

# Randomized Quicksort and its Variants

Analysis and Benchmark Report

## Abstract

We study three instrumented QuickSort implementations (vanilla, smaller-subtree, and cutoff-16), compare them against Merge Sort and the C++ standard library’s `std::sort`, and relate empirical results to classical time and space bounds. Benchmarks cover synthetic and real-world (NIFTY 1-minute) integer datasets, recording wall-clock time and swap counts. All code lives in a single C++17 benchmark driver; plots are produced by accompanying Python scripts.

## 1 Introduction

QuickSort is a comparison sort that partitions an array around a pivot, recursively sorting the resulting subarrays. It is prized for its in-place memory footprint, cache-friendly sequential scans, and strong expected-time guarantees under randomized pivot selection. Practical libraries often combine QuickSort with other strategies (e.g., insertion sort on small segments, heap sort fallbacks), motivating a careful look at variants and their constant factors.

## 2 Algorithms and Complexity

### 2.1 QuickSort procedure (conceptual)

On an input array  $A[lo..hi]$ :

1. Choose a pivot index uniformly at random from  $[lo, hi]$ ; let  $p$  be its value.
2. Partition  $A[lo..hi]$  in one linear pass, maintaining a store index  $s$ . Scan  $i$  from  $lo$  to  $hi - 1$ ; whenever  $A[i] < p$ , swap  $A[i]$  with  $A[s]$  and increment  $s$ . Finally swap  $A[s]$  with the pivot. The pivot ends at index  $s$ ; elements left of  $s$  are  $< p$ ; elements right of  $s$  are  $\geq p$ .
3. Recursively sort the left subarray  $[lo, s - 1]$  and the right subarray  $[s + 1, hi]$  in any fixed order (or via tail recursion elimination where noted).

Only comparisons and swaps inside the partition pass plus the recursive calls contribute to running time; no auxiliary arrays are used.

### 2.2 Vanilla QuickSort

The pivot is sampled uniformly from the current subarray; after partitioning, both sides are recursed on.

**Expected time (full derivation).** Let  $T(n)$  be the expected number of primitive operations (or comparisons) on an array of size  $n$ . Let  $c > 0$  be the partition-pass constant. Conditioning on the pivot rank  $k$  (with  $k$  elements on the left,  $n - 1 - k$  on the right) and applying the law of total expectation:

$$\begin{aligned} T(n) &= \frac{1}{n} \sum_{k=0}^{n-1} \left( T(k) + T(n-1-k) + cn \right) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} T(k) + \frac{1}{n} \sum_{k=0}^{n-1} T(n-1-k) + \frac{1}{n} \sum_{k=0}^{n-1} cn \\ &= \frac{1}{n} \sum_{k=0}^{n-1} T(k) + \frac{1}{n} \sum_{j=0}^{n-1} T(j) + cn \quad (\text{rename } j = n-1-k) \\ &= \frac{2}{n} \sum_{j=0}^{n-1} T(j) + cn. \end{aligned}$$

Define the prefix sum  $S(n) = \sum_{j=0}^n T(j)$ . Then for  $n \geq 1$ :

$$\begin{aligned} nT(n) &= 2S(n-1) + cn^2, \\ S(n) &= S(n-1) + T(n). \end{aligned}$$

Multiply the second equation by  $n$  and substitute  $T(n) = S(n) - S(n-1)$  into the first:

$$\begin{aligned} n(S(n) - S(n-1)) &= 2S(n-1) + cn^2, \\ nS(n) - nS(n-1) &= 2S(n-1) + cn^2, \\ nS(n) &= (n+2)S(n-1) + cn^2, \\ \frac{S(n)}{n+1} &= \frac{S(n-1)}{n} + \frac{cn}{n+1}. \end{aligned}$$

Unrolling this telescoping form from  $n$  down to 1 and noting  $S(0) = T(0) = \Theta(1)$ :

$$\begin{aligned} \frac{S(n)}{n+1} &= \frac{S(0)}{1} + c \sum_{i=1}^n \frac{i}{i+1} \\ &= \Theta(1) + c \sum_{i=1}^n \left( 1 - \frac{1}{i+1} \right) \\ &= \Theta(1) + c \left( n - \sum_{i=1}^n \frac{1}{i+1} \right) \\ &= \Theta(1) + c(n - (H_{n+1} - 1)), \end{aligned}$$

where  $H_m = \sum_{j=1}^m 1/j$  is the  $m$ -th harmonic number. Multiplying by  $n+1$  and using  $T(n) = S(n) - S(n-1)$ , we obtain

$$T(n) \leq 2(n+1)H_n - 4n + O(n),$$

so  $T(n) = \Theta(n \log n)$  in expectation because  $H_n = \Theta(\log n)$ .

