rightarrow$ n° of possible digits

Permutations with indistinguishable value.

Fix the double counting with similar objects.
ex:

1 1	0 0 0

These 2 groups are indistinct, the elements inside the set could not be distinguished from each other. so there will be in the Total amount of permutations a certain value of similar group. I need to find how many similar group there are so:

$5! = \frac{120}{3!}$ = i find the n° of groups with the same 0 positions that are distincted from the other

In these groups there are also the case where there are similar groups with the value '1'

So in this case i need to calculate:

$$\frac{120}{3!\cdot 2} = \frac{\overset{\cancel{60}\ \cancel{30}\ 10}{\cancel{120}}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{2}} = 10 \longrightarrow$$ There are 10 distinct possible combinations between all the possibilities.

ex with ABOB:

$$\frac{4!}{2!} = \frac{24}{2} = 12$$

The sort of the semi-distinct object is a 2 phases process:

permutation of distinct object = permutation considering same object are indistinct × permutation of just the indistinct objects

General approach to counting permutations

When there are n object s.t.:

- n_1 are the same indistinct object
- n_2 are the same indistinct objects
- .
- .
- .
- n_n are the same indistinct objects

The number of indistinct permutations are:

$$\boxed{\frac{n!}{n_1!\, n_2! \dots n_n!}}$$

ex:
How many 6 digits password i can generate using 5 distinct number

$$\# = 5 \times \frac{6!}{2!}$$

$6!$ → total amount of different digits

$2!$ → repeated digits

5 → Here i decide which of the 5 distinct digit will be repeated so every time I have a different digit repeated ex2 :

42 : 01

if i don't have to repeat 1 digit

$$\# = \binom{10}{6} = \frac{10!}{6!\,(10-6)!}$$

total amount of combinations

removing the sets that could not happen because I have 6 digits

The order doesn't count I remove the combinations with different order but same values

If the order counts at the denominator i'll not have the "6!"

Combinations

ex:

There are 20 people.
How many ways can we choose $K = 5$ people to get cake?

A combination is an unordered selection of K objects from a set of n distinct objects:

$$\boxed{\# = \frac{n!}{K!\,(n-K)!}}$$

total amount of possibilities

Given that the order doesn't count i remove all the groups with the same elements but different order

remove the groups that isn't possible to compose if i open the formula:

$$\# = \frac{n!}{k!(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1 way to adjust the selected group (in the case of the digits this is 5 due to the fact that i can decide among 5 digits)

all the possible combinations

the ordered in the subset doesn't count

the element out of the group doesn't count

$n!$ = order n distinct objects

1 = take first k as chosen

$1/k!$ = overcounted: any ordering of the same group is same choice

$1/(n-k)!$ = overcounted: any ordering of unchosen group is same choice

$$\boxed{\# = \binom{n}{k}}$$

Binomial coefficient

ex:

choose 3 element of a set of indistinct objects:

⟶ Unique outcome

ex 2:

How many ways are there to choose 3 books from a set of 6 distinct books?

$$\# = \binom{6}{3} = 6! \times 1 \times \frac{1}{3!} \times \frac{1}{(6-3)!}$$

$$= \frac{6!}{3!\,3!} = \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{6}} = 20$$

ex 3:

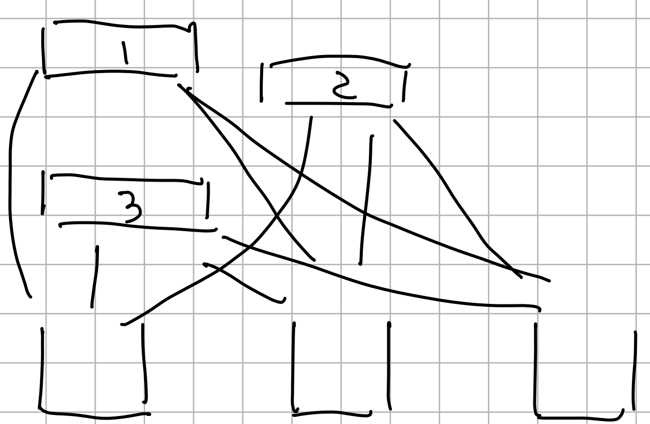
How many unique hands of 5 cards are there in a 52 card deck

