Linear Algebra

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Assignment 5

1)

- $N_T = \{\vec{\alpha} \in V \mid T\vec{\alpha} = \vec{0}\}$ denotes the null space of T.
- $R_T = \{T\vec{\alpha} \in W \mid \vec{\alpha} \in V\}$ denotes the range of T.
- Need to prove: $\dim N_T + \dim R_T = \dim V$.
- Let $\{\vec{\alpha_1}, \dots, \vec{\alpha_k}\}$ denote an arbitrary basis of $\cancel{N_T}$.
- $\exists \{\vec{\alpha}_{k+1}, \dots, \vec{\alpha_n}\}$ s.t $\{\vec{\alpha_1}, \dots, \vec{\alpha_n}\}$ is a basis of V.
- We will show that $\{\vec{\alpha}_{k+1}, \dots, \vec{\alpha_n}\}$ is a basis of R_T
- Consider $\vec{\beta} \in R_T$. $\vec{\beta} = T\vec{\alpha}$ for some $\vec{\alpha} \in V$.

  $\vec{\alpha} = c_1\vec{\alpha_1} + c_2\vec{\alpha_2} + \dots + c_n\vec{\alpha_n}$ (can write in this form because $\{\vec{\alpha_1}, \dots, \vec{\alpha_n}\}$ is a basis of V.)
- $T\vec{\alpha} = T(c_1\vec{\alpha_1}) + \dots + T(c_n\vec{\alpha_n})$

  $= c_1(T\vec{\alpha_1}) + \dots + c_n(T\vec{\alpha_n})$
- Thus, the set $\{T\vec{\alpha_1}, \dots, T\vec{\alpha_n}\}$ ~~is a basis of R_T~~ spans R_T.
- $T\vec{\alpha_i} = \vec{0}$ for $1 \le i \le k$, so the set $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha_n}\}$ spans R_T

- Need to show that $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha}_n\}$ is linearly independent.
- SFC, $\exists c_{k+1}, \dots, c_n \in F$ s.t $c_{k+1}(T\vec{\alpha}_{k+1}) + \dots + c_n(T\vec{\alpha}_n) = \vec{0}$ and $\exists k+1 \leq i \leq n$ s.t $c_i \neq 0$.
- $T(c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n) = \vec{0}$
- This implies that $c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n \in N_T$, so it can be written as a linear combination of $\{\vec{\alpha}_1, \dots, \vec{\alpha}_k\}$, because $\{\vec{\alpha}_1, \dots, \vec{\alpha}_k\}$ is a basis of N_T.
- $\exists c_1, \dots, c_k \in F$ s.t:

$$c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n = c_1\vec{\alpha}_1 + \dots + c_k\vec{\alpha}_k$$

$$\Rightarrow -c_1\vec{\alpha}_1 + \dots + -c_k\vec{\alpha}_k + c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n = \vec{0}$$

- However, the set $\{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$ is linearly independent, so the only sol^n to the equation above is

$$\cancel{c_1 = c_2 = \dots} \quad c_1 = \dots = c_n = 0; \cancel{\text{Contradiction}}$$

$$\Rightarrow c_{k+1} = \dots = c_n = 0; \text{ Contradiction.}$$

- Thus, the set $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha}_n\}$ is a basis of R_T.
- $\dim N_T = k$, $\dim R_T = n-k$, $\dim V = n$.
- $\dim N_T + \dim R_T = k + (n-k) = n = \dim V$

$$\Rightarrow \dim N_T + \dim R_T = \dim V$$

- QED. //

2)a). Need to show that:

$$(T+U)(c\vec{\alpha}+\vec{\beta}) = c(T+U)(\vec{\alpha}) + (T+U)(\vec{\beta}),$$

where $\vec{\alpha}, \vec{\beta} \in V$, $c \in F$.

