Differential Equation

An equation that involves derivatives of one or more dependent variable w.r.t one or more independent variable.

Dependent variable → one/more ordinary
Independent variable → one/more differential eqn

Ordinary differential eqn :- Any differential eqn which has ordinary derivatives and given fxn has such that it will have only one Independent variable.

eg.

1. $\frac{dy}{dx} = y$

2. $\frac{dy}{dx} = \cos x$

3. $\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + 3 = \sin x$

4. $\frac{dy}{dx} + xy = x^2$

Applications:-

1. Free falling of stone.

$$y'' = \frac{d^2y}{dx^2} = g$$

2. Vibrating mass on spring.

$$m\frac{d^2y}{dx^2} + Ky = 0$$

3. $LI'' + RI' + \frac{1}{C} I = E'$

4. $L'' + g\sin\theta = 0$

Order :- It is the highest derivative in the differential eqn.
A general
n^{th} order ordinary differential equation can be written as

$$F(x, y, y', y'', \ldots y^n) = 0$$

I^{st} order ordinary

Degree :-

It is the power of highest order derivation.

→ Linear → An eqn is the said to linear if
→ it satisfy following 2 condition

1. Every dependent Variable and derivatives involved occur to the 1st degree only?.

2. Product of derivatives and/or dependent variable do not occur.

eg. $\frac{dy}{dx} = \cos x$, $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = \sin x$

classmate

Date
Page

Non-linear :-

1. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + \ldots = 0$

Solution of DE :-

$$F(x, y, y') = 0 \Leftrightarrow y'$$

A solution of the 1st order DE and ~~for~~ on some open interval (a, b) is a function $y = h(x)$ such that its derivative exists

This should satisfy the differential eqn.

$$xy' = 2y \quad \forall \; x \in \mathbb{R}$$

$y = x^2$ is a solution

$y = h(x)$ → Explicit solution

Implict soln :- A function $H(x, y) = 0$ is said to be an implict solution if $H(x, y) = 0$ gives at least one explicit solution say ⇒ $y = h(x)$

$H(x,y) = 0 \Leftrightarrow x^2 + y^2 = 1$ is an implict soln.

$$x + y\frac{dy}{dx} = 0$$

General Soln :- A solution that constains as ~~many~~ that contains as many arbitrary constants as the order of D.E is called general solution.

$$y' = \cos x$$

$$\frac{dy}{dx} = \cos x$$

$$\int dy = \int \cos x \, dx + C$$

$$y = \sin x + C$$

Particular Solution :- A solution that is obtained by choosing particular value of arbitary constant in a general soln.

Singular Solution :-

A solution that cannot be obtained from the general solution by assigning any arbitrary value to constants.

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$$

$$y = cx + c^2$$

Date
Page

$$y = -\frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{4}2} = \frac{-x}{2}$$

$$= \frac{x^2}{4} + x\left(-\frac{x}{2}\right) + \frac{x^2}{4} =$$

$$= 0$$

This is called Singular Sol^n.

Q.

IVP :- The problem of determining a solution $y = y(x)$ of the first order diff eq^n which satisfies the condition

$$y(x_0) = y_0 = \text{constant.}$$

$$y' = f(x, y)$$

Geometrical applications :- of 1st order DE.

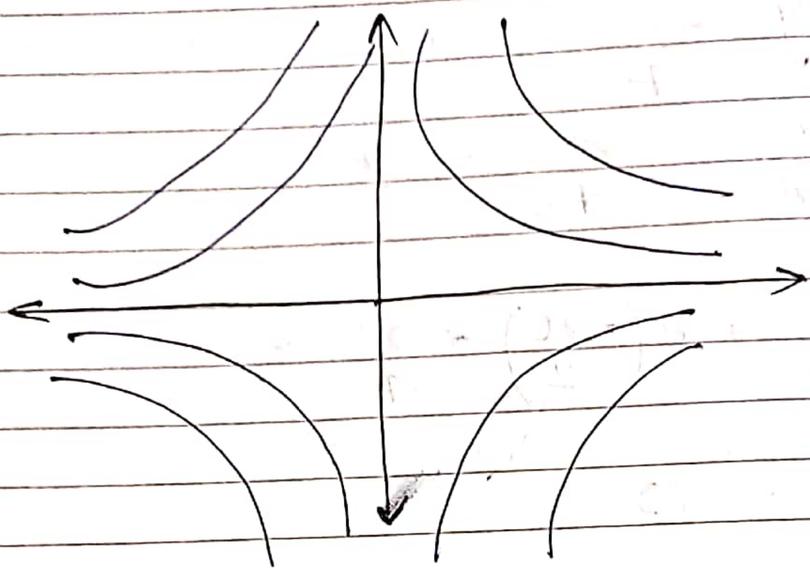
Que:- Find the curve throug $(1,1)$ in xy plane having each of it's point $= -\frac{y}{x}$

