Laplace Transformation

Defn :- Let $f(t)$ be a given function defined for $t \geq 0$. We multiply $f(t)$ by e^{-st} and integrate with respect to 't' in $[0, \infty)$ then the resulting integral exist it will a function of s, say $F(s)$ mathematically given function

Hence $$\int_0^\infty f(t)\, e^{-st}\, dt = F(s) = L\{f(t)\}$$

$$\int_0^\infty f(t)\, e^{-st}\, dt = \lim_{u \to \infty} \int_0^u f(t)\, e^{-st}\, dt$$

Inverse laplace transform, If $F(s)$ is the laplace transform of $f(t)$, we say $f(t)$ is the inverse laplace transform of $F(s)$.

~~Expla~~

Examples:-

(a) Let $f(t) = 1$

$$L\{f(t)\} = L\{1\} = \int_0^\infty 1.e^{-st}\, dt$$

$$= \left. \frac{-e^{-st}}{s} \right|_0^\infty$$

$$= \frac{-0}{s} + \frac{1}{s}$$

$c \to \frac{c}{s}$

$$= \frac{1}{s}$$

b. $L\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st} \cdot dt$

$$= \int_0^\infty (e^{(a-s)t}) \cdot dt$$

$$= \frac{1}{(a-s)} e^{(a-s)t} \Big|_0^\infty \quad (s > a)$$

$$= \frac{1}{s-a}$$

(c) $L\{t\} = \int_0^\infty t \cdot e^{-st} \cdot dt$

$$= \frac{t \cdot e^{-st}}{-s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} \cdot dt$$

$$= \frac{-t}{s} + \frac{1}{s}\left[\frac{-e^{-st}}{s}\right]_0^\infty$$

$$= -\frac{t}{c} + \frac{1}{s}\left[+\frac{1}{s}\right]$$

$$= \frac{1}{s^2}$$

(d) $L\{t^n\} = \int_0^\infty t^n e^{-st} \cdot dt$

$$= -\frac{1}{s} e^{-st} t^n \Big|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} \cdot t^{n-1} \cdot dt$$

$$= \frac{n}{s} L\{t^{n-1}\}$$

$$= \frac{n!}{s^{n+1}}$$

(e) $L\{\cos at\} = \int_0^\infty \cos at \, e^{-st} \cdot dt$

$$= \cancel{\cos} \int_0^\infty e^{-st} \cos at \cdot dt$$

$$= \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right]\Big|_0^\infty$$

$$= -\frac{1}{s^2+a^2}\left[-s\right]$$

$$= \frac{s}{s^2+a^2}$$

(f) $L\{\sin at\} = \int_0^\infty e^{-st} \sin at = \frac{e^{-st}}{s^2+a^2}\left[-s\sin ax - a\cos ax\right]_0^\infty = \frac{-1}{s^2+a^2}\left[-a\right] = \frac{a}{s^2+a^2}$

Theorem (Linearity property) :-

The laplace transform is a linear operation that is for any function $f(t)$ and $g(t)$ where laplace transform exist and for any constant a and b.

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + b\{g(t)\}$$

Prove: Let $f(t)$ & $g(t)$ be any two $f^{n's}$

L.H.S $\Rightarrow L\{af(t) + bg(t)\}$

$$= \int_0^\infty \{af(t) + bg(t)\} e^{-st} \cdot dt$$

$$= \int_0^\infty a e^{-st} f(t)\, dt + \int_0^\infty bg(t) e^{-st} \cdot dt$$

$\Rightarrow$ R.H.S.

