ORDINARY DIFFERENTIAL EQUATIONS

Differential equations: A diff eq^n is an eq^n involving derivatives of one/more dependent variables wrt. one/more independent variables.

Ordinary differential equations: An ODE is an eq^n in which there is only one independent variable.

Order: The order of a diff. eq^n is the order of the highest derivative.

Degree: The degree of a diff. eq^n is the degree/power of the highest order derivative occuring in the eq^n.

Linear differential eq^n: A diff. eq^n is linear when the dependent variable and its derivative occur only with the first degree & no products of the dependent variable & its derivative / of various order derivative occurs.

$$y^n + c_1 y^{n-1} + \cdots + c_{n-1} y' + c_n y^0 = f(x)$$

* Non-linear - which are not linear eq^{n}s.

eg: $y' = 6x^2 \longrightarrow$ not linear.

Methods to solve ODE:

* **Variable separable form:**

The differential eq^n is of the form:

$$N(y)\,dy = M(x)\,dx$$

are called eq^n s. with separable variables, the sol^n of which are obtained by direct integration thus its sol^n is given by

$$\int N(y)\,dy = \int M(x)\,dx + c$$

is arbitrary constant.

* No. of constants = degree

Q. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

$\Rightarrow \displaystyle\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$\Rightarrow \tan^{-1}(y) = \tan^{-1}x + c$

$\Rightarrow \tan^{-1}y - \tan^{-1}x = c$

$\Rightarrow \tan^{-1}\left(\dfrac{y-x}{1+xy}\right) = c$

$\Rightarrow \boxed{\dfrac{y-x}{1+xy} = \tan c}$

Q. $x(1+y^2)\,dx - y(1+x^2)\,dy = 0$

$\Rightarrow x(1+y^2)\,dx = y(1+x^2)\,dy$

$\Rightarrow \displaystyle\int \frac{2y\,dy}{2(1+y^2)} = \int \frac{2x\,dx}{2(1+x^2)}$

$\Rightarrow \frac{1}{2}\ln(1+y^2) = \frac{1}{2}\ln(1+x^2) + c$

$\Rightarrow \ln(1+y^2)^{1/2} - \ln(1+x^2)^{1/2} = c$

$\Rightarrow \ln\left(\dfrac{1+y^2}{1+x^2}\right)^{1/2} = c$

$\Rightarrow \boxed{\dfrac{1+y^2}{1+y^2} = e^{2c}}$

Q. $x\dfrac{dy}{dx} = \sqrt{1-y^2}$

$\Rightarrow \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x}$

$\Rightarrow \sin^{-1}y = \ln x + c$

$\Rightarrow \sin^{-1}y - \ln x = c$

Q. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow \dfrac{\sec^2 x \, dx}{\tan x} = -\dfrac{\sec^2 y \, dy}{\tan y}$ $\qquad \dfrac{\cos x}{\sin x} \quad \dfrac{\sin x}{\cos x}$

$\Rightarrow \int \dfrac{1+\tan^2 x}{\tan x} dx = -\int \dfrac{(1+\tan^2 y)}{\tan y} dy$

$\Rightarrow \int (\cot x + \tan x) \, dx = -\int (\cot y + \tan y) \, dy$

$\Rightarrow \ln \sin x - \ln \cos x = -[\ln \sin y - \ln \cos y] + c$

$\Rightarrow \ln \sin x - \ln \cos x + \ln \sin y - \ln \cos y = c$

$\Rightarrow \ln \tan x + \ln \tan y = c$

$\Rightarrow \boxed{\tan x \tan y = e^c}$

Homogeneous eq^{n}s :

The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ —— (1) is called homogeneous if both $M(x,y)$ & $N(x,y)$ are homogeneous function of same degree.

Eg: $(x^2 + xy)\,dx + y^2 dy = 0$

The eq^n (1) can be written in form,

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y) \quad \text{—— (2)}$$

To solve eq^n (2), we use the fact that every homogeneous eq^n can be reduced to a separable eq^n by the transformation $y = vx$. This can be established as follows,

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$y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——— (3)}$$

from eqn (2) & (3) we get.

$$v + x\frac{dv}{dx} = -\frac{M(x,vx)}{N(x,vx)} = f(v)$$

$\Rightarrow v + x\frac{dv}{dx} = f(v)$ → now separate x & v.

$\Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$ which is in the form of the eqn with separable variable.

Q. $x^2y\,dx - (x^3+y^3)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3+y^3} = \frac{y/x}{1+(y/x)^3}$$

$y = xt \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$\Rightarrow t + x\frac{dt}{dx} = \frac{t}{1+t^3}$$

$$\Rightarrow x\frac{dt}{dx} = \frac{t}{1+t^3} - t = \frac{t-t-t^4}{1+t^3}$$

$$\Rightarrow x\frac{dt}{dx} = \frac{-t^4}{1+t^3}$$

$$\Rightarrow \frac{1+t^3}{t^4}dt = -\frac{dx}{x}$$

$$\Rightarrow \int(t^{-4}+t^{-1})dt = -\int\frac{dx}{x}$$

$$\Rightarrow \frac{-1}{3t^3} + \ln t = -\ln x - \ln c$$

$$\Rightarrow \frac{-1}{3t^3} = -\ln xct$$

$$\Rightarrow \ln xc \cdot \frac{y}{x} = \frac{x^3}{3y^3}$$

$$\Rightarrow \boxed{y = ce^{x^3/3y^3}}$$

Q. $x^2\frac{dy}{dx} - 3xy - 2y^2 = 0$

$\Rightarrow x^2\frac{dy}{dx} = 3xy + 2y^2$

$\Rightarrow \frac{dy}{dx} = 3\left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2$

$\Rightarrow t + x\frac{dt}{dx} = 3t + 2t^2$

$\Rightarrow x\frac{dt}{dx} = 2t + 2t^2$

$\Rightarrow \int \frac{dt}{t(1+t)} = 2\int \frac{dx}{x}$

$\Rightarrow \int\left(\frac{1}{t} - \frac{1}{1+t}\right)dt = 2\int\frac{dx}{x}$

$\Rightarrow \ln t - \ln(1+t) = 2\ln x + \ln c$

$\Rightarrow \ln\left(\frac{t}{1+t}\right) = \ln x^2 c$

$\Rightarrow \frac{y}{x+y} = x^2 c \Rightarrow \boxed{y = (x+y)x^2 c}$

Q. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ Let $y/x = t$

$\Rightarrow t + x\frac{dt}{dx} = t + \tan t$

$\Rightarrow x\frac{dt}{dx} = \tan t$

$\Rightarrow \int\frac{dt}{\tan t} = \int\frac{dx}{x}$

$\Rightarrow \log(\sin t) = \log xc$

$\Rightarrow \sin t = xc$

$\Rightarrow \boxed{y = x\sin^{-1}(xc)}$

Q. $x\frac{dy}{dx} = y + 2xe^{-y/x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + 2e^{-y/x}$

$\Rightarrow x\frac{dt}{dx} + t = t + 2e^{-t}$

$\Rightarrow \frac{dt}{2e^{-t}} = \frac{dx}{x}$

$\Rightarrow \frac{1}{2}\int e^t\,dt = \int\frac{dx}{x}$

$\Rightarrow \frac{1}{2}e^t = \log xc$

$\Rightarrow \boxed{e^{y/x} = \log x^2c^2}$

First order ordinary differential eq^n:

① Variable separable form:

$$f(x)\,dx + g(y)\,dy = 0$$

eg: $3e^x \tan y\,dx + (1+e^x)\sec^2 y\,dy = 0$

$\Rightarrow$ $\cancel{\frac{3e^x}{\sec^2 y}dx}$ $3e^x \tan y\,dx = -(1+e^x)\sec^2 y\,dy$

$\Rightarrow$ $\displaystyle\int \frac{3e^x}{1+e^x}dx = \int \frac{-\sec^2 y}{\tan y}dy$

$\tan y = t$

$\sec^2 y\,dy = dt$

$\Rightarrow$ $\cancel{\frac{3e^x+3-3}{1+e^x}dx}$

$\Rightarrow$ $3\ln(1+e^x) = -\ln(\tan y) + \ln c$

$\Rightarrow$ $\ln(1+e^x)^3 + \ln(\tan y) = \ln c$

$\Rightarrow$ $\boxed{(1+e^x)^3 \tan y = c}$

② Homogeneous form:

$$\frac{dy}{dx} = f(y/x) \qquad \text{let } y/x = v$$

eg: $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$

$\Rightarrow$ $-\dfrac{dy}{dx} = \dfrac{x\tan y/x - y\sec^2 y/x}{x\sec^2 y/x}$

$$\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \left(v + x\frac{dv}{dx}\right) = v - \frac{\tan v}{\sec^2 v}$$

$$\Rightarrow \int \frac{\sec^2 v}{\tan v}\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \ln(\tan v) = \ln x + \ln c$$

$$\Rightarrow \ln\left(\frac{\tan v}{x}\right) = \ln c \Rightarrow \tan y/x = xc$$

$$\Rightarrow \boxed{y = x \tan^{-1}(xc)}$$

3 Linear form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

IF: Integrating factor $= e^{\int P(x)dx}$

Q. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\Rightarrow 1+y^2 = (\tan^{-1}y - x)\frac{dy}{dx} \Rightarrow (1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\Rightarrow \cancel{1+y^2 = \tan^{-1}y\frac{dy}{dx} - x\frac{dy}{dx}} \Rightarrow (1+y^2)\frac{dx}{dy} + x = \tan^{-1}y$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \text{———} (1)$$

$$e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\Rightarrow$ multiplying IF to eqn (1)

$$e^{\tan^{-1}y}\frac{dx}{dy} + \frac{x}{1+y^2}e^{\tan^{-1}y} = \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}$$

$$\Rightarrow \frac{d}{dy}\left(x e^{\tan^{-1}y}\right) = \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}$$

$$\Rightarrow \int d\left(x e^{\tan^{-1}y}\right) = \int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}\,dy$$

$$\tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = dt$$

$$= \int \underset{I}{t}\cdot \underset{II}{e^t}\,dt$$

$$= te^t - \int 1\cdot e^t\,dt = te^t - e^t$$

$$x e^{\tan^{-1}y} = (\tan^{-1}y - 1)\,e^{\tan^{-1}y}$$

$$\Rightarrow \cancel{x = (\tan^{-1}y - 1)} \quad \Rightarrow \left((x+1) - \tan^{-1}y\right)e^{\tan^{-1}y} = c$$

$$\Rightarrow \cancel{\tan^{-1}y = x+1}$$

$$\Rightarrow \cancel{y = \tan(x+1)}$$

Bernoulli's eq^n:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Q. $\frac{dy}{dx} + y\tan x = y^3 \sec x$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x$

$\frac{1}{y^2} = t \Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{-dt}{2dx} + t\tan x = \sec x$

$\Rightarrow \frac{dt}{dx} - 2t\tan x = -2\sec x$

$IF = e^{-2\int \tan x\, dx} = e^{-2\ln \sec x\, dx}$

$= \cos^2 x$

$\Rightarrow \cos^2 x \frac{dt}{dx} - 2t\cos^2 x \tan x = -2\sec x \cos^2 x$

$\Rightarrow \cos^2 x \frac{dt}{dx} - t\sin 2x = -2\cos^3 x$

$\Rightarrow \frac{d}{dx}(t\cos^2 x) = -2\cos^3 x$

$\cos^2 x \cos x$

$(1-\sin^2 x)\cos x$

$\cos x - \sin^2 x \cos x$

$\Rightarrow d(t\cos^2 x) = -2\cos^3 x\, dx$

$\Rightarrow \int d(t\cos^2 x) = \int -2\cos x + 2\cos x \sin^2 x\, dx$

$\sin x = v$

$\cos x\, dx = dv$

$\int 2 \cdot v^2 dv$

$2 \cdot \frac{v^3}{3}$

$\Rightarrow t\cos^2 x = -2\sin x + 2 \cdot \frac{\sin^3 x}{3}$

$\Rightarrow \frac{1}{y^2}\cos^2 x + 2\sin x - \frac{2\sin^3 x}{3} = 0$

$\boxed{\frac{1}{y^2} + 2\tan x \sec x - 2\tan^2 x \sin x = 0}$

Exact differential equation :

