Bivariate Extreme Value Distributions and Its Applications

A Project report submitted in partial fulfillment of the requirements for the Degree of M.Sc. (Statistics) with specialization in Industrial Statistics

Submitted by

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(2025 - 2026)

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CERTIFICATE

This is to certify that **Mr. Patil Vinay Sanjay**, **Mr. Patil Sushil Arun** and **Miss. Nikam Nisha Narendra** students of M.Sc.(Statistics) with specialization in Industrial Statistics, at Kavayitri Bahinabai Chaudhari, North Maharashtra University, Jalgaon have successfully completed their project work entitled “Bivariate Extreme Value Distributions and its Application” as a part of M.Sc. (Statistics) program under my guidance and supervision during the academic year 2025-2026.

Asst. Prof. Manoj C. Patil

(Project Guide)

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Signature

Abstract

This project investigates the theoretical development and practical implementation of Bivariate Extreme Value Distributions (BEVDs), which play a fundamental role in the statistical modeling of joint extreme phenomena. While univariate EVT provides tools for analyzing marginal extremes, real-world events frequently involve simultaneous extremes across multiple variables. Accordingly, this work examines the structure, dependence mechanisms, and inferential methods associated with BEVDs, with a particular emphasis on Type A (Gumbel Type) and Type B (Logistic) models as originally proposed by Gumbel.

The study rigorously formulates these distributions via extreme-value copulas, highlighting the importance of the Pickands dependence function in characterizing tail dependence. Both parametric estimation techniques including Maximum Likelihood Estimation and Method of Moments and non-parametric approaches such as empirical copula estimation, kernel-based methods, and Pickands estimators are evaluated. Goodness-of-fit procedures for BEVDs are also reviewed.

Applications to hydrological and environmental data, such as rainfall–river discharge maxima and wind–wave extremes, demonstrate the practical value of BEVDs in capturing joint tail behaviour. Implementation using R software further validates the modeling framework. The findings underscore the necessity of multivariate extreme modeling for effective risk assessment in fields such as hydrology, climatology, finance, and structural engineering.

Acknowledgement

We would like to express our sincere gratitude to Prof. K. K. Kamalja, Head, Department of Statistics, Kavayitri Bahinabai Chaudhari, North Maharashtra University, Jalgaon, for her unwavering support and invaluable comments. Her generous attitude and guidance have been instrumental in making our work much easier and more meaningful.

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Contents

List of Figures

List of Tables

Chapter 1

About Extreme Value Theory (EVT)

1.1 Introduction

Extreme Value Theory (EVT) is a branch of statistics that deals with the stochastic behavior of extreme values observations i.e. maximum or minimum values within datasets. Generally the statistical models focus on average behavior, outcome or etc. of random variables. But EVT is focuses on rare oe extreme events or values that lies in the tail of probability distribution.

The study of extremes is essential in many practical fields, such as hydrology (flood levels and rainfall), meteorology (temperature extremes), finance (market crashes and risk management), and engineering (structural safety and material strength). These extreme events often have a profound impact on systems and society, making it necessary to model and predict their occurrence accurately.

1.2 Importance and Motivation of EVT

In many real-world phenomena, it is not the average or typical observations that cause concern, but rather the rare and extreme events that can lead to severe consequences. For example, a single episode of extreme rainfall may result in devastating floods, or an extreme financial loss may trigger a market collapse. Hence, understanding and modeling the behavior of such extreme occurrences is of great practical and scientific importance.

The traditional statistical models, which primarily focus on the central portion of the data, often underestimate the probability of rare events. Extreme Value Theory (EVT) provides the tools necessary to model and analyze the tail behavior of distributions, allowing for more accurate estimation of the likelihood and magnitude of such extremes.

The motivation for studying EVT arises from its wide range of applications in risk management and decision-making. In hydrology, EVT is used to estimate return levels and return periods of extreme floods or rainfall. In meteorology, it helps to analyze extremes in temperature and wind speed. In engineering, EVT is essential for designing structures that can withstand rare stress levels, while in finance and insurance, it assists in quantifying extreme losses and risk exposure.

Moreover, the increasing frequency of climate-related extremes due to environmental changes has further emphasized the need for reliable models of extreme events. EVT provides a solid theoretical foundation for such modeling, helping policymakers and researchers develop mitigation strategies and safety measures based on probabilistic assessments rather than empirical guesswork.

1.3 About Univariate EVT

Univariate EVT focuses on extremes in a single variable, real-world phenomena often involve multiple factors that interact simultaneously. For example, river discharge and rainfall jointly influence flooding events, and different financial assets may experience extreme losses at the same time. This interdependence leads to the necessity of modeling multivariate extremes, specifically through approaches like the Bivariate Extreme Value Distribution.

Univariate extreme value theory (EVT) Kotz and Nadarajah [2000] is fundamental to understanding the probabilistic behavior of extremes in stochastic processes. This review synthesizes classical results, modern estimation techniques, and prominent applications, providing a comprehensive resource for researchers in statistics, engineering, environmental science, and finance.

Some Univariate EVT

- **Type 1**: Lai and Balakrishnan [2009]
  This is also known as the Gumbel Distribution, and its c.d.f. and p.d.f. are:

$$F(x) = \exp(-e^{-x}), \quad -\infty < x < \infty \tag{1}$$

$$f(x) = e^{-x}\exp(-e^{-x}), \quad -\infty < x < \infty \tag{3}$$

- **Type 2**:
  This is also known as the Fréchet Distribution. For $\alpha > 0$, the c.d.f. is given by

$$F(x) = \exp(-x^{-\alpha}), \quad x \geq 0 \tag{3}$$

  Note that if X has the Fréchet distribution in the above c.d.f., then $Y = X^{-\alpha}$ has an exponential distribution.

- **Type 3**:
  This is related to the Weibull distribution, and its c.d.f. is given by

$$F(x) = \exp\left[-(-x)^{\alpha}\right], \quad x \leq 0 \tag{4}$$

Applications of Univariate EVT

Extensive applications illustrate EVT's broad relevance:

- **Environmental Extremes**: Modeling rare rainfall, wind speeds, and floods, enabling prediction of return periods for natural disasters in various regions. Ever and Silvanos [2024]

- **Finance and Risk**: Estimating probabilities of market crashes and maximal drawdowns.Smith [2003]
- **Engineering**: Evaluating minimal or maximal stress, dike height, and load-sharing system failure thresholds Steven et al. [2002].
- **Human Lifespan**: Analysis suggesting real upper endpoints for expected extreme human ages Einmahl et al. [2019].

These cases demonstrate the practical power and flexibility of EVT in univariate contexts.

1.4 Generalized Extreme Value (GEV) Distributions

Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed (iid) random variables with cumulative distribution function F. The behavior of the maximum $M_n = \max\{X_1, \ldots, X_n\}$ is characterized by the so-called Extremal Types Theorem, establishing that appropriately normalized maxima converge to one of three possible limit distributions:

$$\begin{aligned}
\text{Type I (Gumbel):} \quad & \Lambda(x) = \exp(-\exp(-x)),\ x \in \mathbb{R} \\
\text{Type II (Fréchet):} \quad & \Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ \exp(-x^{-\alpha}) & x \geq 0 \end{cases} \text{ for some } \alpha > 0 \\
\text{Type III (Weibull):} \quad & \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}) & x < 0 \\ 1 & x \geq 0 \end{cases} \text{ for some } \alpha > 0
\end{aligned}$$

These are unified by the **Generalized Extreme Value** (**GEV**) distribution, with location parameter μ, scale $\sigma > 0$, and shape ξ:

$$G(x) = \exp\left(-\left[1 + \xi \frac{x-\mu}{\sigma}\right]^{-1/\xi}\right), \quad \text{defined for } 1 + \xi \frac{x-\mu}{\sigma} > 0$$

1.5 Objectives of the Study

The main objectives of this project are:

1. Introduce and explain the theoretical foundations of the Bi-variate Extreme Value Distribution.
2. To study the dependence structure between extreme events using appropriate statistical methods.
3. Explore estimation techniques for parameters in bivariate extreme models.
4. Apply these methods to real-life data to assess joint extreme risks.
5. Provide insights into the limitations, assumptions, and practical relevance of bivariate models in various fields.

1.6 Scope and Applications

This study is focused on the theoretical development and practical implementation of bivariate extreme value models. The scope includes:

- Deriving conditions for joint tail behavior.
- Exploring different families of bivariate distributions such as Gumbel, logistic, and other parametric models.
- Discussing dependence measures like copulas and spectral representations.
- Applying the models to real datasets, particularly from hydrology or finance.

Applications of the study span multiple domains, including but not limited to:

- Environmental risk analysis.
- Financial risk management.
- Insurance and disaster modelling.

The findings will provide statisticians, researchers, and practitioners with tools to better understand and manage multivariate extremes in practical settings.

Chapter 2

About Bivariate Extreme Value distribution (BEVD)

2.1 Bivariate Extreme Value Models

Bivariate Extreme Value Models (BEVMs) extend univariate extreme value theory to analyse the joint behaviour of extreme events in two dimensions. These models are essential in applications such as hydrology, finance, environmental sciences, and engineering, where understanding the dependence between extreme events is critical.

Applications and Challenges

BEVMs are applied to fields such as flood risk assessment, financial contagion, and climate extremes. However, challenges remain in model selection, estimating tail dependence, and computational complexity, especially in high dimensions Coles et al. [2001], Beirlant et al. [2004a], de Haan and Ferreira [2006].

Table 2.1: Example dataset for Bivariate Extreme Value Models applied to flood risk assessment and climate extremes.

Year	**River Discharge (m^3/s)**	**Precipitation (mm/day)**	**Remarks**
2015	3200	95	Heavy rainfall leading to high river discharge
2016	2800	60	Moderate flood event
2017	4500	110	Severe flood with extreme rainfall
2018	1500	40	Normal conditions
2019	3900	85	Heavy rainfall, significant river rise
2020	5000	120	Extreme event – record rainfall and discharge
2021	3100	70	Flood warning issued
2022	4200	100	River overflow, emergency response

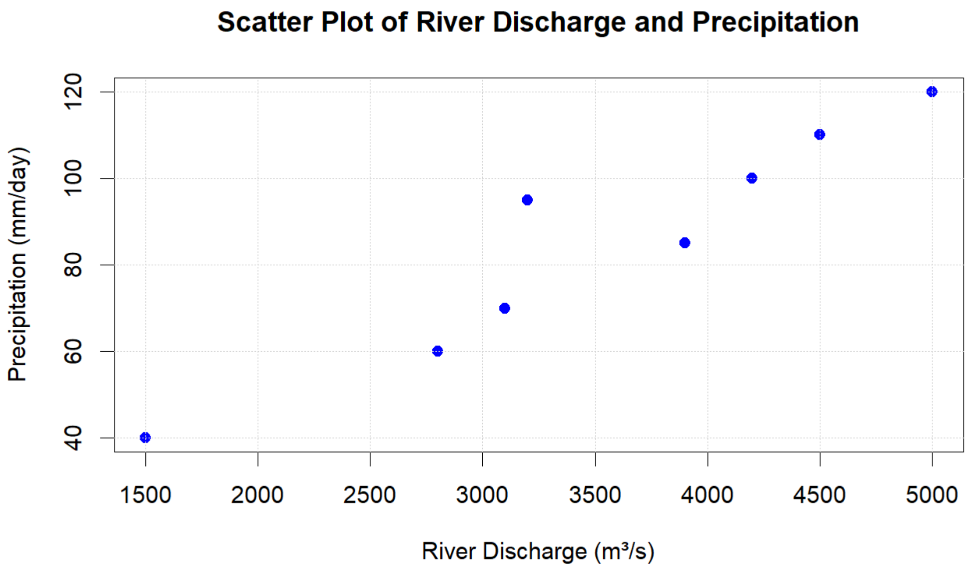

Figure 2.1: Scatter Plot of River Discharge and Precipitation

2.2 Bivariate Extreme Value Distribution

Often interest is in extreme values of more than one variable (Two-variable).e.g.

1. EI-NINO caused by extreme values of sea temperature and air pressure.
2. Floods: Caused by extreme values of rainfall and winds.
3. Hurricanes: caused by extreme values of sea temperature, rainfall and wind speed.

Definition

Lai and Balakrishnan [2009] Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be n pairs of independent bivariate random variables with

$$X_{\max} = \max(X_1, \ldots, X_n), \qquad Y_{\max} = \max(Y_1, \ldots, Y_n).$$

It is possible to find linear transformations.

$$X_{(n)} = a_n X_{\max} + b_n \quad (a_n > 0), \qquad Y_{(n)} = c_n Y_{\max} + d_n \quad (c_n > 0),$$

such that $X_{(n)}$ (and $Y_{(n)}$) is one of the three types of extreme-value distributions as $n \to \infty$. Then, the limiting joint distribution of $X_{(n)}$ and $Y_{(n)}$ is a bivariate extreme-value distribution.

A general definition of a bivariate extreme-value distribution can be presented through a copula Krieger and Pickands III [1981]. Let (X, Y) have a joint bivariate extreme-value distribution with marginals $F(x)$ and $G(y)$; then the associated copula is given by

$$C(u, v) = \Pr\{F(X) \leq u,\, G(Y) \leq v\} = \exp\left(\log(uv)\, A\left(\frac{\log u}{\log(uv)}\right)\right) \tag{2.1}$$

for all $0 \leq u, v \leq 1$, in terms of a convex function A defined on $[0, 1]$ in such a way that

$$\max(t, 1-t) \leq A(t) \leq 1, \qquad \text{for all } 0 \leq t \leq 1.$$

A is known as the *dependence function*, and we will discuss its properties in Section 2.4.