Since Integration of sum of equal sum of individual integrations.

eg. $L\{\cosh at\} = L\{\frac{1}{2}(e^{at} + e^{-at})\}$

$$= \frac{1}{2} L\{e^{at} + e^{-at}\}$$

$$\Rightarrow \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \Rightarrow \frac{s}{s^2-a^2} \quad (s > a > 0)$$

b. $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

c.) $L\{e^{-i\omega t}\}$

$$\Rightarrow \int_0^\infty e^{-i\omega t} e^{-st} \cdot dt = F(s)$$

$$= \int_0^\infty e^{-(s+i\omega)t} \cdot dt$$

$$= \left. \frac{-e^{-(s+i\omega)t}}{(s+i\omega)} \right|_0^\infty$$

$$= \frac{1}{(s+i\omega)} \frac{(s-i\omega)}{(s-i\omega)}$$

$$= \frac{s-i\omega}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2} - i\frac{\omega}{s^2+\omega^2}$$

First shifting theorem:—

If $f(t)$ has the laplace transform $F(s)$ (where $s > k$) then $e^{at} f(t)$ has the laplace transform $F(s-a)$, $s-a > k$. Mathematically we can write

$$\text{If } L\{f(t)\} = f(s)$$

$$\text{then } L\{e^{at} f(t)\} = F(s-a)$$

or

$$L\{e^{-at} f(t)\} = F(s+a)$$

Application:

we know the $L\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2}$

$$L\{e^{at}\cos\omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$$

$$L\{e^{at}\sin\omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

Proof:— $\quad L.H.S. = L\{e^{at} f(t)\}$

$$= \int_0^\infty e^{at} f(t)\, e^{-st}.dt$$

$$= \int_0^\infty e^{-(a+s)t}.f(t).dt \Rightarrow F(s-a)$$

Existence of laplace transform:-

Let $f(t)$ be a function which ~~precise~~ peicewise continous on every finite interval in the range $t \geq 0$ and satisfies $|f(t)| \leq Me^{\alpha t}$, $\forall\ t \geq 0$ for some constant α and M, then the laplace transform of $f(t)$ exists for every $s > \alpha$.

A function is said to be peicewise continous if the interval can be partitioned and brokened into finite no. of sub interval such that in each sub interval the fn is continoaus and ~~lifi~~ a the limit at the end points of each sub-interval should exist.

Proof:- $\because f(t)$ is peicewise continuous then $e^{-st} f(t)$ is continuous.

$e^{-st} f(t)$ is integrable, then for any $s > \alpha$

$$|\mathcal{L}\{f(t)\}| = \left| \int_0^\infty e^{-st} f(t)\, dt \right|$$

$$= \int_0^\infty e^{-st} |f(t)|\, dt$$

$$= M \int_0^\infty e^{(\alpha - s)t} \cdot dt$$

finite $\Rightarrow L\{f\}$ exist

$$= M \int_0^\infty e^{-(s-\alpha)t} \cdot dt$$

$$|F(s)| = |L\{f(t)\}| \leq \frac{M}{s-\alpha} \quad (s > \alpha)$$

$$\boxed{\Gamma\alpha = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt}$$

classmate

Date

Page

The existence condition is sufficient but not necessary.

Consider an example :-

$$f(t) = \frac{1}{\sqrt{t}} \longrightarrow \infty \text{ as } t \to 0$$

$f(t)$ is not bounded by a $Me^{\alpha t}$.
but

$$L\{f(t)\} = \int_0^\infty \frac{1}{\sqrt{t}} \cdot e^{-st} \cdot dt \quad -①$$

$$= -\frac{e^{-st}}{s} \cdot \frac{1}{\sqrt{t}} + \frac{1}{s}\int 2\sqrt{t}\, e^{-st} \cdot dt$$

put $st = x$ in ①
$s\,dt = dx$

$$= \int_0^\infty \left(e^{-x} \cdot x^{-1/2} s^{1/2} \cdot \frac{1}{s}\, dx\right)$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2}\, dx$$

$$= \frac{1}{\sqrt{s}} \Gamma\frac{1}{2}$$

$$= \sqrt{\frac{\pi}{s}}$$

for all fxn $\left(\frac{1}{t^{\alpha}}\right)$, $\alpha \in (0,1)$.