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$

If its exact, then there is some function

$u(x,y)$

If (1) is exact then

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

differential form,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = M dx + N dy = 0$$

↳ sol^n $\boxed{u(x,y) = c}$

$$\frac{\partial u}{\partial x} = M \quad \& \quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial M}{\partial y} \qquad \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

If the second order partial derivative is continuous

then, $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial x \partial y} \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Q. $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy e^{xy^2} - 3y^2)\,dy = 0$

$$\frac{d e^{y^2}}{dy} = e^{xy^2}\cdot 2y$$

$$du = M dx + N dy = 0$$

$$M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xy e^{xy^2} - 3y^2$$

$$\frac{\partial M}{\partial y} = 2y e^{xy^2} + 2y^3 x e^{xy^2} \qquad \frac{\partial N}{\partial x} = 2y e^{xy^2} + 2y^3 x e^{xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, given differential is exact.

$$\frac{\partial U}{\partial x} = M = y^2 e^{xy^2} + 4x^3$$

$$\int \partial U = \int \left(y^2 e^{xy^2} + 4x^3 \right) \partial x$$

$$\Rightarrow \quad U = \frac{\cancel{y^2} e^{xy^2}}{\cancel{y^2}} + x^4 + h(y)$$

$$\Rightarrow \quad U = e^{xy^2} + x^4 + h(y)$$

$$\Rightarrow \quad \frac{\partial U}{\partial y} = 2xy e^{xy^2} + \frac{d}{dy} h(y) = N = 2xy e^{xy^2} - 3y^2$$

$$\therefore \quad \frac{d}{dy} h(y) = -3y^2$$

$$\Rightarrow \quad \int dh(y) = -\int 3y^2 \, dy$$

$$\Rightarrow \quad h(y) = -y^3 + c'$$

$$\therefore \quad U = e^{xy^2} + x^4 - y^3 + c' = c$$

$$\Rightarrow \quad \boxed{e^{xy^2} + x^4 - y^3 = c_1}$$

* Problem set : 1.3 Q2, Q3, Q7, Q9–Q-11, Q-16, Q-17 to Q25.
(Kreyszig 8th edition)

2 assignt. copies. (1-midsem, 1-endsem)
to be submitted for TA marks.

Q. $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$du = Mdx + Ndy$

$M = \left(1+e^{x/y}\right) \qquad N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = \frac{-x}{y^2}e^{x/y} \qquad \frac{\partial N}{\partial x} = \frac{1}{y}e^{x/y}\left(1-\frac{x}{y}\right) + e^{x/y}\left(\frac{-1}{y}\right)$

$= -\frac{x}{y^2}e^{x/y}$

$\frac{\partial u}{\partial x} = M$

$\frac{\partial u}{\partial x} = 1+e^{x/y}$

$\int \partial u = \int\left(1+e^{x/y}\right)dx$

$\Rightarrow u = x + ye^{x/y} + h(y)$

$\frac{\partial u}{\partial y} = 0 + e^{x/y} + ye^{x/y}\left(-\frac{x}{y^2}\right) + \frac{d}{dy}h(y)$

$= e^{x/y}\left(1-\frac{x}{y}\right) + \frac{d}{dy}(h(y)) = N$

$\frac{\partial h(y)}{\partial y} = 0$

$\Rightarrow h(y) = C_1$

$u = x + ye^{x/y} = c$

Q. $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y \, dy = 0$

$M = \sec x \tan x \tan y - e^x$ $\qquad N = \sec x \sec^2 y$

$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y$ $\qquad \frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y$

$$\frac{\partial v}{\partial x} = M = \sec x \tan x \tan y - e^x$$

$$\int \partial v = \int (\sec x \tan x \tan y - e^x)\, dx$$

$$\Rightarrow v = \sec x \tan y - e^x + h(y)$$

$$\frac{\partial v}{\partial y} = \sec x \sec^2 y + \frac{\partial h(y)}{\partial y} = N = \sec x \sec^2 y$$

$$\frac{\partial h(y)}{\partial y} = 0$$

$$\Rightarrow h(y) = c$$

$$\boxed{v = \sec x \tan y - e^x + c}$$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow$ not exact

Then we need to find a IF.

Rules: (for IF)

① If $Mdx + Ndy = 0$ is a homogeneous eq^n in x & y then IF is $\frac{1}{Mx + Ny}$, provided $Mx + Ny \neq 0$

Eg: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$M = x^2y - 2xy^2 \qquad N = (3x^2y - x^3)$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

Given ODE is not exact,

since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Then IF of (1) is $\dfrac{1}{x(x^2y - 2xy^2) + y(3x^2y - x^3)}$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}}$$

$$= \frac{1}{x^2y^2}$$

Multiplying (1) by IF; we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0.$$

$M_1 = \dfrac{1}{y} - \dfrac{2}{x} \qquad N_1 = \dfrac{3}{y} - \dfrac{x}{y^2}$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N_1}{\partial x} = \frac{-1}{y^2}$$

∴ The ODE is exact now,

$$\frac{\partial u}{\partial x} = M_1 = \frac{1}{y} - \frac{2}{x}$$

$$\boxed{\cancel{\int \partial u = \frac{1}{y^2}\int \partial x} \quad \Rightarrow \cancel{u = -\frac{x}{y^2} + h(y)} \quad \cancel{\frac{\partial u}{\partial y}} = \ldots}$$

$$\int \partial u = \int\left(\frac{1}{y} - \frac{2}{x}\right)\partial x$$

$$\Rightarrow u = \frac{x}{y} + \overset{-2\ln x}{\cancel{\frac{2}{x^2}}} + h(y)$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2} + \frac{\partial h(y)}{\partial y}, \quad N = \frac{3}{y} - \frac{x}{y^2}$$

$$\therefore \frac{\partial h(y)}{\partial y} = \frac{3}{y}$$

$$\Rightarrow \int \partial h(y) = 3\int \frac{\partial y}{y}$$

$$\Rightarrow h(y) = 3\ln y + c.$$

So, $$\boxed{u = \frac{x}{y} + \overset{-2\ln x}{\cancel{\frac{2}{x^2}}} + 3\ln y + c}$$

①Q. $(xy\,e^{x/y} + y^2)\,dx - x^2 e^{x/y}\,dy = 0$

$M = xy\,e^{x/y} + y^2 \qquad N = -x^2 e^{x/y}$

$$\frac{\partial M}{\partial y} = x e^{x/y} + xy\left(-\frac{x}{y^2}\right)e^{x/y} + 2y = x e^{x/y} - \frac{x^2}{y}e^{x/y} + 2y$$

$$\frac{\partial N}{\partial x} = -2x e^{x/y} - x^2 \cdot \frac{1}{y} e^{x/y} = -2x e^{x/y} - \frac{x^2}{y} e^{x/y}$$

$$\text{IF} = \frac{1}{x(xye^{x/y}+y^2) - yx^2e^{x/y}} = \frac{1}{xy^2}$$

Now,
$$\frac{1}{xy^2}(xye^{x/y}+y^2)dx - \frac{x^2}{xy^2}e^{x/y}dy = 0$$

$$\Rightarrow \left(\frac{e^{x/y}}{y} + \frac{1}{x}\right)dx - \frac{x}{y^2}e^{x/y}dy = 0.$$

$$M_1 = \frac{e^{x/y}}{y} + \frac{1}{x} \qquad N_1 = -\frac{x}{y^2}e^{x/y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}e^{x/y} + \frac{1}{y}\cdot\left(\frac{-x}{y^2}\right)e^{x/y} \qquad \frac{\partial N}{\partial x} = \frac{-e^{x/y}}{y^2} - \frac{xe^{x/y}}{y^3}$$

$$= \frac{-e^{x/y}}{y^2} - \frac{xe^{x/y}}{y^3}$$

$\therefore$ Now its exact.

$$\frac{\partial U}{\partial x} = M = \frac{e^{x/y}}{y} + \frac{1}{x}$$

$$\Rightarrow \int \partial U = \int\left(\frac{e^{x/y}}{y} + \frac{1}{x}\right)\partial x$$

$$\Rightarrow U = \frac{e^{x/y}}{y}\cdot\frac{1}{y} - \frac{1}{x^2} + h(y)$$

Rule II: If $Mdx + Ndy = 0$ can be written in the form

$$f(xy)y\,dx + f_2(xy)x\,dy = 0$$

then the IF $= \dfrac{1}{Mx - Ny}$ provided $Mx - Ny \neq 0$

(2) Eg: $(2xy+1)y\,dx + (1+2xy - x^3y^3)x\,dy = 0$

$M = 2xy^2 + y$ $\qquad$ $N = x + 2x^2y - x^4y^3$

$\dfrac{\partial M}{\partial y} = 4xy + 1$ $\qquad$ $\dfrac{\partial N}{\partial x} = 1 + 4xy - 4x^3y^3$

$$IF = \frac{1}{x(2xy^2+y) - y(x + 2x^2y - x^4y^3)}$$

$$= \frac{1}{\cancel{2x^2y^2} + \cancel{xy} - \cancel{xy} - \cancel{2x^2y^2} + x^4y^4} = \frac{1}{x^4y^4}$$

$\therefore$ $\dfrac{y}{x^4y^4}(2xy+1)\,dx + \dfrac{x}{x^4y^4}(1 + 2xy - x^3y^3)\,dy = 0$

$\Rightarrow$ $\left(\dfrac{2}{x^3y^2} + \dfrac{1}{x^4y^3}\right)dx + \left(\dfrac{1}{x^3y^4} + \dfrac{2}{x^2y^3} - \dfrac{1}{y}\right)dy = 0.$

$\qquad M_1 \qquad\qquad\qquad N_1$

$\dfrac{\partial M_1}{\partial y} = \dfrac{-4}{x^3y^3} - \dfrac{3}{x^4y^4}$ $\qquad$ $\dfrac{\partial N_1}{\partial x} = \dfrac{-3}{x^4y^4} \cancel{- \dfrac{6}{x^4y^3}} \cancel{\dfrac{1}{x^2y}} - \dfrac{4}{x^3y^3}$

Hence its exact,

$$\frac{\partial U}{\partial x} = M_1 = \frac{2}{x^3y^2} + \frac{1}{x^4y^3}$$

$$\int \partial U = \int\left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right) dx \qquad \frac{2x^{-3}}{y^3} \quad \frac{x^{-2}}{-2} \quad \frac{1}{y^3}\, x^{-4} \quad \frac{-1}{3x^3y^3}$$

$$= -\frac{1}{x^2y^2} - \frac{1}{3x^3y^3}$$

$$\Rightarrow U = \cancel{\frac{-6}{x^4y^2}} \cancel{- \frac{y}{x^5y^3}} \cancel{+h(y)} = N_1$$

③ Q. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$

Problem-set 1.5 (8, 9, 11, 12, 14, 16–21, 23, 26, 30, 31, 33, 35, 37)