- $(T+U)(c\vec{\alpha}+\vec{\beta}) = T(c\vec{\alpha}+\vec{\beta}) + U(c\vec{\alpha}+\vec{\beta})$

$= c(T\vec{\alpha}) + T\vec{\beta} + c(U\vec{\alpha}) + U\vec{\beta}$

$= c(T\vec{\alpha} + U\vec{\alpha}) + (T\vec{\beta} + U\vec{\beta})$

$= c(T+U)(\vec{\alpha}) + (T+U)(\vec{\beta})$

- QED. //

b). Need to show that:

$$(cT)(d\vec{\alpha}+\vec{\beta}) = d[(cT)(\vec{\alpha})] + (cT)(\vec{\beta}),$$

where $\vec{\alpha}, \vec{\beta} \in V$, $c, d \in F$

- $(cT)(d\vec{\alpha}+\vec{\beta}) = c[T(d\vec{\alpha}+\vec{\beta})]$

$= c[d(T\vec{\alpha}) + T\vec{\beta}] = c[d(T\vec{\alpha})] + c(T\vec{\beta})$

$= d[c(T\vec{\alpha})] + c(T\vec{\beta}) = d[(cT)(\vec{\alpha})] + (cT)\vec{\beta}$

c). Need to show that $L(V, W)$ with addition and scalar multiplication as defined in (a) and (b) form a vector space over field F.

- We will show that all axioms of a vector space hold true.

(i) Closure under addition:

- Proven in (a)

(ii) Additive identity:

- Define the zero transformation O ~~[illegible]~~ as: $O(\vec{\alpha}) = \vec{0}$, where $\vec{\alpha} \in V$.
- $(T+O)(\vec{\alpha}) = T(\vec{\alpha}) + O(\vec{\alpha}) = T(\vec{\alpha})$

(iii) Additive inverse:

- Define $(-T)(\vec{\alpha}) = -T(\vec{\alpha})$
- $(T+(-T))(\vec{\alpha}) = T(\vec{\alpha}) - T(\vec{\alpha}) = \vec{0}$

(iv) Commutativity of addition:

- $(T+U)\vec{\alpha} = T(\vec{\alpha}) + U(\vec{\alpha}) = U(\vec{\alpha}) + T(\vec{\alpha}) = (U+T)(\vec{\alpha})$

(v) Associativity of addition:

- $((S+T)+U)(\vec{\alpha}) = (S+T)(\vec{\alpha}) + U(\vec{\alpha}) = S(\vec{\alpha}) + T(\vec{\alpha}) + U(\vec{\alpha})$
$= S(\vec{\alpha}) + ((T+U)(\vec{\alpha})) = (S+(T+U))(\vec{\alpha})$

(vi) Closure under scalar multiplication:

- Proven in (b)

(vii) Distributivity of scalar multiplication over vector addition:

- $(c(T+U))(\vec{\alpha}) = c[(T+U)(\vec{\alpha})] = c[T(\vec{\alpha}) + U(\vec{\alpha})]$

$= c[T(\vec{\alpha})] + c[U(\vec{\alpha})] = (cT)(\vec{\alpha}) + (cU)(\vec{\alpha})$

$= (cT + cU)(\vec{\alpha})$

(viii) Distributivity of scalar multiplication over field addition:

- $[(c+d)T](\vec{\alpha}) = (c+d)[T(\vec{\alpha})] = c[T(\vec{\alpha})] + d[T(\vec{\alpha})]$

$= (cT)(\vec{\alpha}) + (dT)(\vec{\alpha}) = (cT + dT)(\vec{\alpha})$

(ix) Associativity of scalar multiplication:

- $[c(dT)](\vec{\alpha}) = c[(dT)(\vec{\alpha})] = c[d[T(\vec{\alpha})]]$

$\cancel{= d[c[T(\vec{\alpha})]] = d[(cT)(\vec{\alpha})] =}$

$= (cd)[T(\vec{\alpha})] \cancel{= [(cd)T](\alpha} = [(cd)T](\vec{\alpha})$

(x) Multiplicative identity:

- $(1T)(\vec{\alpha}) = 1 \cdot T(\vec{\alpha}) = T(\vec{\alpha})$

- All axioms of a vector space hold true.
- QED. //

3). • Let $B = \{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$ denote an arbitrary basis of V.

• Let $B' = \{\vec{\beta}_1, \dots, \vec{\beta}_m\}$ denote an arbitrary basis of W.

• Let $L(V, W)$ denote the set of all linear transformations from V to W.