Scanned by CamScanner

classmate

Date

Page

Laplace transformation of derivative of function :-

Theorem :- Suppose that $f(t)$ is continuous for all $t \geq 0$ and $f(t)$ satisfies the condition of existence of laplace transform and has a derivative $f'(t)$ that is peicewise continu-ous on every finite interval in the range $t \geq 0$, then the laplace transform of the derivative $f'(t)$ exist for $s > K$ and

$$\boxed{L\{f'\} = s\,L\{f\} - f(0)}$$

Since f is peicewise continuous and also the derivative f' is peicewise, the laplace trans. of f' exist for some $(K > \alpha)$.

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \underbrace{e^{-st} f(t) \Big|_0^\infty}_{\text{found}} + s\int e^{-st} f(t)\,dt$$

$$\boxed{L\{f'(t)\} = f(0) + s\,L\{f\}}$$

$$\boxed{L\{f''\} = s\{L\{f'\}\} - f'(0) = s^2 L\{f\} - sf(0) - f'(0)}$$

Scanned by CamScanner

for n^{th} derivative,

Let $f(t)$ and its derivatives, $f', f'', f''', \ldots f^{n-1}$ exist and continuous $\forall$ $t \geq 0$ and the L.T. of all these exist ~~$\forall$~~ for $s > K$ n^{th} derivative be peicewise continuous on every finite interval $t > 0$, then the laplace transform of n^{th} derivative is given by

$$L\{f^n(t)\} = s^n L\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots\ldots f^{n-1}(0).$$

Applications :-

<u>Find L.T. of some functions :-</u>

eg. $f(t) = t^2$

$f'(t) = 2t$

$f'(0) = 0 = f(0)$

$f''(t) = 2$ $\qquad f''(0) = 2$

$$L\{f''\} = s^2 L\{f\} - s f(0) - f'(0)$$

$$L\{2\} = s^2 L\{t^2\} \Rightarrow L\{t^2\} = \frac{2}{s^3}$$

classmate

Date

Page

eg. $L\{\cos\omega t\} =$

$f'(t) = -\omega\sin\omega t \qquad f'(0) = 0$

$f''(t) = -\omega^2\cos\omega t$

$L\{f''\} = s^2 L\{f\} - s f(0) - f'(0)$

$L\{-\omega^2\cos\omega t\} = s^2 L\{\cos\omega t\} - s$

$-\omega^2 L\{\cos\omega t\} = s^2 L\{\cos\omega t\} - s$

$$\boxed{L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}}$$

eg. $L\{\sin^2 t\}$

$f'(t) = \sin 2t \qquad \Rightarrow f(0) = 0$

$f''(t) = 2\cos 2t \qquad f'(0) = 0$

$L\{f'\} = sL\{f\} - f(0)$

$L\{\sin 2t\} = s L\{\sin^2 t\}$

$$\boxed{L\{\sin^2 t\} = \frac{2}{s(s^2+4)}}$$

$L\{t\sin\omega t\} = ?$

Scanned by CamScanner

classmate

Date

Page

2. To solve a DE $\{$ $y(0) = k_0$

$y'(0) = k_1$

$$y'' + ay' + by = r(t)$$

$L\{y\} = Y(s)$

Step-1 apply L both sides

$$L\{y''\} + aL\{y'\} + bL\{y\} = L\{r(t)\}$$

$\{L\{r(t)\} = R(s)$

~~$s^2 L\{y\} - s y(0) - y'(0)$~~

$$[s^2 Y(s) - s y(0) - y'(0)] + a[s Y(s) - y(0)] + bY(s) = R(s)$$

$$Y(s) = \frac{[(s+a) y(0) + y'(0)]}{s^2 + as + b} + \frac{R(s)}{s^2 + as + b}$$

apply again inverse both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{F_1(s)}{G_1(s)}\right\} + L^{-1}\left\{\frac{R(s)}{G_1(s)}\right\}$$