About Copula

A copula is a mathematical function that joins (or couples) multivariate distribution functions to their one-dimensional marginal distribution function.
Nelsen [2006] It covers definitions, types, properties, and applications in depth.
Definition: A *copula* is a multivariate distribution function with uniform marginals on the interval $[0,1]$. According to Sklar's Theorem, any multivariate joint distribution $F(x_1, x_2, \ldots, x_n)$ can be expressed as:

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

where $F_i(x_i)$ are the marginal cumulative distribution functions and C is the copula function.

Types of Copulas:

- **Gaussian Copula:**

  $$C_\rho(u,v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

  where Φ is the standard normal CDF and Φ_ρ is the bivariate normal CDF with correlation ρ.

- **Student-t Copula:**

  $$C_{\rho,\nu}(u,v) = t_{\rho,\nu}(t_\nu^{-1}(u), t_\nu^{-1}(v))$$

  where t_ν is the univariate Student-t CDF and $t_{\rho,\nu}$ is the bivariate Student-t CDF with correlation ρ and degrees of freedom ν.

- **Clayton Copula:**

  $$C_\theta(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 0$$

  Exhibits strong lower tail dependence.

- **Gumbel Copula:**

  $$C_\theta(u,v) = \exp\left[-\left((-\ln u)^\theta + (-\ln v)^\theta\right)^{1/\theta}\right], \quad \theta \geq 1$$

  Exhibits strong upper tail dependence.

- **Frank Copula:**

  $$C_\theta(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right), \quad \theta \neq 0$$

  Symmetric with no tail dependence.

From the above five types of copulas Gaussian Copulas and Student-t Copula are konwn as Eliptical Copulas and remaing three are known as Archimedian Copulas.

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2.2.1 General Properties of BEVD

Lai and Balakrishnan [2009]

- Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a random sample from a bivariate population with a joint distribution whose copula is C. Let $X_{(n)} = \max\{X_i\}$ and $Y_{(n)} = \max\{Y_i\}$. Then the copula that corresponds to $X_{(n)}$ and $Y_{(n)}$ is

$$C_{(n)}(u,v) = C^n(u^{\frac{1}{n}}, v^{\frac{1}{n}}).$$

  A copula C_* is an extreme-value copula if there exists a copula C such that.

$$C_*(u,v) = \lim_{n\to\infty} C^n(u^{\frac{1}{n}}, v^{\frac{1}{n}});$$

  see [Nelsen, 2006], p. 97.

- Shi [2003] has considered a transformation of variables from the copula above with

$$S = -\log(UV)A\left(\frac{\log U}{\log(UV)}\right), \quad T = \frac{\log U}{\log(UV)}.$$

It has been shown that S and T are "essentially" independent; this leads to some stochastic representation for the bivariate extreme-value distribution.

- In many bivariate distributions (such as the bivariate normal), $X_{\max}$ and $Y_{\max}$ may be asymptotically independent (as the sample size tends to infinity) even if X and Y are correlated. This is so if $\bar{H}(xy)/\{1-H(x,y)\} \to 0$ as $x, y \to \infty$. This result is due to Geffroy [1958, 1959](Let (X, Y) have a bivariate extreme-value distribution. Then, X and Y are PQD.

- Let $H_1(x, y)$ and $H_2(x, y)$ be two bivariate extreme-value distributions, so their weighted geometric mean is

$$[H_1(x, y)]^{\beta}[H_2(x, y)]^{1-\beta}, \quad 0 \leq \beta \leq 1,$$

see Gumbel and Goldstein [1964]

2.2.2 Real-life Example

Joint Modelling of Rainfall and River Discharge
Consider the problem of studying flood risk in a river basin. Two extreme variables of interest are:

- X_i: the annual maximum daily rainfall (mm) in year i,
- Y_i: the annual maximum river discharge (m^3/s) in year i.

We observe these as pairs (X_i, Y_i), $i = 1, 2, \ldots, n$.

Maxima. Define the sample maxima across n years:

$$X_{\max} = \max(X_1, X_2, \ldots, X_n), \qquad Y_{\max} = \max(Y_1, Y_2, \ldots, Y_n).$$

Normalization. To avoid degenerate limits, we apply linear transformations:

$$X_{(n)} = a_n X_{\max} + b_n, \quad a_n > 0, \qquad Y_{(n)} = c_n Y_{\max} + d_n, \quad c_n > 0.$$

Here,

- a_n, c_n: scale parameters (normalise growth rate of extremes),
- b_n, d_n: location parameters (stabilise the maxima).

Limit Distribution. The joint distribution of normalised maxima converges to a bivariate extreme-value distribution:

$$\lim_{n\to\infty} P\left(\frac{M_{n,1}-b_n}{a_n} \le x,\ \frac{M_{n,2}-d_n}{c_n} \le y\right) = G(x,y),$$

where $M_{n,1} = \max(X_1,\dots,X_n)$ and $M_{n,2} = \max(Y_1,\dots,Y_n)$, and $G(x,y)$ is a non-degenerate BEVD.

Copula Representation. Dependence between rainfall and discharge extremes can be expressed via the extreme-value copula:

$$C(u,v) = \Pr(F(X) \le u,\ G(Y) \le v) = \exp(\log(uv)\, A(\frac{\log u}{\log(uv)})),$$

where

- $F(x)$: marginal distribution of rainfall maxima,
- $G(y)$: marginal distribution of discharge maxima,
- $A(t)$: the dependence function, describing how strongly rainfall and Discharge extremes occur together.

Interpretation of Parameters.

- a_n, c_n: adjust for different units (mm vs. m^3/s) and scale of extremes,
- b_n, d_n: shift values so maxima do not diverge,
- $F(x), G(y)$: marginal extreme-value laws (e.g., rainfall may follow a Gumbel distribution, discharge may follow a Fréchet distribution),
- $A(t)$: encodes tail dependence. If rainfall extremes almost always cause discharge extremes, $A(t)$ is close to its upper bound; if weakly related, closer to independence.

- **Left Side– Joint Extremes of Rainfall and Discharge:**
  - This scatter plot shows the joint occurrences of annual maximum rainfall (in mm) and annual maximum discharge (in cubic meters per second).
  - Each blue point represents one year's extreme observation of rainfall and the corresponding river discharge.
  - The spread of points indicates how the extremes of rainfall and discharge co-vary over the observed period.

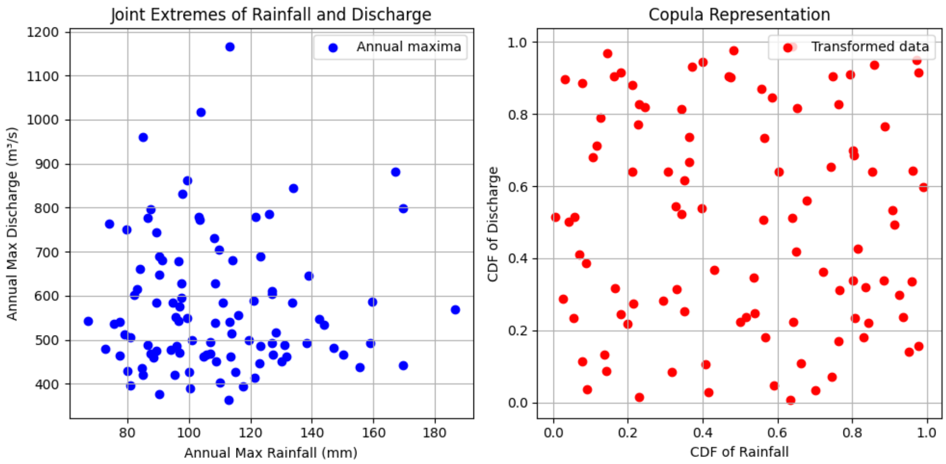

Figure 2.2: Scatter plot Jointly extremes and Copula Representation of Rainfall and Discharge

- A clustering of points at lower values suggests that extreme discharges are often associated with moderate rainfall, but some points with very high discharge values reflect possible flood events driven by intense precipitation.

- **Right Side – Copula Representation:**
  - The transformed data (in red) shows the cumulative distribution function (CDF) values of rainfall and discharge, effectively mapping the joint behaviour into a uniform space.
  - This representation, known as a copula plot, isolates the dependence structure between the two variables, independent of their marginal distributions.
  - The lack of clustering along the edges suggests that the dependence is moderate, with joint extremes not being overly concentrated.
  - Copula models are especially useful in hydrology and risk assessment, as they allow for flexible modelling of dependence structures beyond linear correlations.

Practical Use. Hydrologists use this model to estimate the joint return period of "100-year floods," where extreme rainfall and extreme river discharge occur together.

2.3 Types of BEVDs in General form

Extreme Value Theory can be extended from the univariate case to the bivariate or multivariate case to model the joint behavior of two or more extreme variables, such as rainfall and river discharge. The bivariate extreme value distributions (BEVDs) capture both marginal behavior (each variable individually follows an extreme value law) and the dependence structure between them. The general form of a BEVD can be expressed as:

$$H(x,y) = \exp[-V(x,y)], \quad x,y \in \mathbb{R}, \tag{2.2}$$

where $V(x,y)$ is called the *dependence function* or *exponent measure* that determines the type of dependence between extremes. Different specifications of $V(x,y)$ lead to different families of BEVDs Gumbel [1960], Tiago de Oliveira [1962], Tawn [1988], Coles [2001].

If we see the in the book Lai and Balakrishnan [2009] chapter 12 section 12.3 there are give Genral forms of both Type A and Type B BEVD described by Gumbel (1958,1965).

1. Type A distribution

This distribution is known as Gumbel model

$$H(x,y) = F(x)G(y)\exp\{-\theta\left[\frac{1}{\log F(x)} + \frac{1}{\log G(y)}\right]^{-1}\}, \qquad 1 \le \theta < 1. \tag{2.3}$$

The corresponding copula is

$$C(u,v) = uv\exp\left(\frac{-\theta}{\log(uv)}(\log u \log v)\right). \tag{2.4}$$

Here we can see that Gumbel type copula (Logistic Copula) is used.

2. Type B Distribution

This distribution is known as the logistic model.

$$H(x,y) = \exp\{-\left[(-\log F(x))^m + (-\log G(y))^m\right]^{1/m}\}, \qquad m \ge 1. \tag{2.5}$$

The copula that corresponds to the Type B extreme-value distribution is

$$C(u,v) = \exp\{-\left[(-\log u)^m + (-\log v)^m\right]^{1/m}\}. \tag{2.6}$$

It is an extreme value copula since $C(u^k, v^k) = C^k(u,v)$; in fact, it is the only Archimedian copula that is also an extreme-value copula. It is also known as Gumbel-Hougaard copula.

3. Type C : Negative Logistic Distribution

The Type C model, proposed by Tiago de Oliveira [1962] describes negative dependence between the variables—when one variable takes on an extreme high value, the other tends to take on a less extreme one. It is also known as Negative Logistic Model.

Cumulative Distribution Function (CDF)

$$H(x,y) = \exp\{-\left[(e^{-x})^{-\alpha} + (e^{-y})^{-\alpha} - 1\right]^{-1/\alpha}\}, \quad \alpha > 0. \tag{2.7}$$

Probability Density Function (PDF)

$$\begin{aligned} h(x,y) = H(x,y)\, e^{-(x+y)} \left[(e^{-x})^{-\alpha} + (e^{-y})^{-\alpha} - 1\right]^{-(1+2/\alpha)} \\ \times \left[(1+\alpha)(e^{-x}e^{-y})^{-\alpha} - \alpha(e^{-x})^{-2\alpha} - \alpha(e^{-y})^{-2\alpha}\right]. \end{aligned} \tag{2.8}$$

We can see here the Negative Logistic Copula is used here.

4. Type D : Marshall–Olkin Distribution

The Type D model, introduced by Marshall and Olkin [1967] allows for asymmetric dependence, meaning one variable may dominate in extreme behavior.

Cumulative Distribution Function (CDF)

$$H(x,y) = \exp\left[-\left(e^{-x} + e^{-y} - \lambda e^{-\max(x,y)}\right)\right], \quad 0 \le \lambda \le 1. \tag{2.9}$$

Probability Density Function (PDF)

For $x < y$:

$$h(x,y) = H(x,y)e^{-(x+y)}(1 - \lambda e^{x-y}), \tag{2.10}$$

and for $x > y$:

$$h(x,y) = H(x,y)e^{-(x+y)}(1 - \lambda e^{y-x}). \tag{2.11}$$

5. Type E : Hüsler–Reiss Distribution

The Hüsler–Reiss model Hüsler and Reiss [1989] arises as a limit of certain elliptical dependence structures and is suitable for modeling weak dependence.

Cumulative Distribution Function (CDF)

$$H(x,y) = \exp\{-\frac{1}{x}\Phi\left(\frac{\lambda}{2} + \frac{1}{\lambda}\ln\frac{y}{x}\right) - \frac{1}{y}\Phi\left(\frac{\lambda}{2} + \frac{1}{\lambda}\ln\frac{x}{y}\right)\}, \tag{2.12}$$

where $\Phi(\,)$ denotes the standard normal distribution function and $\lambda > 0$ is the dependence parameter.

2.4 Dependence Function

In the context of Extreme Value Theory (EVT), especially for bivariate extreme value distributions (BEVD), the dependence function A(t) plays a central role in modeling the joint behavior of extremes. Here's how it works:
Definition: The dependence function A(t), also called the **Pickands Dependence Function**, is a convex function defined on the interval $t \in [0,1]$ that satisfies: $\max(t, 1-t) \leq A(t) \leq 1$
Purpose: It governs the copula of the BEVD, which describes how the marginal extreme value distributions are linked together.

