LA Assignment 5

February 21, 2025

1. **(Rank-Nullity Theorem)** Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V to a vector space W. We define $rank(T) := dim(im(T))$ and $nullity(T) := dim(ker(T))$. Show that

$$rank(T) + nullity(T) = dim(V)$$

2. Let V, W be two vector spaces defined over a field F. Define the set $\mathcal{L}(V, W)$ as the set of all linear transformations from V to W. Let, $\forall \vec{\alpha} \in V$, the vector addition of elements $T, U \in \mathcal{L}(V, W)$ be defined as

$$(T + U)\vec{\alpha} := T\vec{\alpha} + U\vec{\alpha}$$

and the scalar multiplication of an element c of F with an element T of $\mathcal{L}(V, W)$ be defined as

$$cT(\vec{\alpha}) := c(T\vec{\alpha})$$

Show that these two operations give a linear transformation in $\mathcal{L}(V, W)$ as their output. Furthermore show that $\mathcal{L}(V, W)$ along with these two operations forms a vector space over F.

3. Let V, W be finite-dimensional vector spaces and let $\mathcal{L}(V, W)$ be defined as in the previous question. Show that

$$dim(\mathcal{L}(V, W)) = dim(V) \times dim(W)$$