**Worst and best cases.** Worst-case pivot ranks yield  $T(n) = T(n-1) + cn = \Theta(n^2)$ . Perfectly balanced splits give  $T(n) = 2T(n/2) + cn = \Theta(n \log n)$ . Stack depth is  $O(n)$  in the worst case.

## 2.3 QuickSort with smaller-subtree recursion

After partitioning, the algorithm recurses on the smaller side and iterates on the larger. The running-time recurrence matches vanilla QuickSort, so the same bounds hold:  $\Theta(n \log n)$  expected,  $\Theta(n^2)$  worst case. Stack depth satisfies

$$D(n) \leq 1 + D\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

giving  $D(n) = O(\log n)$  for all inputs.

**Stack depth.** We prove  $D(n) \leq \lfloor \log_2 n \rfloor + 1$  by induction on  $n \geq 1$ . Base:  $n = 1$  gives  $D(1) = 1$ . Inductive step: assume the claim holds for all sizes  $< n$ . The smaller side has size at most  $\lfloor (n-1)/2 \rfloor$ , so

$$D(n) \leq 1 + D\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) \leq 1 + \left\lfloor \log_2 \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor + 1 \leq \lfloor \log_2 n \rfloor + 1.$$

Thus depth grows logarithmically regardless of pivot outcomes.

**Expected time.** The pivot rank is still uniform, so the conditional recurrence is identical to the vanilla case:

$$T_{\text{small}}(n) = \frac{1}{n} \sum_{k=0}^{n-1} (T_{\text{small}}(k) + T_{\text{small}}(n-1-k)) + cn.$$

Define  $S_{\text{small}}(n) = \sum_{j=0}^n T_{\text{small}}(j)$ . Then for  $n \geq 1$ :

$$\begin{aligned} n T_{\text{small}}(n) &= 2 S_{\text{small}}(n-1) + cn^2, \\ S_{\text{small}}(n) &= S_{\text{small}}(n-1) + T_{\text{small}}(n). \end{aligned}$$

Substitute  $T_{\text{small}}(n) = S_{\text{small}}(n) - S_{\text{small}}(n-1)$ :

$$\begin{aligned} n(S_{\text{small}}(n) - S_{\text{small}}(n-1)) &= 2S_{\text{small}}(n-1) + cn^2, \\ nS_{\text{small}}(n) &= (n+2)S_{\text{small}}(n-1) + cn^2, \\ \frac{S_{\text{small}}(n)}{n+1} &= \frac{S_{\text{small}}(n-1)}{n} + \frac{cn}{n+1}. \end{aligned}$$

Unrolling from  $n$  to 1 gives

$$\frac{S_{\text{small}}(n)}{n+1} = \Theta(1) + c \sum_{i=1}^n \frac{i}{i+1} = \Theta(1) + c(n - (H_{n+1} - 1)),$$

so  $T_{\text{small}}(n) = S_{\text{small}}(n) - S_{\text{small}}(n-1) = \Theta(n \log n)$ , matching the vanilla expectation.

## 2.4 QuickSort with cutoff 16

When a subarray has size  $\leq 16$ , the implementation immediately applies insertion sort and stops recursing on that branch. Partitioning proceeds as in A1. Costs consist of:

- Partitioning down to size 16: still  $\Theta(n \log n)$ .
- Insertion sort on many tiny segments: each element participates in at most one small-segment insertion pass, contributing  $O(n)$ .

Thus the overall bound remains  $\Theta(n \log n)$  expected,  $\Theta(n^2)$  worst case, but constant factors drop because insertion sort is efficient on very small arrays. Stack depth follows A1 when the tail-recursive strategy is used.

