Dt: 03/02/2020

Vector Space:

Let $\{F, +, \cdot\}$ be a field. A nonempty set V is called a vector space over the field F, if

(a) There is a binary composition called vector addition +, satisfying the following;

① $\alpha + \beta \in V \quad \forall \alpha, \beta \in V$ (closure property)

② $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in V$ – (commutative property)

③ $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \forall \alpha, \beta, \gamma \in V$ – (associative property)

④ There exist an element θ (zero vector) such that $\alpha + \theta = \alpha \quad \forall \alpha \in V$ (existence of identity)

⑤ For each element $\alpha \in V$, there exists an element $(-\alpha)$ in V such that
$\alpha + (-\alpha) = \theta$, (existence of inverse)

(b) There is an external composition '$\cdot$' of the field F with V satisfying the following

(i) $c \cdot \alpha \in V, \quad \forall \alpha \in V$ and $c \in F$

(ii) $(c_1 \cdot c_2) \cdot \alpha \in V \quad \forall \alpha \in V$ & $c_1, c_2 \in F$

(iii) $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta \quad \forall \alpha, \beta \in V$ & $c \in F$

(iv) $(c_1+c_2)\alpha = c_1\cdot\alpha + c_2\cdot\alpha \;\forall\; \alpha \in V$ & $c_1, c_2 \in F$

(v) $1\cdot\alpha = \alpha \;\forall\; \alpha \in V$ & 1 is the indentity element of 'F'

eg: $\{(a_1, a_2) \mid a_i \in R\}$

$\mathbb{R}^2$

$\{(a_1, a_2, a_3) \mid a_i \in R\}$

$\mathbb{R}^3$

$\{(a_1, a_2, a_3 \cdots a_n) \mid a_i \in R\}$

$R^n \longrightarrow$ real vector space.

Eg: $\{(a_{ij})_{m\times n} \mid a_{ij} \in R\}$ This will form a vector space.

Eg: The set of all polynomials

$\{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in R\}$

Subspace:

Let V be a vector space over the field F, A non-empty set 'W' is called a subspace of V if W is a subset of V and it is also a vector space over the field F wrt.

binary composition vector addition and scalar multiplication on V.

This external composition is also called scalar multiplication. The elements of V are called vectors & of F are called scalars.

Theorem: A non-empty set W is a subspace of V over the field F. If

(i) $\alpha \in W$ & $\beta \in W \Rightarrow \alpha+\beta \in W$

(ii) $c \in F$ & $\alpha \in W \Rightarrow c \cdot \alpha \in W$

Q. Let $S = \{(x,0,0) \mid x \in R\}$ then show that S is a subspace of R^3.

S is a non-empty set since $(0,0,0) \in S$ (atleast one element)

(i) Let $\alpha = (a,0,0) \in S$

& $\beta = (b,0,0) \in S$

where $(a,b) \in R$

$\alpha+\beta = (a+b,0,0) \in S$

since $a+b \in R$

(ii) Let $c \in R$ and $\alpha = (x,0,0) \in S$

where $x \in R$

$c \cdot \alpha = (c \cdot x, 0, 0) \in S$

since $c \cdot x \in R$

Hence S is a subspace of R^3

Q. Let $S = \{(x,y,z) \mid x^2+y^2 = z^2\}$ check whether S is a subspace of $\mathbb{R}^3$.

S is a non-empty set since, $(0,0,0) \in S$

(i) Let $\alpha = \cancel{(a,b,c)}$ $(x_1,y_1,z_1) \in S$ where $x_1^2+y_1^2 = z_1^2$

& $\beta = (x_2,y_2,z_2) \in S$ $x_2^2+y_2^2 = z_2^2$

$\alpha+\beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

may not be equal to

$(x_1+x_2)^2 + (y_1+y_2)^2 \neq (z_1+z_2)^2$

$\alpha+\beta \notin S$ since

$\therefore$ S is not a subspace of $\mathbb{R}^3$

Q. Check whether the following space is a subspace of $\mathbb{R}^3$ or not.

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \mid x+y-z=0 \text{ \& } 2x-y+z=0 \right\}$$

S is a non empty set as $(0,0,0) \in S$,

(i) Let $\alpha = (x_1,y_1,z_1) \in S$ where, $x_1+y_1-z_1=0$

$2x_1-y_1+z_1=0$

& $\beta = (x_2,y_2,z_2) \in S$ where, $x_2+y_2-z_2=0$

$2x_2-y_2+z_2=0$

$\alpha+\beta = (x_1+x_2, y_1+y_2, z_1+z_2)$

$(x_1+x_2)+(y_1+y_2)-(z_1+z_2) = 0$

$2(x_1+x_2)-(y_1+y_2)+z_1+z_2 = 0$

$\alpha+\beta \in S$ since

(ii) Let $c \in R$

$c \cdot \alpha = (c \cdot x_1, c \cdot y_1, c \cdot z_1) \in S$

since $c x_1 + c y_1 - c z_1$

$\Rightarrow c(x_1 + y_1 - z_1) = 0$

Hence S is a subspace of R^3.

