Vector Space

Let V be a non-empty set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every U, V, W in V and scalars c and d then V is called vector space over the set of real numbers

① **Addition**

ⓐ $U+V \in V$ for all $U, V \in V$

ⓑ $U+V = V+U$ for all $U, V \in V$

ⓒ $(U+V)+W = U+(V+W)$ for all $U, V, W \in V$

ⓓ There is a zero vector θ such that

$$U+\theta = U \quad \text{for all } U \in V$$

ⓔ for every $U \in V$, there exists a vector $-U \in V$ such that

$$U+(-U) = \theta.$$

② **Multiplication**

ⓐ $cU \in V$ for all $U \in V$

ⓑ $c(U+V) = cU + cV$ for all $U, V \in V$

(c) $c(du) = (cd)u$ for all $u \in V$

(d) $(c+d)u = cu + du \quad \forall u \in V$

(e) $1 \cdot u = u \quad \forall u \in V$

Ex Let V be a set of all fifth-degree polynomials with standard operations. Is it a vector space?

$$f_1(x) = x^5 + x^4 + 2$$

$$f_2(x) = -x^5$$

$$f_1(x) + f_2(x) = x^4 + 2 \notin V$$

So, it is not a vector space.

Ex $V = \{(x,y) : x \geq 0,\ y \geq 0\}$ with standard operations. Is it a vector space.

$$(-2,-3) + (2,3) = (0,0) \in V.$$

But $(-2,-3) \notin V$.

So, it is not a vector space

Ex Let $V = \{(x, \frac{1}{2}x) : x \in R\}$ with standard operations. Is it a vector space?

Yes.

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Subspace

Let V be an arbitary vector space defined over the set of real numbers. A non-empty subset W of V is called a subspace if W forms a vector space over the set of real numbers $\mathbb{R}$.

Theorem

A non empty subset W of a vector space V is a subspace if and only if

(i) $\alpha \in W,\ \beta \in W \Rightarrow \alpha+\beta \in W$

(ii) $\alpha \in W,\ c \in \mathbb{R} \Rightarrow c\alpha \in W$

Ex Let S be the subset of $\mathbb{R}^3$ defined by

$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=z^2\}$. Is S a subspace?

$(-3,4,5) \in \mathbb{R}^3$ $\quad (3,-4,5) \in \mathbb{R}^3$.

but $\alpha+\beta = (0,0,10) \notin \mathbb{R}^3$.

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Ex

$$S = \{(x, y, z) \in R^3 : x+y+z=1\}$$

$(0,0,0) \notin S$

So, S is not a subspace.

Definition

Let V be a vector space. Let $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n \in V$.

A vector β in V is said to be a linear combination of the vectors $\alpha_1, \cdots, \alpha_n$ if β can be written as

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$$

for some scalars $c_1, \cdots c_n \in R$

Ex In R^3, $\alpha = (4,3,5)$, $\beta = (0,1,3)$ and $\gamma = (2,1,1)$,

$\delta = (4,2,2)$

Examine if

(i) α is a linear combination of β and γ

(ii) β is a linear combination of γ and δ

Spanning Set

Def^n Let S be a subset of vector space V. and suppose that every element in V can be obtained as linear combination of the elements taken from S. Then S is said to be a spanning set of the vector space V.

or,

we say that S spans V.

Ex Let V be a vector space of 2×2 real matrices. Show that the sets

(i) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(ii) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

span V

Sol^n Let us consider any 2×2 real matrices of the vector space V as

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in R$

condition $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$c_1 = a$ $\quad c_3 = c$ $\quad$ as $a, b, c, d \in R$.