① other half:

$$\frac{\partial U}{\partial x} = M_1 = \frac{e^{x/y}}{y} + \frac{1}{x}$$

$$\int \partial U = \int\left(\frac{e^{x/y}}{y} + \frac{1}{x}\right) \partial x$$

$$\Rightarrow U = \frac{e^{x/y}}{\cancel{y}} \cdot \cancel{y} + \ln x + h(y) =$$

$$\therefore U = e^{x/y} + \ln x + h(y)$$

$$\frac{\partial U}{\partial y} = \frac{-xe^{x/y}}{y^2} + \cancel{\frac{1}{x}}^{0} + \frac{\partial h(y)}{\partial y} = N_1 = \frac{-x}{y^2}e^{x/y}$$

$$\frac{\partial h(y)}{\partial y} = 0 \Rightarrow h(y) = c_1$$

$$\therefore \boxed{U = e^{xy} + \ln x + c_1}$$

$$U = -\frac{1}{x^2y^2} - \frac{1}{3x^3y^3} + h(y)$$

$$\frac{\partial U}{\partial y} = \frac{2}{x^2y^3} + \frac{1}{x^3y^4} + \frac{d\,h(y)}{\partial y} = N = \frac{2}{x^2y^3} + \frac{1}{x^3y^4} + \frac{1}{y}$$

$$\frac{\partial h(y)}{\partial y} = \frac{1}{y} \Rightarrow \int \partial h(y) = \int \frac{\partial y}{y}$$

$$\Rightarrow h(y) = \ln y + c$$

$$\therefore \boxed{U = \frac{-1}{x^2y^2} - \frac{1}{3x^3y^3} + \ln y + c}$$

3) $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$

$\quad\downarrow \qquad\qquad\qquad\qquad \downarrow$

$\quad M \qquad\qquad\qquad\qquad N$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \qquad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

This is not exact.

$$I_f = \frac{1}{x(xy^2 + 2x^2y^3) - y(x^2y - x^3y^2)}$$

$$= \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3}$$

$$= \frac{1}{3x^3y^3}$$

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Now, $\frac{1}{3x^3y^3}(xy^2+2x^2y^3)\,dx+\frac{1}{3x^3y^3}(x^2y-x^3y^2)\,dy=0$

$$\Rightarrow \underbrace{\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)}_{M_1}dx+\underbrace{\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)}_{N_1}dy=0$$

$$\frac{\partial M_1}{\partial y}=\frac{-1}{3x^2y^2} \qquad \frac{\partial N_1}{\partial x}=\frac{-1}{3x^2y^2}$$

$$\frac{\partial v}{\partial x}=M_1=\frac{1}{3x^2y}+\frac{2}{3x}$$

$$\int \partial v=\int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)\partial x$$

$$\Rightarrow v=\frac{-1}{3xy}+\frac{2}{3}\ln x+h(y)$$

$$\frac{\partial v}{\partial y}=\frac{1}{3xy^2}+\cancel{\frac{2}{3x}}^{0}+\frac{\partial h(y)}{\partial y}=N_1=\frac{1}{3xy^2}-\frac{1}{3y}$$

$$\therefore \frac{\partial h(y)}{\partial y}=\frac{-1}{3y}$$

$$\Rightarrow \int \partial h(y)=\frac{-1}{3}\int\frac{\partial y}{y}$$

$$\Rightarrow h(y)=-\frac{\ln y}{3}+c$$

$$\therefore \boxed{v=\frac{-1}{3xy}+\frac{2}{3}\ln x-\frac{\ln y}{3}+c}$$

Rule III:

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x, say $f(x)$, then

$e^{\int f(x)dx}$ is an IF of $Mdx + Ndy = 0$

eg: $(x^2+y^2+1)dx - 2xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$$

Now, $$\frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x)$$

$$IF = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\left(\frac{x^2+y^2+1}{x^2}\right)dx - \frac{2xy}{x^2}dy = 0$$

$$\Rightarrow \left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}dy = 0.$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

$$\frac{\partial U}{\partial x} = 1+\frac{y^2}{x^2}+\frac{1}{x^2} \qquad \frac{\partial U}{\partial y} = 0 - \frac{2y}{x} + \frac{\partial h(y)}{\partial y}$$

$$U = \int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx$$

$$= x - \frac{y^2}{x} - \frac{1}{x} + h(y) \quad \therefore U = x - \frac{y^2}{x} - \frac{1}{x} + c$$

Q. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2xy \qquad \dfrac{\partial N}{\partial x} = -2xy$

Now, $\dfrac{4xy}{-x^2y} = \dfrac{-4}{x} \qquad IF = e^{-4\int \frac{1}{x}dx} = \dfrac{1}{x^4}$

$\therefore \cancel{\left(-4y^2 + \dfrac{4e^{1/x^3}}{x}\right)dx + 4xy\,dy = 0}$

$\cancel{\dfrac{\partial M_1}{\partial y} = -8y} \qquad \partial.$

$\therefore \left(\dfrac{y^2}{x^3} - \dfrac{e^{1/x^3}}{x^4}\right)dx - \dfrac{y}{x^2}dy = 0$

$\dfrac{\partial M_1}{\partial y} = \dfrac{2y}{x^3} \qquad \dfrac{\partial N_1}{\partial x} = \dfrac{2y}{x^3}$

$\therefore \dfrac{\partial U}{\partial x} = \dfrac{y^2}{x^3} - \dfrac{e^{1/x^3}}{x^4}$

$\dfrac{1}{x^4} = t$

$e^t \qquad \dfrac{1}{x^3} = t$

$\dfrac{-3}{x^4}dx = dt$

$\int \partial U = \int\left(\dfrac{y^2}{x^3} - \dfrac{e^{1/x^3}}{x^4}\right)\partial x$

$= \dfrac{-y^2}{2x^2} + \dfrac{1}{3}e^{1/x^3} + h(y)$

$\dfrac{\partial U}{\partial y} = \dfrac{-y}{x^2} + \dfrac{\partial h(y)}{\partial y} = \dfrac{-y}{x^2}$

$\Rightarrow h(y) = C$

So, $\boxed{U = \frac{-y}{x^2} + \frac{e^{1/x^3}}{3} = C}$

Rule : 4: If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a f^n of y, say $f(y)$

then the IF is $e^{\int f(y)\,dy}$

eg: $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

$\frac{\partial M}{\partial y} = 3xy^2 + 1$ $\qquad$ $\frac{\partial N}{\partial x} = 4xy^2 + 2$

Now, $\frac{4xy^2 + \cancel{2} - 3xy^2 - \cancel{1}}{xy^3 + y} = \frac{xy^2 + 1}{xy^3 + y}$

$= \frac{1}{y}$

$IF = e^{\int \frac{1}{y} dy} = y$

$(xy^4 + y^2)\,dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0$

$\frac{\partial M_1}{\partial y} = 4xy^3 + 2y$ $\qquad$ $\frac{\partial N_1}{\partial x} = 4xy^3 + 2y$

$\frac{\partial U}{\partial x} = xy^4 + y^2$ $\qquad$ $\frac{\partial h(y)}{\partial y} = 2y^5$

$\partial U = \int (xy^4 + y^2)\,dx$ $\qquad$ $h(y) = 2\frac{y^6}{6_3} = \frac{y^6}{3}$

$U = \frac{x^2y^4}{2} + xy^2 + h(y)$

$\boxed{U = \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3}}$

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Q. $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

$$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$$

Now, $f(y) = \dfrac{y^3-4-4y^3-2}{y^4+2y}$

$$= \frac{-3y^3-6}{y^4+2y} = \frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{-3}{y}$$

$$IF = \frac{1}{y^3}$$

$$\therefore \left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

$$\frac{\partial M_1}{\partial y} = 1-\frac{4}{y^3} \qquad \frac{\partial N_1}{\partial x} = 1-\frac{4}{y^3}$$

$$\frac{\partial v}{\partial x} = y+\frac{2}{y^2}$$

$$\int \partial v = \int \left(y+\frac{2}{y^2}\right)\partial x \qquad \frac{\partial h(y)}{\partial y} = 2y$$

$$\Rightarrow v = xy + \frac{2x}{y^2} + h(y) \qquad \int \partial h(y) = \int 2y\,\partial y$$

$$\partial h(y) = y^2$$

$$\frac{\partial v}{\partial y} = x - \frac{4}{y^3} + \frac{\partial h(y)}{\partial y}$$

Riccati Differential equation:

$$y'(x) = a_0(x) + a_1(x)y(x) + a_2(x)y(x)^2 \quad \text{—} \quad ①$$

If $a_0(x) = 0$ then eqⁿ ① reduces to Bernoulli's eqⁿ

If $a_2(x) = 0$ then eqⁿ ① reduces to linear diff eqⁿ.

$\therefore$ $a_0(x) \neq 0$ $\quad a_2(x) \neq 0$.

but $a_1(x) \rightarrow$ can be zero

Problem set : 1.6 Q-44

$$y' = x^3(y-x)^2 + \frac{y}{x}$$

$$\frac{x}{x-y} = \frac{x^5}{5} + c$$

$$v = y - x$$

$$\frac{dv}{\partial x} = \frac{\partial y}{\partial x} - 1$$

$$y' = x^3 v^2 + \frac{v+x}{x}$$

$$\frac{\partial v}{\partial x} + \cancel{1} = x^3 v^2 + \frac{v}{x} + \cancel{1}$$

$$\frac{-1}{x-y}$$

Let $\frac{-1}{v} = u$

$$\frac{\partial v}{\partial x} = v^2 x^3 + \frac{v}{x}$$

$$\Rightarrow \frac{+1}{v^2}\frac{dv}{dx} = \frac{du}{dx}$$

$$\frac{1}{v^2}\frac{dv}{dx} = x^3 + \frac{1}{vx}$$

$$\frac{1}{v^2}\frac{dv}{dx} - \frac{1}{vx} = x^3$$

IF $= x$

$$\Rightarrow \frac{du}{dx} + \frac{u}{x} = x^3$$

$$\Rightarrow x\frac{du}{dx} + u = x^4 \quad \Rightarrow \quad x\,du + u\,dx = x^4 dx$$

$$\Rightarrow d(xu) = x^4 dx \Rightarrow xu = \frac{x^5}{5} + c$$

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$\Rightarrow \quad xv = \frac{x^5}{5} + c$

$\Rightarrow \quad \boxed{\frac{x}{x-y} = \frac{x^5}{5} + c}$

Clairaut's equation:

$y = px + f(p)$ where $p = \frac{dy}{dx}$

$y = xy' + f(y')$

$y = px + \frac{a}{p}$ 2

diff. wrt x, $p = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$

$\Rightarrow \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = y$

$\frac{dp}{dx} = 0 \Rightarrow p = c$

The general soln is

$y = cx + \frac{a}{c}$, $x = \frac{a}{p^2} \Rightarrow p^2x = a$

$py = p^2x + a = 2a$

$\Rightarrow p = \frac{2a}{y}$

$\therefore \quad y = \frac{2ax}{y} + \frac{a \cdot y}{2a} = \frac{2ax}{y} + \frac{y}{2}$

$\Rightarrow \quad \boxed{y^2 = 4ax}$

Prob set : 1.6 Q - 48.