H.W

Q. Let $W = \{(x, 2y, 3z) \mid (x, y, z) \in R\}$

show that W is a subspace of R^3 but if

$W = \{(x, y, z) \mid (x, y, z) \in Q\}$

then show that $W \neq$ subspace of R^3.

Linear Combination:

Let V be a vector space over the field F. A vector β in V is called linear combination of the vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ in V if β can be expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n$

for some scalars, $c_1, c_2, \dots, c_n \in F$.

Linear Span:

Let V be a vector space over the field F. A non-empty finite subset 'S' of V is called linear span of 'V' if 'S' contains all the possible linear combinations of the vectors in 'S'.

* It is denoted by $L(S)$

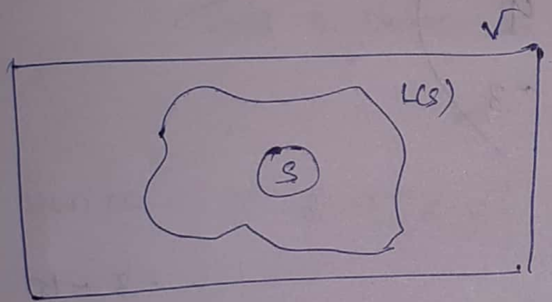
$S \subset V$

$S = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$

$$L(S) = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n\}$$
$$\forall\ c_1, c_2 \cdots c_n \in F$$

$L(S)$ is generated or spanned by 'S' & S is the generator of $L(S)$

$L(S)$ is also a subspace of V which is the smallest subspace of V containing the finite set 'S'.



Q. Show that the vector $\alpha = (2,-5,3)$ is not in the subspace of R^3 generated by the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$.

Let $S = \{(1,-3,2), (2,-4,-1), (1,-5,7)\}$

$$L(S) = \{c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7)$$
$$\text{such that } c_1, c_2, c_3 \in \mathbb{R}\}$$

If possible, $\alpha \in L(S)$

Then $\underset{(2,-5,3)}{\alpha} = c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7)$

$c_1 + 2c_2 + c_3 = 2$

$-3c_1 - 4c_2 - 5c_3 = -5$

$2c_1 - c_2 + 7c_3 = 3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right)$$

$R_2' \leftarrow R_2 + 3R_1 \qquad R_3' \leftarrow R_3 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & -2 & 4 \\ 0 & -5 & 5 & -3 \end{array}\right)$$

$R_3' \leftarrow R_3 + \frac{5}{2}R_2$

$-3 - \frac{15}{2} = \frac{-21}{2}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & -21/2 \end{array}\right)$$

System is inconsistent

↳ no solution.

∴ Our assumption is wrong.

So $\alpha \notin L(S)$

Q. In R

exam

If

Q. In R^3 $\alpha = (1,3,0)$ & $\beta = (2,1,-2)$. Determine $L(\alpha,\beta)$ examine if $\gamma = (-1,3,2)$ & $\delta = (4,7,-2)$ Adding $L(\alpha,\beta)$.

If possible, let $\gamma \in L\{\alpha,\beta\}$

$$L\{\alpha,\beta\} = \{c_1(1,3,0) + c_2(2,1,-2) \mid c_1, c_2 \in R\}$$

$$(-1,3,2) = c_1(1,3,0) + c_2(2,1,-2)$$

$c_1 + 2c_2 = -1 \Rightarrow c_1 = 1$

$3c_1 + c_2 = 3 \Rightarrow c_1 = 2/3$

$-2c_2 = 2 \Rightarrow \boxed{c_2 = -1}$

This system is inconsistent

so no sol^n for c_1 & c_2

Hence $\gamma \notin L\{\alpha,\beta\}$

If possible, let $\delta \in L\{\alpha,\beta\}$

$$(4,7,-2) = c_1(1,3,0) + c_2(2,1,-2)$$

$c_1 + 2c_2 = 4$ $\boxed{c_1 = 2}$

$3c_1 + c_2 = 7$ $c_1 = 2$

$-2c_2 = -2 \Rightarrow \boxed{c_2 = 1}$

$\therefore c_1 = 2, \ c_2 = 1$

$\delta \in L\{\alpha,\beta\}$

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PS: 6.6 - 14, 16, 18, 19

PS: 6.7 - 1, 3, 5, 7, 8, 13

PS: 6.8 - 1, 2, 3, 5-8, 10

Dt: 04/02/2020

Basis: **Linear Dependence:**

Let V be a vector space over a field F, a set of vectors $(\alpha_1, \alpha_2 \cdots \alpha_n)$ is called linearly dependent if there exists some scalars $(c_1, c_2 \cdots c_n)$ not all zero. Such that the following condition are verified.