$c_2 = b$ $\quad c_4 = d$. $\quad \Rightarrow c_1, c_2, c_3, c_4 \in R$

(11) $a = c_1 + c_2 + c_3 + c_4$

$b = c_2 + c_3 + c_4$

$c = c_3 + c_4$

$d = c_4$

$c_1 = a - b$

$c_2 = b - c$

$c_3 = c - d$

$c_4 = d.$

Linearly independent vectors

$\alpha_1, \alpha_2 \text{ --- } \alpha_n \in V$

$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$

$\alpha_1, \alpha_2 \cdots \alpha_n$ are linearly independent

Ex let $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$, $v_3 = (0,0,1)$ be the ~~ele~~ elements of R^3. show that the vectors are linearly independent

$\rightarrow c_1(1,-1,0) + c_2(0,1,-1) + c_3(0,0,1) = 0.$

$\Rightarrow c_1 = 0 \qquad c_1 = c_2, \qquad c_3 = 0.$

$\Rightarrow c_2 = 0$

$c_1 = c_2 = c_3 = 0$

Ex let $v_1 = (1,-1,0)$, $v_2 = (0,1,-1)$, $v_3 = (0,2,1)$, $v_4 = (1,0,3)$ be the elements of R^3 check the independence of the given vector.

$$c_1(1,-1,0) + c_2(0,1,-1) + c_3(0,2,1) + c_4(1,0,3) = 0$$

$\Rightarrow \quad c_1 + c_4 = 0 \quad \Rightarrow c_4 = -c_1$

$\Rightarrow \quad -c_1 + c_2 + 2c_3 = 0$

$\Rightarrow \quad -c_2 + c_3 + 3c_4 = 0$

$\Rightarrow \quad c_3 - 3c_1 - c_2 = 0$

$\quad\; 2c_3 + c_2 - c_1 = 0$

$\quad\; 3c_3 - 4c_1 = 0$

$c_2 = \frac{4c_1}{3} - 3c_1$

$c_2 = -\frac{5c_1}{3}$

$\Rightarrow \quad c_3 = \frac{4c_1}{3}$

$$-c_1 - \frac{5c_1}{3} + \frac{8c_1}{3} = 0.$$

Basis

Let V be a vector space over the set of real numbers. A set S of vectors of V is said to be a basis of V if.

(i) S is linearly independent in V.

(ii) S spans V

Dimension.

The number of elements of the basis known as the dimension of the vector space.

Ex. Show that the set

$S = \{(1,0),(0,1)\}$ is a basis of R^2

Ex: Show that the set.

$S = \{(1,0,1),(0,1,1),(1,1,0)\}$ is a basis of R^3.

$c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0) = 0.$

$\Rightarrow c_1 + c_3 = 0$

$c_2 + c_3 = 0$

$c_1 + c_2 = 0$

$c_1 = c_2.$

$c_2 = c_3.$

$c_1 = c_2 = c_3 = 0$

$(a,b,c) = c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0)$

$a = c_1 + c_3$

$b = c_2 + c_3$

$c = c_1 + c_2.$

$a+b = c_1 + c_2 + 2c_3.$

$a+b-c = 2c_3$

$\Rightarrow c_3 = \dfrac{a+b-c}{2}.$

$c_2 = \dfrac{b+c-a}{2}.$

$c_1 = \dfrac{a+c-b}{2}$

as $a, b, c \in \mathbb{R}.$

$\Rightarrow c_1, c_2, c_3 \in \mathbb{R}$

so, S is a basis of $\mathbb{R}^3$

Ex Find a basis and the dimension of subspace W of $\mathbb{R}^3$, where

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid x+y+z = 0 \}$$

$$(x, y, z) = (x, y, -x-y)$$
$$= x(1, 0, -1) + y(0, 1, -1)$$

$S = \{ (1,0,-1), (0,1,-1) \}$

since all vectors canbe written inform of S.

so, S spans W

5cts

Rank of Matrix

Elementary row operations

(i) Interchang of two rows.

(ii) Multiply a row by non zero constant

(iii) Multiply a row by non-zero constant and add it to another row, replacing that row.

Row-echelon form

(i) If there is a row of all zeros then it must be at the bottom of the matrix

(ii) The first non-zero element of any row is one. That element is called $\boxed{\text{leading one}}$

(iii) The leading one of any row is to the right of the leading ~~row~~ one of the previous row, not necessarily to the immediate right.

Reduced Row-echelon form

(iv) All elements above and below the leading one are zero.

Rank : Rank of a matrix A is the positive integer r such that there exists at least one r-rowed square matrix with non-vanishing determinant while every $(r+1)$ or more rowed matrices have vanishing determinant.

Note : (i) The rank of a matrix A is the maximum number of linearly independent row vectors of A.

(ii) rank of A and A^T are same.