1 Q. $y = xy' - \frac{y'}{\sqrt{1+y'^2}}$ general solⁿ: $y = cx + f(c)$

$$\frac{dy}{dx} = y' + xy'' + \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]$$

$x^{1/2 - 1}$ $\frac{-1}{2x^{3/2}}$

~~let $1+y'^2 = t$~~

~~$2y'y'' dx = dt$~~

$$= y' + xy'' + \frac{y''\sqrt{1+y'^2} + \frac{y'}{2(1+y'^2)^{3/2}}}{(1+y'^2)}$$

$$= y' + xy'' + \frac{2y''(1+y'^2) + y'}{2(y'^2+1)^{5/2}}$$

Prob. set : 1.6   Q - 7, 12, 13, 14, 16, 17, 19, 20, 22, 33, 34, 35, 36, 38, 44, 46, 47, 48

Q.1. $\frac{dy}{dx} = \frac{x-y+1}{x+y-3}$

$\Rightarrow (x+y-3)\,dy - (x-y+1)\,dx = 0$

$\Rightarrow (x-y+1)\,dx - (x+y-3)\,dy = 0$

$\frac{\partial M}{\partial y} = -1$ $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial U}{\partial x} = x-y+1 \Rightarrow \int \partial U = \int (x-y+1)\,\partial x$

$\Rightarrow U = \frac{x^2}{2} - yx + x + h(y)$

$\frac{\partial U}{\partial y} = -x + \frac{\partial h(y)}{\partial y}$

$\therefore \frac{\partial h(y)}{\partial y} = 3-y$

$h(y) = 3y - \frac{y^2}{2} + c$

$\therefore \boxed{U = \frac{x^2}{2} - yx + x + 3y - \frac{y^2}{2} + c = 0}$

$(x-y)^3 = c(x+y+2$

2. $(2x+y+3)\frac{dy}{dx} = x+2y+3$

$\cancel{(2x+y+3)\,dy}$ $(x+2y+3)\,dx - (2x+y+3)\,dy = 0$

$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = -2$

not exact.

$$IF = \frac{1}{x(x+2y+3) - y(2x+y+3)}$$

$$= \frac{1}{x^2+\cancel{2xy}+3x + \cancel{2xy} - y^2 - 3y}$$

$$= \frac{1}{\cancel{x^2+3x+y^2+3y+4xy}} = \frac{1}{x^2-y^2+3x-3y}$$

$$\frac{x+2y+3}{x^2+3x+y^2-3y\ \cancel{+4xy}}\,dx - \frac{(2x+y+3)}{x^2+y^2+3x+3y}$$

No

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+3} = \cancel{1+2y/x+3/}$$

$\cancel{x = X+h}$ $y =$ . . $x = X+h$

$y = Y+k$

$$= \frac{X+h+2Y+2k+3}{2X+2h+Y+k+3}$$

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

$h+2k+3=0 \Rightarrow 2h+4k+6=0$

$2h+k+3=0 \qquad 2h+k+3=0$

$3k+3=0$

$\boxed{h=-1}$ $\boxed{k=-1}$

Let $Y = vX$ $\qquad v = \frac{Y}{X} = \frac{y+1}{x+1}$

$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$

$\Rightarrow v + X\frac{dv}{dX} = \frac{1+2v}{2+v}$

$\Rightarrow X\frac{dv}{dX} = \frac{1+\cancel{2v}-\cancel{2v}-v^2}{2+v}$

$\Rightarrow \frac{2+v}{1-v^2} = \frac{dX}{X}$

$\quad 1-v^2 = t$

$\quad -2v\,dv = dt$

$\Rightarrow \frac{2+v}{(1+v)(1-v)} = \frac{dX}{X}$

$\Rightarrow \frac{2}{(1+v)(1-v)} + \frac{v}{(1+v)(1-v)} = \frac{dX}{X}$

$\Rightarrow \int\frac{(1+v)+(1-v)}{(1+v)(1-v)} - \frac{1}{2}\int\frac{dt}{t} = \int\frac{dX}{X}$

$\Rightarrow \ln(1-v) + \ln(1+v) - \frac{1}{2}\ln(1-v^2) = \ln XC$

$\Rightarrow \ln(1-v^2) - \ln(1-v^2)^{1/2} = \ln XC$

$\Rightarrow \ln\left(\frac{1-v^2}{\sqrt{1-v^2}}\right) = \ln XC$

$\Rightarrow \ln\sqrt{1-v^2} = \ln XC$

$\Rightarrow \sqrt{1-v^2} = XC$

$\Rightarrow 1-\left(\frac{y+1}{x+1}\right)^2 = (x+1)^2C^2 \quad \Rightarrow \boxed{\frac{(y+1)^2}{(1+x)^2} = 1 - C_1^2(x+1)^2}$

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Q. $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial u}{\partial x} = y\cos x + \sin y + y$$

$$u = y\sin x + x\sin y + xy + h(y)$$

$$\frac{\partial u}{\partial y} = \sin x + x\cos y + x + \frac{\partial h(y)}{\partial y}$$

$$\frac{\partial h(y)}{\partial y} = 0 \Rightarrow h(y) = c$$

$$\therefore \boxed{u = y\sin x + x\sin y + xy + c}$$

Q. $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

$$f = \cancel{\frac{x^2 - 4xy - 6xy + 3x^2}{3x^2y - x^3}} , \quad \cancel{\frac{4x^2 - 10xy}{3x^2y - x^3}}$$

$$IF = \frac{1}{x(x^2y - 2xy^2) + y(3x^2y - x^3)} = \frac{1}{x^3y - 2x^2y^2 + 3x^2y^2 - x^3y}$$

$$= \frac{1}{x^2y^2}$$

$$\therefore \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2},$$

$$\frac{\partial v}{\partial x} = \frac{1}{y} - \frac{2}{x} \Rightarrow v = \frac{x}{y} - 2\ln x + h(y)$$

$$\frac{\partial v}{\partial y} = \frac{-x}{y^2} + \frac{dh(y)}{dy}$$

$$\frac{\partial h(y)}{\partial y} = \frac{3}{y} \Rightarrow h(y) = 3\ln y + c$$

$$\therefore \quad v = \frac{x}{y} - 2\ln x + 3\ln y + c = 0$$

$$\Rightarrow \boxed{\frac{x}{y} + \ln\frac{y^3}{x^2} = C}$$

Q. $y(x^2y^2 + xy + 1)\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$

$$= (x^2y^3 + xy^2 + y)\,dx + (x^3y^2 - x^2y + x)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \qquad \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$$

$$IF = \frac{1}{x(x^2y^3 + xy^2 + y) - y(x^3y^2 - x^2y + x)} = \frac{1}{\cancel{x^3y^3} + x^2y^2 + \cancel{xy} - \cancel{x^3y^3} + x^2y^2 - \cancel{xy}}$$

$$= \frac{1}{2x^2y^2}$$

$$\therefore eq^n: \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = 0$$

$$\frac{\partial v}{\partial x} = \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2 y}$$

$$v = \frac{xy}{2} + \frac{1}{2}\ln x - \frac{1}{2xy} + h(y)$$

$$\frac{\partial v}{\partial y} = \frac{x}{2} + \frac{1}{2xy^2} + \frac{\partial h(y)}{dy}$$

$$\frac{\partial h(y)}{\partial y} = \frac{-1}{2y}$$

$$h(y) = \frac{-1}{2}\ln y + c.$$

$$\therefore\ v = \frac{xy}{2} + \frac{1}{2}\ln x - \frac{1}{2xy} - \frac{1}{2}\ln y + c$$

$$= \boxed{xy - \frac{1}{xy} + \log\frac{x}{y} = c}$$

Q5. $(x^2+y^2)\,dx - 2xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$$

$$f(x) = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x}$$

$$IF = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$\frac{x-2y}{-1} = -y$

$$\frac{\partial U}{\partial x} = 1+\frac{y^2}{x^2} \quad \Rightarrow \quad U = x - \frac{y^2}{x} + h(y)$$

so $$\frac{\partial U}{\partial y} = -\frac{2y}{x} + \frac{\partial h(y)}{\partial y}$$

$$h(y) = c$$

$$\therefore \quad U = x - \frac{y^2}{x} + c = 0$$

$$\Rightarrow \boxed{x^2 - y^2 = cx}$$

Q. $\frac{dy}{dx} + \frac{1}{x}y = 3x$

$xy = t$

$$e^{\int \frac{1}{x}dx} = x$$

$y = ctx$

dy

$$x\frac{dy}{dx} + y = 3x^2$$

$$\Rightarrow x\,dy + y\,dx = 3x^2\,dx$$

$$\Rightarrow \int d(xy) = \int 3x^2\,dx$$

$$\Rightarrow \boxed{xy = x^3 + c}$$

$\log x = t$

$$e^{\int \frac{1}{x\log x}dx} = \log x$$

Q. $x\log x \frac{dy}{dx} + y = 2\log x$

~~$\Rightarrow \log x \frac{dy}{}$~~

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

$\frac{y}{x} = t$

$$\Rightarrow \log x \frac{dy}{dx} + \frac{1}{x}\cdot y = \frac{2}{x}\log x$$

$$\log x \frac{dy}{dx} + \frac{y}{x} = \frac{2}{x}\log x$$

$$\Rightarrow \int d(y\log x) = \int \frac{2}{x}\log x$$

$\Rightarrow \boxed{y\log x = (\log x)^2 + c}$ $= \frac{1}{x}\log x$

$= t\,dt$
t^2

Q. $\frac{dy}{dx} + 5y = 3e^x \quad y(0) = 1$

$$\text{IF} = e^{\int 5dx} = e^{5x}$$

$$e^{5x}\frac{dy}{dx} + 5e^{5x}y = 3e^x \cdot e^{5x}$$

$$\Rightarrow \int \frac{d}{dx}(ye^{5x}) = \int 3e^{6x}\,dx$$

$$\Rightarrow ye^{5x} = \frac{1}{2}e^{6x} + c$$

$$\Rightarrow y = \frac{e^x}{2} + ce^{-5x}$$

$$1 = \frac{1}{2} + ce^0 \Rightarrow c = \frac{1}{2}$$

$$\Rightarrow \boxed{y = \frac{1}{2}(e^x + e^{-5x})}$$

Q1.
\# $y = xy' - \frac{y'}{\sqrt{1+y'^2}}$

Let $y' = p \Rightarrow \frac{dy}{dx} = p$

$$\Rightarrow y = xp - \frac{p}{\sqrt{1+p^2}}$$

$$dy = xdp + pdx - \frac{dp\sqrt{1+p^2} - \frac{p^2 dp}{\sqrt{1+p^2}}}{(\sqrt{1+p^2})^2}$$

$$dy = xdp + pdx - \frac{(1+p^2)dp - p^2 dp}{(1+p^2)^{3/2}}$$

$$dy = xdp + pdx - \frac{dp}{(1+p^2)^{3/2}}$$

Now, $dy = \cancel{xdp}\ pdx$

$$\cancel{xdp} = \overset{xdp}{\cancel{xdp}} + \cancel{pdx} - \frac{dp}{(1+p^2)^{3/2}}$$

$$\Rightarrow xdp - \frac{dp}{(1+p^2)^{3/2}} = 0$$

$$\Rightarrow \left(x - \frac{1}{(1+p^2)^{3/2}}\right)dp = 0 \Rightarrow dp = 0 \Rightarrow p = c$$

$\therefore$ General solution: $y = cx + \sqrt{1+c^2}$

& $x - \frac{1}{(1+p^2)^{3/2}} = 0$

$\Rightarrow (1+p^2)^{3/2} = \frac{1}{x}$

$\Rightarrow (1+p^2) = \frac{1}{x^{2/3}}$

$\Rightarrow p^2 = \frac{1-x^{2/3}}{x^{2/3}}$

$\Rightarrow p = \sqrt{\frac{1}{x^{2/3}} - 1}$

~~$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1}{x^{2/3}} - 1}$~~

~~$\Rightarrow \int dy = \int \left(\frac{1}{x^{2/3}} - 1\right)^{1/2} dx$~~

corresponding parameter for y:

$$y = \frac{p}{(1+p^2)^{3/2}} - \frac{p}{(1+p^2)^{1/2}} = \frac{p - p(1+p^2)}{(1+p^2)^{3/2}} = \frac{p - p - \ldots}{(1+p^2)^{3/2}}$$