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$$

Linear Independent:

Let V be a vector space over the field F, a set of vectors $(\alpha_1, \alpha_2, \cdots \alpha_n)$ is called linearly independent if $c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n = 0$ if & only if $c_1 = c_2 = c_3 \cdots c_n = 0$

P.S 6.4 Q-5 check $[2, -4], [1, 9], [3, 5]$

Let $\alpha_1 = [2, -4]$

$\alpha_2 = [1, 9]$

$\alpha_3 = [3, 5]$

then $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$\Rightarrow c_1[2,-4] + c_2[1,9] + c_3[3,5] = [0,0]$

$\Rightarrow 2c_1 + c_2 + 3c_3 = 0 \quad \rightarrow 4c_1 + 2c_2 + 6c_3 = 0$

$-4c_1 + 9c_2 + 5c_3 = 0 \quad -4c$

$$\begin{bmatrix} 2 & 1 & 3 & 10 \\ -4 & 9 & 5 & 10 \end{bmatrix}$$

$R_2' \leftarrow R_2 + 2R_1$

$$= \begin{bmatrix} 2 & 1 & 3 & 10 \\ 0 & 11 & 11 & 10 \end{bmatrix} \quad \text{Rank} = 2 < 3$$

$\Rightarrow \infty$ sol^{n}s

there exist many non-trivial sol^{n}s.

Hence given vectors are linearly dependent.

P.S 6.4 Q.7 $[1,9,9,8]$, $[2,0,0,3]$, $[2,0,0,8]$

Let $\alpha_1 = [1,9,9,8]$

$\alpha_2 = [2,0,0,3]$

$\alpha_3 = [2,0,0,8]$

then $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$\Rightarrow c_1[1,9,9,8] + c_2[2,0,0,3], + c_3[2,0,0,8] = [0,0,0]$

$c_1 + 2c_2 + 2c_3 = 0 \quad\quad c_1 = 0 \quad c_2 = -c_3$

$9c_1 + 0c_2 + 0c_3 = 0 \quad\quad 3c_2 - 8c_2 = 0$

$9c_1 = 0 \quad\quad \Rightarrow c_2 = 0 = c_3$

$8c_1 + 3c_2 + 8c_3 = 0$

Scanned by CamScanner

$c_1 = c_2 = c_3 = 0$

$\Rightarrow$ Linearly independent.

Q. Show that the vectors : $[1,2,1], [3,1,5], [3,-4,7]$ are linearly dependent in $\mathbb{R}^3$.

$\alpha_1 = [1,2,1] \quad \alpha_2 = [3,1,5] \quad , \quad \alpha_3 [3,-4,7]$

$c_1 \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3 = 0$

$\Rightarrow c_1 [1,2,1] + c_2 [3,1,5] + c_3 [3,-4,7] = [0,0,0]$

$c_1 + 3c_2 + 3c_3 = 0$

$2c_1 + c_2 - 4c_3 = 0$

$c_1 + 5c_2 + 7c_3 = 0$

$$A \begin{pmatrix} 1 & 3 & 3 & 10 \\ 2 & 1 & -4 & 10 \\ 1 & 5 & 7 & 10 \end{pmatrix}$$

$R_2' \leftarrow R_2' - R_1 \qquad R_3' \leftarrow R_3 - R_1$

$\Rightarrow \det(A) = 0$

Rank < 3

$\Rightarrow \infty$ sol^{n}s $\rightarrow$ given vectors are linearly dependent.

Basis :

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Basis:

Let V be a vector space over a field 'F' then non-empty set S is called basis of V if .

(i) S is linearly independent

(ii) S generates V i.e. $V = L(S)$

$L(S) \subseteq V$

$V \subseteq L(S)$

$L(S) = V$

If the no. of vectors of a basis of V is finite, then V is called finite dimensional vector space. or else V is called infinite dimensional.

