138. Equation of Line, 3d, and Hyperplane

Equation of a Straight Line, 3D Plane, and Hyperplane

Summary

- **Linear equations** serve as the foundation for machine learning algorithms like **Logistic Regression** and **Support Vector Machines (SVM)**.
- A **2D straight line** can be represented mathematically as $y = mx + c$ or in vector notation as $w^T x + b = 0$.
- This concept extends to 3 dimensions as a **3D Plane** and to n-dimensions as a **Hyperplane**, maintaining the general vector equation $w^T x + b = 0$.
- When a line or plane passes through the **origin**, the intercept b becomes zero, simplifying the equation to $w^T x = 0$.
- Geometrically, the weight vector w is always **perpendicular (orthogonal)** to the plane or line passing through the origin.

Equation of a Straight Line (2D)

Understanding the equation of a straight line is a fundamental prerequisite for analyzing linear machine learning models.

Standard Notation

In a 2D space with axes x and y, a straight line is commonly defined by the equation:

$$y = mx + c$$

- m **(Slope)**: Represents the unit movement in the y-axis with respect to a unit movement in the x-axis.
- c **(Intercept)**: The point where the line crosses the y-axis when $x = 0$.

Alternative notations include $ax + by + c = 0$ or $y = \beta_0 + \beta_1 x$. These can be algebraically rearranged to match the slope-intercept form .

Machine Learning Notation

In machine learning, variables are often denoted as features (x_1, x_2) rather than axes x, y to accommodate higher dimensions. The equation is rewritten using **weights** (w) and **bias** (b):

$$w_1 x_1 + w_2 x_2 + b = 0$$

This can be expressed compactly using **vector notation**:

$$w^T x + b = 0$$

- w: A vector of coefficients (weights).
- x: A vector of feature inputs.

- b: The intercept (bias).

Equation of a 3D Plane

When extending the concept to three dimensions (x_1, x_2, x_3), a line becomes a **3D Plane**.

The equation expands to include a third weight and feature:

$$w_1x_1 + w_2x_2 + w_3x_3 + b = 0$$

Despite the added dimension, the general vector representation remains consistent:

$$w^Tx + b = 0$$

Here, the vectors are defined as:

- $w = [w_1, w_2, w_3]^T$
- $x = [x_1, x_2, x_3]^T$

Hyperplane (n-Dimensions)

In an n-dimensional space, the geometric structure is referred to as a **Hyperplane**. The equation generalizes for n features:

$$w_1x_1 + w_2x_2 + ... + w_nx_n + b = 0$$

Or simply:

$$\pi : w^Tx + b = 0$$

This universal equation applies regardless of the number of dimensions.

Geometric Interpretation and the Origin

A critical property of these linear equations arises when considering lines or planes that pass through the **origin** $(0, 0)$.

Intercept at the Origin

If a line or plane passes through the origin, the intercept b **equals zero**. The equation simplifies to:

$$w^Tx = 0$$

Orthogonality of the Weight Vector

Using **Linear Algebra**, the dot product w^Tx can be expressed in terms of magnitudes and the angle θ between the vectors:

$$w^Tx = ||w|| \cdot ||x|| \cdot \cos(\theta) = 0$$

For this product to equal zero (assuming non-zero magnitudes), $\cos(\theta)$ must be zero. This occurs when $\theta = 90^\circ$.

Key Geometric Conclusion: The weight vector w is always **perpendicular (orthogonal)** to the hyperplane (or line) passing through the origin. This geometric relationship holds true for any point x lying on the plane.