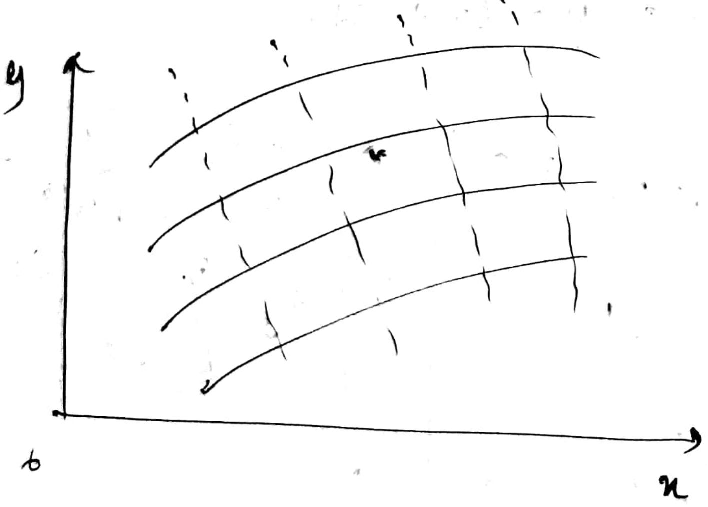
$$y = \frac{-p^3}{(1+p^2)^{3/2}}$$

Now $\quad x^2 + y^2 = \dfrac{1}{(1+p^2)^3} + \dfrac{p^6}{(1+p^2)^3} = \dfrac{1+p^6}{(1+p^2)^3}$

$$x^2 + y^2 = \frac{\cancel{(1+p^2)}(1 - p^2 + p^4)}{(1+p^2)^{\cancel{3}\,2}} = \frac{1+p^4-p^2}{(1+p^2)^2}$$

Orthogonal Trajectory:

2 family of chords are said to be orthogonal trajectory if every member of either family intersects each no. of chords of either family ~~intersects~~ at 90°.



$$\begin{cases} f(x,y,c) = 0 \\ g(x,y,c_1) = 0 \end{cases}$$

$$f(x,y,c) = 0$$

$$x + y + c = 0$$

$$\underline{\quad\quad\quad\quad}$$

$$x^2 + y^2 + c = 0$$

given a family of curves $f(x,y,c) = 0$ which can be represented by a differential eq^n.

$$\frac{dy}{dx} = F(x,y)$$

then, the slope of a curve of the family of curves at the point (x_0, y_0) is $F(x_0, y_0)$. The slope of the corresponding orthogonal trajectory at that point (x_0, y_0) should be $\frac{-1}{F(x_0, y_0)}$ since the two tangents to the curves at the intersection point (x_0, y_0) are perpendicular to each other.

The differential eq^n of the orthogonal trajectory is

$\frac{dy}{dx} = \frac{-1}{F(x,y)}$ hence, the eq^n of orthogonal trajectory can be obtained by solving this differential eq^n.

Q. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$ where a being a varying parameter.

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{\sqrt{4ax}} = \frac{\sqrt{a}}{x} = \frac{2a}{y} \quad \text{—②}$$

①

$$\cancel{\frac{dy}{dx} = \frac{-x}{\sqrt{a}} \Rightarrow \int \sqrt{a}\,dy = \int x\,dx}$$

$$\cancel{\Rightarrow \sqrt{a}\,y = -\frac{x^2}{2}}$$

eliminating a from ① & ②,

$$\frac{dy}{dx} = \frac{y}{2x}$$

Diff. eqn of orthogonal trajectory is

$$\frac{dy}{dx} = -\frac{dx}{dy} = \frac{y}{2x}$$

$$\Rightarrow \quad -\int 2x\,dx = \int y\,dy$$

$$\Rightarrow \quad -x^2 = \frac{y^2}{2} + c$$

$$\Rightarrow \quad \boxed{2x^2 + y^2 = c}$$

Q. Find the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$

$$\Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$-\frac{dx}{dy} = \cancel{-}\left(\frac{y}{x}\right)^{1/3}$$

$$\int x^{1/3}\,dx = \int y^{1/3}\,dy$$

$$\Rightarrow \quad \frac{3x^{4/3}}{4} = \frac{3y^{4/3}}{4} + c$$

$$\Rightarrow \quad \boxed{\frac{3}{4}x^{4/3} - \frac{3}{4}y^{4/3} = c} \Rightarrow \boxed{x^{4/3} - y^{4/3} = c_1}$$

Q. Find the orthogonal trajectory of the family of curves:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1 \qquad \lambda \rightarrow \text{varying parameter.}$$

$$\frac{y^2}{b^2+\lambda}=1-\frac{x^2}{a^2}$$

$$\Rightarrow b^2+\lambda=\frac{a^2y^2}{a^2-x^2}$$

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x}{a^2}\frac{(b^2+\lambda)}{2y}=-\frac{x}{y}\left(\frac{b^2+\lambda}{a^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}\left(\frac{\cancel{b^2}+\frac{a^2y^2}{a^2-x^2}-\cancel{b^2}}{a^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y}\left(\frac{\cancel{a}y^2}{a^2-x^2}\right)=\frac{-\cancel{a}xy}{a^2-x^2}$$

Now

$$-\frac{dx}{dy}=\frac{\cancel{a}xy}{a^2-x^2}$$

$$\Rightarrow \int\frac{a^2-x^2}{x}\,dx=\int \cancel{a}y\,dy$$

$$\Rightarrow \int\left(\frac{a^2}{x}-x\right)dx=\int \cancel{a}y\,dy$$

$$\Rightarrow a^2\ln x-\frac{x^2}{2}=\frac{\cancel{a}y^2}{2}+c$$

$$\Rightarrow \boxed{\frac{x^2}{2}+\frac{\cancel{a}y^2}{2}-a^2\ln x=c}$$

Q. Prove that the family of con-focal & co-axial para...
$y^2 = 4a(x+a)$ is self orthogonal;

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$y^2 = 4ax + 4a^2$$

$$\Rightarrow 4a^2 + 4ax - y^2 = 0$$

$$a = \frac{-4x \pm \sqrt{16x^2+16y^2}}{8} = \frac{-4x \pm 4\sqrt{x^2+y^2}}{8}$$

$$= \frac{-2x \pm 2\sqrt{x^2+y^2}}{2}$$

$$a = \frac{-x + \sqrt{x^2+y^2}}{2}$$

$$\therefore \frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y}$$

$$y^2 = 4\frac{yy'}{2}\left(x + \frac{yy'}{2}\right) \Rightarrow y = 2xy' + yy'^2 \quad (3)$$

Replacing $y' = \frac{-1}{y'}$

$$y = -\frac{2x}{y'} + \frac{y}{y'^2} \Rightarrow yy'^2 = -2xy' + y \quad (4)$$

$$\Rightarrow y = yy'^2 + 2xy'$$

From eq^n (3) & (4)
we observed that

Now, (3) = (4)

eq^n (3) represents the family of curves given in eq^n (1) & eq^n (4) is a differential eq^n representing the corresponding family of orthogonal trajectory.

→ Hence it is self orthogonal.

Q P.T. the family of con-focal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$

$\lambda \rightarrow$ variable is self orthogonal.

$$\frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0$$

$$\frac{2yy'}{b^2+\lambda}=\frac{-2x}{a^2+\lambda}=\frac{1}{k} \text{ say}$$

$$y'=-\frac{x}{y}\left(\frac{b^2+\lambda}{a^2+\lambda}\right)$$

$\Rightarrow \quad a^2+\lambda=kx \qquad b^2+\lambda=-kyy'$

$\Rightarrow \quad a^2-b^2=kx+kyy'$

$\Rightarrow \quad a^2-b^2=k(x+yy')$ —— (3)

$\Rightarrow \quad k=\frac{a^2-b^2}{x+yy'}$

$\Rightarrow \quad a^2+\lambda=\frac{(a^2-b^2)x}{(x+yy')} \qquad b^2+\lambda=\frac{-yy'(a^2-b^2)}{(x+yy')}$

$$\therefore \quad y'=\frac{-x}{y}\left(\frac{-yy'\cancel{(a^2-b^2)}}{\cancel{(x+yy')}}\times\frac{\cancel{(x+yy')}}{\cancel{(a^2-b^2)}\,x}\right)$$

$\Rightarrow \quad y'=y'$

$$\frac{x^2}{kx}-\frac{y^2}{kyy'}=1 \Rightarrow x-\frac{y}{y'}=k$$

Now $\quad x-\frac{y}{y'}=\frac{a^2-b^2}{x+yy'} \Rightarrow a^2-b^2=\left(x-\frac{y}{y'}\right)(x+yy')$

Hence is orthogonal.

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Dt: 14/01/20

Linear differential eqⁿ of 2nd order:

① → $y'' + p(x)y' + q(x)y = r(x)$ defined on open interval I.

where $p(x)$, $q(x)$ and $r(x)$ are continuous fⁿ in open interval

If $r(x) = 0$, ① reduces to

$$y'' + p(x)y' + q(x)y = 0 \quad \text{— ②}$$

② is called homogeneous eqⁿ

Superposition principle / Linearity property:

If $y_1(x)$ & $y_2(x)$ are two solutions of homogeneous eqⁿ ② on open interval I, then any linear combination of y_1 & y_2 is also a solⁿ of eqⁿ ② on open interval I.

Eg: $c_1y_1(x) + c_2y_2(x)$ is also a solⁿ of eqⁿ ② on open interval I.

Eg: $y'' + y = 0$ then

$y_1(x) = \cos x$ $\qquad$ $y_2(x) = \sin x$

$c_1y_1(x) + c_2y_2(x) = c_1\cos x + c_2\sin x$

eg: $y'' - y = 0$ with initial condⁿ $y(0) = 4$, $y'(0) = -1$

$\frac{d^2y}{dx^2} = y$ Let $y_1(x) = e^x$ $y_2(x) = e^{-x}$

$c_1 y_1(x) + c_2 y_2(x) = c_1 e^x + c_2 e^{-x}$ → also a solⁿ

$y(0) = c_1 + c_2 = 4$ $y'(x) = c_1 e^x - c_2 e^{-x}$

$c_1 - c_2 = -1$ $y'(0) = c_1 - c_2 = -1$

$2c_1 = 3$

$c_1 = 3/2$ $c_2 = 5/2$.

$$\boxed{y(x) = \frac{3e^x + 5e^{-x}}{2}}$$

General solⁿ:

The general solⁿ of homogeneous eqⁿ ② is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 & c_2 are arbitrary constants & $y_1(x)$ & $y_2(x)$ are not proportional to each other.

$y_1(x)$ & $y_2(x)$ are both solⁿs of eqⁿ ② on I but not proportional to each other.

Independent Solutions:

Two solⁿs $y_1(x)$ & $y_2(x)$ are said to be linearly independent if there exists constants k_1 & k_2 such that

$k_1 y_1(x) + k_2 y_2(x) = 0$ holds iff $k_1 = k_2 = 0$

Dependent sol^ns:

Two sol^ns $y_1(x)$ & $y_2(x)$ are said to be linearly depen[dent] if there exist some const. k_1 & k_2 not all zero such that $k_1 y_1(x) + k_2 y_2(x) = 0$ holds.

Suppose $k_1 \neq 0$

$$\Rightarrow y_1(x) = -\frac{k_2}{k_1} y_2(x)$$

this implies $y_1(x)$ & $y_2(x)$ are proportional to each other

Suppose, $k_2 \neq 0$

$$\Rightarrow y_2(x) = \frac{-k_1}{k_2} y_1(x)$$

If the 2 sol^ns $y_1(x)$ & $y_2(x)$ are linearly dependent then they are proportional to each other. If linearly independent then not proportional to each other.

$$k_1 y_1'(x) + k_2 y_2'(x) = 0$$

If $W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$ then the 2 sol^ns $y_1(x)$ & $y_2(x)$ are linearly independent

Wronskian

or else linearly dependent.

↳ can be extended to higher orders.

Basis

A basis ... independen...

Method

We ... other ... eqn ... $y_2(x)$ ... a fin...

Eg:

Now,

∴

$\Rightarrow$

$\Rightarrow$

Basis sol^n:

or fundamental system

A basis sol^n of the homogeneous eq^n (2) is a pair of linearly independent sol^ns $y_1(x)$ & $y_2(x)$ of homogeneous eq^n (2) on open interval I.