$$\# = \binom{52}{5} = 52! \times 1 \times \frac{1}{5!} \times \frac{1}{(52-5)!} = \frac{52! \cdot 5}{5! \cdot 47!}$$

$$= \frac{\cancel{52}^{13} \cdot \cancel{51}^{17} \cdot 50 \cdot 49 \cdot \cancel{48}^{24}}{5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = \frac{13 \cdot 17 \cdot 50 \cdot 49 \cdot 24}{5} \simeq 2{,}6 \cdot 10^6$$

Hash tables and distinct strings

How many ways are there to hash n distinct objects to r buckets?



Since each bucket can contain all the 3 string or could be empty the total amount of possible combination is:

First string:
n possible buckets

2nd string:
n possible buckets

⋮

Result = $n \times n \times n$

$$\boxed{\# = n^r}$$

distinct buckets

distinct strings

With indistinct objects and distinct buckets

- The divider method

The number of ways to distribute n indistinct object into r buckets is equivalent to the n° of ways to permut n+r-1 objects such that:

n are indistinct object

r-1 are indistinct dividers :

$$\boxed{\# = \frac{(n+r-1)!}{n!\,(r-1)!}}$$

overcounted ways based on the order of the dividers

overcounted ways based on the order of the objects

$$\boxed{\# = \binom{n+r-1}{r-1}}$$

Resume

Counting Operations on n objects:

- Permutation:
  - $n!$ = Number of unique ordering giving a set of n distinct objects
  - $\frac{n!}{n_1!\,n_2!\dots n_n!}$ = Number of unique ordering giving a set of n distinct object and n° set of indistinct objects.
- Combinations:
  - $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ = set of k distinct objects among a set of n elements.

- Hash tables :
  - n^r : r distinct objects distributed inside n distinct buckets
  - $\binom{n+r-1}{r-1}$: n indistinct objects distributed in r indistinct buckets.

What is probability?

If i want to pick all the unique couples among a set of n elements without taking into consideration the elements with themselves i need to do a combination $\binom{n}{2}$.
By looking trough a different perspective

	A	B	C
A	x	o	o
B	x	x	o
C	x	x	x

I'm taking into consideration actually $\binom{n}{2}$ elements.

Sample Space

Set of possible outcomes coming from an experiment.
ex:

coin flip : $S = \{ \text{testa}, \text{croce} \}$

Event Space

The set E is a subset of S that respects a certain condition
ex:

coinflip is testa : $E = \{ \text{testa} \}$

The probability is a number between 0 and 1

Pr(E) = our belief that an event E occurs.

$$\boxed{P(E) = \lim_{n \to \infty} \frac{n(E)}{n}}$$

Imagine to repeat an infinite time the experiment, what is the ratio between the n° of times that the condition of an event is respected over the total amount of events

Axioms of probability:

Axiom 1 = $0 \leq P(E) \leq 1$

Axiom 2 = $P(S) = 1$

Axiom 3 = if event E and F are mutually exclusive
then $P(E \cup F) = P(E) + P(F)$

Given S = set of all the possible outcomes
E = set of possible outcomes under a certain condition

Core rules of probability:

Axiom 1 = $0 \leq P(E) \leq 1$

Axiom 2 = $P(S) = 1$

Identity = $P(E^c) = 1 - P(E)$
↳ complementare di evento E

Equally Likely outcomes

Some sample spaces have equally likely outcomes for ex a coinflip
if we have equally likely outcomes, then $P(\text{each outcome}) = \frac{1}{|S|}$

Therefore $\boxed{P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}}$ by axiom 3

I just sum all the occurences in which i obtain E until I have $|E|$

When i calculate the probability of something i always do a counting of the sample space and then I count the event space

In the example with dices i need to take into consideration that dices are distinct otherwise the outcome {1,2} would be equal to the outcome {2,1} and so the sample space would be halfed

ex:

4 cows and 3 pigs in a toy box, 3 drawn.
What is the probability to draw 1 cow and 2 pigs?