**Expected time.** Let  $T_c(n)$  be the expected time with cutoff  $c = 16$ . For  $n > c$ ,

$$T_c(n) = \frac{1}{n} \sum_{k=0}^{n-1} (T_c(k) + T_c(n-1-k)) + cn,$$

identical to the vanilla QuickSort recurrence for the partitioning portion. For  $n \leq c$ ,  $T_c(n) = O(c^2)$  due to insertion sort. By induction on  $n$ ,

$$T_c(n) \leq T_{\text{vanilla}}(n) + O(n),$$

where  $T_{\text{vanilla}}(n)$  is the solution to the vanilla recurrence. Inductive step: if  $n > c$ ,

$$T_c(n) \leq \frac{2}{n} \sum_{j=0}^{n-1} (T_{\text{vanilla}}(j) + O(j)) + cn = T_{\text{vanilla}}(n) + O(n),$$

using the same prefix-sum manipulation as before and the inductive hypothesis on subproblem sizes  $< n$ . If  $n \leq c$ ,  $T_c(n) = O(c^2) = O(n)$ . Therefore  $T_c(n) = \Theta(n \log n)$  in expectation.

**Stack depth.** Because the algorithm recurses only on the smaller side and iterates on the larger, the depth recurrence matches the smaller-subtree case:

$$D_c(n) \leq 1 + D_c\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

with base  $D_c(n) = 1$  for  $n \leq c$ . The same induction as before yields  $D_c(n) \leq \lfloor \log_2 n \rfloor + 1$ . Worst-case pivot choices still yield  $T_c(n) = \Theta(n^2)$  time, but stack depth remains  $O(\log n)$  because the larger side is never recursed.

## 2.5 Merge Sort

The top-down implementation recurses on halves and merges with a temporary buffer of size  $n$ . The recurrence  $T(n) = 2T(n/2) + cn$  yields  $\Theta(n \log n)$  in all cases; extra space is  $\Theta(n)$  for the buffer.

## 2.6 std::sort

Library `std::sort` is an introspective hybrid (QuickSort + heap sort fallback + insertion sort on small partitions), guaranteeing  $O(n \log n)$  worst-case time while remaining in-place. It serves as a realistic production baseline.

## 2.7 Summary of Bounds

Algorithm	Best	Expected	Worst (time / stack)
QuickSort (vanilla)	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n^2)$ time, $O(n)$ stack
QuickSort (smaller-subtree)	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n^2)$ time, $O(\log n)$ stack
QuickSort (cutoff 16)	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n^2)$ time, $O(\log n)$ stack
Merge Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$ , $O(n)$ buffer
<code>std::sort</code>	$\Theta(n \log n)$	$\Theta(n \log n)$	$O(n \log n)$ (introsort)

Table 1: Asymptotic behavior of evaluated algorithms.

## 3 Experimental Workflow

The entire benchmark lives in `src/main.cpp`; a compiled binary `quicksort_bench` is driven by `scripts/run_benchmarks.py`.

### 3.1 Data Generation

- **Synthetic distributions:** deterministic given  $(n, \text{seed})$ . Sorted uses  $a_i = i$ ; almost-sorted applies  $k = \lfloor 0.05n \rfloor$  random swaps; uniform draws from  $[-10^9, 10^9]$ ; normal draws from  $\mathcal{N}(0, 1000^2)$  with clamping to  $[-10^9, 10^9]$ .
- **Stock data:** preprocessed Kaggle NIFTY 1-minute closes ( $10^6$  integers) stored in `data/nifty_1m_int_1M.txt`. The benchmark reads the first  $n$  values for the `stock` distribution.
- **Randomness:** all generators and pivot choices use `std::mt19937_64` seeded with `seed_base + rep`, where `seed_base` is deterministic per configuration (see `compute_seed_base` in the Python script).

### 3.2 Benchmark Driver

- **Invocation:** run `./quicksort_bench` with arguments `<algo_id>`, `<dist_id>`, `n`, `seed_base`, `reps`, `stock_path`.
- **Metrics:** each repetition prints `<time_ns> <swaps>`; swap count is  $-1$  for `std::sort`.
- **Sizes and reps:** `run_benchmarks.py` sweeps  $n = 2^{10}, \dots, 2^{20}$  (1024 to 1,048,576), plus the full stock file size 1,007,339; repetitions  $R = 5$ .
- **Outputs:** rows are aggregated into `results/raw_results.csv`.  
Plots produced by `plot_results.py` reside in `results/plots/` and are embedded below.