(iii) rank of a zero matrix is 0

(iv) For a rectangular matrix A of order $m \times n$,

$$\text{rank}(A) \leq \min(m, n)$$

Ex : Determine the rank of the following matrix

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$$

$|A| = 0 \quad \text{Rank}(A) \neq 3$

$\text{Rank}(A) \neq 2$.

$$\boxed{\text{Rank}(A) = 1}$$

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$$\begin{pmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{pmatrix} \xrightarrow{R_1^* = R_1/4} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{4} \\ 8 & 4 & 6 \\ -2 & -1 & -1.5 \end{bmatrix}$$

$$R_3^* = R_3 + 2R_1 \mid R_2^* = R_2 - 8R_1$$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex (H.W)

$$A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ 2 & 3 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

find the rank of A, B and $A+B$.

Ex Determine the value of x such that rank of A is 3.

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ x & 2 & 2 & 2 \\ 9 & 9 & x & 3 \end{pmatrix}$$

$$1\begin{vmatrix} 4 & -3 & 1 \\ 2 & 2 & 2 \\ 9 & x & 3 \end{vmatrix} - \begin{vmatrix} 4 & -3 & 1 \\ x & 2 & 2 \\ 9 & x & 3 \end{vmatrix} - \begin{vmatrix} 4 & 4 & 1 \\ x & 2 & 2 \\ 9 & 9 & 3 \end{vmatrix}$$

$$= \left[4(6-2x) + 3(6-18) + (2x-18)\right] - \left[4(6-2x) + 3(3x-18) + (x^2-18)\right]$$

$$- \left[4(6-18) - 4(3x-18) + (9x-18)\right]$$

$$= 24 - 8x - 36 + 2x - 18 - \left[24 - 8x + 9x - 54 + x^2 - 18\right]$$

$$- \left[-48 - 12x + 72 + 9x - 18\right]$$

$-30 - 6x - x^2 - x + 48 + 3x + 6 = 0$

$-x^2 - 4x + 24 = 0$

$x^2 + 4x - 24 = 0$

$$x = \frac{-4 \pm \sqrt{16 + 24 \times 4}}{2}$$

$= \cancel{-4 \pm \sqrt{16+96}}$ $\quad \frac{-4 \pm \sqrt{112}}{2}$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Coefficient Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = C$$

Augmented Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix}$$

if Rank (C) = Rank (A).

solution exists.

else solution doesnot exist.

Rank (C) = Rank(A) = no. of variables.

$\Rightarrow$ Unique solution

Gauss - Elimination Method.

Ex

$$8x + 2y - 2z = 8$$
$$2x + y + 9z = 12$$
$$x - 8y + 3z = -4$$

$$A = \begin{pmatrix} 8 & 2 & -2 & 8 \\ 2 & 1 & 9 & 12 \\ 1 & -8 & 3 & -4 \end{pmatrix}$$

$\downarrow R_1^* = \frac{1}{8}R_1$

$$\begin{pmatrix} 1 & 1/4 & -1/4 & 1 \\ 2 & 1 & 9 & 12 \\ 1 & -8 & 3 & -4 \end{pmatrix}$$

$\downarrow$ $R_2^* = R_2 - 2R_1$
$R_3^* = R_3 - R_1$

$$\begin{pmatrix} 1 & 1/4 & -1/4 & 1 \\ 0 & 1/2 & 19/2 & 10 \\ 0 & -\frac{33}{4} & 13/4 & -5 \end{pmatrix}$$

$\downarrow R_2^* = 2R_2$

$$\begin{pmatrix} 1 & 1/4 & -1/4 & 1 \\ 0 & 1 & 19 & 20 \\ 0 & -33/4 & 13/4 & -5 \end{pmatrix}$$

$\downarrow R_3^* = R_3 + \frac{33}{4} \cdot R_2$

$$\begin{pmatrix} 1 & 1/4 & -1/4 & 1 \\ 0 & 1 & 19 & 20 \\ 0 & 0 & 160 & 160 \end{pmatrix}$$

$\downarrow R_3^* = \frac{1}{160} R_3$

$$\begin{pmatrix} 1 & 1/4 & -1/4 & 1 \\ 0 & 1 & 19 & 20 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$ $\} \to$ ~~row~~ ech

By back-substitution

$z = 1$

$y + z \cdot 19 = 20$

$x + 1/4 \, y + -1/4 \, z = 1$

$\Big\} \to$ $\underline{\underline{x = y = z = 1}}$

Gauss - Jordan Method.