Pickands Dependence Function

Lai and Balakrishnan [2009]
Here

$$\vec{H}(x,y) = \exp\left[-(x+y)A\left(\frac{y}{x+y}\right)\right], \quad x,y>0 \tag{2.13}$$

where the function $A(w)$ is given by:

$$A(w) = \int_0^1 \max[(1-w)q, w(1-q)]\frac{dB}{dq}dq \tag{2.14}$$

In which B is positive function on $[0,1]$.In order to have unit exponential mariginals, we need

$$1 = \int_0^1 q\frac{dB}{dq}dq = \int_0^1 (1-q)\frac{dB}{dq} \tag{2.15}$$

[To deduce this, we successively set $x=0$ and $y=0$ in (2.13). We then find that $A(0)$ and $A(1)$ must both be 1 and put these values into (2.15).] It follows from (2.15) that $\frac{1}{2}B$ is the distribution function of random variable with mean $\frac{1}{2}$. We call A the dependence function of (X,Y), in accordance with usage of Pickands [1981] and Tawn [1988].
This condition implies that B is the distribution function of a random variable with mean $\frac{1}{2}$. Specifically, setting $x=0$ and $y=0$ in equation (2.13) leads to the requirement that $A(0) = A(1) = 1$, which follows from equation (2.15).

Properties of Pickands Dependence Function

1. $A(0) = A(1) = 1$.

2. $max(w, 1-w) \leq A(w) \leq 1, 0 \leq w \leq 1$.

3. $A(w) = 1$ implies that X and Y are independent. $A(w) = max(w, 1 - w)$ implies that X and Y are equal, i.e. $Pr(X = Y) = 1$.

4. A is convex; i.e. $A[\lambda x + (1 - \lambda) y] \leq \lambda A(x) + (1 - \lambda) A(y)$.

5. If Ai are dependence function, so is $\sum_{i=1}^{n} \alpha_i A_i$ where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} = 1$.

Summary Table

Table 2.2: Summary of Common Types of Bivariate Extreme Value Distributions

Type	Distribution Name	Parameter Range	Dependence Nature
A	Gumbel Type	$0 < \theta < 1$	Positive Dependence
B	Logistic	$0 < \alpha \leq 1$	Symmetric Positive
C	Negative Logistic	$\alpha > 0$	Negative Dependence
D	Marshall–Olkin	$0 \leq \lambda \leq 1$	Asymmetric Dependence
E	Hüsler–Reiss	$\lambda > 0$	Weak/Asymptotic Dependence

These models provide flexible frameworks for describing the joint behavior of extreme events across multiple variables. They differ in the form of dependence they capture—ranging from perfect positive dependence to negative or asymmetric relationships—and have wide applications in hydrology, climatology, and finance Coles [2001], Tawn [1988], Gumbel [1960].

Out of these BEVD's, we are going to study only two types are given as follows:

- Type A: Gumbel Type Distribution
- Type B: Logistic Distribution

2.5 Core Methods for Modeling Extremes

Extreme Value Theory (EVT) uses two primary methods to analyze and model extreme observations from a dataset, both of which are foundational to Bivariate Extreme Value Distribution (BEVD) modeling .

2.5.1 Block Maxima Method (BMM)

The Block Maxima Method (BMM) is a widely used approach to extract and model the behavior of the largest (or smallest) values of a series of observations over fixed, non-overlapping blocks of time (for example, maximum daily rainfall per year) .

Consider a sequence of bivariate observations $\{(X_t, Y_t)\}_{t=1}^{n}$, which is partitioned into m blocks of equal size k (so that $n = m \times k$). The method then uses the componentwise maxima from each block, defined as

$$M_j = \left(\max_{t \in B_j} X_t, \max_{t \in B_j} Y_t \right), \quad j = 1, 2, \ldots, m,$$

where B_j denotes the index set of the j-th block .

This procedure reduces the dataset while focusing entirely on the extreme behavior of the variables, which is often of primary interest in risk assessment and design studies Coles et al. [2001].

Block Size Selection and Practical Considerations

The choice of block size k is a critical practical step in the BMM.

If k is too small, the extracted block maxima may not be truly "extreme", which can lead to poor tail estimation and high bias in parameter estimates . On the other hand, if k is too large, the number of blocks m becomes small, reducing the amount of data available for inference and increasing the variance of estimators .

In practice, the block size is chosen to balance this bias–variance trade-off, often supported by graphical diagnostics such as mean residual life plots and other exploratory tools .

Table 2.3: Wind and Wave Data (1991–2020)

Year	Wind Speed	Wave Height
1991	142.0	5.2
1992	128.4	4.7
1993	175.6	7.8
1994	110.9	3.9
1995	158.3	6.6
1996	135.2	5.1
1997	190.7	8.9
1998	162.0	6.9
1999	98.7	3.2
2000	150.5	6.0
2001	123.2	4.5
2002	180.1	8.1
2003	140.4	5.0
2004	129.9	4.6

Year	Wind Speed	Wave Height
2005	198.6	9.4
2006	168.7	7.2
2007	153.8	6.1
2008	116.3	4.0
2009	138.9	5.3
2010	185.2	8.4
2011	104.8	3.5
2012	160.6	6.8
2013	147.1	5.7
2014	192.0	9.1
2015	118.5	3.8
2016	170.9	7.5
2017	176.4	8.0
2018	125.8	4.9
2019	181.9	8.6
2020	155.0	6.2

The above dataset represents the annual block maxima of two correlated environmental variables. Applying the Block Maxima method, the marginal distributions for wind speed and wave height can be modeled using the Generalized Extreme Value (GEV) distribution, while their joint dependence can be characterized through the Type A BEVD. This modeling framework provides valuable insights into the likelihood of joint extreme events (e.g., simultaneous strong winds and high waves), which are critical for coastal risk assessment and structural safety evaluation.

2.5.2 Limiting Distribution: The Generalized Extreme Value (GEV)

Under suitable regularity conditions, the distribution of the normalized block maxima converges to a Generalized Extreme Value (GEV) distribution as the block size k increases Coles et al. [2001].

The GEV distribution $G(z)$ unifies the classical Type I (Gumbel), Type II (Fréchet), and Type III (Weibull) extreme value distributions through a common parametrization in terms of a location parameter μ, a scale parameter $\sigma > 0$, and a shape parameter ξ Fisher and Tippett [1928]. It is given by

$$G(z) = \exp\left(-\left[1+\xi\frac{z-\mu}{\sigma}\right]^{-1/\xi}\right), \quad \text{for } 1+\xi\frac{z-\mu}{\sigma} > 0$$

2.5.3 Peaks Over Threshold (POT) Approach

The Peaks Over Threshold (POT) method is an alternative to the BMM that considers all observations exceeding a suitably high threshold u Coles et al. [2001].

This approach focuses on the excesses over the threshold, defined as $Y_t = X_t - u$ for all observations with $X_t > u$.

Because it uses all threshold exceedances rather than only one maximum per block, the POT method often utilizes the available information more efficiently, especially when extreme events are rare Beirlant et al. [2004a].

Selection of the threshold u is crucial: it must be high enough for the limiting approximation to be valid, yet low enough to retain a sufficient number of exceedances for reliable statistical inference Coles et al. [2001].

2.5.4 Generalized Pareto Distribution (GPD)

Under appropriate conditions, the distribution of exceedances over a high threshold converges to the Generalized Pareto Distribution (GPD) as the threshold u increases. Pickands [1975].

The GPD $G(y)$ is defined for $y > 0$ by

$$G(y) = 1 - \left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi}, \quad \text{for } 1 + \frac{\xi y}{\beta} > 0$$

where ξ is the shape parameter and $\beta > 0$ is the scale parameter Balkema and de Haan [1974]. This family provides a flexible framework for modeling diverse tail behaviors encountered in practical extreme value applications Coles et al. [2001].

Real Life Application

The Peaks Over Threshold (POT) approach is used to model and analyze observations that exceed a pre-specified threshold. This method is particularly effective in applications where extreme events are rare but their frequency and magnitude are of concern.

Example: Insurance Claim Severity

In the insurance industry, large claim amounts arising from catastrophic events are of particular interest. By using the POT method, insurers can model claims exceeding a threshold amount, such as $50,000. The exceedances are then fitted to the Generalized Pareto Distribution (GPD) to assess tail risk and improve pricing strategies [Pickands, 1975, Balkema and de Haan, 1974, Coles et al., 2001].

Table 2.4: Exceedances Over Threshold in Insurance Claims

Observation No.	Claim Amount (in $1000)	Exceedance over Threshold (in $1000)
1	65	15
2	72	22
3	48	–
4	55	5
5	80	30
6	53	3
7	90	40
8	47	–
9	60	10
10	75	25

Interpretation: For a threshold of $50,000, the claims exceeding this value are modeled using the GPD. The exceedances range from $3,000 to $40,000 over the threshold. By fitting the GPD, insurers can better understand the frequency and severity of extreme claims, allowing for more informed decision-making in risk management and policy pricing.

2.6 Methods of Estimation

In the bivariate extreme value distribution there are several methods of estimation of parameters. These methods are divided into two types such as Parametric and Non-Parametric Methods of estimation. In parametric methods there are main two methods such as Maximum Likelihood Estimation (MLE) and Method of Moment Estimator (MOM). Whereas the non-parametric methods for BEVD are Kernal Density approach (KDE), Ranked Based Methods.

2.6.1 Parametric Methods

Estimation of parameters in BEVD by parametric methods we use two most popular techniques names as:

- Maximum likelihood Estimation (MLE)
- Method of Moments (MOM)

In this section, we provide a detailed derivation of both methods and discuss their applications.

2.6.2 Non-Parametric Methods

Non-parametric estimation techniques for bivariate extreme value distributions (BEVD) are used when the underlying distribution is unknown or difficult to model parametrically. These approaches rely on data-driven methods and make minimal assumptions.

1. Empirical Copula-Based Estimation

The copula approach separates the marginal distributions from the dependence structure. The empirical copula is defined by transforming the observations into ranks and using the uniform empirical distribution function.

Let (X_i, Y_i) for $i = 1, \dots, n$ be observations. The empirical distribution functions are:

$$\hat{F}_X(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \le x), \quad \hat{F}_Y(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(Y_i \le y), \tag{2.16}$$

where $\mathbf{1}(\cdot)$ is the indicator function.

The empirical copula is then given by:

$$\hat{C}(u, v) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left(\hat{F}_X(X_i) \le u, \hat{F}_Y(Y_i) \le v\right), \tag{2.17}$$

for $u, v \in [0, 1]$ Genest and Favre [2007].

2. Kernel Density Estimation (KDE)

Kernel Density Estimation is a smoothing technique used to estimate the joint pdf without assuming a parametric form:

$$\hat{f}(x, y) = \frac{1}{n h_x h_y}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_x}\right) K\left(\frac{y - Y_i}{h_y}\right), \tag{2.18}$$

where $K(\cdot)$ is a kernel function, and h_x, h_y are bandwidth parameters Silverman [1986].

KDE is particularly useful when modeling the joint density of extreme events, though it requires careful selection of bandwidths to balance bias and variance.

3. Non-Parametric Pickands Dependence Function Estimation

The Pickands dependence function $A(t)$ characterizes the dependence structure in BEVD. Non-parametric estimators of $A(t)$ can be obtained by:

$$\hat{A}(t) = \max\{\frac{\ln \hat{C}(1 - t, t)}{\ln(1 - t) + \ln t}, t, 1 - t\}, \tag{2.19}$$

where $\hat{C}(u, v)$ is the empirical copula Genest and Rivest [1993].

This estimator is used when the dependence structure is complex and cannot be adequately modeled by parametric families.

4. Rank-Based Methods

Rank correlation measures such as Spearman's rho and Kendall's tau are widely used to estimate dependence non-parametrically:

$$\hat{\rho} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}, \tag{2.20}$$

$$\hat{\tau} = \frac{2(P-Q)}{n(n-1)}, \tag{2.21}$$

where d_i is the difference in ranks, P is the number of concordant pairs, and Q is the number of discordant pairs.

These measures are robust to outliers and distributional assumptions and can be used as inputs in constructing non-parametric dependence models Nelsen [2007].

Real-Life Application:

Modeling Electricity Demand Based on Temperature Scenario: During peak summer months, electricity demand surges due to increased usage of cooling systems. Understanding the dependence between ambient temperature and electricity consumption is crucial for grid reliability and energy planning.

Simulation Setup:

- **Temperature** (T) is simulated uniformly between 30°C and 40°C to represent hot weather conditions.
- **Electricity Demand** (D) is modeled as a linear function of temperature with added Gaussian noise:

$$D = 4000 + 100 \cdot T + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, 50)$$

Copula-Based Analysis:

- **Empirical Copula:** Captures the joint behavior of (T, D), especially in the tails, which is vital for extreme event prediction.
- **Kernel Density Estimation (KDE):** Provides a smooth estimate of the joint distribution, helping visualize the likelihood of simultaneous high temperature and demand.

- **Pickands Estimator:** Evaluates tail dependence, useful for assessing risk during heatwaves.
- **Rank-Based Measures:** Spearman's ρ and Kendall's τ quantify monotonic dependence, aiding in forecasting and demand response strategies.

Real-World Relevance:

- **Grid Management:** Helps utilities anticipate peak loads and prevent outages.
- **Renewable Integration:** Supports planning for solar and wind energy, which are weather-sensitive.
- **Climate Resilience:** Assists policymakers in preparing for future energy needs under changing climate conditions.
- **Smart Pricing:** Enables dynamic pricing models to incentivize reduced consumption during peak hours.

