Iterative Quantum Amplitude Estimation.

We approximate $\mathbb{P}[|1\rangle] = \sin^2((2k+1)\theta_a)$
for the last qubit in

$$Q^k A|0\rangle_n|0\rangle$$ for different powers of k.

Suppose a confidence interval $[\theta_l, \theta_u] \subseteq [0, \frac{\pi}{2}]$
for θ_a and a power k of Q as well as an
estimate for
$\sin^2((2k+1)\theta_a)$ $\rightarrow$ $\boxed{\text{Probability}}$

As ~~$\sin^2$~~ $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

we translate our estimates for $\sin^2[(2k+1)\theta_a]$
into estimates for $\cos[(4k+2)\theta_a]$.

Invertibility of Cosine alone is enough to ~~...~~ estimate

Cosine is invertible only if domain is $[0, \pi]$
or $[\pi, 2\pi]$ because

eg if $\cos x = 0.5$

$x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \ldots\ldots$

$\frac{\pi}{3}$ $\downarrow$ lower half; $\frac{5\pi}{3}$ $\searrow$ upper half

So, unique solution in lower and upper if domain is restricted.

So, our function is invertible if we have prior knowledge ~~that~~ x lies in lower or ~~half~~ upper half plane.

As $P[|1\rangle] = \dfrac{1 - \cos[(4k+2)\theta_a]}{2}$

~~the ... ...~~ measurement only gives us information about $\cos[(4k+2)\theta_a]$.

We have no way to get the sign of the original sine term [whether positive or not] or any phase information.

The act of squaring discards this information

→ Difference between Kitaev's QPE where ~~real and ...~~ real (cosine) and imaginary (sine) parts of a phase are estimated, which cost a lot of resources.

IQAE → only estimate cosine term.

Finding next K subroutine is what makes this algorithm so efficient.

⇒ Finds next measurement setting K ~~that~~ is not too small or too large to give us the maximum information possible in the next step without creating ambiguities

Main goal : Unambiguous Inversion

To solve the cosine invertibility problem.

We have a confidence interval $[\theta_l, \theta_u]$, which is our current best guess for true angle θ_a.
To shrink this interval and improve our estimate, we apply Q k times, which effectively multiplies angle by $\boxed{K' = 4k+2}$

$\cos(4k+1)\theta_a$
$\cos(k'\theta_a)$

~~If K is~~

If k' is too small: we don't zoom in very much on the angle. New estimate will be more precise but not by much.

If k' is too large: The scaled interval of uncertainty $[k'\theta_l, k'\theta_u]$ becomes wider than π

$\Rightarrow K[\theta_u - \theta_L] > \pi$!

Here $\cos(K\theta_a)$ will get ambiguous and we can't use this to narrow our ~~bound~~ estimate.

<u>If K' is just right</u>:

we get the largest possible k ~~that~~ keeps the entire scaled interval

$[K'\theta_l, K'\theta_u]$ within $[0, \pi]$ upper half

or

$[K'\theta_l, K'\theta_u]$ within $[\pi, 2\pi]$ lower half

$\Rightarrow$ Unambiguous inversion and maximum reduction in width of confidence interval.

We can also look at it by looking at Fisher information

$$I(\theta_a) \propto N_{shots} \cdot K'^2 \quad \text{but why } K'^2 ?$$

$$P(\theta_a) = \sin^2[(2k+1)\theta_a]$$

$$\frac{dP}{d\theta_a} = \sin[2[2k+1]\theta_a](2k+1)$$

~~$I(\theta_a)$~~

$I(\theta_a) \propto N_{shots}$ because each shot is an independent measurements. By collecting more data our estimation is more accurate. So, N_{shots} is N_{shots} times that amount of information.

$$I(\theta_a) = N_{shots} \cdot \frac{1}{P[1-P]} \left(\frac{dP}{d\theta_a}\right)^2$$

$$= N_{shots} \cdot \frac{1}{P[1-P]} \cdot \left[\frac{(2K+1) \times 2}{2}\right]^2$$

$$\boxed{I(\theta_a) \propto N_{shots} \, k^{\prime 2}}$$

Doubling iterations quadruples the information we gain.