### 3.3 Implementation Notes

All implementations operate on `std::vector<int>` and use a Mersenne Twister (`std::mt19937_64`) seeded deterministically per run. The global counter `g_swap_count` tracks swaps for instrumented algorithms (QuickSort variants and Merge Sort); `std::sort` leaves it at  $-1$ .

## 4 Results

Figures 1–5 show mean wall-clock time (ms) versus  $n$  on log-scaled  $x$  and swap counts for Quick-Sort/Merge Sort variants.

Distribution: sorted

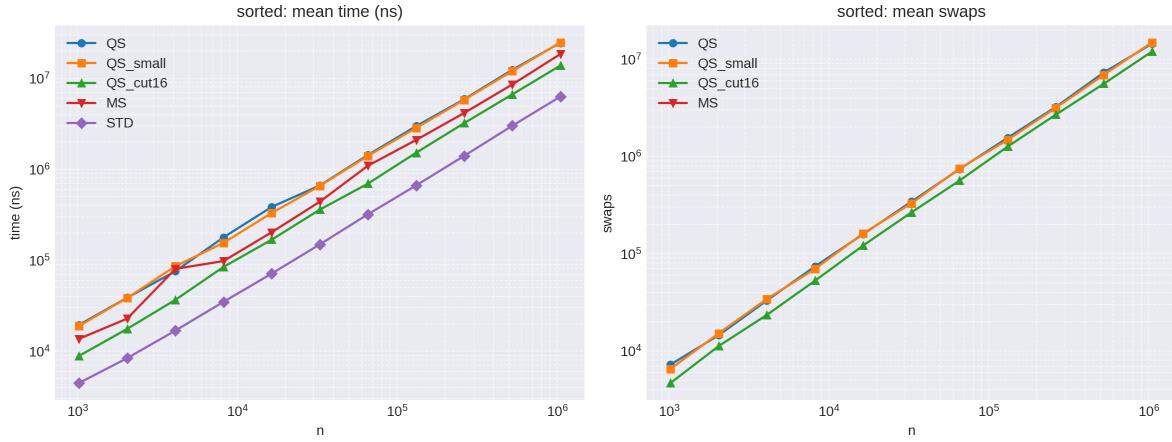


Figure 1: Sorted input.

Distribution: almost\_sorted

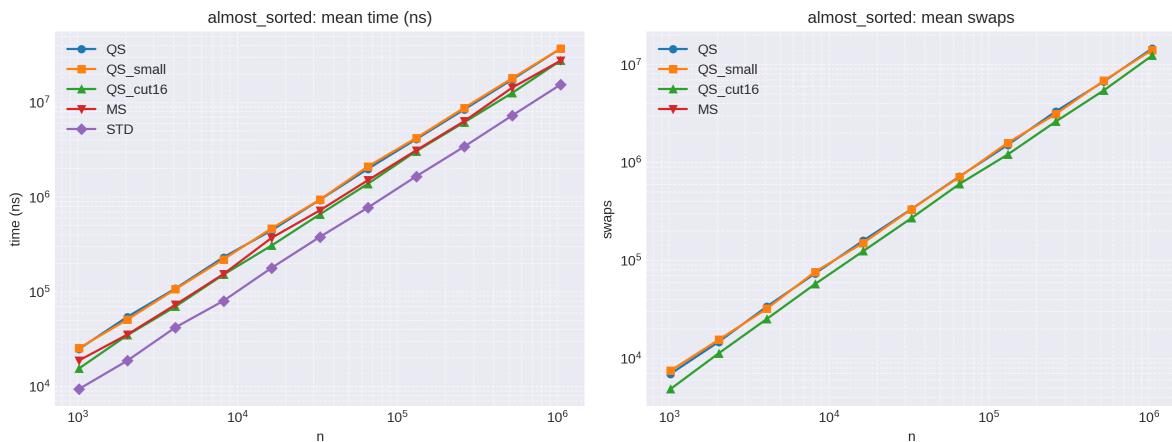


Figure 2: Almost-sorted input (5% swaps).