$$\Rightarrow \boxed{y(t) = g(t) + h(t)}$$

Que:- Solve the $y'' - y = t$. $y(0) = 1$, $y'(0) = 1$

$$L\{y''\} - L\{y\} = L\{t\}$$

$$s^2 Y(s) - s y(0) - y'(0) - Y(s) = \frac{1}{s^2}$$

$$s^2 Y - s - 1 - Y = \frac{1}{s^2}$$

$$Y(s^2-1) - s - 1 = \frac{1}{s^2}$$

$$Y = \frac{\frac{1}{s^2}+s+1}{s^2-1}$$

$$Y = \frac{1+s^3+s^2}{s^2(s^2-1)}$$

$$Y(s) = \frac{s+1}{s^2-1} - \frac{1}{s^2(s^2-1)}$$

$$L^{-1}Y(s) = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^t - \sinh t - t$$

Laplace transformation of integral of a function :-

Theorem :-

Let $F(s)$ be the laplace transform of $f(t)$. If $f(t)$ is peicewise continuous and satisfies the existence condition of $L\{f(t)\}$, then

$$L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

Or alternatively $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau$

Date
Page

Proof :-

f is peicewise continuous and bounded

Let $g(t) = \int_0^t F(\tau)\, d\tau$

$g(t)$ is continous $\forall$ for any $t \geq 0$

$(s>\alpha)$

$$|g(t)| \leq \int_0^t |f(\tau)|\, d\tau \leq M \int_0^t e^{\alpha\tau}\, d\tau = \frac{M}{\alpha}(e^{\alpha t}-1)$$

$$\leq \underbrace{\frac{M}{\alpha}}_{M^*} e^{\alpha t}$$

$$= M^* e^{\alpha t}$$

Hence $g(t)$ satisfies conditions of existence of Laplace transformation.
So L.T. of $g(t)$ exists.

Now, $g'(t) = f(t)$, $g(0) = 0$

$$L\{f(t)\} = L\{g'(t)\}$$

$$= s\, L\{g(t)\} - \cancel{g(0)}^{\,0}$$

$$= s\, L\{g(t)\}$$

$$L\{f(t)\} = s\, L\{g(t)\}$$

$$F(s) = s \cdot L\left\{\int_0^t F(\tau)\, d\tau\right\}$$

$$\boxed{L\left\{\int_0^t F(\tau)\,d\tau\right\} = \frac{F(s)}{s}} \quad s>\alpha$$

Application :-

Find the inverse laplace transform $\frac{1}{s(s^2+\omega^2)}$.

or if $L\{f\} = \frac{1}{s(s^2+\omega^2)}$. Find f.

$$L^{-1}\left\{\frac{1}{s}F(s)\right\} = L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau$$

$$= \frac{1}{\omega^2}(1-\cos\omega t)$$

Que:- Solve the IVP by using Laplace transformation.

$$y''+y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$y'\left(\frac{\pi}{4}\right) = 2-\sqrt{2}$$

$$L\{y''\} + L\{y\} = L\{2t\}$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0) - Y(s)\right] = \frac{2}{s^2}$$

If initial point is t_0, we do the substitution

$$t = \tilde{t} + t_0$$

$$y(\tilde{t} + t_0) = y(\tilde{t})$$

$$y'(t) = y'(\tilde{t})$$

$$y''(t) = y''(\tilde{t})$$

$$t_0 = \frac{\pi}{4} \Rightarrow t = \tilde{t} + \frac{\pi}{4}$$

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{\pi}{4}\right)$$

$$\tilde{y}(0) = \frac{\pi}{2}$$

$$\tilde{y}'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

$$s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0) + \tilde{Y} = \frac{2}{s^2} + \frac{\pi}{2}\cdot\frac{1}{s}$$

$$s^2\tilde{Y} - s\left(\frac{\pi}{2}\right) - 2 + \sqrt{2} + \tilde{Y} = \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\tilde{Y}(s) = \frac{2/s^2 + \pi/2s + s(\pi/2) + 2 - \sqrt{2}}{(s^2+1)}$$

$$\tilde{Y}(s) = \frac{2}{s^2(s^2+1)} + \frac{2-\sqrt{2}}{s^2+1} + \frac{\pi}{2s(s^2+1)} + \frac{s\pi}{2(s^2+1)}$$