Our goal is to maximize Fisher information on a given iteration in a greedy manner

~~Theorem~~ Theorem-1: Suppose a confidence level $1-\alpha \in (0,1)$ a target accuracy $\varepsilon > 0$, and a Number of shots $N_{shots} \in \{1, \dots N_{max}(\varepsilon, \alpha)\}$, where

$$\boxed{N_{max}(\varepsilon,\alpha) = \frac{32}{\left[1-2\sin\left[\frac{\pi}{14}\right]\right]^2} \log\left[\frac{2}{\alpha}\log_2\left(\frac{\pi}{4\varepsilon}\right)\right]}$$

~~Three~~ 3 Key promises about the algorithm.

1.) **It will finish**:- It will terminate after a maximum number of "rounds", a number that depends logarithmically on target accuracy ε. A round is defined as a series of measurements that all use the same power, k_i, of the grover operator.

2) **It will be correct**: The algorithm outputs an interval $[a_l, a_u]$ that contains the true value a with a probability $1-\alpha$. The width of this interval is guaranteed to be less than & equal to 2ε,

$\Rightarrow$ estimate $\boxed{\hat{a} = \frac{a_l + a_u}{2}}$ is within ε

of the true value.

3.) **It will be efficient (quadratic speedup):**
There is a strict upper bound on the number of times you need to call the oracle.

Proof: $a = \sin^2(\theta_a) = P[|1\rangle]$

$$\frac{|a_U - a_L|}{2} = \frac{|\sin^2\theta_U - \sin^2\theta_l|}{2} = \frac{|\sin(\theta_U+\theta_l)||\sin(\theta_U-\theta_l)|}{2}$$

$$\frac{|a_U - a_L|}{2} = \frac{|\sin(\theta_U+\theta_l)||\sin(\theta_U-\theta_l)|}{2}$$

$$\le \frac{1}{2}\sin(\theta_U-\theta_L) \quad [\because |\sin x| \le 1]$$

$$\frac{a_U - a_L}{2} \le \frac{|\theta_U-\theta_L|}{2} \quad [\because |\sin x| \le |x|]$$

$$\boxed{|a_U - a_L| \le |\theta_U - \theta_L|} \quad \text{— (i)}$$

$\Rightarrow$ If we can show θ bound is small, then probability of error is very small.

Chernoff Hoeffding bound

~~Img~~ Imagine you have a biased coin, but you don't know ~~its~~ its bias. Your goal is to find the true probability p of getting heads.

Let's flip the coin N times (let's take $N = 1000$) and count number of heads [Let's take 600 heads]

Estimate $(\tilde{p})$ is our best guess ~~and~~ for the true probability. Our guess will be the number of heads by total flips as the ~~best~~ best guess.

$$\tilde{p} = \frac{600}{1000} = 0.6$$

But How are we sure that true probability p is exactly 0.6?

$\boxed{\text{Highly Unlikely}}$

Maybe 0.59 & 0.61, etc.

So how do we quantify this uncertainty?

Chernoff Hoeffding bound!

↳ What is the probability that my estimate $\tilde{p}$ is wrong by more than a certain amount $\tilde{\epsilon}$?

$$\boxed{\mathbb{P}\left(|\tilde{p} - p| \geq \varepsilon\right) \leq 2 e^{-2N\tilde{\varepsilon}^2}}$$

The probability ~~that~~ our estimate $\tilde{p}$ is off from the true value (p) by more than ~~err~~ error margin ($\tilde{\varepsilon}$) is ~~ext~~ extremely small, at it gets exponentially fast with the number of flips.

In our algorithm, we use this formula backwards. We decide on an acceptable, tiny probability of being wrong, and the formula tells us the error margin $\tilde{\varepsilon}$ we can guarantee.

This error margin is $\tilde{\varepsilon}_{a_i}$, which is used to create the confidence interval

$$\left[\tilde{a}_i - \tilde{\varepsilon}_{a_i} , \tilde{a}_i + \tilde{\varepsilon}_{a_i}\right]$$

But what happens in one round of IQAE?