Method of Reduction Of Order:

We often get a sol^n, $y_1(x)$ or we can obtain $y_1(x)$ by some other method. Now to obtain a basis sol^n of homogeneous eq^n (2) we need another linearly independent sol^n $y_2(x)$ on I. That sol^n can be obtained by solving a first order linear differential eq^n by method of reduction of order.

$$y_2(x) = v(x)\, y_1(x)$$

eg: $y'' - y' = 0$ —(1)

Let $y_1(x) = e^x$ this is a sol^n of (1)

Now, according to method of reduction of order,

$$y_2(x) = v(x)\, e^x$$

$$\therefore \quad y_2'' - y_2' = 0$$

& $y_2' = e^x v(x) + v'(x) e^x$

$y_2'' = e^x v(x) + e^x v'(x) + v''(x) e^x + e^x v'(x)$

$\Rightarrow v''(x) e^x + e^x v'(x) = 0.$

$\Rightarrow v'' + v' = 0$

Let $v = u'$

then $v' + v = 0$

$\Rightarrow \quad v = e^{-x}$ $\Rightarrow \quad u = -e^{-x}$

$\Rightarrow \quad u' = e^{-x}$

General sol^n is

$$\cancel{y(x) = c_1 e^{x} + c_2 e^{x} e^{-x}}$$

$$\cancel{y(x) = c_1 e^{x} + c_2}$$

~~Basis sol^n,~~ $\cancel{y_1(x) = e^{x}}$

$\cancel{y_2(x) = 1}$

$y(x) = c_1 e^{x} - c_2 e^{x} \cdot e^{-x}$

$= c_1 e^{x} - c_2$

Basis sol^n, $y_1(x) = e^{x}$

$y_2(x) = -1$

Q. Find basis sol^n for following

$x^2 y'' - x y' + y = 0$

Let $y_1(x) = x$ is a sol^n.

$y_2(x) = x\,v(x)$

Now, $x^2 y_2'' - x y_2' + y_2 = 0$

$y_2' = v + x v'$

$y_2'' = v' + v' + x v''$

$\therefore \quad x^2 (2v' + x v'') - x(v + x v') + x v = 0$

$\Rightarrow \quad 2x^2 v' + x^3 v'' - \cancel{xv} - x^2 v' + \cancel{xv} = 0$

$\Rightarrow \quad x^3 v'' + x^2 v' = 0$

$\Rightarrow \quad xv'' + v' = 0$

Let $v' = U$

$\Rightarrow \quad xU' + U = 0$

~~$U = x$~~

General: $c_1 x + c_2 x \ln|x|$

$U = 1/x$

$v = \ln x$

$y_2 = x\ln|x|$ } Basis solⁿ.
$y_1 = x$

Q. $x^2y'' - 5xy' + 9y = 0$ Given $y_1(x) = x^3$

Let $y_1(x) = x^3$

$y_2(x) = x^3 v$

Now, $x^2 y_2'' - 5xy_2' + 9y = 0$

$y_2' = 3x^2 v + x^3 v'$

$y_2'' = 6xv + 3x^2v' + 3x^2v' + x^3v''$

$= x^3v'' + 6x^2v' + 6xv$

$x^2(x^3v'' + 6x^2v' + 6xv) - 5x(3x^2v + x^3v') + 9(x^3v) = 0$

$\Rightarrow x^5v'' + 6x^4v' + 6x^3v - 15x^3v - 5x^4v' + 9x^3v = 0$

$\Rightarrow x^5v'' + x^4v' - \cancel{6x^3v} + \cancel{6x^3v} = 0$

$\Rightarrow x^5v'' + x^4v' = 0$

Let $v' = U$

$\Rightarrow xU' + U = 0$

$U = 1/x \Rightarrow v = \ln x$

$\therefore y_2 = x^3 \ln|x|$

$$\boxed{\begin{array}{l}\text{Basis sol}^n\text{:}\\ y_1 = x^3 \\ y_2 = x^3\ln|x| \\ \text{General sol}^n\text{,} \\ c_1x^3 + c_2x^3\ln|x|\end{array}}$$

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P-1.7 7, 12, 15, 16, 19,

P-1.8 10, 12, 14, 15, 17, 18

Linear differential equation of higher order:

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad \text{--- (1)}$$

let $y = A e^{mx}$ is the trail soln of (1)

$(A \neq 0)$

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + P_3 m^{n-3} + \cdots + P_{n-1} m + P_n = 0 \quad \text{--- (2)}$$

The eqn (2) is called characteristic eqn / auxillary equation

Rule I If the roots of the characteristic eqn (2) say $m_1, m_2, \ldots, m_n$ are all distinct then the general soln of homogeneous eqn (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Rule II If the 2 roots of the characteristic eqns m_1 & m_2 are equal & equal to (m) say

Then corresponding part of general soln of homo eqn (1) is $y = (c_1 + c_2 x) e^{mx}$

If the 3 roots of the characteristic eq^n (2) are equal say $m_1 = m_2 = m_3 = m$ then the corresponding part of general sol^n is $y = (c_1 + c_2 x + c_3 x^2) e^{mx}$

Rule-III If the characteristic eq^n (2) has a pair of imaginary roots say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ then the corresponding part of homo eq^n (1) is $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Rule IV If the pair of imaginary roots say $\alpha \pm i\beta$ repeats twice / occurs twice, then the corresponding part of homo eq^n (1) is $e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x]$

Rule V If the characteristic eq^n (2) has a pair of real roots say $\alpha \pm \sqrt{\beta}$ then the corresponding part of homo eq^n (1) is $e^{\alpha x}[c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x]$

Q. $y''' - 3y'' + 4y = 0$

char. eq^n: $m^3 - 3m^2 + 4 = 0$

$\Rightarrow \cancel{m^3 - 4m^2 + m^2 + 4 = 0}$ $\Rightarrow m = -1$

$\Rightarrow \cancel{m^2(m-4) + m^2 + 4}$ $m = 2, 2$

General eq^n is $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

$y^{iv} + 8y'' + 16y = 0$

Char eq^n: $m^4 + 8m^2 + 16 = 0$

$\Rightarrow m^4 + 4m^2 + 4m^2 + 16 = 0$

$\Rightarrow m^2(m^2+4) + 4(m^2+4) = 0$

$\Rightarrow (m^2+4)(m^2+4) = 0$

$m = \pm 2i, \pm 2i$

$$y = e^0 \left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$$

$y'' - 4y' + y = 0$

Char eq^n: $m^2 - 4m + 1 = 0$

$\Rightarrow m = \dfrac{4 \pm \sqrt{16-4}}{2} = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

$$y = e^{2x}\left[c_1 \cosh \sqrt{3}x + c_2 \sinh \sqrt{3}x\right]$$

Cauchy - Euler eq^n:

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} x \frac{dy}{dx} + P_n y = 0$$

$P_1, P_2 \cdots\cdots P_n \longrightarrow$ are all constants

The name of this type of eq^n is Cauchy-Euler eq^n.

Q. P-2.6

Q. P-2.6 60. $x^2y''+xy'+9y=0$ $\quad y(1)=2$, $y'(1)=0$

Let $x=e^z \Rightarrow z=\ln x$

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx}=\frac{dy}{dz} \quad \text{—— (1)}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$=\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$=\frac{d}{dz}\left(\frac{1}{e^z}\frac{dy}{dz}\right)\frac{1}{x}$$

$$\Rightarrow \left(\frac{1}{e^z}\frac{d^2y}{dz^2}-e^{-z}\frac{dy}{dz}\right)\frac{1}{x} \Rightarrow \frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$\therefore$ $$\frac{d^2y}{dz^2}-\cancel{\frac{dy}{dz}}+\cancel{\frac{dy}{dz}}+9y=0$$

$$\Rightarrow \frac{d^2y}{dz^2}+9y=0$$

Char. eqn $\Rightarrow m^2+9=0$

$\Rightarrow m=\pm 3i$

General eqn:

$$y=(c_1\cos 3z+c_2\sin 3z)$$

$$\Rightarrow y=(c_1\cos 3\ln x+c_2\sin 3\ln x)$$

$$\Rightarrow \boxed{2=c_1} \Rightarrow y'=\left(\frac{3c_1}{x}\cos 3\ln x+\frac{3c_2}{x}\sin 3\ln x\right)$$

$$\Rightarrow 0=3c_1\times 0+c_2$$

$$\Rightarrow \boxed{c_2=0}$$

$\therefore$ $$\boxed{y=2\cos 3\ln x}$$

H.W P-2.1 Q. 3, 5, 6, 7, 9-12, 17, 19

P-2.2 Q. 5-9, 11, 13, 15, 16, 20, 22, 23-25

Q. $y' + 4xy + xy^3 = 0$ → bernoulli's eqn.

$y' + x(4y + y^3) = 0$ ⇒ IF = $e^{\int (4y+y^3)\,dy}$

⇒ $\frac{dy}{dx} = -x(4y+y^3)$

$y' + 4xy = -xy^3$

⇒ $\cancel{\frac{dy}{4y+y^3} = -x\,dx}$

$\frac{1}{y}\frac{dy}{dx} + 4x = -xy^2$

⇒ $\cancel{\frac{dy}{y(4+y^2)}}$

⇒ $\frac{1}{y^3}\frac{dy}{dx} + \frac{4x}{y^2} = -x$

$t = \frac{1}{y^2}$ ⇒ $\frac{-2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

⇒ $\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dt}{dx}$

∴ $-\frac{1}{2}\frac{dt}{dx} + 4xt = -x$

⇒ $\frac{dt}{dx} - 8xt = 2x$

IF = $e^{\int -8x\,dx} = e^{-4x^2}$

⇒ $e^{-4x^2}\frac{dt}{dx} - 8xe^{-4x^2}t = 2xe^{-4x^2}$

$\Rightarrow e^{-4x^2}\,dt - 8x e^{-4x^2} t\,dx = 2x e^{-4x^2}\,dx$

$\Rightarrow \int d(e^{-4x^2} t) = \int 2x e^{-4x^2}\,dx$ $\qquad e^{t}$ $\qquad e^{-4x^2} = t$

$-4x^2 = t$

$-8x\,dx = dt$

$2x\,dx = -\frac{1}{4}\,dt$

$\Rightarrow e^{-4x^2} t = -\frac{1}{4}\int e^{t}\,dt = -\frac{1}{4} e^{-4x^2} + c$

$\Rightarrow t = -\frac{1}{4} + c e^{4x^2}$

$\Rightarrow \boxed{\frac{1}{y^2} = c e^{4x^2} - \frac{1}{4}}$

Q. $\frac{dy}{dx} - y = y^2(\sin x + \cos x)$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \sin x + \cos x$

Let $-\frac{1}{y} = t \Rightarrow +\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + t = \sin x + \cos x$ $\qquad IF = e^{\int dx} = e^{x}$

$\Rightarrow \cancel{(dt + t\,dx) = (\sin x + \cos x)\,dx}$

$\Rightarrow \cancel{\int d(tx) = \int (\sin x + \cos x)\,dx}$

$\Rightarrow \cancel{tx =}$ $\Rightarrow \int e^{x}\,dt + e^{x} t\,dx = \int e^{x}(\sin x + \cos x)\,dx$

$\Rightarrow \int d(te^{x}) = e^{x}\sin x + c$

$\Rightarrow -\frac{e^{x}}{y} = e^{x}\sin x + c$

$\Rightarrow \boxed{y = \frac{1}{ce^{-x} - \sin x}}$

Q. $2xy' = 10x^3y^5 + y$
Q. $y' + y = xy^{5/3}$
} Bernoulli H.W

Q. Find the orthogonal trajectories of the hyperbola

(a) $x^2 - y^2 = C$

$\Rightarrow 2x - 2y\frac{dy}{dx} = 0$

$\Rightarrow 2y\frac{dy}{dx} = 2x$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\frac{-dx}{dy} = \frac{x}{y}$