2.6.3 Goodness-of Fit (GOF) for BEVD

Goodness-of-fit (GOF) assessment is a crucial step when modelling extremes because parameter estimates and tail-related risk measures are sensitive to model misspecification. For bivariate extreme-value distributions (BEVDs) a GOF procedure usually has two parts: (i) marginal diagnostics (fit of the univariate extreme-value/Gev/Generalized Pareto models for each margin) and (ii) dependence diagnostics for the max-stable / extreme-value dependence structure (often assessed via EV copulas or max-stable models). See Coles et al. [2001], Beirlant et al. [2004b] for general extreme-value methodology and Genest et al. [2009] for copula-based GOF methods.

1. Cramer–von Mises Test for Bivariate Extreme Value Copulas

The Cramesr–von Mises (CvM) test is one of the most widely used goodness-of-fit tests for BEVD, particularly for testing the Pickands dependence function. The test statistic measures the distance between a parametric estimate of the Pickands dependence function and its nonparametric estimator.

The test statistic is given by:

$$\mathrm{CvM} = n\int_0^1 \left[\hat{A}_n(t) - A(t)\right]^2 H(dt)$$

where

- $\hat{A}_n(t)$ is the nonparametric estimator of the Pickands dependence function,
- $A(t)$ is the parametric dependence function under the null hypothesis,
- $H(dt)$ is a suitable weight function,
- n is the sample size.

This test compares the empirical dependence function to the theoretical one, with smaller values of the statistic indicating better fit.

2. Anderson-Darling Test

The Anderson-Darling test provides an alternative approach for testing extreme value distributions with composite hypotheses, applicable to both marginal and joint distributions. The test statistic accounting for parameter estimation uncertainty is:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(x_{(i)}) + \ln\left(1 - F(x_{(n+1-i)})\right)\right]$$

Practical Considerations

- Goodness-of-fit tests should be complemented with graphical diagnostics such as Q-Q plots and P-P plots.
- Small sample sizes reduce the power of tests; consider multiple complementary methods.
- Tail dependence assessment is critical in extreme value applications.
- Software implementations in python include `evd`, `copula`, and `eva` packages.

See Lai and Balakrishnan [2009] section 12.4 **Classical BEVD with Gumbel Marginals** in this section three special types are considered Type A, Type B and type C all having Gumbel Marginals. Out of this these three types we are going to study Type A and type B Distributions. There C.D.F's and P.D.F's are given as follows:

1. Type A: Gumbel Type Distribution

Cumulative Distribution Function C.D.F

The joint distribution function is

$$H(x,y) = \exp\{-\left(e^{-x}+e^{-y}\right)+\theta\left(e^{x}+e^{y}\right)^{-1}\}, \qquad \theta \le 1 \tag{2.22}$$

which is an increasing function of θ.

joint density function (P.D.F.)

The joint density function is

$$\begin{aligned} h(x,y) = e^{-(x+y)}[1-\theta(e^{2x}+e^{2y})(e^{x}+e^{y})^{-2}+2\theta e^{2(x+y)}(e^{x}+e^{y})^{-3} \\ +\theta^{2}e^{2(x+y)}(e^{x}+e^{y})^{-4}]\exp[-e^{-x}-e^{-y}+\theta(e^{x}+e^{y})^{-1}]. \end{aligned} \tag{2.23}$$

2. Type B: Logistic Distribution

Lai and Balakrishnan [2009]The Type B is known as Logistic Model

Cumulative Distribution Function (CDF)

$$H(x,y) = \exp\{-\left[(e^{-mx}+e^{-my})^{m}\right]\}, m \ge 1. \tag{2.24}$$

Since

$$\lim_{m\to\infty}(e^{-mx}+e^{-my})^{1/m} = \max(e^{-x}, e^{-y}),$$

we obtain

$$\lim_{m\to\infty} H(x,y) = \min\left[\exp(-e^{-x}), \exp(-e^{-y})\right] = \min\left(F(x), G(y)\right).$$

It is clear that, for $m = 1$, X and Y are independent.

Probability Density Function (PDF)

$$\begin{aligned} h(x,y) = e^{-m(x+y)}[(e^{-mx}+e^{-my/})^{\frac{1}{m}-2}]\{m-1 \\ +(e^{-mx}+e^{-my})^{\frac{1}{m}}\}.\exp[-(e^{-mx}+e^{-my})^{\frac{1}{m}}] \end{aligned} \tag{2.25}$$

Chapter 3

Type A Distributions

Formula for the C.D.F

The joint distribution function is

$$H(x,y) = \exp\{-\left(e^{-x}+e^{-y}\right)+\theta\left(e^{x}+e^{y}\right)^{-1}\}, \qquad \theta \le 1 \tag{3.1}$$

which is an increasing function of θ.

Formula for joint density function

The joint density function is

$$\begin{aligned} h(x,y) = e^{-(x+y)}\,[1 - \theta(e^{2x}+e^{2y})(e^{x}+e^{y})^{-2} + 2\theta e^{2(x+y)}(e^{x}+e^{y})^{-3} \\ + \theta^2 e^{2(x+y)}(e^{x}+e^{y})^{-4}]\exp[-e^{-x}-e^{-y}+\theta(e^{x}+e^{y})^{-1}]. \end{aligned} \tag{3.2}$$

3.1 Derivation of the Joint PDF

The joint distribution function is given by

$$H(x,y) = \exp\{-(e^{-x}+e^{-y})+\theta\,(e^{x}+e^{y})^{-1}\}, \qquad \theta \le 1.$$

Let

$$A(x,y) = -(e^{-x}+e^{-y})+\theta\,(e^{x}+e^{y})^{-1}, \qquad H(x,y) = e^{A(x,y)}.$$

We use the formula

$$\frac{\partial^2}{\partial x \partial y} e^{A} = e^{A}(A_{xy}+A_x A_y).$$

First derivatives of $A(x,y)$

$$\begin{aligned} A_x &= \frac{\partial}{\partial x}\left(-e^{-x}+\theta(e^{x}+e^{y})^{-1}\right) \\ &= e^{-x}-\theta\, e^{x}(e^{x}+e^{y})^{-2}, \\ A_y &= e^{-y}-\theta\, e^{y}(e^{x}+e^{y})^{-2}. \end{aligned}$$

Mixed second derivative

$$A_{xy} = \frac{\partial}{\partial y}(A_x) = \frac{\partial}{\partial y}(-\theta e^x (e^x + e^y)^{-2}) = 2\theta \frac{e^{x+y}}{(e^x + e^y)^3}.$$

Now, Product $A_x A_y$

$$\begin{aligned} A_x A_y &= (e^{-x} - \theta e^x (e^x + e^y)^{-2})(e^{-y} - \theta e^y (e^x + e^y)^{-2}) \\ &= e^{-(x+y)} - \theta \frac{e^{-x} e^y + e^x e^{-y}}{(e^x + e^y)^2} + \theta^2 \frac{e^{x+y}}{(e^x + e^y)^4}. \end{aligned}$$

Notice that

$$e^{-x} e^y = e^{-(x+y)} e^{2y}, \qquad e^x e^{-y} = e^{-(x+y)} e^{2x}.$$

So

$$A_x A_y = e^{-(x+y)} \left[1 - \theta \frac{e^{2x} + e^{2y}}{(e^x + e^y)^2} + \theta^2 \frac{e^{2(x+y)}}{(e^x + e^y)^4} \right].$$

Combine them both $A_{xy} + A_x A_y$

$$A_{xy} + A_x A_y = e^{-(x+y)} \left[1 - \theta \frac{e^{2x} + e^{2y}}{(e^x + e^y)^2} + 2\theta \frac{e^{2(x+y)}}{(e^x + e^y)^3} + \theta^2 \frac{e^{2(x+y)}}{(e^x + e^y)^4} \right].$$

Now we get the Final joint pdf

$$\begin{aligned} h(x, y) &= \frac{\partial^2}{\partial x \partial y} H(x, y) \\ &= H(x, y)\,(A_{xy} + A_x A_y) \\ &= \exp\{-e^{-x} - e^{-y} + \theta (e^x + e^y)^{-1}\}\, e^{-(x+y)} \\ &\quad \times \left[1 - \theta \frac{e^{2x} + e^{2y}}{(e^x + e^y)^2} + 2\theta \frac{e^{2(x+y)}}{(e^x + e^y)^3} + \theta^2 \frac{e^{2(x+y)}}{(e^x + e^y)^4} \right]. \end{aligned}$$

$$\begin{aligned} h(x, y) = e^{-(x+y)} \exp[&-e^{-x} - e^{-y} \\ &+ \theta (e^x + e^y)^{-1}]\,[1 - \theta \frac{e^{2x} + e^{2y}}{(e^x + e^y)^2} + 2\theta \frac{e^{2(x+y)}}{(e^x + e^y)^3} + \theta^2 \frac{e^{2(x+y)}}{(e^x + e^y)^4}] \end{aligned} \tag{3.3}$$

This matches the formula shown in the reference.

3.2 Univariate Properties:

The marginal distribution function of X is $F(x) = \exp[-e^{-x}]$, $-\infty \leq$x$\leq \infty$ and similar for $G(y)$. That is, both marginals are type I extreme-value distributions. Note that the type I extreme value distribution is also known as the Gumbel distribution. In fact, it is the distribution commonly referred to in the discussion of univariate extreme value distributions.

Median and Modes

The median of the common distribution of X and Y is

$$\mu = -log(log2) = 0.36651 \tag{3.4}$$

So that $F(\mu)G(\mu) = \frac{1}{4}$
and
So that $F(\mu)G(\mu) = \frac{1}{4}$, and

$$H(\mu,\mu) = \exp\left(-2e^{-\mu} + \frac{1}{2}\theta e^{-\mu}\right) = \left(\frac{1}{4}\right)^{1-\frac{\theta}{4}} \tag{3.5}$$

The value $\tilde{\mu}$, such that $H(\tilde{\mu},\tilde{\mu}) = \frac{1}{4}$, satisfies the equation:

$$\left(2 - \frac{1}{2}\theta\right) e^{-\mu} \tag{3.6}$$

and so

$$\tilde{\mu} = log\left(1 - \frac{1}{4}\theta\right) - log(log2) = log\left(1 - \frac{1}{4}\theta\right) + 0.3665, \tag{3.7}$$

Since $0 \leq \theta \leq 1$, $0.3665 - log\left(\frac{4}{3}\right) = 0.0787 \leq \tilde{\mu} \leq 0.3665$. **mode** The mode of the common distribution of X and Y is at zero. The mode of joint distribution is at

$$x = y$$

$$= \log[\frac{(2-\theta)(4-\theta)}{2\theta}\{\sqrt{\frac{1}{2} + \frac{2}{(2-\theta)^2}} - 1\}] \tag{3.8}$$

Correlation Coefficients

The expression for the product-moment correlation is quite Complex. However, the Spearman rho (a grade correlation) is simpler and is given by

$$\rho s = 3\,(2 - \frac{1}{4}\theta)^{-1}\,[1 + 2\,(2\theta - \frac{1}{4}\theta)^{2}\,\tan^{-1}\{(2\theta - \frac{1}{4}\theta^{2})^{\frac{1}{2}}\,(2 - \frac{1}{2}\theta)^{-1}\}] - 3. \quad (3.9)$$

3.3 Estimation of Parameters

1. Method of Maximum Likelyhood Estimation

Bivariate extreme value distributions (BEVD) are used to model the joint behavior of extremes of two random variables. In practice, the estimation of parameters is crucial for fitting the model to data. One of the most widely used methods is the Maximum Likelihood Estimation (MLE) [Coles et al., 2001].

Model Specification

Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be independent and identically distributed observations from a bivariate extreme value distribution with joint cumulative distribution function (c.d.f.) $H(x, y)$ and corresponding marginal distributions $F(x)$ and $G(y)$.

Type A BEVD

Lai and Balakrishnan [2009]

The joint c.d.f. is given by:

$$H(x, y) = F(x)G(y)\exp\{-\theta\left[\frac{1}{\log F(x)} + \frac{1}{\log G(y)}\right]^{-1}\}, \quad 1 \le \theta < 1 \qquad (3.10)$$

To find the MLE of θ, differentiate $\ell(\theta)$ with respect to θ:

$$\frac{\partial \ell(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \theta} h(x_i, y_i; \theta)}{h(x_i, y_i; \theta)}. \qquad (3.11)$$

Equating to zero:

$$\frac{\partial \ell(\theta)}{\partial \theta} = 0. \qquad (3.12)$$

The joint probability density function **(PDF)** of the Type A bivariate extreme value distribution is given by

$$h(x,y;\theta) = \exp[-e^{-x} - e^{-y} + \theta(e^x + e^y)^{-1}]\, e^{-(x+y)}\,[1 - \theta\frac{e^{2x} + e^{2y}}{(e^x + e^y)^2} + 2\theta\frac{e^{2(x+y)}}{(e^x + e^y)^3} + \theta^2\frac{e^{2(x+y)}}{(e^x + e^y)^4}] \tag{3.13}$$

where θ is the dependence parameter $(0 \le \theta \le 1)$.