Distribution: uniform

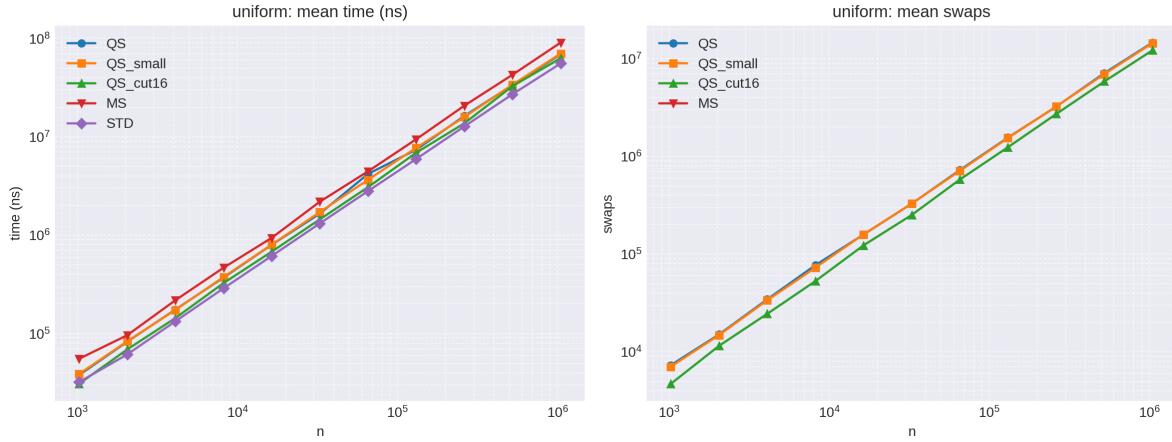


Figure 3: Uniform random input.

Distribution: normal

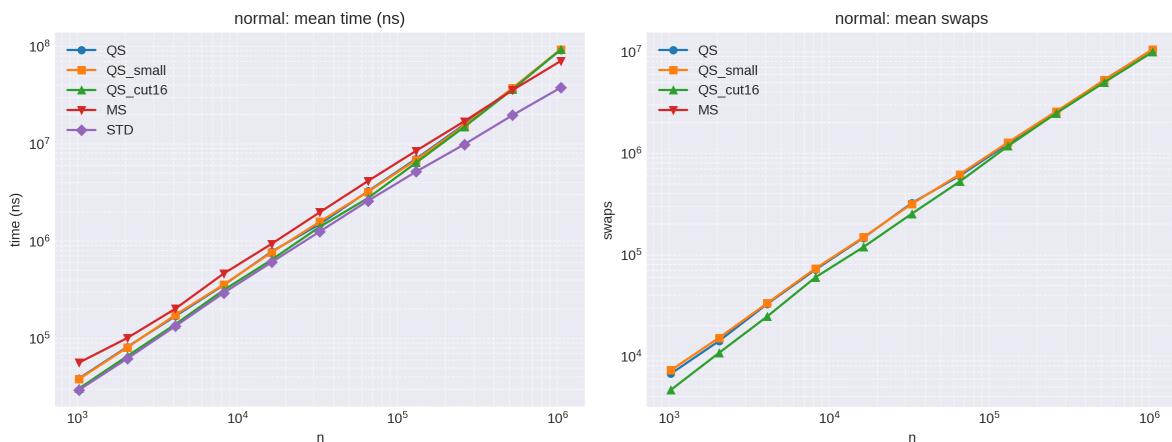


Figure 4: Normal distribution input.