Taking I.L.T both side.

$$= 2t - \sin t + \cos t$$

Inverse laplace Transform :-

Target is to get inverse of a function

$$\boxed{H(s) = \frac{F(s)}{G(s)}}$$

depending upon the nature of factors in $G(s)$, we have the following 4 case:

1. Simple factors and real say $(s-a)$.
2. Repeated factors in $G(s)$, $(s-a)^m$.
3. Complex factors in $G(s)$, $(s-a)(s-\bar{a})$
4. Repeated complex factos in $G(s) = [(s-a)(s-\bar{a})]^2$.

Que:- Solve the DE, $y'' + y' - 6y = 1$, $y(0) = 0$, $y'(0) = 1$.

$$s^2 Y(s) - s\,Y(0) - Y'(0) - 6Y = \frac{1}{s}$$

$$s^2 Y - \cancel{s(0)} - 1 - 6Y = \frac{1}{s}$$

$$Y = \frac{1/s + 1}{s^2 - 6} = \frac{s(s+1)}{s(s^2-6)}$$

$$[s^2 Y - sY(0) - Y'(0)] + 1\,[sY - Y(0)] - 6\,[Y$$

$$Y(s) = \frac{s+1}{s(s-2)(s+3)} = \frac{F(s)}{G(s)}$$

$$Y(s) = \frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$A = \left.\frac{s+1}{(s-2)(s+3)}\right|_{s=0}$$

$$A = -\frac{1}{6}$$

$$B = \left.\frac{s+1}{s(s+3)}\right|_{s=2}$$

$$= \frac{3}{2(5)} = \frac{3}{10}$$

$$C = \left.\frac{s+1}{s(s-2)}\right|_{s=-3}$$

$$= \frac{-2}{-3(-5)}$$

$$C = -\frac{2}{15}$$

$$= \frac{-1}{6s} + \frac{3}{10(s-2)} + \frac{-2}{15(s+3)}$$

$$y(t) = -\frac{1}{6}\cancel{t} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

classmate
Date
Page

Case : 2 Repeated Real roots in $G(s) \neq F$

$$H(Y) = \frac{F(s)}{G(s)}$$

That is terms of the form $(s-a)^2, (s-a)^3 \ldots$

To handle the situation $\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2}$

eg. Solve the ivp $y'' - 3y' + 2y = 4t$, $y(0) = 1$
$y'(0) = -1$

$$\left(s^2 Y(s) - s(1) + 1\right) - 3(sY - 1) + 2Y(s) = \frac{4}{s^2}$$

$$Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)} = \frac{F(s)}{G(s)}$$

$$\frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$B = \left.\frac{s^3 - 4s^2 + 4}{s^2(s-1)}\right|_{s=2}$$

$$B = \frac{8 - 16 + 4}{4(1)} = -1$$

$$C = \left.\frac{s^3 - 4s^2 + 4}{s^2(s-2)}\right|_{s=1}$$

$$= \frac{1 - \cancel{4} + \cancel{4}}{1(1-2)} = -1$$

$$s^3 - 4s^2 + 4 = A_2(s-2)(s-1) + A_1 s(s-1)(s-2)$$
$$+ B(s^2)(s-1) + C s^2(s-2)$$

$= A$.