$$\frac{dy}{dx} = \frac{-y}{x} \qquad xy = c$$

$$y = \frac{c}{x} \Rightarrow c = 1$$

$$-\frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow -\ln y = \ln x + c$$



gen. solⁿ of 1st order DE gives family of curves.

Methods of solution:-

Variable seperable differential equation:-

Let $F(x, y, y') = 0$ $\quad y' = \frac{dy}{dx}$

$$\Leftrightarrow y' = f(x, y) \quad — (1)$$

If the equation (1) can be written in the form $f_1(y)\, d(y) = f_2(x)\, dx$ then we say the differential eqⁿ (1) is in Variable seperable form.

The solution of DE can be obtained by directly integrating.

$$\boxed{\int f_1(y)\, dy = \int f_2(x)\, dx + C}$$

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Date
Page

★ ~~Ho~~ Sometimes, we need to substitute a new variable to get the diff. eqn into variable seperable form.

eg. Solve $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1} y = \tan^{-1} x + c$$

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

$$\tan^{-1} \frac{y-x}{1+xy} = \tan^{-1} c$$

$$\boxed{\frac{y-x}{1+xy} = c}$$

eg. Solve $\int x \frac{dy}{dx} = \sqrt{1-y^2}$

$$\sin^{-1} y = \ln x + \ln c^*$$

{c^* is any arbitrary constant)

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Date
Page

eg. Solve $(x+y)dx + dy = 0$

$$x+y = z \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$$

① $$(x+y) + \frac{dy}{dx} = 0$$

$$z + \frac{dz}{dx} - 1 = 0$$

$$\frac{dz}{dx} = 1 - z$$

$$\int \frac{dz}{1-z} = \int dx + C^{*}$$

$$-\ln(1-z) = x + C$$

$$-\ln(1-x-y) = x + C$$

② $$z(dz-dy) + dy = 0$$

$$z\,dz + (1-z)\,dy = 0$$

$$c + \int \frac{z\,dz}{z-1} = \int dy$$

$$y = \int \frac{z\,dz}{z-1} + c$$

$$y = z + \ln(z-1) + C$$

$$\cancel{y} = x + \cancel{y} + \ln(x+y-1) + C$$

$\downarrow$ arbitrary consta

Equations that can be solved by variable seperable form i.e. Homogenous eqn

Homogenous function :- A function $f(x,y)$ is said to be homogenous of degree n, if

$$f(tx, ty) = t^n f(x,y) \quad t \in \mathbb{R}$$

Homogenous differential eqn :- The differential eqn $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogenous if both $M(x,y)$ and $N(x,y)$ are homogenous of same degree.

eg. $(x^2 + xy)\,dx + y^2\,dy = 0$

Once we identify that the differential eqn is homogenous then we solve as below

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Also $y' = g(y/x) \Rightarrow \dfrac{dy}{dx} = g(v)$

classmate
Date
Page

$$v + x\frac{dv}{dx} = g(v) \Rightarrow \int \frac{dv}{g(v)-v} = \int \frac{dx}{x} + C$$

Que:- $x^2 y\,dx - (x^3+y^3)\,dy = 0$

$$y = vx$$

$$x^2(vx) - (x^3 + v^3x^3)\left(v + x\frac{dv}{dx}\right) = 0$$

$$v - (1+v^3)\left(v + x\frac{dv}{dx}\right) = 0$$

$$\frac{v}{1+v^3} = v + x\frac{dv}{dx}$$

$$\frac{v}{1+v^3} - v = x\frac{dv}{dx}$$

$$\frac{\cancel{v} - \cancel{v} - v^4}{1+v^3} = x\frac{dv}{dx}$$

$$\frac{-v^4}{1+v^3} \quad -\frac{dx}{x} = \frac{dv(1+v^3)}{v^4}$$

$$+\frac{dx}{x} = -\left(\frac{1}{v^4} + \frac{1}{v}\right)dv$$

$$\ln x = \frac{-1}{3v^3} - \ln v + \ln C$$

$$\cancel{\ln}\ \frac{vx}{C} = e^{\frac{-1}{3v^3}}$$

$$\frac{vx}{C} = e^{-1} \cdot e^{\frac{1}{3v^3}} \Rightarrow y = Ce^{\frac{x^3}{3y^3}}$$

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Special types of differential equations which can be solved by homogenous method :-

Type-1 $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ here, $\frac{a}{a'} = \frac{b}{b'}$

a, b, c, a', b', c' are constant.