Distribution: stock

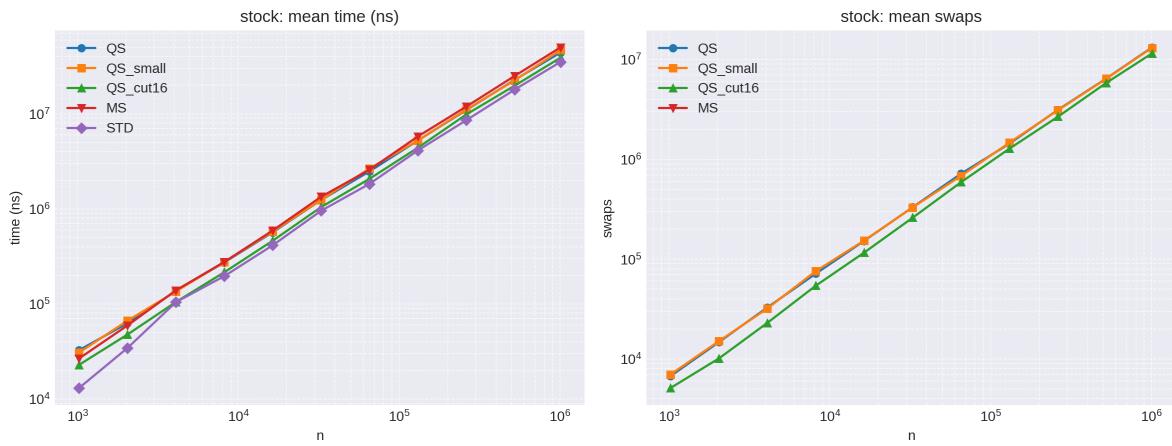


Figure 5: Real stock data input.

## 4.1 Observations on Figures

- **QS\_cut16 leads among custom variants:** Across all distributions, QS\_cut16 is the fastest of the instrumented QuickSorts. On uniform input with  $n = 2^{20}$ , QS\_cut16 averages  $\approx 62.9$  ms versus  $\approx 68.8$  ms for QS and  $\approx 70.1$  ms for QS\_small; swap counts drop from  $\approx 14.6M$  (QS) to  $\approx 12.2M$  (QS\_cut16).
- **Effect of tail recursion control:** QS\_small and QS have nearly identical times (e.g., sorted  $n = 2^{20}$ :  $\approx 24.9$  ms vs  $\approx 24.7$  ms) while QS\_small caps stack depth at  $O(\log n)$ .
- **QS vs Merge Sort:** Merge Sort trails the QuickSort variants on noisy data (70–90 ms at  $n = 2^{20}$  for uniform/normal) and requires  $\Theta(n)$  extra space; on structured data (sorted/almost-sorted) it closes the gap but still lags QS\_cut16.
- **Real stock mirrors synthetic:** On the full NIFTY slice ( $n = 1,007,339$ ), QS\_cut16 is the quickest custom option at  $\approx 38.9$  ms, ahead of QS ( $\approx 44.1$  ms) and QS\_small ( $\approx 46.9$  ms); patterns match the synthetic distributions.
- **Why QS\_cut16 excels:** Insertion sort on tiny partitions eliminates extra partition passes and reduces swaps; tail recursion keeps stack shallow. These micro-optimizations shrink constant factors without altering the  $\Theta(n \log n)$  expectation, yielding measurable speedups on both ordered and random inputs.

## 5 Why QuickSort Works Well in Practice

- **In-place:** Partitioning needs only  $O(1)$  auxiliary memory, which preserves cache locality and avoids large allocations compared to Merge Sort’s  $O(n)$  buffer.
- **Cache-friendly scans:** Sequential reads/writes during partitioning yield few cache misses, visible in the tight constants of A2 and `std::sort`.
- **Randomization:** Uniform pivot sampling delivers  $\Theta(n \log n)$  expected time on non-adversarial inputs—a good match for market and logging data that lack worst-case structure.
- **Hybridization payoff:** Adding insertion sort on tiny ranges (A2) trims swap counts and improves time without changing asymptotic bounds; introsort in `std::sort` pushes this idea further with heap-sort fallbacks.
- **Empirical alignment:** The real-stock results match the synthetic curves, reinforcing that QuickSort-style strategies handle both clustered (sorted/almost-sorted) and noisy (uniform/normal) data effectively.

## 6 Real-world Use Cases

- **C++ standard library:** `std::sort` is an introspective QuickSort hybrid used pervasively in systems code (cppreference).
- **Java primitive arrays:** `Arrays.sort` for primitives uses a dual-pivot QuickSort (Oracle JDK docs).
- **Database engines:** PostgreSQL uses a QuickSort-based tuplesort for many in-memory sorts before spilling to disk (source code).
- **Systems utilities:** The classic C `qsort` in libc remains a QuickSort-style routine, making the algorithm a default choice for general-purpose array sorting.

## 7 Conclusion

QuickSort’s combination of in-place operation, cache-friendly access, and strong expected bounds makes it a natural first choice for large in-memory datasets. Our experiments confirm that modest engineering—tail recursion elimination and tiny-segment insertion sort—improves constant factors without altering complexity. Production-grade hybrids such as `std::sort` remain the performance ceiling, but tuned QuickSort variants deliver predictable, efficient behavior on both synthetic and real financial data.