The no. of vectors of a finite dimensional vector space is called dimension of a vector space V.

& denoted by dim V.

Theorem: Let V be a vector space of dimension n over a vector field 'F' then any linearly independent set of n vectors of V is a basis of V.

Q. Show that the vectors $\overset{\alpha_1}{[1,0,-1]}$, $\overset{\alpha_2}{[1,2,1]}$, $\overset{\alpha_3}{[0,-3,2]}$ form a basis of $\mathbb{R}^3$.

$c_1[1,0,-1] + c_2[1,2,1] + c_3[0,-3,2] = [0,0,0]$

$\Rightarrow$

$c_1 + c_2 = 0$

$2c_2 - 3c_3 = 0$

$-c_1 + c_2 + 2c_3 = 0$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

$= 1(4+3) - 1(-3) = 10 \neq 0$

∴ The vectors are linearly independent.

Since R^3 is a real vector space of dim = 3 and S is a linearly independent set containing 3 vectors of R^3 therefore · S generates / is a basis of R^3

Q. Let $S = \{(x,y,z) \in R^3 \mid 3x - y + z = 0\}$ show that S is a subspace of R^3. Hence find a basis of 'S'.

Since, $(0,0,0) \in S$ ∴ non-empty set

① Let $\alpha = (x_1, y_1, z_1)$ such that $3x_1 - y_1 + z_1 = 0$

$\beta = (x_2, y_2, z_2)$ such that $3x_2 - y_2 + z_2 = 0$

$\alpha + \beta = (x_1 + x_2, (y_1 + y_2), z_1 + z_2) \in S$

As ⇒ $3(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = 0$

∴ $\alpha + \beta \in S$

② Let $c \in R$ then $c \cdot \alpha = (cx_1, cy_1, cz_1) \in S$

As $3cx_1 - cy_1 + cz_1 = 0$

∴ S is a subspace of R^3

↳ vector space over the field R.

Let (a,b,c) be any arbitrary vector of 's'.

then we have $3a-b+c=0$

$\Rightarrow b = 3a+c$

$(a,b,c) = (a, 3a+c, c)$

$= a(1,3,0) + c(0,1,1)$

That means $(1,3,0), (0,1,1)$ generates S.

i.e. $S = L\{(1,3,0), (0,1,1)\}$

To examine linear independence,

let consider the relation,

$c_1(1,3,0) + c_2(0,1,1) = (0,0,0)$

$c_1 = 0$

$3c_1 + c_2 = 0 \Rightarrow c_2 = 0$

$\therefore$ These vectors are linearly independent

$\Rightarrow (1,3,0), (0,1,1)$ is a basis of S

dim $S = 2$, as it has only 2 lin independent sol^ns

Pg. 6.4 Q.19. Consider all the vectors in $\mathbb{R}^3$ such that $2V_1 + 3V_3 = 0$ is the given set of vectors a vector space. If possible determine the dimension & find a basis.

Ans: Let $S = \{(V_1, V_2, V_3) \in \mathbb{R}^3 \mid 2V_1 + 3V_3 = 0\}$ S is non-empty $(0,0,0) \in S$

① Let $\alpha = (x_1, y_1, z_1) \Rightarrow 2x_1 + 3z_1 = 0$

$\beta = (x_2, y_2, z_2) \Rightarrow 2x_2 + 3z_2 = 0$

$\alpha+\beta = \{(x_1+x_2), y_1+y_2, z_1+z_2\} \in S$

As $2(x_1+x_2)+3(z_1+z_2)=0$

$\therefore \alpha+\beta \in S$

(2) Let $c \in R$ then $c\cdot\alpha = (cx_1, cy_1, cz_1) \in S$

As, $2cx_1+3cz_1=0$

Hence, S is a subspace of $\mathbb{R}^3$.

$\therefore$ S is a vector space over the field R

Let (a,b,c) be an arbitrary element of S

$\Rightarrow 2a+3c=0$

$\Rightarrow c = -2a/3$

$(a,b,c) = (a, b, -2a/3)$

$= a(1,0,-2/3) + b(0,1,0)$

That means $(1,0,-2/3)$ & $(0,1,0)$ generates S.

i.e. $S = L\{(1,0,-2/3), (0,1,0)\}$

To examine linear dependence

Lets consider, $c_1(1,0,-2/3)+c_2(0,1,0)=(0,0,0)$

$c_1=0$
$c_2=0$ } linearly independent.

$\Rightarrow$ $(1, 0, -\frac{2}{3})$, $(0, 1, 0)$ is a basis of S

$\boxed{\dim S = 2}$