Let x_i, y_i be samples for $i = 1, 2, \ldots, n$. The probability density function is given by:

$$h(x_i,y_i;\theta) = \exp[-e^{-xi} - e^{-yi} + \theta(e^x + e^y)^{-1}]\, e^{-(xi+yi)}\,[1 - \theta\frac{e^{2xi} + e^{2yi}}{(e^{xi} + e^{yi})^2} + 2\theta\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^3} + \theta^2\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^4}] \tag{3.14}$$

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} h(x_i, y_i;\theta)$$

Taking the logarithm:

$$\log L(\theta) = \sum_{i=1}^{n} \log h(x_i, y_i;\theta)$$

$$= \sum_{i=1}^{n} [(-e^{-xi} - e^{-yi} + \theta(e^x + e^y)^{-1}) - (xi + yi) + log[1 - \theta\frac{e^{2xi} + e^{2yi}}{(e^{xi} + e^{yi})^2} + 2\theta\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^3} + \theta^2\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^4}]] \tag{3.15}$$

Let:

$$g(\theta) = \left[1 - \theta\frac{e^{2xi} + e^{2yi}}{(e^{xi} + e^{yi})^2} + 2\theta\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^3} + \theta^2\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^4}\right]$$

Then the derivative of the log-likelihood is:

$$\ell(\theta) = \frac{\partial}{\partial\theta} log(g_i(\theta)) = \frac{g_i'(\theta)}{g_i(\theta)}$$

$$\therefore \ell(\theta) = -\frac{e^{2xi} + e^{2yi}}{(e^{xi} + e^{yi})^2} + 2\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^3} + 2\theta\frac{e^{2(xi+yi)}}{(e^{xi} + e^{yi})^4}$$

We can see that the derivative is **highly non-linear in θ,** analytical solution is not feasible. So we go through numerical approach for MLE of θ.

Newton-Raphson Update Rule

$$\theta^{(t+1)} = \theta^{(t)} - \frac{\ell'(\theta^{(t)})}{\ell''(\theta^{(t)})} \tag{3.16}$$

Where:

$$\ell''(\theta) = \frac{\partial^2 \ell(\theta)}{\partial \theta^2}$$

And:

$$\begin{aligned}\frac{\partial^2 \ell(\theta)}{\partial \theta^2} &= \sum_{i=1}^{n} \left[\frac{g_i(\theta) g_i''(\theta) - (g_i'(\theta))^2}{(g_i(\theta))^2} \right] \\ &= \sum_{i=1}^{n} \left[-\frac{g_i(\theta) g_i''(\theta) - (g_i(\theta))^2}{g_i(\theta)^2} \right]\end{aligned}$$

2. Method of Moments Estimator (MOM)

Nadarajah and Tawn [2010] For the method of moments, we equate the theoretical moments with the sample moments.

- The first theoretical moment is

$$\mu_1 = E[X] \quad \text{or} \quad E[Y],$$

  which depends on θ.

- The corresponding sample moment is

$$m_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{or} \quad \frac{1}{n} \sum_{i=1}^{n} y_i.$$

- Similarly, the covariance (or dependence) moment

$$\mu_{XY} = E[XY] - E[X]E[Y]$$

  is equated with the sample covariance

$$m_{XY} = \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}.$$

By solving the system of equations

$$\mu_1 = m_1, \quad \mu_{XY} = m_{XY},$$

we obtain the moment estimator $\hat{\theta}_{MOM}$.

Conclusion

- The MLE of θ is obtained by solving the score equation

$$\frac{\partial \ell(\theta)}{\partial \theta} = 0.$$

- The MOM estimator of θ is obtained by matching theoretical moments with sample moments.

3.4 Real-Life Example:

Rainfall and River Discharge Extremes A practical application of the Type A Bivariate Extreme Value Distribution (BEVD) can be seen in modelling the joint extremes of rainfall and river discharge. In hydrology, heavy rainfall often leads to significant increases in river discharge, and understanding their dependence structure is crucial for flood risk management and water resource planning.

Marginals. Each marginal distribution is assumed to follow a Gumbel distribution, suitable for modeling extreme events. For a Gumbel$(0,1)$ distribution, the theoretical values are:

$$\text{Median} = -\log(\log 2) \approx 0.3665, \quad \text{Mode} = 0.$$

Dependence.

The joint behaviour of rainfall (X) and discharge (Y) is captured using the Type A BEVD copula. The degree of association between the two variables can be quantified using Kendall's τ or Pearson's correlation coefficient. For the simulated dataset, the empirical correlation coefficient was found to be approximately.

$$\rho(X,Y) \approx 0.42 \quad (\text{for } \theta = 0.5).$$

Simulation Study.

A sample of $n = 5000$ observations was generated using the Type A BEVD with parameter $\theta = 0.5$. The simulated data reflect the positive dependence

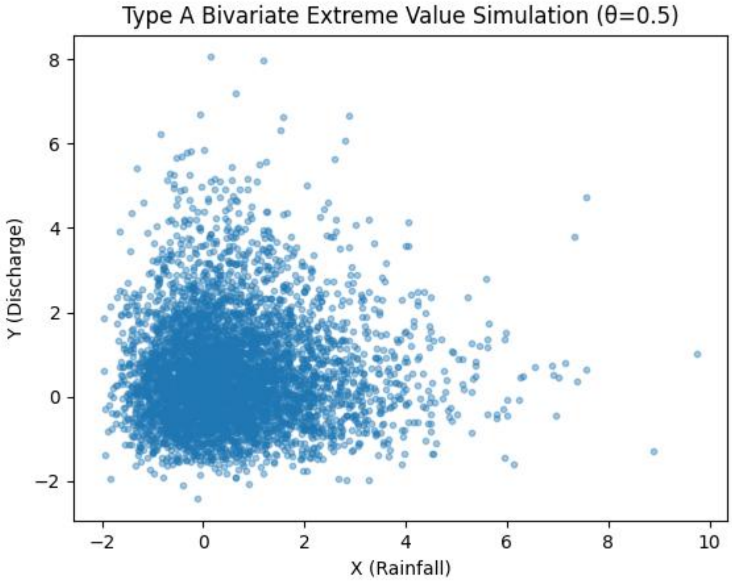

Figure 3.1: Simulated extremes from Type A BEVD (n = 5000, θ = 0.5)

between rainfall maxima and river discharge extremes. The following figure shows a scatter plot of the simulated values.
Median (theoretical): 0.36651292058166435
Mode (theoretical): 0
Sample correlation (Pearson): 0.008666803273959198
This example highlights how the Type A BEVD can effectively capture the dependence structure between environmental extremes, providing insights for designing flood control systems and managing hydrological risks.

Chapter 4

Type B Distribution

Type B bivariate extreme value distribution is known as the logistic model. Lai and Balakrishnan [2009]

Formula for the C.D.F

The Joint C.D.F. is

$$H(x,y) = \exp\left[-(e^{-mx} + e^{-my})^{1/m}\right], m \geq 1 \tag{4.1}$$

Since,

$$\lim_{m\to\infty}\left(e^{-mx} + e^{-my}\right)^{1/m} = max(e^{-x}, e^{-y}), \tag{4.2}$$

, we obtain

$$\lim_{m\to\infty} H(x,y) = \min\left[\exp(-e^{-x}), \exp(-e^{-y})\right] \tag{4.3}$$

$$= \min\left(F(x), F(y)\right) \tag{4.4}$$

It is clear that for $m = 1$, X and Y are independent.

Formula for joint density function

$$h(x,y) = \left[e^{-m(x+y)}\left(e^{-mx} + e^{-my}\right)^{-2+\frac{1}{m}}\right] \cdot \left(m - 1 + \left(e^{-mx} + e^{-my}\right)^{\frac{1}{m}}\right) \cdot \exp\left[-\left(e^{-mx} + e^{-my}\right)^{\frac{1}{m}}\right] \tag{4.5}$$

, for $m \geq 1$

4.1 Derivation of the joint pdf

The joint cumulative distribution function (Type B, logistic) is

$$H(x,y) = \exp\{-(e^{-mx} + e^{-my})^{1/m}\}, \qquad m \geq 1.$$

Define

$$S(x,y) = e^{-mx} + e^{-my}, \qquad \alpha = \frac{1}{m} \in (0,1],$$

so that $H(x,y) = \exp(-S^{\alpha})$.

Put $A(x,y) = -S^{\alpha}$, hence $H = e^{A}$. We use the identity

$$\frac{\partial^2}{\partial x \partial y} e^{A} = e^{A}(A_{xy} + A_x A_y).$$

First derivatives of A Since $A = -S^{\alpha}$,

$$A_x = -\alpha S^{\alpha-1} S_x, \qquad S_x = \frac{\partial}{\partial x}(e^{-mx} + e^{-my}) = -me^{-mx}.$$

Therefore

$$A_x = -\alpha S^{\alpha-1}(-me^{-mx}) = \alpha m e^{-mx} S^{\alpha-1}.$$

By symmetry,

$$A_y = \alpha m e^{-my} S^{\alpha-1}.$$

Mixed second derivative A_{xy},

Differentiate A_x w.r.t. y.

Note that e^{-mx} is independent of y, so

$$A_{xy} = \alpha m e^{-mx} \cdot (\alpha-1) S^{\alpha-2} S_y = \alpha m e^{-mx} (\alpha-1) S^{\alpha-2}(-me^{-my}).$$

Thus

$$A_{xy} = -\alpha(\alpha-1) m^2 e^{-m(x+y)} S^{\alpha-2}.$$

Product $A_x A_y$

$$A_x A_y = (\alpha m)^2 e^{-m(x+y)} S^{2(\alpha-1)} = \alpha^2 m^2 e^{-m(x+y)} S^{2\alpha-2}.$$

Factor $S^{\alpha-2}$:

$$A_x A_y = \alpha^2 m^2 e^{-m(x+y)} S^{\alpha-2}\, S^{\alpha}.$$

Combine $A_{xy} + A_x A_y$ Factor out $m^2 e^{-m(x+y)} S^{\alpha-2}$:

$$\begin{aligned} A_{xy} + A_x A_y &= m^2 e^{-m(x+y)} S^{\alpha-2}[-\alpha(\alpha-1) + \alpha^2 S^{\alpha}] \\ &= m^2 e^{-m(x+y)} S^{\alpha-2}\, \alpha(1-\alpha+\alpha S^{\alpha}). \end{aligned}$$

Now substitute $\alpha = 1/m$. Note that $\alpha m^2 = m$ and $\alpha S^{\alpha} = \dfrac{1}{m} S^{1/m}$. Multiplying through gives a very simple form:

$$A_{xy} + A_x A_y = e^{-m(x+y)} S^{\frac{1}{m}-2}((m-1) + S^{1/m}).$$

(Verification: multiplying out the factors yields the same algebraic expression.)

Final joint density Recall $H = \exp(-S^{1/m})$. Hence

$$\boxed{\begin{aligned} h(x,y) &= \frac{\partial^2}{\partial x \partial y} H(x,y) = H(x,y)(A_{xy} + A_x A_y) \\ &= \exp(-S^{1/m})\, e^{-m(x+y)}\, S^{\frac{1}{m}-2}\,((m-1)+S^{1/m}), \end{aligned}}$$

where $S = e^{-mx} + e^{-my}$. Writing out S explicitly,

$$\boxed{\begin{aligned} h(x,y) = e^{-m(x+y)}(e^{-mx}+e^{-my})^{\frac{1}{m}-2}\left[(m-1)+(e^{-mx}+e^{-my})^{1/m}\right] \\ \cdot \exp\{-(e^{-mx}+e^{-my})^{1/m}\}. \end{aligned}}$$

This is the joint pdf for the Type-B (logistic) bivariate extreme value distribution for $m \geq 1$. One can check that for $m \to \infty$ this tends to independence structure and for $m = 1$ it reduces to a particular simple logistic case (you may verify those limits by substitution).

4.2 Univariate properties

Lai and Balakrishnan [2009] The marginal distributions are both Type-I extreme value distributions

Median and Modes

With the uniariate median μ defined as $F(\mu) = G(\mu) = \frac{1}{2}$, we find for type B distributions,

$$H(\mu,\mu) = \left(\frac{1}{4}\right)^{1/m} \tag{4.6}$$

and

$$H(0,0) = (e^{-2})^{1/m} \tag{4.7}$$

Compare these with 3.12 and 3.13. The values of $\tilde{\mu}$ such that $H(\tilde{mu},\tilde{mu}) = \frac{1}{4}$ satisfies the equation

$$\exp\left[-2^{1/m} e^{-\tilde{mu}}\right] = \frac{1}{4}, \tag{4.8}$$

and so

$$\tilde{\mu} = -log(log2) - \frac{m-1}{m} log2. \tag{4.9}$$

for $m \geq 1$, 0.3665-$log2 = -0.32665 \leq \tilde{\mu} \leq 0.3665$. **The mode of joint distribution is at**

$$x = y = (1+m)^{-1} log2 - \log\left[\sqrt{(m-1)^2)+4} - m + 3\right] \tag{4.10}$$

Correlation Coefficient

The Pearson product moment correlation coefficient is $\rho = 1 - m^{-2}$

Other Properties

The expression $X - Y$ has logistic distribution with

$$Pr = (X - Y \leq t) = \left(1 + e^{-mt}\right)^{-1} \tag{4.11}$$

4.3 Estimation of Parameters

1. Method of MLE

The joint PDF of type B distribution is

$$\boxed{\begin{aligned} h(x,y) = e^{-m(x+y)}(e^{-mx}+e^{-my})^{\frac{1}{m}-2}\left[(m-1)+(e^{-mx}+e^{-my})^{1/m}\right] \\ \cdot \exp\{-(e^{-mx}+e^{-my})^{1/m}\}. \end{aligned}}$$

Let (xi, yi) be the sample of n bivariate observations, $i = 1, 2 \ldots n$, and the only parameter to estimate is m.