I have to choose my sample space considering that i can choose the objects in an ordered way or in an unordered way or or consider all the objects distinct from the other or undistinct. → all the cows and pigs are distinct from each other
If i consider the unordered distinct:

Among all the possible group of 3 animals, what is the probability to obtain what i mentioned before:

$$\frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}}$$

→ i can multiply here because, given that they are unordered, the outcome of an event doesn't affect the outcome of the other event

→ I multiply the results of picking 2 pigs from a group of the 3 and 1 cow from the group of the 4. These 2 event must happen at the same time over all the possible groups of 3 among the 7 elements.

if i want to considered order items:

$$\binom{7}{3} = \frac{7!}{(4!)} = 7\cdot 6\cdot 5 = 210$$

Now i want to pick the right things

$$\binom{4}{1} = \frac{4!}{(4-1)!} = 4,\quad \binom{3}{2}\ \frac{3\cdot 2}{(3-2)!} = 3,\quad \binom{2}{2} = \frac{2!}{2\cdot(2-2)!}$$

$$4\cdot 3\cdot 2$$

$$\frac{\{4\cdot 3\cdot 2\} + \{3\cdot 4\cdot 2\} + \{3\cdot 2\cdot 4\}}{210}$$

Here I have to sum cause the order matter so the pos of the cow matters and allows a comb to be specific

When the order counts like in this case the events are mutually exclusive so i can sum then over the total n° of possibilities

A good way to make the problem easier is to make things distinct so there will be equally likely outcomes

ex:

5 cards from a deck of 52, and the 5 cards must be consecutive

$$\frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}}$$

I can start and finish with an ace
I can start from 10 different version
and the result would be the same

ex:

n chips crafted, 1 of which is defective
k random chips selected from n for testing
- What is the probability to catch the defective chip in the selected k chips.

$$P = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

↳ When 2 events are independent i need to multiply their probability because for every possible scenario of the first event i need to check all the other possible scenarios of the other event ↳

Faster way:

if I have a group of n elements and groups of k and i need to find the corrupted element i can think about the fact that the corrupted element is inside the group of k elements and so

The final result will be $\frac{k}{n}$

Condition probabilities and Bayes

Prove that $P(E^c) = 1 - P(E)$ by using the 3 axioms giving that they are mutually exclusive.
Starting from the 3rd axiom i can say $P(E \cup E^c) = P(E) + P(E^c)$
Since everything is or E or not E the entire sample space will be $P(S) = P(E) + P(E^c)$
Given this i can say with the help of the 2nd axiom that
$1 = P(E) + P(E^c)$ and so $P(E^c) = 1 - P(E)$

ex:
Stand Ford people $= 17000$
room : 268 people.
What is the prob to find n of your friend?

$$P(E) = \frac{\binom{17000-n}{268}}{\binom{17000}{268}}$$

With $n = 100$ I have 20% of probability to find some one i know. The rest 80% is the prob that i don't know anyone.

Conditional probability

ex:

Roll 2 dices, the sum needs to be 4
what is $P(D_1 + D_2) = 4$?

$|S| = 36$
$E = \{(1,3),(2,2),(3,1)\} = 3/36 = 1/12$

What if the result of a certain dice is already given?
The total outcomes are no more 36 but $\{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$
The event space in this case is $E = \{(2,2)\}$

$P(E) = 1/6$

The conditional probability of E given F is the probability that E occurs given that F has already occured. This is known as conditioning on F:

Written as: $P(E|F)$ • F is alredy occured

The sample space S shrinks $= S \cap F$
The outcome must be consistent with $F = E \cap F$

With EQUALLY LIKE OUTCOMES:

$$Pr(E|F) = \frac{\#\text{ of outcomes in } E \text{ consistent with } F}{\#\text{ of outcomes in } S \text{ consistent with } F}$$

$$= \frac{|EF|}{|SF|} = \frac{|EF|}{|F|}$$

Considering that S is the total space of outcomes an intersection between something and the total space will be that something thus $|S \cap F| = |SF| = |F|$

Conditional probability in general

General definition:

• These properties hold also when the outcomes are not equally likely

$$P(E|F) = \frac{P(EF)}{P(F)}$$

if $P(F) = 0$ then $P(E|F)$ will be undefined.

ex:

What is the prob that a user will watch life is beautiful?