To solve this differential equation, we observe that $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are intersecting each other

Let (h, k) be the point of intersection

$ah + bk + c = 0$, $a'h + b'k + c' = 0$

we can solve these two equation to get the value of (h, k). Now shifting the origin to (h, k) we $x = h + X$, $y = k + Y$

$$\frac{dy}{dx} = \frac{dY}{dX} \quad \text{--- (1)}$$

$$\frac{dy}{dx} = \frac{a(h + X) + b(k + Y) + c}{a'(h + X) + b'(k + Y) + c'}$$

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

Now substitute $Y = vX$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\frac{dY}{dX} = \frac{aX + bvX}{a'X + b'vX} = \frac{a + bv}{a' + b'v} \quad \text{--- } (**)$$

$$V + x\frac{dv}{dx} = \frac{a+bV}{a'+b'V}$$

this can be solved by Variable seperable with we get the sol.

Type-2 $\quad \frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ wher $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{\lambda}$

$$ax+by+c=0 \ \& \ a'x+b'y+c'=0$$

$$\frac{dy}{dx} = \frac{ax+by+c}{a\lambda x+b\lambda y+c'} = \frac{ax+by+c}{\lambda(ax+by)+c'}$$

$$= \frac{z+c}{\lambda z+c'}$$

we substitute

$$z = ax+by$$

$$\frac{dz}{dx} = a+b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx}-a\right) \text{—} (*)$$

equating (*) and (**) we get,

$$\boxed{\frac{1}{b}\left(\frac{dz}{dx}-a\right) = \frac{z+c}{\lambda z+c}}$$

This can be solved by variable seperdy. finally s.

Solve the following differential eqns :-

$$\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$$

$$\frac{a}{a'} \neq \frac{b}{b'}$$

$h = 1$

$k = -5$

let the pair of straight lines $x+y+4=0$, $x-y-6=0$ will intersect at (h, k). $h=1$ $k=-5$

$x = X+1$, $y = Y-5$

$$\frac{dy}{dx} = \frac{X+\cancel{1}+Y-\cancel{5}+\cancel{4}}{X+\cancel{1}-Y+\cancel{5}-\cancel{6}}$$

$$\frac{dY}{dX} = \frac{X+Y}{X-Y}$$

Put $Y = VX$

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$\cancel{V+} X\frac{dV}{dX} = \frac{1+V}{1-V} - V$$

$$X\frac{dV}{dX} = \frac{1+\cancel{V}-\cancel{V}+V^2}{1-V}$$

$$\frac{1-V}{1+V^2}\,dV = \frac{dX}{X}$$

$$\tan^{-1}(V) - \frac{1}{2}\log_e(1+V^2) = \log_e X + \ln C$$

$$\tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left(1+\frac{Y^2}{X^2}\right) = \ln X + \ln C$$

Classmate

Date

Page

$$\tan^{-1}\left(\frac{y+5}{x-1}\right)$$

$$\tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln(X^2+Y^2) = \ln c$$

$$\tan^{-1}\left(\frac{Y}{X}\right) = \ln c + \ln\sqrt{X^2+Y^2}$$

$$e^{\tan^{-1}\frac{Y}{X}} = c\sqrt{X^2+Y^2}$$

$$e^{\tan^{-1}\left(\frac{y+5}{x-1}\right)} = c\sqrt{(x-1)^2+(y+5)^2}$$

Que:- $\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$

here,

Pair of lines are $\parallel^{el}$ $\quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$