$s = 0$, $\boxed{A_2 = 2}$

$1 = A_1 + B + C$

$1 = A_1 + (-1) + (-1)$

$\boxed{A_1 = 3}$

$$= \frac{3}{s} + \frac{2}{s^2} + \frac{-1}{s-2} + \frac{-1}{s-1}$$

Taking inverse

$$\Rightarrow y(t) = 3 + 2t - e^{2t} - e^{t}$$

Case 3: Complex factors (unrepeated) in $G(s)$, that is $(s-a)(s-\bar{a})$. we can write.

$$a = \alpha + i\beta$$
$$\bar{a} = \alpha - i\beta$$

we can write,

$$\frac{F(s)}{(s-a)(s-\bar{a})} = \frac{As + B}{(s-a)(s-\bar{a})}$$

Que:- Solve: $y'' + 2y' + 2y = r(t)$

$$r(t) = \begin{cases} 10\sin 2t, & \text{if } 0 < t < \pi \\ 0, & \text{if } t > \pi \end{cases}$$

$y(0) = 1,\ y'(0) = -5$

Apply Laplace transform both sides.

$$[s^2Y - s(1) - (-5)] + 2[sY - 1] + 2Y = \int_0^{\pi} 10\sin 2t\, e^{-st}\,dt + \int_{\pi}^{\infty} 0.e^{-st}\,dt$$

$$s^2Y - s + 5 + 2sY - 2 + 2Y = \frac{10.e^{-st}}{4+s^2}\left[-s\sin 2t - 2\cos 2t\right]_0^{\pi}$$

$$= \frac{10e^{-\pi s}}{4+s^2}[-2] - \frac{10}{4+s^2}[-2]$$

$$= \frac{20}{s^2+4}\left[1-e^{-\pi s}\right]$$

$$Y(s) = \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}$$

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}$$

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4)$$

$0 = A + M$

$0 = 2A + B + N$

$0 = 2A + 2B + 4M$ $\quad A = -2, \quad B = -2$

$20 = 2B + 4N$ $\quad M = +2. \quad N = 6$

$$= \frac{-2-2s}{s^2+4} + \frac{2s+6}{s^2+2s+2}$$

= taking inverse

$$= \frac{-2(s+1)}{s^2+4} + \frac{2(3+s)}{s^2+2s+2}$$

$$= \frac{-2s}{s^2+4} - \frac{2}{s^2+4} + \frac{6}{s^2+2s+2} + \frac{2s}{s^2+2s+2}$$

$$= -2\cos 2t - \sin 2t + \frac{6}{(s+1)^2+1} + \frac{2(s+1-1)}{(s+1)^2+1}$$

$$= -2\cos 2t - \sin 2t + 6 \cdot e^{-t}\sin t + 2e^{-t}\cos t - 2e^{-t}\sin t$$

Case 4: Repeated complex factors, in $G(s)$ that is $[(s-a)(s-\bar{a})]^2$

In this case we write $\frac{F(s)}{[(s-a)(s-\bar{a})]^2} = \frac{As+B}{[(s-a)(s-\bar{a})]^2}$

classmate

Date ______
Page ______

(9) ~~Q. A62~~. $L^{-1}\left\{\frac{2}{(s^2+4)^2}\right\}$

$$\frac{2}{(s^2+4)^2} = \qquad (s^2+4)^2 = \left((s+2i)(s-2i)\right)^2$$

$$L^{-1}\left\{\frac{1}{(s-2i)^2}\right\} = e^{2it}\,t$$

<u>Heaviside expansion formula for inverse:-</u>

Suppose $F(s) = \frac{F_1(s)}{F_2(s)}$ {degree of $F_1(s)$ < deg. of $F_2(s)$}

<u>Case I.</u> The function $F_2(s) \to$ ~~Now~~ Non repeated factors, say

$(s-a_1)(s-a_2)\ldots\ldots(s-a_n)$

$$F(s) = \frac{F_1(s)}{(s-a_1)(s-a_2)\ldots(s-a_n)}$$

Now, $F(s) = \frac{A}{s-a_1} + \frac{A_2}{s-a_2} + \ldots \frac{A_r}{s-a_r} + \ldots \frac{A_n}{s-a_n}$

Target to determine the coefficient of A_r multiplying $(s-a_r)$ both sides

$$(s-a_r)\,F(s) = \frac{s-a_r}{s-a_1}A_1 + \frac{s-a_r}{s-a_2} + \ldots \frac{s-a_r}{s-a_n}A_n.$$