This round is one complete "measurement-and-update" cycle

At the start of the round, we know

θ_a is somewhere in $[\theta_L, \theta_U]$

Let's say eg interval is $[20^\circ, 25^\circ]$. This is our confidence interval.

Example Case:

~~Step~~ we apply ~~of~~ grover operator Q a total of k_i times. This has the effect of multiplying our hidden angle by a large number.

$$K_i' = 4k_i + 2$$

Example Case :-

Let's assume $k_i' = 10$ ~~0~~

Our new angle is $10 \cdot \theta_a$

our range of uncertainty is $[200°, 250°]$

This is the scaled interval $[\theta_i^{min}, \theta_i^{max}]$

$$P[|1\rangle] = a_i = \frac{1 - \cos(k_i' \theta_a)}{2}$$

We run the quantum circuit N_{shots} times and count the outcomes. ~~[illegible]~~

This is ~~a~~ analogous to flipping coin N_{shots} times

This gives us estimated probability $\tilde{a}_i$. Then we use the chernoff Hoeffding bound to calculate error margin $\tilde{\epsilon}_{a_i}$

So $a_i := \frac{1 - \cos(k'_i \theta_a)}{2}$

$$a_i^{min} := \max(0, \tilde{a}_i - \tilde{\varepsilon}_{a_i})$$

$$a_i^{max} := \min(1, \tilde{a}_i + \tilde{\varepsilon}_{a_i}),$$

$$\varepsilon_{a_i} := \frac{|a_i^{max} - a_i^{min}|}{2}$$

Where $\tilde{\varepsilon}_{a_i}$ is a half width of a confidence interval, calculated from Chernoff-Hoeffding bound for ~~D~~ Independent Identically distributed ~~D~~ Trials with N_{shots} samples

$\Rightarrow$

B3 — $$\boxed{P\left[a_i \in [a_i^{min}, a_i^{max}]\right] \leq 2e^{-2N_{shots}\tilde{\varepsilon}_{a_i}^2}}$$

Actual error ε_{a_i} can be lower that $\tilde{\varepsilon}_{a_i}$. If this scenario is realized, then $\theta_i^{max} \in \{\pi, 2\pi\}$ or $\theta_i^{min} \in \{0, \pi\}$

ie confidence interval touches the ~~lower~~ ~~low~~ boundary of the upper or lower half circle.

The "actual" half-width ε_{a_i} will be smaller than the calculated $\tilde{\varepsilon}_{a_i}$ if and only if one of these clippings occurs. This happens if our measured probability $\tilde{a}_i$ is so close to 0 or 1 that error margin crosses the boundary. For example, if $\tilde{a}_i = 0.05$ and $\tilde{\varepsilon}_{a_i} = 0.1$, the raw interval is $[-0.05, 0.15]$, which gets clipped to $[0, 0.15]$

The goal is to ensure the total probability of failure for the entire algorithm is less than α (e.x = 5%).

The proof's strategy is to divide this total "error budget" among all the bounds.

Let T be the maximum possible number of rounds the algorithm could ever run.

Then the probability of failure in any single round

$$\leq \frac{\alpha}{T}$$

$$\Rightarrow \quad \text{(B4)} \quad 2e^{-2N_{max}\tilde{\varepsilon}_{a_i}^2} \leq \frac{\alpha}{T} \quad \forall\, i \in \{1, \ldots, t\},$$

Then the union bound asserts that the main ~~bound~~ condition of the theorem ~~is~~ "I_θ will be correct" that it is guaranteed to be bounded.