$\Rightarrow -\int\frac{dx}{x} = \int\frac{dy}{y}$

$\Rightarrow -\ln x = \ln y + \ln c$

$\Rightarrow \boxed{xy = c}$

Q.(b) family of circles passing through the points $(0,2)$ & $(0,-2)$

$= x^2 + (y+2)(y-2) = 0$

$\Rightarrow x^2 + y^2 - 4 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

Now, $\frac{-dx}{dy} = \frac{-x}{y} \Rightarrow \int\frac{dx}{x} = \int\frac{dy}{y}$

$\Rightarrow \ln x = \ln y + \ln c$

$\Rightarrow \ln\frac{x}{y} = \ln c \Rightarrow \boxed{\frac{x}{y} = c}$

Q. Show that e^x, e^{-x} and linear combination $c_1e^x + c_2e^{-x}$ are solution of the homogeneous equation $y'' - y = 0$

Let $y_1(x) = e^x$ is a soln

then $y_2(x) = e^x\, v(x)$

$\Rightarrow y_2' = e^x v + e^x v'$

$y_2'' = e^x v + e^x v' + e^x v' + e^x v''$

now

$y_2'' - y_2 = 0$

$\Rightarrow \cancel{e^x v} + 2e^x v' + e^x v'' - \cancel{e^x v} = 0$

$\Rightarrow 2v' + v'' = 0$

Let $v' = U$

$\Rightarrow U' + 2U = 0$

$y_1'' = e^x$

Now $y_1'' - y_1 = e^x - e^x = 0$

$y_2'' = e^x$

& $y_2'' - y_2 = e^x - e^x = 0$

Q. $y = px + \frac{a}{p}$ Let $\frac{dy}{dx} = p$

$\Rightarrow dy = p\,dx + x\,dp - \frac{a}{p^2}\,dp$

$\Rightarrow \cancel{dy} = \cancel{p\,dx} + x\,dp - \frac{a}{p^2}\,dp$

$\Rightarrow x\,dp - \frac{a}{p^2}\,dp = 0$

$\Rightarrow dp\left(x - \frac{a}{p^2}\right) = 0$

$\Rightarrow dp = 0 \Rightarrow p = c \Rightarrow \boxed{y = xc}$

Now $x - \frac{a}{p^2} = 0 \Rightarrow x = \frac{a}{p^2} \Rightarrow p^2 = \frac{a}{x}$

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$\Rightarrow \quad p = \frac{\sqrt{a}}{\sqrt{x}}$ $\qquad \Rightarrow \int dy = \sqrt{a}\int \frac{dx}{\sqrt{x}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$ $\qquad \Rightarrow \quad y = 2\sqrt{a}\sqrt{x} + c$

$\Rightarrow \quad y = 2\sqrt{ax} + c$

Q. Solve: $\quad y''' - 3y'' + 4y = 0$

$m^3 - 3m^2 + 4 = 0 \qquad \Rightarrow m = -1, 2, 2$

$\Rightarrow \cancel{m^3 - 4m^2 + m^2 + 4 = 0}$

$\Rightarrow \quad y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$

Q. $y'' - y' - 6y = 0$ verify that the sol^ns are linearly independent.

Ans: $\quad y'' - y' - 6y = 0$

$m^2 - m - 6 = 0 \quad \Rightarrow \cancel{m(m-1) = 0}$

$\Rightarrow \cancel{m = 0, 1}$

$\Rightarrow \quad m = 3, -2$

$y(x) = c_1 e^{-2x} + c_2 e^{3x}$

$y_1 = e^{-2x} \qquad y_2 = e^{3x}$

find wronskian.

$$W = \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix} \Rightarrow 3e^{x} + 2e^{x} = 5e^{x} \neq 0$$

is linearly independent

Q. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ $\quad x(0) = 2, \; x'(0) = 0$

$x'' - 3x' + 2x = 0$

$\Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

$x(t) = c_1 e^{t} + c_2 e^{2t}$

$x(t) = c_1 e^{t} + c_2 e^{2t}$

$x'(t) = c_1 e^{t} + 2c_2 e^{2t}$

$x(0) = c_1 + c_2 = 2 \qquad x'(0) = c_1 + 2c_2 = 0$

$\therefore$

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + c_2 &= 2 \\ \hline c_2 &= -2 \end{aligned}$$

$c_1 = 4$

$\therefore \boxed{x(t) = 4e^{t} - 2e^{2t}}$

MATRIX THEORY:

Let $A = (a_{ij})_{m \times n}$

elementary row operation - It's an operation of 3 types

1. Interchange of rows.
2. Multiplication of row with non-zero no.
3. Addition of a row multiplied with a no. with another row

Row equivalent matrices:

If a matrix B can be obtained from matrix A after applying a finite no. of elementary row operations then 2 matrices are said to be row equivalent matrices.

Similarly matrix B can be obtained from A after applying a finite no. of elementary column operations then 2 matrices are said to be column equivalent matrices.

Echelon Matrix:

A matrix A is called echelon matrix if it follows following cond^{n}s:

- All 0 rows follow all non zero rows.
- The no. of zeroes preceding the 1st non-zero element in a row increases as we go row to row downwards.

Eg:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

Theorem 1: Any matrix can be made row equivalent to a echelon matrix.

Rank of A Matrix:

It is a no. k (real, k∈N) if

① There exists atleast one non-singular square sub matrix of order 'k' of matrix 'A'.

② every square sub matrix of A of order greater than 'k' is singular.

Since there exists atleast one sq. sub matrix of order 2 $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ which is non-singular and every sq. sub matrix of order 3 is singular, so rank of A = 2.

Theorem-2: If an echelon matrix A has r no. of non-zero rows then rank of A is "r".

* 2 row equivalent matrices have same rank.

Q. Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \quad \begin{matrix} R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix} \quad \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

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$R_3' \leftarrow R_3 + R_2$ $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$

$R_3 \leftrightarrow R_4 = R_{34}$ $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$R_3 \rightarrow R_3 + \frac{5}{2} R_2$ $\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$ say

B is an echelon matrix (row reduced)

A & B are row equivalent matrices.

$\therefore$ Accd^n to theorem 2/3 rank of A = rank of B

$= \boxed{3}$ (non-zero rows)

Sol^n of linear system of equations:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

The coefficient matrix = $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ coefficient vector $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{n \times 1}$

If the system ① has atleast one solution, then the system of linear eq^n's ① is called consistent. on the other hand, if the system has no solution, then the system is called inconsistent.

In matrix form $A\underset{\sim}{x} = \underset{\sim}{b}$ ②.

The augmented matrix is,

$$\bar{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]_{m \times (n+1)}$$

① If rank of A = Rank $\bar{A}$
then the system① is consistent

② If Rank A = Rank $\bar{A}$ = n
then the system ① is consistent. There exists a unique soln of ①.

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③ Rank A = Rank $\bar{A}$ $< n$ → no. of unknowns

then there exists an infinite no. of soln of system of linear eqn ①

In this case the no. of independent solns =

$\boxed{n - \text{rank } A}$

Q. ④ If rank $A \neq$ Rank $\bar{A}$

→ inconsistent, no solution

Q. Examine if the following system is consistent or not. If consistent then solve it.

$x_1 + 2x_2 - x_3 = 3$

$3x_1 - x_2 + 2x_3 = 1$

$2x_1 - 2x_2 + 3x_3 = 2$

$x_1 - x_2 + x_3 = -1$

$$\bar{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2' \leftarrow R_2 - 3R_1$ $\quad R_3' \leftarrow R_3 - 2R_1$ $\quad R_4' \leftarrow R_4 - R_1$

$$\bar{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

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$R_3 \leftarrow R_3 - \frac{6}{7}R_2$ $\quad$ $R_4' \leftarrow R_4 - \frac{3}{7}R_2$

$$\bar{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$$

$R_4' \leftarrow R_4 + \frac{1}{5}R_3$

$$\bar{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \text{ say}$$

A

since B is a row reduced echelon matrix, hence Rank B = 3

$\bar{A}$ & B are row equivalent matrices

$\therefore$ Rank $\bar{A}$ = rank B = 3

Rank A = 3

$\therefore$ Rank A = rank $\bar{A}$ = 3

The given system of linear eq^n is consistent & has an unique sol^n.

The equivalent system of linear eq^n is

$x_1 + 2x_2 - x_3 = 3$ $\qquad$ $x_1 + 8 - 4 = 3 \Rightarrow \boxed{x_1 = -1}$

$-7x_2 + 5x_3 = -8$ $\qquad \Rightarrow -7x_2 = -8 - 20 \Rightarrow \boxed{x_2 = 4}$

$5/7\, x_3 = 20/7$ $\qquad \Rightarrow \boxed{x_3 = 4}$

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Q. For what values of λ the following system is consistent? Also solve in each consistent cases.

$x - y + z = 1$

$x + 2y + 4z = \lambda$

$x + 4y + 6z = \lambda^2$

If $\lambda = 1, 2/3$ then Rank A = Rank $\bar{A}$ = $2 < n$ then the given system is consistent & there exists ∞ no. of solutions otherwise the given system has no soln.

$$\bar{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 6 & \lambda^2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & \lambda-1 \\ 0 & 5 & 5 & \lambda^2-1 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - \frac{5}{3}R_2$

$$\bar{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & \lambda-1 \\ 0 & 0 & 0 & (\lambda^2-1)-\frac{5}{3}(\lambda-1) \end{bmatrix}$$

$$(\lambda^2-1) - \frac{5}{3}(\lambda-1) = 0$$

$\Rightarrow 3(\lambda^2-1) - 5(\lambda-1) = 0$

$\Rightarrow 3\lambda^2 - 3 - 5\lambda + 5 = 0$

$\Rightarrow 3\lambda^2 - 5\lambda + 2 = 0$

$\Rightarrow 3\lambda^2 - 3\lambda - 2\lambda + 2 = 0$

$\Rightarrow 3\lambda(\lambda-1) - 2(\lambda-1) = 0$

$\boxed{\lambda = 1, 2/3}$

$\lambda = 1$

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - y + z = 1$

$3y + 3z = 0$ let $z = c_1$ (any real no.)

$\Rightarrow y + z = 0$

$\Rightarrow y = -c_1$

$\boxed{x = 1 - 2c_1}$ sol^n: $(1-2c_1, -c_1, c_1)$

$\lambda = 2/3$

$$B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - y + z = 1$

$3y + 3z = -1/3$ let $z = c_1$

$\Rightarrow y + z = -1/9$ $x + c_1 + \frac{1}{9} + c_1 = 0$

$\Rightarrow y = -c_1 - 1/9$ $\Rightarrow x = -\frac{1}{9} - 2c_2$

$\therefore$ sol^n: $\left(-\frac{1}{9} - 2c_2, -\frac{1}{9} - c_2, c_2\right)$

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Q. For what values of λ & μ the following system of linear eqn has,

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

1. no soln
2. unique soln
3. more than 1 soln (∞)

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(1) $\left.\begin{matrix} \lambda = 3 \\ \mu \neq 10 \end{matrix}\right\} \longrightarrow$ no soln

(2) $\lambda \neq 3$, ~~$\mu \neq 10$~~ μ = anyvalue

(3) $\lambda = 3$, $\mu = 10$

Assignment P.S-2.3 Q- 5, 6, 8, 10, 12, 14, 16-19, 21, 22

PS-2.6 Q- 2, 5, 8-10, 12, 15, 17

Soln of system of homogeneous eqn's:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} (1)$$

In matrix form, $A \cdot \underset{\sim}{x} = 0$

coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

* Obvious soln is 0.

But (1) may have non-zero soln's also,

The homogeneous system of eqn (1) has non-trivial soln if Rank $A < n$ (no. of unknowns)

No. of independent soln's $= n -$ Rank A.

Set of all non-trivial soln's of (1) along with trivial soln $(0, 0, 0)$ will form a vector space of dimension $\boxed{n - r}$ ($r =$ Rank A)

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The name of this vector space is 'soln space of A'. Also called ~~'nullity of A'~~. 'null space of A'.