$\boxed{\lambda = 1}$

$$x+y = z$$

$$1+\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} \cancel{-1} = \frac{z+4}{z-6} + 1$$

$$\frac{dz}{dx} = \frac{z+4+z-6}{z-6}$$

$$\frac{dz}{dx} = \frac{2z-2}{z-6}$$

$$\frac{z-6}{2z-2}\,dz = dx$$

$$\frac{z-6}{z-1}\,dz = 2dx$$

$$\left(1 - \frac{5}{z-1}\right) dz = 2dx$$

$$z - 5\ln|z-1| = 2x + C$$

$$x + y - 5\ln|x+y-1| = 2x + C$$

$$y = x + 5\ln|x+y-1| + C$$

Exact differential equation :–

A differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is called , if there exist a function $u(x,y)$,

$$du(x,y) = M(x,y)\,dx + N(x,y)\,dy$$

If the eq^n is exact the Sol^n is given by :–

Theorem :– The differential eq^n $M\,dx + N\,dy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof :– Let differ^n Eq^n $M\,dx + N\,dy = 0$ is exact as per def^n of exact DE we must find a fn^n $u(x,y)$

$$d\,u(x,y) = M\,dx + N\,dy$$

Date

Page

$$d(\psi(x,y)) = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = Mdx + Ndy$$

$$\Rightarrow M = \frac{\partial \psi}{\partial x}, \quad N = \frac{\partial \psi}{\partial y} \quad \text{———} \quad (*)$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y}$$

Because of continuity of the solⁿ, we have

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for the differential eqⁿ $Mdx + Ndy = 0$

We need to show that that $Mdx + Ndy$ is exact i.e. $\exists\ \psi$ such that

$$d(\psi) = Mdx + Ndy$$

let, $V(x,y) = \int M dx$

$$\frac{\partial V}{\partial x} = M, \quad \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) \quad \text{———} \quad (*)$$

Integrating w r t $x \Rightarrow$

$N = \frac{\partial V}{\partial y} + \phi'(y)$ where ϕ' some funcⁿ

Now $M\,dx + N\,dy = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \phi'(y)\,dy$

$= d\,[\underbrace{v + \phi(y)}_{u(x,y)}]$ Thus the diff eqn is exact

Que:- $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Exact

Ques:- Method of Solution for exact differential eqn:-

If the DE $M\,dx + N\,dy = 0$ is exact i.e, we can find a function $u(x,y)$ st $du = M\,dx + N\,dy$

We know $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$

Integrating wrt x we get,

$$u(x,y) = \int M\,dx + \phi(y) \quad \text{———} \quad (\star)$$

$$\frac{\partial u}{\partial y} = N = \frac{\partial}{\partial y}\int M\,dx + \phi'(y) \Rightarrow$$

$$\phi'(y) = N - \frac{\partial}{\partial y}\int M\,dx$$

Integrating w.r.t y

$$\phi(y) = \int\left(N - \frac{\partial}{\partial y}\int M\,dx\right)dy + C$$

Now substituting this value of ϕ in u, we get

$$\boxed{u(x,y) = \int M\,dx + \int \underbrace{\left(N - \frac{\partial}{\partial y}\int M\,dx\right)}\,dy + C}$$

here 'x' term cancelled

$$u(x,y) = \int M\,dx + \int N\,dy$$

Working Rule:— If the DE $M\,dx + N\,dy = 0$ is exact then its solution is given by

$$\underbrace{\int M\,dx}_{\substack{\text{integrating wrt } x \\ \text{(treating } y \text{ as constant)}}} + \underbrace{\int N\,dy}_{\substack{\text{(integrate terms involving only } y \\ \text{or the terms which do not contain } x\text{)}}} = C$$

classmate

Date

Page

Solve:- $xe^{x^2+y^2}dx + y(e^{x^2+y^2}+1)dy = 0 \quad y(0)=0$

$c = \frac{1}{2}$

$M = xe^{x^2+y^2}$

$N = y(e^{x^2+y^2}+1)$

$\frac{\partial M}{\partial y} = xe^{x^2+y^2}.2y$

$\frac{\partial N}{\partial x} = y(e^{x^2+y^2}.2x)$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Exact

$\int xe^{x^2+y^2}dx + \int y(e^{x^2+y^2}+1)dy = c$

$\frac{e^{y^2}.e^{x^2}}{2} + \frac{y^2}{2} = c$

$e^{x^2+y^2} + y^2 = 2c$

Que:- Solve $e^y dx + (xe^y + 2y)dy = 0$

$M = e^y \qquad N = xe^y + 2y$

$\frac{\partial M}{\partial y} = e^y \qquad \frac{\partial N}{\partial x} = e^y$

exact

$\int e^y dx + \int (xe^y + 2y)dy = C$

$e^y x + \cancel{\frac{2y^2}{2}} = c$

Integrating factor:— If the differential eq^n $Mdx + Ndy = 0$ is not exact we can choose a function say $\mu(x,y)$ such that if we multiply $\mu(x,y)$ to the eq^n $Mdx + Ndy = 0$, the DE become exact. This function is known as integrating factor.