Total error probability ~~being~~ is bounded by α.

$$\mathbb{P}\left[a \notin [a_l, a_u]\right]$$

$$\leq \mathbb{P}\left[\exists i\{1,\dots,t\} : a_i \notin [a_i^{min}, a_i^{max}]\right]$$

$$\leq \sum_{i=1}^{t} \mathbb{P}\left[a_i \notin [a_i^{min}, a_i^{max}]\right]$$

$$\leq \sum_{i=1}^{t} 2e^{-2N_{max}\tilde{\varepsilon}_{a_i}^2}$$

$$\leq \sum_{i=1}^{t} \frac{\alpha}{T}$$

$$\leq \frac{t\alpha}{T} \leq \alpha$$

From $\boxed{B4} \rightarrow$

$$2e^{-2N_{max}\tilde{\varepsilon}_{a_i}^2} \leq \frac{\alpha}{T} \quad \forall i \in \{1,\dots,t\}$$

As $\varepsilon_{a_i} \leq \tilde{\varepsilon}_{a_i}$

$$\varepsilon_{a_i}^2 \leq \tilde{\varepsilon}_{a_i} = \frac{1}{2N_{max}} \log\left[\frac{2T}{\alpha}\right] \quad \forall i \in \{1,\dots,t\} \quad \boxed{B6}$$

Variables:

$\varepsilon \rightarrow$ target error

$t \rightarrow$ final round number. The algorithm runs for

$1, 2 \ldots t-1, t$ round.

$$\boxed{K = K_1 + K_2 \ldots K_{t-1} + K_t}$$

K_i : magnification factor used in round i.

K_{t-1} : factor for the last second round,

$$\boxed{K_t > K_{t-1}}$$

$\varepsilon_{\theta,i}$: After measurement in round i, this is the error (Confidence interval width) for magnified angle $K_i \theta_a$.

L_{max} and L_{min} : These are universal constants derived in proof. These represent the absolute max and ~~max minimum~~ min possible error for any magnified angle measurement, $\varepsilon_{\theta,i}$, regardless of the round. So, for any round i, we know

$$L_{min} \leq \varepsilon_{\theta,i} \leq L_{max}$$

$Error_i$: This is the overall error for our ~~final~~ final answer θ_*, at end of round i.

$$\boxed{Error_i = \frac{\varepsilon_{\theta i}}{k_i}}.$$

$$\boxed{\varepsilon < \frac{L_{max}}{K_{t-1}}}.$$

At end of round $t-1$, the overall error is

$$E_{t-1} = \frac{\varepsilon_{\theta, t-1}}{K_{t-1}}$$

We consider the worst case scenario.

Since measurement error $\varepsilon_{\theta, t-1}$ can be atmost L_{max}, the algorithm overall error could be as large as $\frac{L_{max}}{K_{t-1}}$.

This inequality states that the worst case scenario error was still ~~gred~~ greater than target error ε.

~~Because~~ Because we aren't guaranteed to have met our target accuracy, it runs for one more round.

Now, ~~algorithm~~ algorithm ~~for~~ finished round t, using a new, larger magnification factor K_t. The error is now less than equal to ε. We reach a state where error is guaranteed to be less than ε, ie

$$\boxed{\frac{L_{min}}{K_t} < \varepsilon < \frac{L_{max}}{K_t}} \longrightarrow \boxed{B7}$$

Overshooting :

In the final rounds, K_t becomes large. If the algorithm kept the same default number of shots, the the work in final round $\propto N_t K_t$ would be wasteful.

So

$$\boxed{N_t K_t \leq \frac{N_{max} L_{max}}{\varepsilon}} \longrightarrow \boxed{B9}$$

N_t is the shots at last round.

$N_t K_t$ $\propto$ total computational cost for the final round

But,

K_t and K_{t-1} depend upon the random outcomes for the measurements in each round. We don't know what they will be before we run the algo.

We will consider worst case growth model.

In each round, the magnification factor grows by at least some rate, γ.

$$\Rightarrow \quad K_i \geq K_{i-1} \cdot \gamma$$

So, in the slowest possible ~~growth~~ run would be increasing geometric series. [Exponentiation Method] ★

Round 1 K_1

R 2 : $K_2 = K_1 \cdot \gamma$

$K_3 = K_1 \cdot \gamma^2$

$\vdots$

$$\boxed{K_t = K_1 \gamma^{t-1}}$$

Now, we can predict t.

$$\Rightarrow \quad \varepsilon < \frac{L_{max}}{K_{t-1}}$$

$$\boxed{\varepsilon < \frac{L_{max}}{K_1 \gamma^{t-2}}}$$

Also,

$$\frac{L_{max}}{K_t} \leq \varepsilon$$

$$\boxed{B_9} \text{ — } \boxed{\frac{L_{max}}{K_1 \gamma^{t-1}} \leq \varepsilon \leq \frac{L_{max}}{K_1 \gamma^{t-2}}}$$

We define

$$T := \left\lceil \log_\gamma (\gamma L_{max} / 2\varepsilon) \right\rceil \quad [\text{let } K_1 = 2]$$

The actual number of rounds t will never be larger than T.

