Linear Algebra

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Assignment 5

1)

- $N_T = \{\vec{\alpha} \in V \mid T\vec{\alpha} = \vec{0}\}$ denotes the null space of T.
- $R_T = \{T\vec{\alpha} \in W \mid \vec{\alpha} \in V\}$ denotes the range of T.
- Need to prove: $\dim N_T + \dim R_T = \dim V$.
- Let $\{\vec{\alpha_1}, \dots, \vec{\alpha_k}\}$ denote an arbitrary basis of N_T.
- $\exists \{\vec{\alpha}_{k+1}, \dots, \vec{\alpha_n}\}$ s.t $\{\vec{\alpha_1}, \dots, \vec{\alpha_n}\}$ is a basis of V.
- We will show that $\{\vec{\alpha}_{k+1}, \dots, \vec{\alpha_n}\}$ is a basis of R_T.
- Consider $\vec{\beta} \in R_T$. $\vec{\beta} = T\vec{\alpha}$ for some $\vec{\alpha} \in V$.

  $\vec{\alpha} = c_1\vec{\alpha_1} + c_2\vec{\alpha_2} + \dots + c_n\vec{\alpha_n}$ (can write in this form because $\{\vec{\alpha_1}, \dots, \vec{\alpha_n}\}$ is a basis of V.)
- $T\vec{\alpha} = T(c_1\vec{\alpha_1}) + \dots + T(c_n\vec{\alpha_n})$

  $= c_1(T\vec{\alpha_1}) + \dots + c_n(T\vec{\alpha_n})$
- Thus, the set $\{T\vec{\alpha_1}, \dots, T\vec{\alpha_n}\}$ ~~is a basis of R_T~~ spans R_T.
- $T\vec{\alpha_i} = \vec{0}$ for $1 \le i \le k$, so the set $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha_n}\}$ spans R_T

- Need to show that $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha}_n\}$ is linearly independent.
- SFC, $\exists c_{k+1}, \dots, c_n \in F$ s.t $c_{k+1}(T\vec{\alpha}_{k+1}) + \dots + c_n(T\vec{\alpha}_n) = \vec{0}$ and $\exists k+1 \le i \le n$ s.t $c_i \ne 0$.
- $T(c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n) = \vec{0}$
- This implies that $c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n \in N_T$, so it can be written as a linear combination of $\{\vec{\alpha}_1, \dots, \vec{\alpha}_k\}$, because $\{\vec{\alpha}_1, \dots, \vec{\alpha}_k\}$ is a basis of N_T.
- $\exists c_1, \dots, c_k \in F$ s.t:

$$c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n = c_1\vec{\alpha}_1 + \dots + c_k\vec{\alpha}_k$$

$$\Rightarrow -c_1\vec{\alpha}_1 + \dots + -c_k\vec{\alpha}_k + c_{k+1}\vec{\alpha}_{k+1} + \dots + c_n\vec{\alpha}_n = \vec{0}$$

- However, the set $\{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$ is linearly independent, so the only sol^n to the equation above is

$$\cancel{c_1 = c_2 = \dots} \quad c_1 = \dots = c_n = 0; \ \cancel{\text{Contradiction}}$$

$$\Rightarrow c_{k+1} = \dots = c_n = 0; \text{ Contradiction.}$$

- Thus, the set $\{T\vec{\alpha}_{k+1}, \dots, T\vec{\alpha}_n\}$ is a basis of R_T.
- $\dim N_T = k$, $\dim R_T = n-k$, $\dim V = n$.
- $\dim N_T + \dim R_T = k + (n-k) = n = \dim V$

$$\Rightarrow \dim N_T + \dim R_T = \dim V$$

- QED. //