* The dimension of the null space is called nullity of A. $(= n-r)$

$\boxed{\text{nullity } A = n - \text{Rank } A}$

$\boxed{\text{Rank } A + \text{nullity } A = n}$ # Rank nullity theorem

$\boxed{\underset{\sim}{x} \in \mathbb{R}^n}$

Q. Solve the following homogeneous eq^ns:

$$x_1 - x_2 + 2x_3 - 3x_4 = 0$$

$$3x_1 + 2x_2 - 4x_3 + x_4 = 0$$

$$5x_1 - 3x_2 + 2x_3 + 6x_4 = 0$$

Ans:

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 5 & -3 & 2 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1 \qquad R_3 \to R_3 - 5R_1$

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

$R_3' \leftarrow R_3 - 2/5 R_2$

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 0 & -4 & 17 \end{bmatrix} = B \text{ say}$$

Rank B = 3

since A & B are now equivalent matrices

Rank A = Rank B = 3 < 4 (n)

$\therefore$ The given homogeneous system of linear eq^n has ∞ no. of non-trivial sol^{n}s.

$x_1 - x_2 + 2x_3 - 3x_4 = 0$

$5x_2 - 10x_3 + 10x_4 = 0$

$-4x_3 + 17x_4 = 0$

Let $\boxed{x_4 = c}$ (c = any real no.)

$\boxed{x_3 = \frac{17c}{4}}$

$5x_2 - \frac{170c}{4} + 10c = 0$

$\boxed{x_2 = 13/2\, c}$ $\boxed{x_1 = c}$

$\therefore$ Sol^n vector : $c \begin{pmatrix} 1 \\ 13/2 \\ 17/4 \\ 1 \end{pmatrix}$

Q. solve :
$x_1 + 3x_2 + 2x_3 = 0$
$2x_1 - x_2 + 3x_3 = 0$
$3x_1 - 5x_2 + 4x_3 = 0$
$x_1 + 17x_2 + 4x_3 = 0$

$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$

Rank B = 2 = Rank A < 3

$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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$\therefore \quad x_1 + 3x_2 + 2x_3 = 0$

$\quad\quad -7x_2 - x_3 = 0$

let $x_3 = c$ $\quad\quad x_1 = -\frac{11c}{7}$

$\Rightarrow x_2 = -c/7$

Soln: $c \begin{pmatrix} -11/7 \\ 1/7 \\ 1 \end{pmatrix}$

H.W

Determine the values of λ for which the following system of eqn may posses non-trivial soln:

$3x_1 + x_2 - \lambda x_3 = 0$

$4x_1 - 2x_2 - 3x_3 = 0$

$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

For each possible value of λ, determine the general soln.

Gauss-Elimination Method:

$A\underset{\sim}{x} = \underset{\sim}{b}$

The augmented matrix,

$[A|\underset{\sim}{b}] \xrightarrow{\text{Gauss-elimination}} [U|\underset{\sim}{g}]$

Q. $x_1 + x_2 + x_3 = 9$

$2x_1 - 3x_2 + 4x_3 = 13$

$3x_1 + 4x_2 + 5x_3 = 40$

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Step 1:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$$

$R_2' \leftarrow R_2 - 2R_1 \qquad R_3' \leftarrow R_3 - 3R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right)$$

Step - 2:

$R_3' \leftarrow R_3 + \frac{R_2}{5}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12/5 & 12 \end{array}\right)$$

$x_1 + x_2 + x_3 = 9 \Rightarrow \boxed{x_1 = 1}$

$-5x_2 + 2x_3 = -5 \Rightarrow \boxed{x_2 = 3}$

$12/5\, x_3 = 12 \Rightarrow \boxed{x_3 = 5}$

Q. $x_1 + x_2 + x_3 = 6$

$3x_1 + 3x_2 + 4x_3 = 20$

$2x_1 + x_2 + 3x_3 = 13\cancel{0}$

Step1:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

In gauss method, in each step, diagonal element should be non-zero, if its zero, can't apply gauss-elimination method.

We need to apply, partial pivoting method.

Apply in step 1

$$R_{12} \quad \left(\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right)$$

$R_2 \leftarrow R_2 - \frac{1}{3} R_1$ $\quad R_3 \leftarrow R_3 - \frac{2}{3} R_2$

$$\left(\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{array}\right)$$

(PS: see magnitude only not sign

step-2

$$\left(\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{array}\right)$$

$3x_1 + 3x_2 + 4x_3 = 20$ $\quad \boxed{x_1 = 3}$

$-x_2 + x_3/3 = -1/3$ $\quad \boxed{x_2 = 1}$

$-\frac{1}{3} x_3 = -\frac{2}{3} \Rightarrow \boxed{x_3 = 2}$

Gauss Jordan Method:

$A\underset{\sim}{x} = \underset{\sim}{b}$ $\quad$ $(\because A \to$ square matrix$)$

$[A \mid \underset{\sim}{b}] \xrightarrow{\text{Gauss jordan}} [I \mid \underset{\sim}{d}]$ $\quad$ $A = [a_{ij}]_{n\times n}$

$\underset{\sim}{x} = \underset{\sim}{d}$

* In gauss elimination method $(n-1)$ steps are req. for finding soln if there are n-unknown.
* In gauss jordan method, it req. n steps to find soln if there are n-unknowns.

Q. $2x_1 + 4x_2 + x_3 = 3$

$3x_1 + 2x_2 - 2x_3 = -2$

$x_1 - x_2 + x_3 = 6$

Partial Pivot

$$A = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 3 \\ 3 & 2 & -2 & -2 \\ 1 & -1 & 1 & 6 \end{array}\right] \xrightarrow{R_{12}} = \left[\begin{array}{ccc|c} 3 & 2 & -2 & -2 \\ 2 & 4 & 1 & 3 \\ 1 & -1 & 1 & 6 \end{array}\right]$$

$R_1' \leftarrow R_1'/3$ $\quad$ $R_2' \leftarrow R_2 - 2R_1$, $\quad R_3' \leftarrow R_3 - R_1$

$$A = \left[\begin{array}{ccc|c} 1 & 2/3 & -2/3 & -2/3 \\ 2 & 4 & 1 & 3 \\ 1 & -1 & 1 & 6 \end{array}\right] \to \left(\begin{array}{ccc|c} 1 & 2/3 & -2/3 & -2/3 \\ 0 & 8/3 & 7/3 & 13/3 \\ 0 & -5/3 & 5/3 & 20/3 \end{array}\right)$$

Step-2

$$R_2' \leftarrow \frac{3}{8} R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2/3 & -2/3 & -2/3 \\ 0 & 1 & 7/8 & 13/8 \\ 0 & -5/3 & 5/3 & 20/3 \end{array}\right)$$

$$R_1' \leftarrow R_1 - \frac{2}{3} R_2 \qquad R_3' \leftarrow R_3 + \frac{5}{3} R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5/4 & -7/4 \\ 0 & 1 & 7/8 & 13/8 \\ 0 & 0 & 25/8 & 75/8 \end{array}\right)$$

Step-III

$$\xrightarrow{\frac{8}{25} R_3} \left(\begin{array}{ccc|c} 1 & 0 & -5/4 & -7/4 \\ 0 & 1 & 7/8 & 13/8 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$R_1' \leftarrow R_1 + \frac{5}{4} R_3 \qquad R_2' \leftarrow R_2 - \frac{7}{8} R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

∴ The required solⁿ is $\underset{\sim}{x} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

i.e. $x_1 = 2$, $x_2 = -1$, $x_3 = 3$

$-\frac{2}{3} - \frac{2}{3}($

$\frac{5}{3} + \frac{5}{3} \times \frac{7}{8}$

$\frac{20}{3} + \frac{5}{3} \times \frac{13}{8}$

$-\frac{7}{4} + \frac{5}{4}$

$8/4$

Gauss jordon can be applied for finding the inverse of a matrix.

$A = (a_{ij})_{n \times n}$

$$[A \mid I] \xrightarrow[\text{method}]{\text{Gauss jordan}} [I \mid X]$$

$$\boxed{X = A^{-1}}$$

Assgnt P.S - 6.3 3, 4, 5, 7, 9, 11, 13, 14

P.S. - 6.4 2, 3, 6, 8, 9, 13-15, 17, 18, 20, 24

Q. Find rank :

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{pmatrix} \qquad \begin{matrix} R_2' \leftarrow R_2 - 2R_1 \\ R_3' \leftarrow R_3 + 2R_1 \end{matrix}$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{pmatrix} \qquad R_3' \leftarrow R_3 + 2R_2$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{pmatrix} \qquad \therefore \text{ Rank} = 2$$

$$\begin{array}{r} 32 \\ 17 \\ \hline 15 \end{array}$$

Q. $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{pmatrix}$$

$R_2' \leftarrow R_2 - 2R_1$

$R_3' \leftarrow R_3 - R_1$

$R_4' \leftarrow R_4 - 8R_1$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -45 & -10 & -15 \end{pmatrix}$$

$R_3' \leftarrow R_3 + R_2$ $\quad$ $R_4' \leftarrow R_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\text{Rank} = 2}$$

Q. Solve:

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{pmatrix}$$

$R_2' \leftarrow R_2 - \frac{R_1}{2}$ $\quad$ $R_3' \leftarrow R_3 + \frac{R_1}{2}$

$$\bar{A} = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{pmatrix}$$

$4 - \frac{4 \times 5}{3}$

$4 \times \left(-\frac{2}{3}\right)$

$-\frac{3}{2} + \frac{5}{3} \times \frac{5}{2}$

$-\frac{3}{2} + \frac{25}{6} = \frac{-9+25}{6}$

$= \frac{16}{6} = \frac{8}{3}$

$R_3' \leftarrow R_3 + \frac{5}{3}R_2$

$$\bar{A} = \begin{pmatrix} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{pmatrix}$$

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$$2x_1 + x_2 - x_3 = 4 \qquad \boxed{x_1 = 1}$$

$$-\frac{3}{2}x_2 + \frac{5}{2}x_3 = -4 \Rightarrow \boxed{x_2 = 1}$$

$$+\frac{8}{3}x_3 = -8/3 \Rightarrow \boxed{x_3 = 1}$$

Q. solve the following; (Gauss elimination)

$$4x - 3y - 9z + 6w = 0$$

$$2x + 3y + 3z + 6w = 6$$

$$4x - 21y - 39z - 6w = -24$$

step 1

$$\left(\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right)$$

$R_2' \leftarrow R_2 - R_1/2 \qquad R_3' \leftarrow R_3 - R_1$

$$\left(\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & -18 & -30 & -12 & -24 \end{array}\right)$$

$R_3' \leftarrow R_3 + 4R_2$

$-30 + \cancel{4}^{2}(15/2)$

$-12 +$

$$\left(\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$4x - 3y - 9z + 6w = 0$$

$$\frac{9}{2}y + \frac{15}{2}z + 3w = 6 \Rightarrow 9y + 15z + 6w = 12$$

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let $\boxed{z = c_1 \quad w = c_2}$

$9y + 15c_1 + 6c_2 = 12$

$9y = 12 - 15c_1 - 6c_2$

$y = \dfrac{12 - 15c_1 - 6c_2}{9} = \boxed{\dfrac{4 - 5c_1 - 2c_2}{3}}$

$4x - \left(\dfrac{4 - 5c_1 - 2c_2}{3}\right) - 9c_1 + 6c_2 = 0$

$4x - 4 + 5c_1 + 2c_2 - 9c_1 + 6c_2 = 0$

$4x - 4 - 4c_1 + 8c_2 = 0$

~~$x = 1 + c_1 - 2c_2$~~ $\Rightarrow \boxed{x = 1 + c_1 - 2c_2}$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 + c_1 - 2c_2 \\ \dfrac{4 - 5c_1 - 2c_2}{3} \\ c_1 \\ c_2 \end{pmatrix}$$

Q. Using gauss jordan method, find inverse of A.

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

$$(A \mid I) = \left(\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$