To prove $t \leq T$.

$$\boxed{B_{10}} \text{ — } \boxed{\gamma^{T-1-j} > \prod_{i=1}^{t-2-j} q_i} \quad \forall j \in \{0, \dots t-2\}$$

We compare the worst case growth to actual growth

LHS is the slow growth, magnification factor multiplied by γ each time

RHS is the actual growth of the magnification factor

$\boxed{B10}$ implies $\gamma^{T-1} > \gamma^{t-2}$. Since γ is greater than 1, this is true only if left exponent is larger than right, ~~~~ ie

$$T-1 > t-2$$

$$\Rightarrow T > t-1$$

$$\Rightarrow \boxed{T \geq t}$$

Hence Proved

From $\boxed{B6}$

$$\tilde{\varepsilon}^2_{a_i^o} = \frac{1}{2N_{max}} \log\left[\frac{2T}{\alpha}\right], \quad \tilde{\varepsilon}_{a_i^o} = \sqrt{\frac{1}{2N_{max}} \log\left[\frac{2T}{\alpha}\right]}$$

~~Let~~ $\cancel{2\sqrt{\frac{1}{2N_{max}} \log\left[\frac{2T}{\alpha}\right]} = \sin^2(L)}$

Let $\boxed{\sin^2(L) = 2\tilde{\varepsilon}_{a_i^o}}$ — $\boxed{B11}$

because $\tilde{\varepsilon}_{a_i^o}$ is the half width of error of statistical estimate of probability $\tilde{a}_i^o$.

$$\tilde{\varepsilon}_{a_i^o} \leq 0.5, \quad 2\tilde{\varepsilon}_{a_i^o} \leq 1$$

$$\Rightarrow \frac{\sin^4[L]}{4} = \frac{1}{2N_{max}}\left[\log\left[\frac{2T}{\alpha}\right]\right],$$

$$N_{max}(\varepsilon, \alpha) = \frac{2}{\sin^4[L]} \log\left[\frac{2T(\varepsilon)}{\alpha}\right]$$

$$\cancel{\leq \frac{2}{\sin^4[L]} \log\left[\frac{2}{\alpha} \log_\gamma\left(\frac{\gamma^2 L_{max}}{2\varepsilon}\right)\right]}$$

From B9

$$T \approx \log_\gamma\left(\frac{\gamma^2 L_{max}}{2\varepsilon}\right)$$

$\boxed{B12}$ —

$$\boxed{N_{max} \leq \frac{2}{\sin^4[L]} \log\left[\frac{2}{\alpha} \log_\gamma\left[\frac{\gamma^2 L_{max}}{2\varepsilon}\right]\right]}$$

Lemma. 1

For any $L \leq L^*$; $L^* = \sin^{-1}\left[\frac{1}{2}\sqrt{1 - 2\sin\left(\frac{\pi}{14}\right)}\right]$

the number of shots $N_{max}(L)$, defined by $\boxed{B12}$, is sufficient to ensure

$$\forall \ \gamma \in (1, 3] \quad a_i \geq \gamma$$

Moreover $L_{max} = L$ & $L_{min} = \sin^{-1}\left[\sin^2(L)\right]$

The entire performance ~~lags~~ hinges on K_i, growing in each round. The growth factor is $q_i = \frac{K_{i+1}}{K_i}$. If this is ~~a~~ closer to 1, the algorithm stalls.

Lemma ~~1~~ 's purpose is to ~~a~~ calculate the maximum tolerable error in our measurements that still guarantees progress.

It answers how many measurement shots?