Ex.

$2x + 2y + 4z = 18$

$x + 3y + 2z = 13$

$3x + y + 3z = 14$

H.W

$$2x + 4y + 6z = 18$$
$$4x + 5y + 6z = 24$$
$$3x + y - 2z = 4$$

Finding Inverse of a Matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 5 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow R_1^* = R_1/2$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 1 & 1 & 5 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 0 & -1/2 & 3 & -1/2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -6 & 1 & -2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow R_3^* = R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -6 & 1 & -2 & -0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right]$$

$$\Big\downarrow R_3^* = R_3 \times \left(-\frac{1}{6}\right)$$

$$\begin{bmatrix} 1 & 3/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -6 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1/3 & -1/6 \end{bmatrix}$$

H.W

$$\begin{bmatrix} 2 & 4 & 6 & 18 \\ 4 & 5 & 6 & 24 \\ 3 & 1 & -2 & 4 \end{bmatrix}$$

$$\Big\downarrow R_1^* = R_1/2$$

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 3 & 1 & -2 & 4 \end{bmatrix} \xrightarrow[R_3^* = R_3 - 3R_1]{R_2^* = R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & -5 & -11 & -23 \end{bmatrix}$$

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$R_2^* = R_2^*(-1/3)$

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & -5 & -11 & -23 \end{bmatrix}$$

$R_3^* = R_3 + 5R_2$, $R_1^* = R_1 - 2R_2$

$\cancel{1 \quad 2 \quad 3 \quad 9}$
$\cancel{0}$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_2^* = R_2 + 2R_3$, $R_1^* = R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_3^* = (-1)R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore x = 4$

$y = -2$

$z = 3$

The null space of a Matrix

The null space of $m \times n$ matrix. A denoted by Null(A), is the set of all solutions to the ~~two~~ homogenous equation

$$AX = 0$$

Ex. find the null space of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix}$$

$$R_1^* = \left(-\frac{1}{3}\right) R_1$$

$$\begin{matrix} 1 & -2 & \frac{1}{3} & -\frac{1}{3} & \frac{7}{3} \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{matrix}$$

Sol :)
$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 6x_2 - x_3 + x_4 - 7x_5 = 0$$

$$x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 = 0$$

$$2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 = 0$$

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix}$$

$R_1^* = R_1\left(\frac{-1}{3}\right)$

$$\begin{bmatrix} 1 & -2 & \frac{1}{3} & -\frac{1}{3} & \frac{7}{3} & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}$$

$R_2^* = R_2^* - R_1$
$R_3^* = R_3 - 2R_1$

$$\begin{bmatrix} 1 & -2 & \frac{1}{3} & -\frac{1}{3} & \frac{7}{3} & 0 \\ 0 & 0 & \frac{5}{3} & \frac{10}{3} & \frac{4}{3} & 0 \\ 0 & 0 & \frac{13}{3} & \frac{26}{3} & \frac{2}{3} & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$2x_4 + \cancel{2x_5} = -2r + 4s - 2t$

$2x_4 - \cancel{2x_5} = -t$

$4x_4 = -2r + 4s - 3t \Rightarrow x_4 = \frac{1}{4}\left[-2r + 4s - 3t\right]$

$x_5 = -\frac{1}{2}r + s - t/4$

$(r, s, t \in R)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = r \overset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix}} + s \overset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}} + t \overset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -3/4 \\ -1/4 \end{pmatrix}} . \quad r, s, t \in R$$

Nullity

It is the dimension of the null space of the matrix

$S = \{(e_1, e_2, e_3)\}$

$c_1 e_1 + c_2 e_2 + c_3 e_3 = 0.$

$c_1 + 0 + 0 = 0 \Rightarrow c_1 = 0$

$0 + c_2 + 0 = 0 \Rightarrow c_2 = 0$

$0 + 0 + c_3 = 0 \Rightarrow c_3 = 0$

so, e_1, e_2 & e_3 are linearly independent

Rank - Nullity Theorem

for any $m \times n$ matrix A.