$$\begin{aligned} h(xi,yi) = e^{-m(xi+yi)}(e^{-mxi}+e^{-myi})^{\frac{1}{m}-2}\left[(m-1)+(e^{-mxi}+e^{-myi})^{1/m}\right] \\ \cdot \exp - \left[(e^{-mxi}+e^{-myi})^{1/m}\right] \end{aligned}$$

Now take Likelyhod function is

$$L(m) = \prod_{i=1}^{n} h(xi, yi; m)$$

The log-likelyhood function is

$$l(m) = logL(m) = \sum_{i=1}^{n} logh(xi, yi; m)$$

Let $Ai = e^{-mxi} + e^{-myi}$

Now take partial derivative of $logL(m)$ and equate them to zero.

$$\therefore \frac{\partial}{\partial m}\log L(m) = \sum_{i=1}^{n}\{-(x_i+y_i) - \frac{\partial}{\partial m}[(2-\frac{1}{m})\log A_i] + \frac{1}{m-1+A_i^{1/m}}\frac{\partial}{\partial m}(m-1+A_i^{1/m}) - \frac{\partial}{\partial m}(A_i^{1/m})\} \tag{4.12}$$

where the partial derivatives are:

$$(1) \implies \frac{\partial}{\partial m}\left[\left(2-\frac{1}{m}\right)logA_i\right] = \frac{1}{m^2}logA_i + \left(2-\frac{1}{m}\right)\frac{1}{A_i}\frac{\partial A_i}{\partial m}$$

where

$$\frac{\partial A_i}{\partial m} = -xie^{xi} - yie^{-myi}$$

$$(2) \implies \frac{\partial}{\partial m}(A_i^{\frac{1}{m}}) = Ai^{\frac{1}{m}}\left(-\frac{lnA_i}{m^2} + \frac{1}{mA_i}.\frac{\partial A_i}{\partial m}\right)$$

$$(3) \implies \frac{\partial}{\partial m}(m-1+A_i^{\frac{1}{m}}) = 1 + A_i^{\frac{1}{m}}\left(-\frac{lnA_i}{m^2} + \frac{1}{mA_i}.\frac{\partial A_i}{\partial m}\right)$$

Substituting these into the score equation and equating to zero gives:

$$\frac{\partial \ln L(m)}{\partial m} = 0 \tag{4.13}$$

Since the above equation is nonlinear in m, an analytical solution is not possible. Hence, the MLE of m is obtained numerically using an iterative method such as the **Newton–Raphson Method**.

4.4 Real life application:

Maximum Temperature and Electricity Demand:
In a city, high daily maximum temperatures often cause an increase in electricity demand due to air-conditioning usage. Extreme high temperatures and extreme high electricity demands tend to occur together.
Variables:
X = Daily maximum temperature (°C)
Y = Daily electricity demand (MW)
Since,

- Both X and Y are extreme (block maxima) over daily observations.

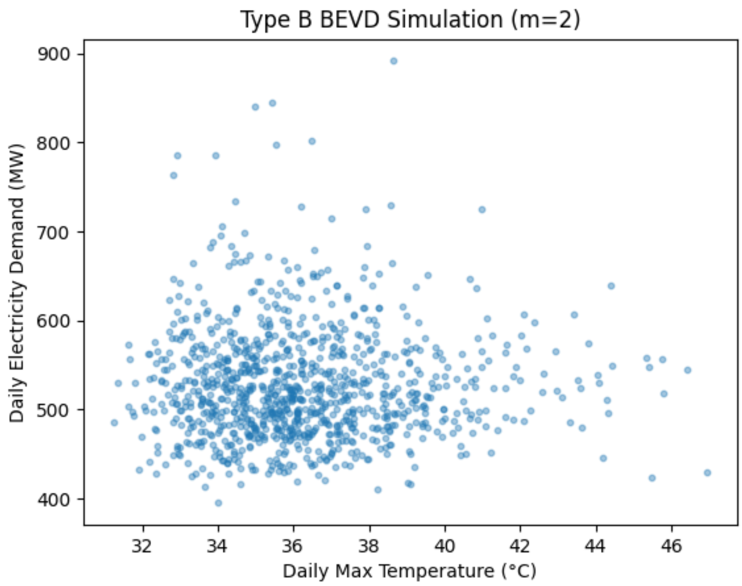

Figure 4.1: Scatterplot of Maximum Temp v/s Electricity demand

- They are positively dependent (high temperatures $\to$ high electricity demand).
- The marginals can be approximated by Gumbel (Type-I extreme value) distributions.
- Pearson correlation can be modelled via $\rho = 1 - m^2$

Appendix

In the R software there is some packages to fit the Extreme value Distribution name as "evd". This package provides functions for extreme value distributions, extending simulation, distribution, quantile, and density functions to univariate and multivariate parametric extreme value distributions. It also offers fitting functions for calculating maximum likelihood estimates for univariate and bivariate maxima models, as well as univariate and bivariate threshold models.

1. Fitting of Type A BEVD By Using R

We take following real life example as follows :
Scenario: Coastal engineering study of storm extremes.

- Block = 1 year. For each year we keep:
- MaxWind(km/h) = maximum observed instantaneous wind speed in a year (km/h) at a coastal station, and
- MaxWave(m) = maximum significant wave height (m) observed in that same year at an offshore buoy.

```
library(evd)
library(readxl)
library(ggplot2)

# 1) Load data
df <- read_excel("D:/Project/BlockMax.data.xlsx")

# 2) Fit marginal GEVs
fit_wind <- fgev(df$MaxWind_kmh)
fit_wave <- fgev(df$MaxWave_m)
print(fit_wind)
print(fit_wave)

# 3) Transform to uniform via parametric CDFs
u <- pgev(df$MaxWind_kmh, loc = fit_wind$estimate["loc"], scale = fit_
    wind$estimate["scale"], shape = fit_wind$estimate["shape"])
```

```r
v <- pgev(df$MaxWave_m, loc = fit_wave$estimate["loc"], scale = fit_wave$estimate["scale"], shape = fit_wave$estimate["shape"])
u <- pmin(pmax(u, 1e-12), 1 - 1e-12)
v <- pmin(pmax(v, 1e-12), 1 - 1e-12)

# 4) Transform to standard Gumbel
x_gumbel <- -log(-log(u))
y_gumbel <- -log(-log(v))

# 5) Type-A joint density function
typeA_density <- function(x, y, theta) {
  ex <- pmin(exp(x), 1e100)
  ey <- pmin(exp(y), 1e100)
  s <- ex + ey
  s[s == 0] <- 1e-300
  term1 <- exp(-(x + y))
  termA <- 1 - theta * (ex^2 + ey^2) / (s^2)
  termB <- 2 * theta * exp(2 * (x + y)) / (s^3)
  termC <- theta^2 * exp(2 * (x + y)) / (s^4)
  pref <- term1 * (termA + termB + termC)
  exp_arg <- -exp(-x) - exp(-y) + theta / s
  dens <- pref * exp(exp_arg)
  pmax(dens, 0)
}

# 6) Negative log-likelihood
negloglik_theta <- function(theta, xg, yg) {
  if (theta > 1) return(1e10)
  dens <- typeA_density(xg, yg, theta)
  if (any(dens <= 0)) return(1e8 + sum(dens <= 0) * 1e6)
  return(-sum(log(dens)))
}

# 7) Optimize theta
res <- optim(par = 0.0, fn = negloglik_theta, xg = x_gumbel, yg = y_gumbel, method = "L-BFGS-B", lower = -50, upper = 1)
theta_hat <- res$par
cat("Estimated theta (R) =", theta_hat, "\n")

# 8) Profile log-likelihood
theta_grid <- seq(-50, 1, length.out = 200)
loglik_values <- sapply(theta_grid, function(th) -negloglik_theta(th, x_gumbel, y_gumbel))
grid_max_theta <- theta_grid[which.max(loglik_values)]

profile_df <- data.frame(theta = theta_grid, loglik = loglik_values)

# 9) Plot
ggplot(profile_df, aes(x = theta, y = loglik)) +
  geom_line(color = "blue") +
  geom_vline(xintercept = theta_hat, linetype = "dashed", color = "red") +
  geom_vline(xintercept = grid_max_theta, linetype = "dotted", color = "purple") +
  labs(
    title = "Profile Log-Likelihood for $\theta$ (Type-A BEVD)",
    subtitle = paste("Grid MLE =", round(grid_max_theta, 4),
                     "| Optimized MLE =", round(theta_hat, 4)),
```

```
    x = expression(theta),
    y = "Log-Likelihood"
  ) +
  theme_minimal()
```

Output

Table 4.1: GEV Parameter Estimates for Marginal Distributions

Parameter	Estimate	Standard Error	Distribution
Location (μ)	143.028	5.9824	Wind Speed
Scale (σ)	28.774	4.7210	Wind Speed
Shape (ξ)	-0.433	0.1687	Wind Speed
Location (μ)	5.555	0.3853	Wave Height
Scale (σ)	1.737	0.3056	Wave Height
Shape (ξ)	-0.291	0.2184	Wave Height

The optimization process converged successfully for both marginals, with deviances of 282.19 (wind speed) and 118.76 (wave height).

$$\hat{\theta} = 1.00 \tag{4.14}$$

The value $\hat{\theta} = 1$ indicates a strong dependence between the annual maxima of wind speed and wave height, suggesting that extreme wind events are likely to coincide with extreme sea states in this coastal region.

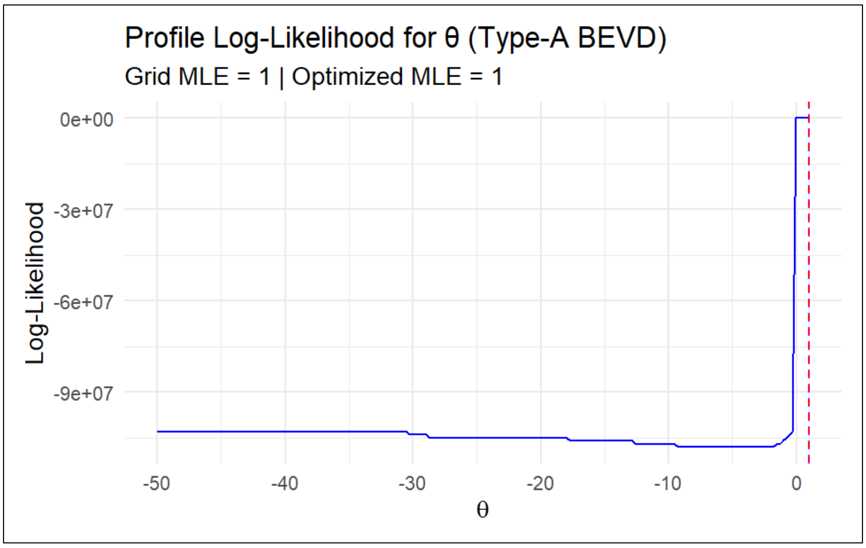

Figure 4.2: Profile Log-Likelyhood for θ

2. Fitting of Type B BEVD by using R

Listing 4.1: Type B Fitting (Logistic) Parameter Estimation using Newton-Raphson

```

# ============================================================
# 1. Simulate sample data (x, y)
# ============================================================
set.seed(123)

# True parameter
m_true <- 2.5

# Generate marginal data (Gumbel-like)
n <- 100
x <- -log(-log(runif(n)))
y <- -log(-log(runif(n)))

# Add dependence via logistic structure
# Here we create pseudo-correlated sample (for demonstration)
u <- exp(-exp(-x))
v <- exp(-exp(-y))
x <- -log(-log(u))
y <- -log(-log(v))

# ============================================================
# 2. Define log-likelihood function for Type B BEVD
# ============================================================
loglik_Btype <- function(m, x, y) {
  if (m <= 1) return(-Inf)
  A <- exp(-m*x) + exp(-m*y)
  log_h <- (-m*(x+y)
            - (2 - 1/m)*log(A)
            + log(m - 1 + A^(1/m))
            - A^(1/m))
  sum(log_h)
}

# ============================================================
# 3. Define first derivative (score function)
# ============================================================
score_Btype <- function(m, x, y) {
  A <- exp(-m*x) + exp(-m*y)
  dA_dm <- -x*exp(-m*x) - y*exp(-m*y)

  term1 <- -(x + y)
  term2 <- -(1/m^2)*log(A) - (2 - 1/m)*(1/A)*dA_dm
  term3 <- (1/(m - 1 + A^(1/m))) *
    (1 + A^(1/m)*(-log(A)/m^2 + (1/(m*A))*dA_dm))
  term4 <- -A^(1/m)*(-log(A)/m^2 + (1/(m*A))*dA_dm)

  sum(term1 + term2 + term3 + term4)
}

# ============================================================
# 4. Newton Raphson Algorithm for MLE of m
# ============================================================
newton_Btype <- function(x, y, start = 2, tol = 1e-6, maxit = 100) {
```

```r
  m <- start
  for (i in 1:maxit) {
    f1 <- score_Btype(m, x, y)
    # numerical second derivative
    h <- 1e-5
    f2 <- (score_Btype(m + h, x, y) - score_Btype(m - h, x, y)) / (2*h)
    m_new <- m - f1 / f2
    if (abs(m_new - m) < tol) {
      cat("Converged in", i, "iterations\n")
      return(m_new)
    }
    m <- m_new
  }
  warning("Did not converge")
  return(m)
}

# ==========================================================
# 5. Estimate m
# ==========================================================
m_hat <- newton_Btype(x, y, start = 1.5)
cat("Estimated M (Newton-Raphson):", m_hat, "\n")

# ==========================================================
# 6. Plot log-likelihood curve
# ==========================================================
m_seq <- seq(1.01, 5, by = 0.05)
logL_values <- sapply(m_seq, loglik_Btype, x = x, y = y)

plot(m_seq, logL_values, type = "l", lwd = 2, col = "blue",
     xlab = "Parameter m", ylab = "Log-Likelihood",
     main = "Log-Likelihood Function for Type B BEVD")
abline(v = m_hat, col = "red", lty = 2, lwd = 2)
text(m_hat, max(logL_values), labels = paste("MLE =", round(m_hat, 3)),
     pos = 4, col = "red")

# ==========================================================
# 7. Output Summary
# ==========================================================

cat("Sample size (n):", n, "\n")
cat("True m (if known):", m_true, "\n")
cat("Estimated m (Newton-Raphson):", round(m_hat, 4), "\n")
cat("Maximum Log-Likelihood:", round(max(logL_values), 4), "\n")
```