It introduces a "speed limit" for our error L^* such that if $L \le L^*$, we will ~~p~~ make progress.

Proof $\Rightarrow$ From ~~B~~ $\boxed{B1}$

$$\boxed{\varepsilon_{a_i} = \frac{a_U^i - a_L}{2} = \frac{\left|\sin\left(\frac{\theta_i^{max} + \theta_i^{min}}{2}\right)\right| \left|\sin\left(\frac{\theta_i^{max} - \theta_i^{min}}{2}\right)\right|}{2}}$$

$$\boxed{\varepsilon_{a_i} = \frac{|\sin(\theta_i)|\,|\sin(\varepsilon_{\theta_i})|}{2}} \;—\; \boxed{B15}$$

$$\boxed{\theta_i := \frac{\theta_i^{max} + \theta_i^{min}}{2}}$$

Using $\boxed{B15}$ and $\boxed{B11}$, we can calculate ε_{θ_i} from ε_{a_i} or from L.

We define

$$g_L(\theta_i) := \begin{cases} \min\left(\sin^{-1}\left[\frac{\sin^2(L)}{\sin(\theta_i)}\right], \theta_i, \pi - \theta_i\right), & \theta \in \mathrm{dom}^u(g_L) \\ \min\left(\sin^{-1}\left[\frac{\sin^2(L)}{\sin(\theta_i - \pi)}\right], \theta_i - \pi, 2\pi - \theta_i\right) & \theta \in \mathrm{dom}^d(g_L) \end{cases}$$

domain of g_L

$g_L(\theta_i)$ is not defined for all angles in $[0, 2\pi)$.

The domain represents the range of plausible measurement outcomes for a given number of shots.

$$\frac{\sin^2[L]}{|\sin(\theta_i)|} \leq 1 \Rightarrow |\sin(\theta_i)| \geq \sin^2[L]$$

The inequality implies that for fixed num of shots it is statistically impossible for measured angle θ_i to be extremely close to boundaries $\{0, \pi, 2\pi\}$ where $|\sin(\theta_i)|$ is very small.

Behaviour of g_L : Finding L_{min} and L_{max}

The shape of the function $g_L[\theta_i]$ within its valid domain tells us about the worst and best case scenarios ~~occur~~.

L_{min}

Error is min where function is most sensitive to angle changes ie Center of half planes, $\boxed{\text{when } \theta_i = \frac{\pi}{2}}$

$$\epsilon_{\theta_i}\left(\text{at } \frac{\pi}{2}\right) \approx \sin^{-1}\left[\frac{\sin^2(L)}{|\sin(\frac{\pi}{2})|}\right] = \sin^{-1}\left[\sin^2 L\right]$$

This is exactly min error and L_{min}

$\underline{L}_{max}$ $\underline{\text{max error}}$

Max error at the very edges of valid domains.
This is where we get least sensitivity θ_i close to 0 & π.

Next factor $k_{i+1} = q_i k_i$ q_i is the growth factor to maximize.

The new interval will be $[q_i \theta_i^{min}, q_i \theta_i^{max}]$

~~The Cond~~ The condition is that our current CI is completely contained within a slice of the circle of size $\frac{\pi}{3}$

Let $\boxed{j=1}$ $\boxed{\theta_i^{min}, \theta_i^{max} \subseteq [0, \frac{\pi}{3}]}$ — $\boxed{B_{19}}$

$\Rightarrow$ $\theta_{min} \geq 0$ & $\theta_i^{max} \leq \frac{\pi}{3}$

If we pick a growth factor $= 3$.
New ~~interl~~ interval is $[3\theta_i^{min}, 3\theta_i^{max}]$.

$\Rightarrow$ lower $= 0$

upper $= \frac{3\pi}{3} = \pi$

So guaranteed to be in upper half.

This logic holds for any of the 6 slices of $\frac{\pi}{3}$ size around

For ~~a~~ a slice of $\frac{\pi}{5}$, our CI is even smaller,

let $j = 1$ $\boxed{[\theta_i^{min}, \theta_i^{max}] \subseteq [0, \frac{\pi}{5}]}$ — $\boxed{B20}$

Now we can be more aggressive. Let's chose $q_i = 5$.