Now, $\lim_{s\to a_r} A_r = \lim_{s\to a_r}\left[(s-a_r)\,F(s)\right] = \lim_{s\to a_r}\left[(s-a_r)\frac{F_1(s)}{F_2(s)}\right]$

$$A_r = F_1(a_r)\lim_{s\to a_r}\left[\frac{s-a_r}{F_2(s)}\right]$$

$$A_r = F_1(a_r) \lim_{s \to a_r} \left[\frac{s-a_r}{F_2(s)} \right] = F_1(a_r) \lim_{s \to a_r} \left[\frac{1}{F_2'(s)} \right]$$

$$= \frac{F_1(a_r)}{F_2'(a_r)} \quad (r = 1, 2, \ldots n)$$

Now, $$F(s) = \sum_{r=1}^{n} \frac{A_r}{s-a_r} = \sum_{r=1}^{n} \frac{F_1(a_r)}{F_2'(a_r)} \times \frac{1}{s-a_r}$$

The laplace inverse is

$$L^{-1}\{F(s)\} = L^{-1}\left\{ \sum_{r=1}^{n} \frac{F_1(a_r)}{F_2'(a_r)} \times \frac{1}{s-a_r} \right\}$$

$$= \sum_{r=1}^{n} \frac{F_1(a_r)}{F_2'(a_r)} L^{-1}\left\{ \frac{1}{s-a_r} \right\}$$

$$= \sum_{r=1}^{n} \frac{F_1(a_r)\, e^{a_r t}}{F_2'(a_r)}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{ \frac{F_1(s)}{F_2(s)} \right\} = \sum_{r=1}^{n} \frac{F_1(a_r)}{F_2'(a_r)} e^{a_r t}$$

Case : 2 Repeating factors in $F_2(s)$ say $(s-a_1)$ is repeated n times,

Say $$F(s) = \frac{F_1(s)}{F_2(s)} = \frac{F_1(s)}{(s-a_1)^n (s-a_2)(s-a_3) \ldots (s-a_n)}$$

$$F(s) = \frac{A_1}{(s-a_1)^n} + \frac{A_2}{(s-a_1)^{n-1}} + \ldots + \frac{A_n}{s-a_1} + \underbrace{\frac{C_2}{s-a_2} + \frac{C_3}{s-a_3} + \ldots + \frac{C_n}{s-a_n}}_{G(s)}$$

$$= \sum_{r=1}^{n} \frac{A_r}{(s-a_1)^{n-r+1}} + G(s)$$

Target to get coefficient A_r, $r = 1, 2, \ldots, n$

let us multiply $(s-a_1)^n$ both sides

$$(s-a_1)^n \, F(s) = \sum_{r=1}^{n} \frac{A_r}{(s-a_1)^{1-r}} + (s-a_1)^n G(s)$$

$$(s-a_1)^n \, F(s) = \sum_{r=1}^{n} (s-a_1)^{r-1} A_r + (s-a_1)^n G(s) \quad — \ (*)$$

Differentiate (*) successively '$m-1$' times

$$\frac{d^{m-1}}{ds^{m-1}} \{(s-a_1)^n F(s)\} = \sum_{r=m}^{n} (r-1)(r-2)\ldots(r-m+1)(s-a_1)^{r-m} A_r + \frac{d^{m-1}}{ds^{m-1}} \{(s-a_1)^n G(s)\}$$

All terms contain $(s-a_1)$

Take limit as $s \to a_1$ (for $r = m$)

$$(m-1)! \, A_m = \lim_{s \to a_1} \left\{ \frac{d^{m-1}}{ds^{m-1}} \{ (s-a_1)^n F(s) \} \right\}$$

$$A_r = \frac{1}{(r-1)!} \lim_{s \to a_1} \left\{ \frac{d^{r-1}}{ds^{r-1}} \{(s-a_1)^n F(s)\} \right\}$$