$\text{rank}(A) + \text{nullity}(A) = n$

H.W Verify Rank - Nullity Theorem for.

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}$$

Ans

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$$\Big\downarrow R_2^* \to R_2 - 2R_1 \mid R_3^* \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

$$\Big\downarrow R_3^* \to R_3 + 10R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$

$$R_3^* \to \frac{R_3}{-12} \Big\downarrow R_2^* \to - R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Rank = 3

$x_1 - 2x_2 + 3x_5 = 0$

$x_2 + 3x_3 + 2x_4 = 0$

$x_3 + v_4 = 0$

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Let $x_1 = k$,

$x_2 = l$

$k - 2l + 3x_5 = 0$

$$\boxed{x_5 = \frac{2l - k}{3}}$$

$$3x_3 + 2x_4 = -l$$
$$3x_3 + 3x_4 = 0$$
$$-\overline{\hspace{6em}}$$
$$-x_4 = +l$$

$$\boxed{x_4 = l}$$

$$\boxed{x_3 = -l}$$

$x_1 = k$

$x_2 = l$

$x_3 = -l$

$x_4 = l$

$x_5 = \frac{2l}{3} - \frac{1}{3}k$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1/3 \end{pmatrix} + l \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 2/3 \end{pmatrix}$$

Thus, nulity = 2

$\therefore$ $2 + 3 = 5$.

Thus rank - nulity thm is satisfied.

Laplace Transformation of periodic function

Th Suppose that $f:[0,\infty] \to \mathbb{R}$ is a periodic function with period $T>0$, that is $f(t+T)=f(t)$ for all $t>0$. If the Laplace transformation of f exists, then

$$F(s) = \frac{\int_0^T f(t) e^{-st}\, dt}{1-e^{-sT}}$$

Ex $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

$T = 2\pi$

$$F(s) = \frac{\int_0^{2\pi} \sin t\, e^{-st}\, dt}{1-e^{-(2\pi)s}}$$

$$I = \int_0^{2\pi} \sin t\, e^{-st}\, dt$$

$$I = \left[-\frac{\sin t \cdot e^{-st}}{s}\right]_0^{2\pi} + \frac{1}{s}\int_0^{2\pi} e^{-st} \cos t\, dt$$

$$= 0 + \frac{1}{s}\left[\left(-\frac{\cos t\, e^{-st}}{s}\right)_0^{2\pi} - \frac{1}{s}\int_0^{2\pi} e^{-st} \sin t\, dt\right]$$

$$sI = \left[\frac{1}{s} - \frac{e^{-2\pi s}}{s} + \frac{1}{s} I\right.$$

$$I(s^2+1) = -e^{-2\pi s} + 1$$

$$\Rightarrow I = \frac{1-e^{-s2\pi}}{s^2+1}$$

$$F(s) = \frac{I}{1-e^{-s2\pi}}$$

$$= \frac{\cancel{(1-e^{-s2\pi})}}{\cancel{(1-e^{-s2\pi})}(s^2+1)} = \frac{1}{s^2+1}$$

Ex $f(t) = \begin{cases} t, & 0 \le t < 1 \\ f(t-1), & t \geq 1 \end{cases}$

Find Laplace transformation of $f(t)$

$T = 1$.

$$F(s) = \frac{\int_0^1 f(t) \cdot e^{-st} dt}{1-e^{-sT}}$$

$$I = \int_0^1 t \cdot e^{-st} dt$$

$$= \left[-\frac{e^{-st}}{s} \cdot t \right]_0^1 + \frac{1}{s}\int_0^1 e^{-st} dt$$

$$= \frac{-e^{-s}}{s} + \frac{1}{s}\left[-\frac{e^{-st}}{s} \right]_0^1$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s}\left[-\frac{e^{-s}}{s} + \frac{1}{s} \right]$$

$$= \frac{-se^{-s}}{s^2} + \frac{1-e^{-s}}{s^2}$$

$$= \frac{1-(1-s)e^{-s}}{s^2}$$

$$F(s) = \frac{1-(1-s)e^{-s}}{s^2(1-e^{-s})}$$

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