• $\forall (p, q)$ where $1 \le p \le m$, $1 \le q \le n$, we define a linear transformation $E^{p,q}$ s.t.:

$$E^{p,q}(\vec{\alpha}_i) = \begin{cases} \vec{0}, & i \ne q \\ \vec{\beta}_p, & i = q \end{cases}$$

$$= \delta_{iq} \vec{\beta}_p$$

NOTE:

$$\delta_{iq} = \begin{cases} 0, & i \ne q \\ 1, & i = q \end{cases}$$

• Note that such a linear transformation $E^{p,q}$ exists for all (p, q) and is always unique due to $\{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$ being a basis of V.

• We will now show that the set of all $E^{p,q}$ forms a basis of $L(V, W)$.

• Let $T: V \to W$ be an arbitrary linear transformation from V to W.

- ~~Need to show that $\exists A_{pq}$ where $1 \leq p \leq m, 1 \leq q \leq n$ s.t.:~~

~~$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}$$~~ ~~(trying to show that T can be written as a linear combination of the linear transformations)~~

- $\exists A_{1i}, \ldots, A_{mi} \in F$ s.t.:

$$T\vec{\alpha_i} = \sum_{p=1}^{m} A_{pi} \vec{\beta_p}$$ (because $\{\vec{\beta_1}, \ldots, \vec{\beta_m}\}$ is a basis of W)

- Consider $$U\vec{\alpha_i} = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\vec{\alpha_i})$$ (linear combination of linear transformation)

$$= \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} \delta_{iq} \vec{\beta_p} = \sum_{p=1}^{m} A_{pi} \vec{\beta_p}$$

- Thus, $U\vec{\alpha_i} = T\vec{\alpha_i}$ holds true $\forall 1 \leq i \leq n$.
- Consider an arbitrary $\vec{\alpha} \in V$.

$$\vec{\alpha} = c_1\vec{\alpha_1} + \ldots + c_n\vec{\alpha_n}$$

- $T\vec{\alpha} = T(c_1\vec{\alpha_1} + \ldots + c_n\vec{\alpha_n}) = c_1(T\vec{\alpha_1}) + \ldots + c_n(T\vec{\alpha_n})$
- $U\vec{\alpha} = U(c_1\vec{\alpha_1} + \ldots + c_n\vec{\alpha_n}) = c_1(U\vec{\alpha_1}) + \ldots + c_n(U\vec{\alpha_n})$
- $T\vec{\alpha_i} = U\vec{\alpha_i}\ \forall 1 \leq i \leq n$, so $T\vec{\alpha} = U\vec{\alpha} \Rightarrow T = U$
- We have managed to write an arbitrary linear transformation T as a linear combination of $E^{p,q}$ $\Rightarrow E^{p,q}$ spans the set of all linear transformations $T: V \to W$.

- Need to show that the set of all $E^{p,q}$ is linearly independent.
- $\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} = 0 \Rightarrow A_{pq} = 0 \ \forall\, 1 \le p \le m, 1 \le q \le n$

This 0 denotes the 0 transformation.

- $\left(\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \right)(\vec{\alpha}) = \vec{0} \ \forall \vec{\alpha} \in V$

$\Rightarrow \left(\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \right)(\vec{\alpha_i}) = \vec{0} \ \forall\, 1 \le i \le n.$

- $\sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\vec{\alpha_i}) = \sum_{p=1}^{m} A_{pi} \overrightarrow{\beta_p}$ (as shown before)

- $\{\overrightarrow{\beta_1}, \dots, \overrightarrow{\beta_m}\}$ is linearly independent, so

$\sum_{p=1}^{m} A_{pi} \overrightarrow{\beta_p} = \vec{0} \Rightarrow A_{pi} = 0 \ \forall\, 1 \le p \le m, 1 \le i \le n.$

- Thus, the set of all $E^{p,q}$ is linearly independent.
- Thus, $E^{p,q}$ is a basis of the set of all linear transformations from V to W denoted by $L(V, W)$
- Size of the set of all $E^{p,q}$ = dim V × dim W.
- Thus, dim $L(V, W)$ = dim V × dim W.
- QED. //