New interval is $[5\theta_i^{min}, 5\theta_i^{max}]$.

$5\theta_i^{min} = 0$

$5\theta_i^{max} = \frac{5\pi}{5} = \pi$

Again new interval is perfectly within the upper half plane.

This demonstrates the core loop:

We use measurement shots to narrow the CI, Once it is narrow enough, make a big jump in magnification.

The same works for $\boxed{[\theta_i^{min}, \theta_i^{max}] \subseteq [\frac{j\pi}{7}, \frac{(j+1)\pi}{7}]}$ — $\boxed{B21}$

Our Next question is, How can we check if our ~~function~~ CI is actually inside one of these slices?

We define <u>distance functions</u>

$f_i(x)$ is the distance from a point x to the nearest endpoint of the $\frac{\pi}{i}$ slice that x is in.

Let's break down the maths for $f_3(x)$. The circle is divided into slices of size $\frac{\pi}{3}$, with endpoints at $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$, etc.

If point x ~~in~~ in the slice $\left[\frac{(j-1)\pi}{3}, \frac{j\pi}{3}\right]$, then:

→ The distance to the left endpoint is $x - \frac{(j-1)\pi}{3}$

→ The distance to the right endpoint is $\frac{j\pi}{3} - x$.

$f_3(x)$ is simply the minimum of these 2 distances.

Visually, $f_3(x)$ looks like a series of triangles or "tents".
Starts at 0, rises to a peak at the midpoint of the slice, and returns to 0 at the next slice boundary.

$\boxed{B22}$ —— $$\boxed{f_i(x) := \min\left(\frac{(j-1)\pi}{i} - x, \frac{j\pi}{i} - x\right)}$$

$\forall j \in \{1, \dots i\}$ on every interval $\left[\frac{j-\pi}{i}, \frac{j\pi}{i}\right]$

This "distance to the boundary" function gives us a powerful way to check if our CI is safely inside a slice.

Our CI is centered at the midpoint θ_i and has half width error of ε_{θ_i}.

$f_i(\theta_i)$ tells us the maximum error ε_{θ_i} we could have at the midpoint θ_i without "spilling over" the edge of the slice.

$\therefore$ Our CI fits inside a slice of size $\frac{\pi}{i}$ iff our actual error is LTE to this Max allowable error.

$$\boxed{\varepsilon_{\theta_i} \leq f_i(\theta_i)}$$

The prof now has a simple test. To see if we can guarantee a growth factor of at least 3, it just needs to check if our actual error $g_L(\theta_i)$ is less than or equal to $f_3(\theta_i)$

<u>To Show</u>

$$\Rightarrow \quad g_L(\theta_i) \leq f_3(\theta_i)$$

<u>Combining the Distance functions</u>

We established 3 conditions that individually guarantee a large jump in the magnification factor:

If our CI fits in $\frac{\pi}{3}$ slice, we get growth factor $q_i \geq 3$.

If it fits in a $\frac{\pi}{5}$ slice, we get $q_i \geq 5$.

If it fits in a $\frac{\pi}{7}$ slice, we get $q_i \geq 7$.

We need our algorithm to satisfy any of these one conditions to guarantee a growth factor $q_i \geq 3$. Simply, the max of the distance boundaries.

$$f_{max}(\theta_i) := \max\left(f_3(\theta_i), f_5(\theta_i), f_7(\theta_i)\right)$$

At any point θ_i on the circle, the value of $f_{max}(\theta_i)$ tells us the maximum possible area our measurement can have ~~tot~~ while still being guaranteed to fit ~~inse~~ inside at least one of the required slice types.

Now the proof imposes its <u>central requirement</u>

$$\boxed{g_L(\theta_i) \le f_{max}(\theta_i)}$$

$g_L(\theta_i)$ is the actual area of our measurement (ε_{θ_i}). It's size depends on the number of shots.