Now, $L^{-1}\{F(s)\} = \sum_{r=1}^{n} L^{-1} \left\{ \frac{A_r}{(s-a_1)^{n-r+1}} \right\} + L^{-1}\{G(s)\}$

$$\boxed{L^{-1}\{f(s)\} = \sum_{r=1}^{n} \frac{A_r e^{a_1 t} \, t^{n-r}}{(n-r)!} + g(t)}$$

Que:- Find the inverse laplace transform of

$$\frac{s^2+2s-3}{s(s-3)(s+2)}$$

Solution:-

$$F_1(s) = s^2+2s-3$$

$$F_2(s) = s(s-3)(s+2)$$

$a_1 = 0$
$a_2 = +3$
$a_3 = -2$

$$F_2'(s) = 3s^2-2s-6$$

$$L^{-1}\{F(s)\} = \sum_{r=1}^{3} \frac{F_1(a_r)}{F_2'(a_r)} e^{a_r t} = \frac{F_1(a_1)}{F_2'(a_1)} e^{a_1 t} + \frac{F_1(a_2)}{F_2'(a_2)} e^{a_2 t}$$

$$+ \frac{F_1(a_3)}{F_2'(a_3)} e^{a_3 t}$$

$$= \frac{1}{2}(1) + \frac{4}{5}e^{3t} - \frac{3}{10}e^{-2t}$$

Que:- Find the inverse laplace trans. of $\frac{s-3}{s^2+2s+2}$

$$= \frac{s-3}{(s+1+i)(s+1-i)}$$

$$F_1(s) = s-3$$

$$F_2(s) = s^2+2s+2$$

$$F_2'(s) = 2s+2$$

$a_1 = -1-i$
$a_2 = -1+i$

$$L^{-1}\{F(s)\} = \frac{-1-i-3}{2(\cancel{-1}-i)+\cancel{2}} e^{(-1-i)t} + \frac{-1+i-3}{2(-1+i)+2} e^{(-1+i)t}$$

$$= \frac{(-1-i)}{-2i} e^{(-1-i)t} + \frac{-1+i}{2i} e^{(-1+i)t}$$

$$= \frac{1+i}{2i} e^{-(1+i)t} + \frac{-1+i}{2i} e^{-(1-i)t}$$

$$= \frac{e^{-t}}{2i} \left[(1+i) e^{-it} + (-1+i) e^{it} \right]$$

$$= \frac{e^{-t}}{2i} \left[i(e^{it} + e^{-it}) - 1(e^{it} - e^{-it}) \right]$$

$$= \frac{e^{-t}}{2i} \left[2i \cos t - 8 i \sin t \right]$$

$$= e^{-t} [\cos t - 1 \sin t].$$

Que L.I. of $\frac{1}{s(s+2)^3} = \frac{A_1}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)} + \frac{C_0}{s}$

$F_1(s) = 1$

$F_2(s) = s(s+2)^3$

$F_2'(s) = \cancel{s}(4s^3 + 18s^2 + 21s + 8$

$$C_0 = \frac{F_1(0)}{F_2'(0)} e^{0 \cdot t} = \frac{1}{8}$$

$$A_1 = \lim_{s \to -2} \left[(s+2)^3 \frac{1}{s(s+2)^3} \right] = -\frac{1}{2}$$

$$A_2 = \lim_{s \to -2} \left[\frac{d}{ds} (s+2)^3 \frac{1}{s(s+2)^3} \right]$$

$$= \lim_{s \to -2} \left[\frac{-1}{s^2} \right] = \frac{-1}{4}$$

$$A_3 = \frac{1}{2} \lim_{s \to -2} \left[\frac{d^2}{ds^2} \left\{ (s+2)^3 \frac{1}{s(s+2)^3} \right\} \right] = \frac{-1}{8}$$

$$L^{-1}\{F(s)\} = \sum_{r=1}^{3} \frac{e^{a_1 t}\, t^{3-r}}{(3-r)!} A_r + \frac{1}{8}$$

$$= \frac{-e^{-2t}}{4}\left(t^2 + t + \frac{1}{2}\right) + \frac{1}{8}$$