$S = \{watch, not\ watch\}$

$E = \{watch\}$

$P(E) = 1/2 \rightarrow$ not correct

ex:

What is the probability to watch life is beautifull given that i watch the film coda?
A good approximation would be:

$$P(E|F) = \frac{P(E|F)}{P(F)} = \frac{\dfrac{\#\text{ people who have watched both}}{\#\text{ people } \cancel{\text{on}} \text{ netflix}}}{\dfrac{\#\text{ people who have wathed coda}}{\#\text{ people } \cancel{\text{on}} \text{ netflix}}}$$

$$= \frac{\#\text{ people who have watched both}}{\#\text{ people who have watche coda}} = 0{,}42$$

It's obvious that $P(E|E^c) = 0$

ex:

prob of E and F simultaneously

$$P(E|F) = \frac{P(EF)}{P(F)} \; ; \; P(E|F)\,P(F) = P(EF) = 0{,}50 \cdot 0{,}50 = P(EF)$$

$$= 0{,}25$$

prob of F

prob E given F is 50%

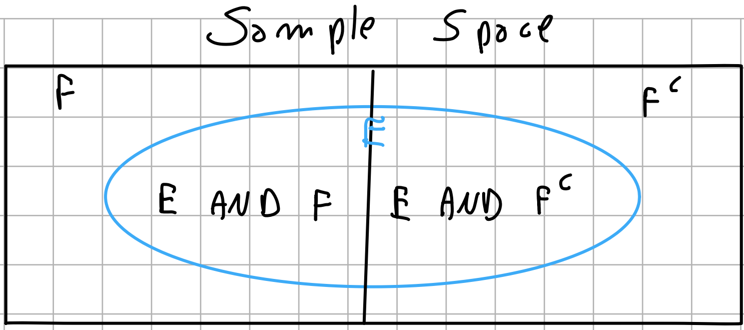
$P(E_1 | E_2, E_3)$: probability that event 1 happens given event that event 2 and 3 happen.

Law of total probability

Say events E and F are in S

$$P(E) = P(EF) + P(EF^c) = P(E|F)\,P(F) + P(E|F^c)\,P(F^c)$$

The total probability of E in this case is given by the sum of the conditional probability of 2 mutually exclusive events.



This can be generalized by:

Theorem:
Let F be an event where $P(F) > 0$. For any event E,

$$\boxed{P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)}$$

Proof:

1 - $E = (EF)$ or (EF^c) since F and F^c are disjoint
2 - $P(E) = P(EF) + P(EF^c)$ probability of or for disjoint
3 - $P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$ chain rule (product rule)

ex:

10% of bacteria have antibiotic resistance
Probability a bacteria survives given it has the mutation = 20%
" it hasn't the " = 1%
What is the probability that a random bacteria survive

$P(S|M) = 20\%$

$P(S|M^c) = 1\%$

$P(M) = 10\%$

$P(M^c) = 1 - P(M) = 90\%$

$P(S) = P(S|M)P(M) + P(S|M^c)(PM^c) =$

$= 0{,}2 \cdot 0{,}1 + 0{,}01 \cdot 0{,}9 = 0{,}029 = 2{,}9\%$

ex:

What is the probability that a bacteria has the mutation given that it survives

$$P(M|S) = \frac{P(SM)}{P_S} = \frac{0{,}025 \cdot 0{,}1}{0{,}025} = 10\%$$

If i have $P(S|M)$ there's a way to obtain $P(M|S)$

Baye's Theorem

It's easier to get the probability of something that i observe given something that i don't observe than the contrary

$$P(F|E) = \frac{P(EF)}{P(E)} \underset{\text{using chain rule}}{=} \frac{P(E|F)\,P(F)}{P(E)} \underset{\text{Using law of total probability}}{=} \frac{P(E|F)P(F)}{P(E|F)\,P(F) + P(E|F^c)\,P(F^c)}$$

theorem:

For any events E and F where $P(E) > 0$ and $P(F) > 0$

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}$$

Proof:

$$= \frac{P(EF)}{P(E)} \underset{\text{using chain rule}}{=} \frac{P(E|F)\,P(F)}{P(E)} \underset{\text{Using law of total probability}}{=} \frac{P(E|F)P(F)}{P(E|F)\,P(F) + P(E|F^c)\,P(F^c)}$$

ex:

60% of all email in 2016 is spam
20% of spam has the word "Dear"
1% of non-spam has the word "Dear"

I get an email with the word "Dear"
What is the probability that the email is spam?

$P(S|D) = ?$ $\qquad P(S) = 0,6$

$P(D|S) = 0,2$

$P(D|S^c) = 0,01$

$P(S^c) = 1 - P(S) = 0,4$

$$P(D|S^c) = \frac{P(DS^c)}{P(S^c)}$$

$$P(S|D) = \frac{P(DS)}{P(D)} = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|S^c)P(S^c)} = \frac{0,2 \cdot 0,6}{0,2 \cdot 0,6 + 0,01 \cdot 0,4}$$

$$= \frac{0,012}{0,012 + 0,04} = \frac{0,012}{0,016} = 75\%$$

I can see this also in this way:

$$P(F|E) = \frac{\overset{\text{likelihood}}{P(E|F)}\ \overset{\text{prior}}{P(F)}}{\underset{\text{normalisation constant}}{P(E)}}$$

ex:

A test is 98% effective at detecting SARS
False positive of 1%
0,5 % of USA population has SARS
Let E = test positive for SARS with this test
Let F = you actually have SARS
What is $P(F|E)$

Bayes theorem = $\dfrac{0{,}98 \cdot 0{,}005}{0{,}98 \cdot 0{,}005 + 0{,}001 \cdot (1-0{,}005)} \approx 0{,}330$

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,PF + P(E|F^c)\,P(F^c)}$$

When i put the condition am testing positive i enter in the domain of people who are tested positive.
Among these people there are also people who are false positive.
The ratio between the people who are actually positive and the false positive is 1/3

$$\{ P(A|B^c) = 1 - P(A^c|B^c) \}$$

Inclusion / exclusion with 3 events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$$

Considering that the events are not mutually exclusive i need to remove the duplicates.

General inclusion/exclusion

$$P(E_1 \cup E_2 \cup E_3 \dots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} Y_r$$

Y_1 = sum of all events on their own : $\sum_i P(E_i)$

Y_2 = Sum of all pairs of events: $\sum_{i,j \text{ s.t. } i \neq j} P(E_i \cap E_j)$

Y_3 = Sum of all triplets of events: $\sum_{i,j,k \text{ s.t. } i\neq j,\, j\neq k,\, i \neq k} P(E_i \cap E_j \cap E_k)$

Where Y_r is the sum, for all combinations of r events, of the probability of the intersection those events.

Independence

2 events A and B are called independent if:

$$\boxed{P(A) = P(A|B)}$$

Knowing that event B happens doesn't change our belief that A will happen
Otherwise they are called dependent events.

Alternative definition

$$P(A,B) = P(A) \cdot P(B|A) \quad \text{chain rule}$$
$$= P(A) \cdot P(B) \quad \text{since B is independent from A}$$

If i am able to show that this is true the 2 events are independent

Independence is reciprocal

$P(A) = P(A|B) \qquad P(B|A) = P(B)$

Proof:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P\,A} \quad \text{Baye's Theorem}$$
$$= \frac{P(A)\,P(B)}{P(A)} \quad \text{Because A is independent from B}$$
$$= P(B)$$

ex :

Show that the probability to obtain a certain result with a dice and another result with another dice are independent.