By analysis, we get

$$f_{max}\left(\frac{8\pi}{21}\right) = \frac{\pi}{21}.$$

$$g_L\left[\frac{8\pi}{21}\right] \le \frac{\pi}{21},$$

$$g_{L^*}\left(\frac{8\pi}{21}\right) = \frac{\pi}{21}$$

$$\Rightarrow L^* = \sin^{-1}\left[\frac{1}{2}\sqrt{1-2\sin\left(\frac{\pi}{14}\right)}\right]$$

Finally,

$$N_{oracle} = \sum_{i=1}^{t} N_i k_i = \sum_{i=1}^{t} N_i \frac{K_i - 2}{4}$$

$N_i k_i \rightarrow$ Cost of single round i.

$K_i = 4k_i + 2$, $\boxed{k_i = \frac{K_i - 2}{4}}$

$$N_{oracle} \leq \frac{N_{max}}{2}\left[1 - t + \frac{K_1}{2}\sum_{i=1}^{t-2}\prod_{j=1}^{i}\right] + \frac{N_{K_t}}{2}$$

$$< \frac{N_{max}}{2}\left[1 - t + \sum_{i=1}^{t-2}\gamma^{T-t+1+i} + \frac{L_{max}}{2\varepsilon}\right]$$

$$< \frac{N_{max}}{2}\left[1 - t + \gamma^{T-t+1}\frac{\gamma^{t-1} - \gamma}{\gamma - 1} + \frac{L_{max}}{2\varepsilon}\right]$$

$$< \frac{N_{max}}{2}\left[\gamma^{T-2}\frac{\gamma^2}{\gamma - 1} + \frac{L_{max}}{2\varepsilon} + 1 - t - \frac{\gamma^{T-t+2}}{\gamma - 1}\right]$$

$$N_{oracle} < \frac{N_{max}}{2}\left[\gamma^{T-2}\frac{\gamma^2}{\gamma - 1} + \frac{L_{max}}{2\varepsilon} + 1 - t - \frac{\gamma^{T-t+2}}{\gamma - 1}\right] <$$

$$\frac{N_{max}}{2}\left[\frac{L_{max}}{2\varepsilon}\frac{\gamma^2}{\gamma - 1} + \frac{L_{max}}{2\varepsilon} + 3 - t - \frac{\gamma^{T-t+2}}{\gamma - 1}\right]$$

$$\boxed{N_{oracle} < N_{max} \frac{L_{max}}{4\varepsilon}\left[1+\frac{\gamma^2}{\gamma-1}\right]}$$

$$\boxed{N_{oracle} < \frac{1}{\varepsilon} \times \frac{L_{max}}{2\sin^4[L_{max}]}\left[1+\frac{\gamma^2}{\gamma-1}\right] \times \log\left[\frac{2}{\alpha}\log_\gamma\left(\frac{\gamma^2 L_{max}}{2\varepsilon}\right)\right]}$$

$\gamma = 2$ minimum growth rate

$L_{max} = L^*$

$L^* < \frac{11\pi}{90}$ ~~from~~ $\cancel{\boxed{B26}}$

$\sin^4[L_{max}] \rightarrow$ from L^* in $\boxed{B26}$

$$L^* = \sin^{-1}\left[\frac{1}{2}\sqrt{1-2\sin\left(\frac{\pi}{14}\right)}\right]$$

$$\sin[L^*] = \frac{1}{2}\sqrt{1-2\sin\left[\frac{\pi}{14}\right]}$$

$$\sin^2[L^*] = \frac{1}{4}\left[1-2\sin\left[\frac{\pi}{14}\right]\right]$$

$$\sin^4[L^*] = \left[\frac{1}{4} - \frac{1}{2}\sin\left[\frac{\pi}{14}\right]\right]^2$$

$$N_{oracle} < \frac{1}{\varepsilon} \times \frac{44\pi}{9\left[1-2\sin\left[\frac{\pi}{14}\right]\right]^2} \log\left[\frac{2}{\alpha}\log_2\left[\frac{\pi}{4\varepsilon}\right]\right]$$

$$\boxed{N_{oracle} < \frac{50}{\varepsilon}\log\left[\frac{2}{\alpha}\log\left[\frac{\pi}{4\varepsilon}\right]\right]}$$ Hence Proved