Output

Sample size (n): 100
True m (if known): 2.5
Estimated m (Newton-Raphson) $\hat{m}$: 0.6614
Maximum Log-Likelihood: -302.1719

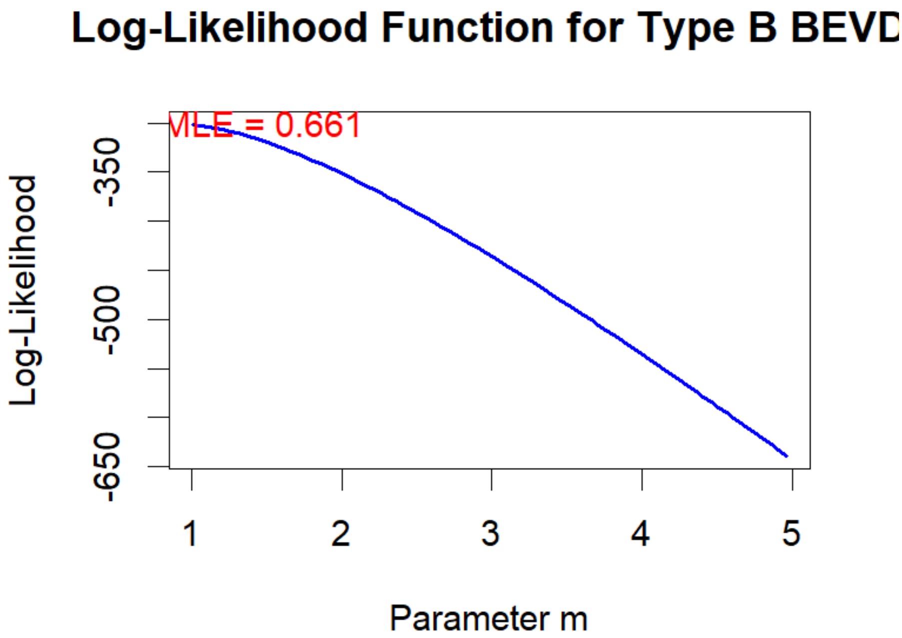

Figure 4.3: Log-likelihood Curve

3. Comparing Type A and Type B BEVD Models

We compare Type A and Type B BEVD using rainfall and river discharge data of India based on longitude and latitude. Data Source: Flood Risk in India (Kaggle).

```
library(readxl)
library(evd)
library(ggplot2)

# ----------------------------
# Read data & marginals
# ----------------------------
df = read_excel("D:/Project/Rainfall_river_Discharge.xlsx")
colnames(df) = c("Rainfall", "Discharge")
X = df$Rainfall
Y = df$Discharge
n = length(X)

# Fit marginals (GEV)
fitX = fgev(X)
fitY = fgev(Y)
print(fitX); print(fitY)

# PIT -> uniforms -> avoid 0/1
u = pgev(X, loc = fitX$estimate["loc"], scale = fitX$estimate["scale"],
     shape = fitX$estimate["shape"])
v = pgev(Y, loc = fitY$estimate["loc"], scale = fitY$estimate["scale"],
     shape = fitY$estimate["shape"])
```

```
u = pmin(pmax(u, 1e-8), 1 - 1e-8)
v = pmin(pmax(v, 1e-8), 1 - 1e-8)

# Gumbel standardization (x,y)
x_g = -log(-log(u))
y_g = -log(-log(v))

# ---------------------------
#  Type A full joint PDF log-likelihood (same structure as earlier)
# Parameter: theta in [0,1]
# Uses the Type A joint pdf
# ---------------------------
loglik_typeA_full = function(theta_vec, x, y) {
  theta = as.numeric(theta_vec[1])
  if(is.na(theta) || theta < 0 || theta > 1) return(1e12)
  ex = exp(x); ey = exp(y)
  # Full joint pdf from project (Type A)
  # Using formula: h(x,y) = e^{-(x+y)} exp[-e^{-x}-e^{-y}+ theta*(ex+ey)^{-1}] * [ ... polynomial ... ]
  termA = exp(-exp(-x) - exp(-y) + theta * (ex + ey)^(-1))
  termB = exp(-(x+y))
  denom1 = (ex + ey)^2
  denom2 = (ex + ey)^3
  denom3 = (ex + ey)^4
  poly = 1 - theta * (ex^2 + ey^2) / denom1 + 2*theta * (ex * ey) / denom2 + theta^2 * (ex * ey) / denom3
  dens = termA * termB * poly
  dens[!is.finite(dens) | dens <= 0] = 1e-12
  return(-sum(log(dens)))
}

resA = optim(par = 0.2, fn = loglik_typeA_full, x = x_g, y = y_g,
              method = "L-BFGS-B", lower = 0, upper = 1, control = list(fnscale = 1))
thetaA_hat = as.numeric(resA$par)
loglikA = -resA$value
cat("Type A: theta_hat =", thetaA_hat, " logLik =", loglikA, "conv=", resA$convergence, "\n")

# ---------------------------
#  Type B full joint PDF log-likelihood (Logistic model)
# Parameter: m >= 1 (we'll enforce lower bound 1)
# Based on formula : H(x,y)=exp(-[(e^{-m x}+e^{-m y})]^{1/m})
#  Use Gumbel-scale x,y.
# ---------------------------
loglik_typeB_full = function(m_vec, x, y) {
  m = as.numeric(m_vec[1])
  if(is.na(m) || m < 1) return(1e12)
  # compute terms
  emx = exp(-m * x)
  emy = exp(-m * y)
  S = emx + emy
  S1m = S^(1/m)   # S^{1/m}
  # PDF formula from project (equation similar to 2.25).
  # h(x,y) = e^{-m(x+y)} * S^{1/m - 2} * { m - 1 + S^{1/m} } * exp( - S^{1/m} )
  # Note: ensure numerically stable for large exponents
  term1 = exp(-m * (x + y))
```

```
  term2 = S^(1/m - 2)
  term3 = (m - 1) + S1m
  dens = term1 * term2 * term3 * exp(-S1m)
  # numeric guards
  dens[!is.finite(dens) | dens <= 0] = 1e-12
  return(-sum(log(dens)))
}

# Optimize m
resB = optim(par = 1.5, fn = loglik_typeB_full, x = x_g, y = y_g,
              method = "L-BFGS-B", lower = 1.0, upper = 50, control =
                 list(fnscale = 1))
mB_hat = as.numeric(resB$par)
loglikB = -resB$value
cat("Type B: m_hat =", mB_hat, " logLik =", loglikB, "conv=", resB$
   convergence, "\n")

# ---------------------------
# Compute Pickands functions
# ---------------------------
t_vals = seq(0.01, 0.99, by = 0.01)

# Empirical Pickands A_emp (same as earlier, using empirical copula C_n
   )
C_hat = function(u0, v0, u, v) mean(u <= u0 & v <= v0)
A_emp = sapply(t_vals, function(t){
  cval = C_hat(1 - t, t, u, v)
  cval = max(cval, 1e-12)
  denom = log(1 - t) + log(t)
  A_est = log(cval) / denom
  max(A_est, t, 1 - t)
})

# Type A model Pickands (from project): A_A(t) = 1 - theta * t * (1 - t
   )
A_typeA = function(t, theta) 1 - theta * t * (1 - t)
A_modA = A_typeA(t_vals, thetaA_hat)

# Type B model Pickands (logistic): A_B(t) = ( t^{1/m} + (1-t)^{1/m} )^
   m
A_typeB = function(t, m) ( t^(1/m) + (1 - t)^(1/m) )^m
A_modB = A_typeB(t_vals, mB_hat)

# CvM on Pickands (approx integral via sum*step)
step = t_vals[2] - t_vals[1]
CvM_A = sum((A_emp - A_modA)^2) * step
CvM_B = sum((A_emp - A_modB)^2) * step
cat("CvM: TypeA =", CvM_A, " TypeB =", CvM_B, "\n")

# ---------------------------
#  AIC / BIC comparison
# We'll use full-model loglik and count parameters = marginal params +
   1 (dependence)
# Marginal params: each GEV has 3 -> total 6; plus one dependence param
    = 7
# (If you prefer dependence-only comparison, use pseudo-likelihood;
   this uses full joint-likelihood.)
# ---------------------------
```

```
k = 7
AIC_A = -2 * loglikA + 2 * k
BIC_A = -2 * loglikA + log(n) * k
AIC_B = -2 * loglikB + 2 * k
BIC_B = -2 * loglikB + log(n) * k

cat("\nModel Selection (full-likelihood):\n")
cat(sprintf("Type A: logLik=%.3f  AIC=%.3f  BIC=%.3f\n", loglikA, AIC_A, BIC_A))
cat(sprintf("Type B: logLik=%.3f  AIC=%.3f  BIC=%.3f\n", loglikB, AIC_B, BIC_B))

# ---------------------------
# STEP 5: Plots - Pickands, Empirical Copula Surface, Copula Contours
# ---------------------------
# Pickands plot comparing empirical vs both models
plot(t_vals, A_emp, type = "l", lwd = 2, col = "black", ylim = c(min(A_emp, A_modA, A_modB), 1),
     main = "Pickands: Empirical vs Type A & Type B", xlab = "t", ylab = "A(t)")
lines(t_vals, A_modA, col = "red", lwd = 2, lty = 2)
lines(t_vals, A_modB, col = "blue", lwd = 2, lty = 3)
legend("topright", legend = c("Empirical", paste0("Type A (Îÿ=", round(thetaA_hat,3),")"),
                              paste0("Type B (m=", round(mB_hat,3),")")),
       col = c("black","red","blue"), lwd = 2, lty = c(1,2,3))

# Empirical copula surface
u_grid = seq(0,1,length.out = 41)
v_grid = seq(0,1,length.out = 41)
C_mat = outer(u_grid, v_grid, Vectorize(function(ug, vg) C_hat(ug, vg, u, v)))
persp(x = u_grid, y = v_grid, z = C_mat, theta = 30, phi = 30, expand = 0.6,
      main = "Empirical Copula Surface", xlab = "u", ylab = "v", zlab = "C_hat(u,v)")

# Parametric copula surfaces for Type A and Type B (copulas defined on u,v)
# We'll compute C_A(u,v) using extreme-value copula form via Pickands A(t):
C_A_model = function(u0, v0, theta){
  # extreme-value copula: C(u,v) = exp( log(uv) * A( log u / log(uv) ) )
  L = log(u0 * v0)
  # handle edge cases
  if(u0<=0 || v0<=0) return(0)
  t = log(u0) / (log(u0) + log(v0))
  A_t = A_typeA(t, theta)
  exp( L * A_t )
}
C_B_model = function(u0, v0, m){
  # logistic extreme-value copula has closed form:
  # C(u,v) = exp( - [ (-log u)^m + (-log v)^m ]^{1/m} )
  exp( - ( (-log(u0))^m + (-log(v0))^m )^(1/m) )
}

```

```
# Build matrices
C_A_mat = outer(u_grid, v_grid, Vectorize(function(ug, vg) C_A_model(max(ug,1e-12), max(vg,1e-12), thetaA_hat)))
C_B_mat = outer(u_grid, v_grid, Vectorize(function(ug, vg) C_B_model(max(ug,1e-12), max(vg,1e-12), mB_hat)))

# Persp for model copulas
persp(x = u_grid, y = v_grid, z = C_A_mat, theta = 25, phi = 20, expand = 0.6,
      main = paste("Type A Copula Surface (Îÿ=", round(thetaA_hat,3), ")", sep=""),
      xlab="u", ylab="v", zlab="C_A(u,v)")
persp(x = u_grid, y = v_grid, z = C_B_mat, theta = 25, phi = 20, expand = 0.6,
      main = paste("Type B Copula Surface (m=", round(mB_hat,3), ")", sep=""),
      xlab="u", ylab="v", zlab="C_B(u,v)")

# Contour plots of model copulas
contour(u_grid, v_grid, C_A_mat, main = "Contour: Type A Copula", xlab="u", ylab="v")
contour(u_grid, v_grid, C_B_mat, main = "Contour: Type B Copula", xlab="u", ylab="v")

# ---------------------------
# STEP 6: Numerical Summary Table and Automatic Interpretation
# ---------------------------
cat("\n\n================= SUMMARY TABLE =================\n")
summary_table = data.frame(
  Model = c("Type A (Gumbel-type)", "Type B (Logistic)"),
  Param = c(round(thetaA_hat,4), round(mB_hat,4)),
  logLik = c(round(loglikA,4), round(loglikB,4)),
  AIC = c(round(AIC_A,4), round(AIC_B,4)),
  BIC = c(round(BIC_A,4), round(BIC_B,4)),
  CvM_Pickands = c(round(CvM_A,6), round(CvM_B,6))
)
print(summary_table)

cat("\n\n=== Automatic interpretation ===\n")
# Compare AIC/BIC
bestAIC = ifelse(AIC_A < AIC_B, "Type A", "Type B")
bestBIC = ifelse(BIC_A < BIC_B, "Type A", "Type B")
bestCvM = ifelse(CvM_A < CvM_B, "Type A", "Type B")

cat(sprintf("By AIC the better model is: %s\n", bestAIC))
cat(sprintf("By BIC the better model is: %s\n", bestBIC))
cat(sprintf(" By Pickands Cramer-von-Mises (lower is better): %s\n", bestCvM))

# Interpret dependence parameter direction
cat("\nDependence interpretation:\n")
if(thetaA_hat > 0.1) cat(sprintf(" Type A theta=%.3f suggests positive tail dependence.\n", thetaA_hat)) else cat("Type A theta is small; weak tail dependence.\n")
if(mB_hat > 1.1) cat(sprintf(" Type B m=%.3f suggests positive dependence (m>1). Larger m -> stronger dependence symmetry.\n", mB_hat)) else cat(" Type B m near 1 indicates near independence.\n")