$P(E) = 1/6$

$P(F) = 1/6$

$P(E|F) = P(F)$

$P(EF) = 1/36$

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{\cancel{1/6} \cdot 1/6}{\cancel{1/6}} = P(E|F) = 1/6 = P(F)$$

$$P(F|E) = \frac{P(FE)}{P(E)} = \frac{1/36}{1/6} = 1/6$$

What if $P(G) = D_1 + D_2 = 5$

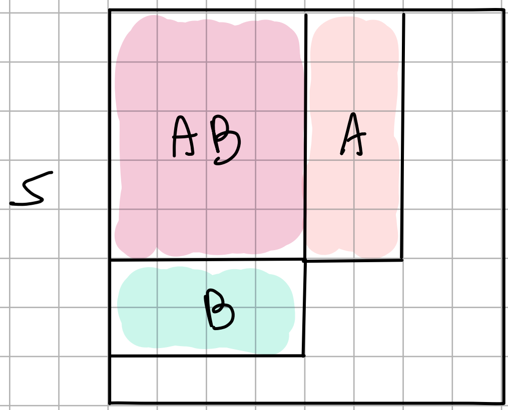
$= 4/36 = 1/9$

$P(E) = 1/6 \quad (D_1 = 1)$

$P(EG) = 1/36$ that is different from $P(E) \cdot P(G) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54}$

Because there's only 1 possibility over 36 that the first dice is 1 and the sum with another makes 5

What's independence looks like



In the figure the ratio between A and B and S and AB and S is the same

Independence definition 1:

$$P(AB) = P(A)\,P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

If the 1 of the 2 definitions isn't respected the 2 events are dependent.

Given independent events A and B, prove that A and B^c are independent.

We want to show that $P(AB^c) = P(A)\,P(B^c)$

$P(AB^c) = P(A) - P(AB)$ By total law of probability.

$= P(A) - P(A)P(B)$ By independence

$= P(A)[1 - P(B)]$ Factoring

$= P(A)\,P(B^c)$ Since $P(B) + P(B^c) = 1$

So if A and B are independent also A and B^c are independent

Generalization of AND when there are more than 2 events. Generalized independence.

General definition of Independence:
Events $E_1, E_2, \dots, E_n$ are independent if for every subset with r elements (where $r \leq n$) it holds that:

$$P(E_1, E_2, \dots, E_r) = P(E_1)P(E_2)\dots P(E_r)$$

Ex:

Outcomes of n separate flips of a coin are all independent of one another
- Each flip in this case is called a trial of the experiment wich mean the same procedure under the same circumstances.

General rule of not independent events

$$P(ABC) = P(A) \cdot P(A|B) \cdot P(C|AB)$$

ex:

3 children
(A, a) parents traits

	A	a
A	AA	Aa
a	Aa	aa

aa → curly hair (25%)

there are 3 children

What is the probability that all the 3 children have curly hair.

$P(E_1 E_2 E_3) = P(E_1)P(E_2)P(E_3) = 0{,}25 \cdot 0{,}25 \cdot 0{,}25$ because they are independent.

ex:

n independent routers each has the probability p_i of functioning
E = Functional path from A to B.

What is $P(E)$

I need to calculate when none of them work and then do the complement.

$$1 - P(\text{none}) = 1 - \left(1 - \prod_{i=1}^{n} 1/p_i\right) = 1 - 1 + \prod_{i=1}^{n} 1/p_i$$

$$P(\text{none}) = 1 - P(\text{all}) = 1 - \prod_{i=1}^{n} 1/p_i$$

$$P(\text{all}) = \prod_{i=1}^{n} 1/p_i$$

Most important core probability question

Suppose to flip a coin n times.
Each flip is independent.
What is the probability of exact k heads.

1- Warmups

If $n = 10$ and $p = 0{,}6$ the probability is $(0{,}6)^{10}$

What if first k head and then $n-k$ tails

$$P = (p_1)^{k} \cdot (1-p)^{n-k}$$

2- Exactly k head

If i flip 10 coins then i get precisely 6 heads.
It's different than before because now the events are dependent.
I can do this with combinations $\binom{n}{k}$

ex:

I flip 2 frisbees, if they land on the same face they are even.
I have $0{,}6$ probability to land on heads.

$$P(E) + P(F) = (0{,}4)^2 + (0{,}6)^2$$

Conditional Independence

2 events are conditionally independent given G if:

$$P(EF \mid G) = P(E \mid G)\, P(F \mid G)$$

or equivalently:

$P(E|FG) = P(E|G)$ Considering that E and F are independent

The causality can make independent events dependent and vice versa

In the conditional paradigm the formulas of probability are preserved
Independence conditions can change with conditioning.

ex:

What is the probability that a user watch 4 particular movies?

13 000 movies on Netflix.
The user watch 30 random titles
E = movies watched include the given 4

$$P = \frac{\binom{4}{4}\binom{12996}{26}}{\binom{13000}{30}} = \approx 10^{-11}$$

First i extract the 4 movies taken into consideration then the other 26 taken from all the others.

Conditional independence is a practical way to decompose hard probability questions.

if E and F are dependent don't mean E and F will be dependent when another event is observed

Random Variables

A random variable is a variable that will have a value but there's uncertainty as to what value.

Properties of random variables.

Probability Mass function:

$P(X=a)$ The probability of every assignment of the random var.

Expectation

$E[X]$

Variance

$Var(X)$

Probability Mass Function (PMF)

The relationship between values a random variables can take on, and the corresponding probability is a function.

Let Y be a random variable

$Y = 2$ is an event when Y takes on a value
$Y \leq 2$ is also an event
It makes sense to say $P(Y \leq 2)$ not $P(Y)$

if $P(Y=2)$ then $P(Y=2)$ is a number between 0 and 1
if instead of $P(Y=2)$ i take $P(Y=k)$ i will have a function and not a probability.

If a random variable is discrete we call the function probability mass function.

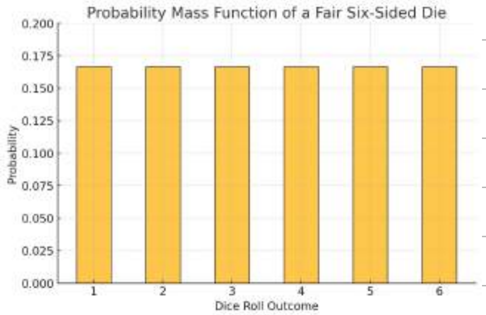
ex:

Let x be a random variable that represents the results of a single dice roll. X can take the results from $\{1, 2, 3, 4, 5, 6\}$
What is the relation between the values i can take on and their probability to appear?

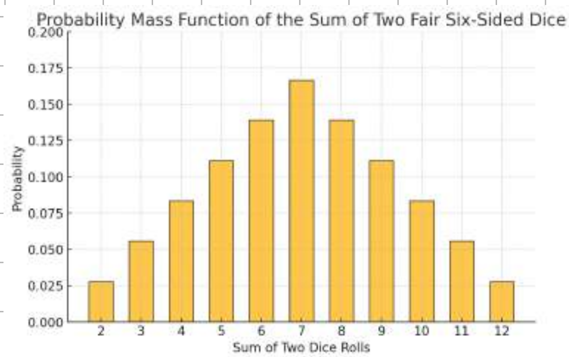
$P(X = x)$

what is the probability that this random variable will assume this specific value

PMF of a roll of a dice



All of the inputs lead to the same probability



This is the same graph but with the sum of the results coming from the rolls of 2 dices

It could be also be represented in the form of an equation:

$$p(X = x) = \begin{cases} \frac{x-1}{36} & \text{if } x \in \mathbb{Z},\ 1 \leq x \leq 6 \\ \frac{13-x}{36} & \text{if } x \in \mathbb{Z},\ 7 \leq x \leq 12 \\ 0 & \text{else} \end{cases}$$

I need to define the function like this because i need to represent when it grows from 1 to 6 but also when it decrese form 7 to 1

check if something is fine.

$\sum_k P(Y = k) \stackrel{?}{=} 1$ This is equivalent to ask the prob of the S

→ The sum of the probabilities that a certain variable takes a certain value is 1 because i'm covering all the possible outcomes.