```

```
# Visual fit note
cat("\nVisual diagnostics: examine Pickands plot (empirical vs models)
    and copula surfaces/contours.\n")
cat(" If empirical Pickands (black) closely follows one model curve,
    that model captures tail dependence better.\n")
cat(" Also check whether the model copula surface resembles empirical C
    _hat surface plotted earlier.\n")
```

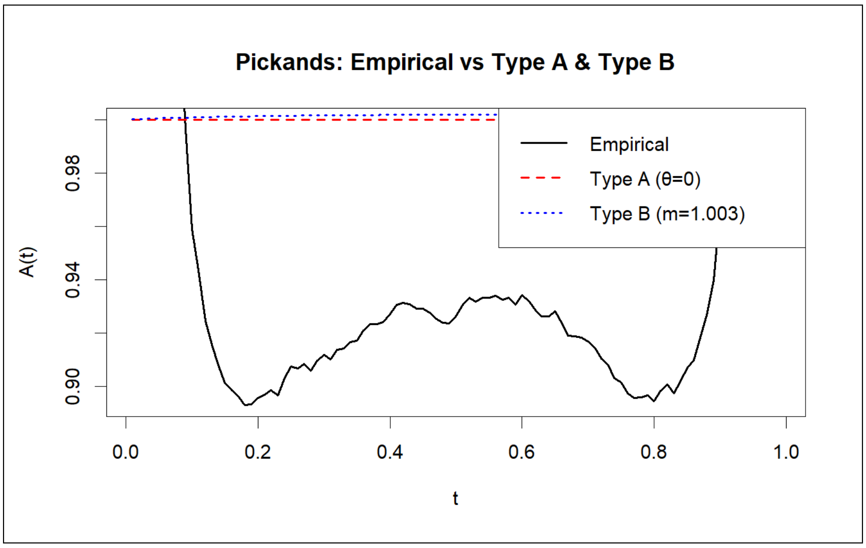

Figure 4.4: Pickands: Empirical vs Type A & Type B

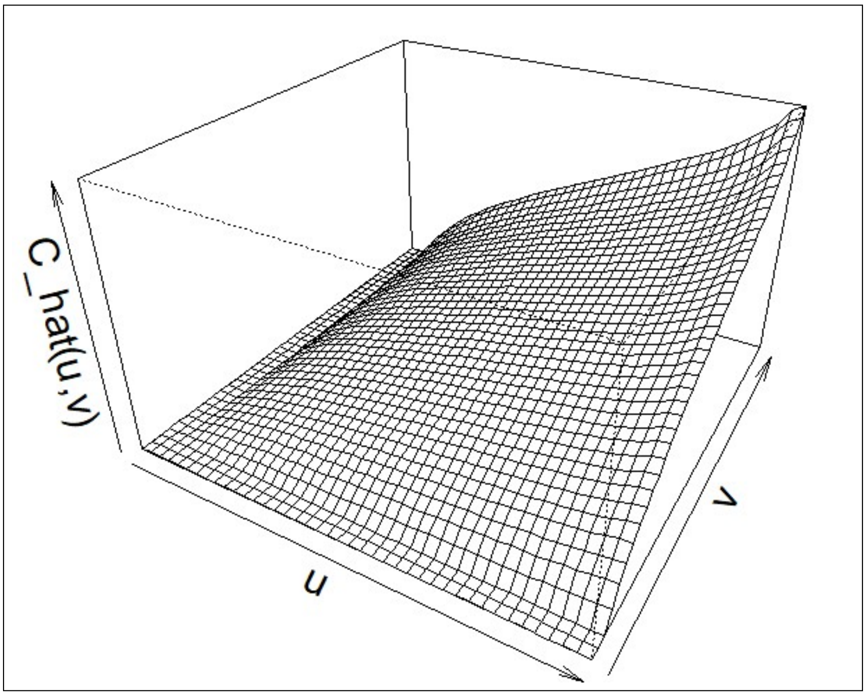

Figure 4.5: Empirical Copula Surface

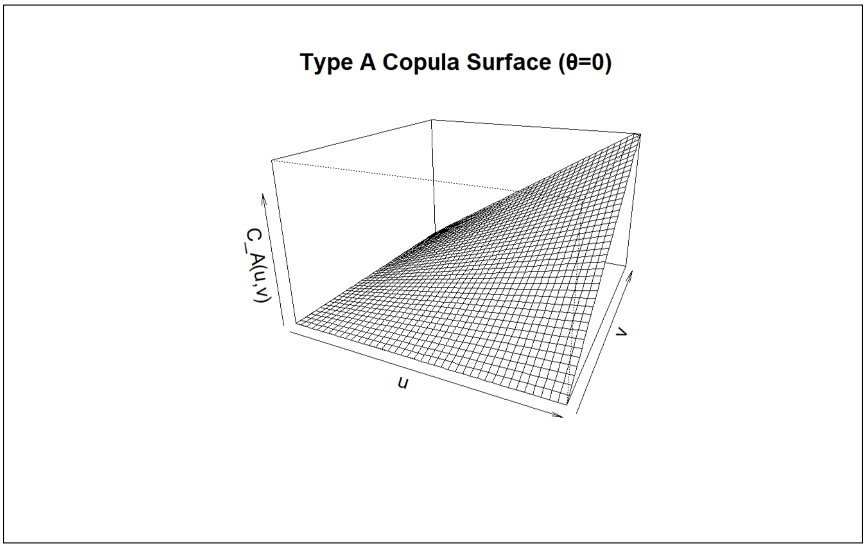

Figure 4.6: Type A Copula Surface

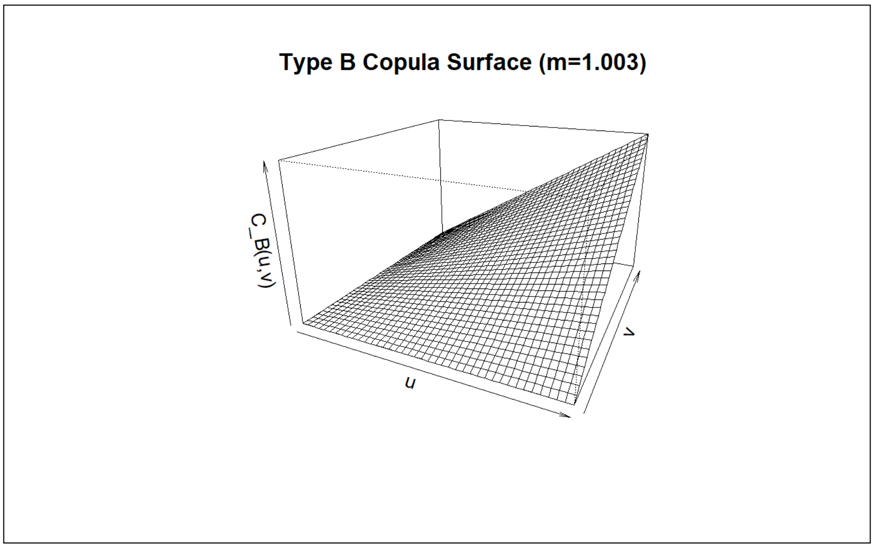

Figure 4.7: Type B Copula

Univariate GEV Marginal Estimation

The marginal distributions of rainfall (X) and river discharge (Y) were fitted using the Generalized Extreme Value (GEV) distribution via maximum likelihood estimation.

GEV Fit for Rainfall (X)

Table 4.2: GEV parameter estimates for Rainfall (X)

Parameter	Estimate	Standard Error
Location (μ_X)	19.9247	0.10270
Scale (σ_X)	8.6691	0.08428
Shape (ξ_X)	-0.4141	0.01137

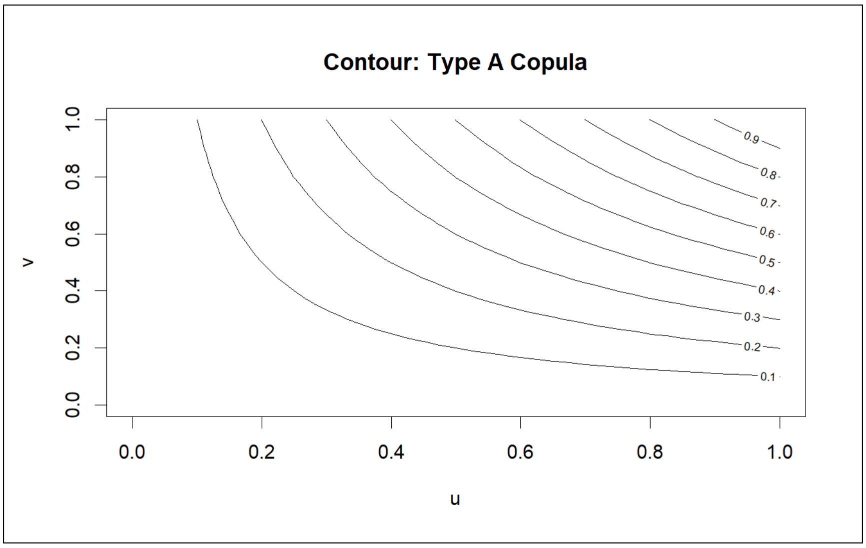

Figure 4.8: Contour: Type A Copula

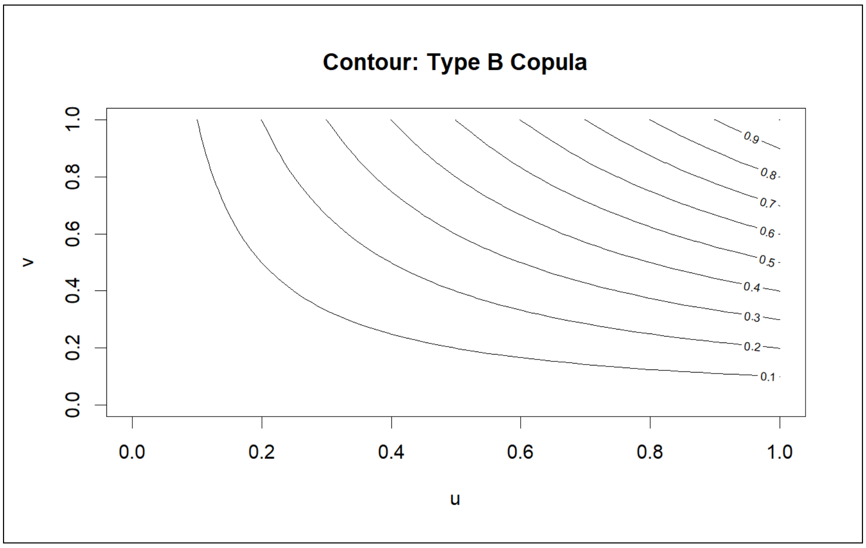

Figure 4.9: Contour: Type B Copula

Deviance: 70334.85
Convergence: Successful

GEV Fit for River Discharge (Y)

Table 4.3: GEV parameter estimates for River Discharge (Y)

Parameter	Estimate	Standard Error
Location (μ_Y)	80.3910	0.10431
Scale (σ_Y)	8.8988	0.08686
Shape (ξ_Y)	-0.4566	0.01111

Deviance: 70365.91
Convergence: Successful

Bivariate Extreme Value Dependence Models

Two bivariate extreme value dependence structures were fitted:

- **Type A**: Logistic BEVD
- **Type B**: Negative Logistic BEVD

Parameter Estimates and Goodness-of-Fit

Table 4.4: Dependence parameter estimates and goodness-of-fit

Model	Parameter	Estimate	Log-Likelihood	CvM Statistic
Type A	$\hat{\theta}$	0.0000	-32209.94	4.580236
Type B	$\hat{m}$	1.0027	-32209.79	4.579933

Model Selection Criteria

Table 4.5: Comparison of Type A and Type B BEVD Models

Model	**Param**	**logLik**	**AIC**	**BIC**	**CvM_Pickands**
Type A (Gumbel-type)	0.0000	-32209.94	64433.89	64484.36	4.580236
Type B (Logistic)	1.0027	-32209.79	64433.58	64484.05	4.579933

Interpretation of Results

Based on AIC, BIC, and the Pickands Cramer–von Mises statistic, the **Type B bivariate extreme value distribution** provides a slightly better fit than Type A.

The estimated dependence parameter for Type A is very small, indicating weak tail dependence. For Type B, the parameter estimate $\hat{m} \approx 1$ suggests a dependence structure close to independence.

Visual diagnostics such as Pickands dependence plots and copula surface comparisons further support the numerical findings.

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