Que:- Solve $xdx + ydy + (x^2+y^2)x^2 dx = 0$, is not exact, if we multiply $\boxed{\mu(x,y) = \frac{1}{x^2+y^2}}$ both sides,

I.F.

$$\frac{xdx + ydy}{x^2+y^2} + x^2 dx = 0$$

$$\Rightarrow \frac{1}{2}\log(x^2+y^2) + \frac{x^3}{3} = c$$

★ The no. of IF for a differential eq^n is infinite.

Proof:- Let $\mu(x,y)$ be an integrating factor of the $Mdx + Ndy = 0$

that is $\mu(x,y)[Mdx + Ndy] = d(\psi(x,y))$ $\} \psi(x,y)$

Let us multiply a function of $\phi(\psi)$

$$\boxed{\phi(\psi)\,\mu(x,y)[Mdx + Ndy] = \phi(\psi)d\psi}$$

Since $\phi(4)$ is any function of 4, $4\,\phi(4)$ is also an IF.

Methods to get IF :-

Thrm ① If the differential eqn $Mdx + Ndy$ is not exact and homogenous such that $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an IF.

Thrm ② If differential eqn $Mdx + Ndy = 0$ is of the form
$$f_1(x,y)\, y\, dx + f_2(xy)\, x\, dy = 0$$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an IF.

Ques:- for Thrm 1

Solve $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0$

$$Mx + Ny = \cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}$$
$$= x^2y^2$$

$$IF = \frac{1}{x^2y^2}$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$$

$$\frac{1}{y}x - 2\ln|(x)| + 3\ln|y| = C$$

Thr^m

Ques:- Solve $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

$$Mx - Ny = \cancel{x^3y^3} + x^2y^2 + \cancel{xy} - \cancel{x^3y^3} + x^2y^2 - \cancel{xy}$$
$$= 2x^2y^2$$

$$\Rightarrow \left(\frac{1}{2} + \frac{1}{2xy} + \frac{1}{2x^2y^2}\right)y\,dx + \left(\frac{1}{2} - \frac{1}{2xy} + \frac{1}{2x^2y^2}\right)x\,dy = 0$$

$$\int\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \int\left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0 + C$$

$$\boxed{xy + \ln x - \frac{1}{xy} - \ln y = C}$$

Thr^m (3) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ N = f(x)$ fn of only x. then

$$I.F. = e^{\int f(x)\cdot dx}$$

Thr^m (4) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/ M = f(y)$

$$I.F. = e^{-\int f(y)\,dy}$$

Ques:- Solve $(x^2+y^2)\,dx - 2xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = -2y$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ -(2xy) = -\frac{\cancel{4y}^{\,2}}{\cancel{2xy}} = \frac{-2}{x} = f(x)$$

$$\text{I.F.} = e^{\int \frac{-2}{x}\,dx}$$

$$= e^{-2\ln x}$$

$$= e^{\ln x^{-2}}$$

$$= \frac{1}{x^2}$$

$$\int\left(1 + \frac{y^2}{x^2}\right)dx - \int \frac{2y}{x}\,dy = 0 + C$$

$$x + y^2\left(-\frac{1}{x}\right) = C$$

$$\frac{x^2 - y^2}{x} = C$$

Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy$

$$\frac{\partial M}{\partial y} = 3x^2(4y^3) + 2x$$

$$\frac{\partial N}{\partial x} = 2y^3(3x^2) - 2x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 12x^2y^3 + 2x - 6x^2y^3 + 2x$$

$$= 6x^2y^3 + 4x$$

$$\frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy} \qquad \frac{6x^2y^3 + 4x}{2x^3y^3 - x^2}$$

$\Downarrow$ $\qquad\qquad \Rightarrow$

$$\frac{2}{y}\frac{\cancel{(3x^2y^4 + 2xy)}}{\cancel{3x^2y^4 + 2xy}}$$

$$f(y) = \frac{2}{y}$$

$$I.F. = e^{-\int \frac{2}{y}\,dy}$$

$$= e^{-2\ln y^2}$$

$$= \frac{1}{y^2}$$

$$\int (3x^2y^6 + 2xy^3)\,dx + \int (2x^3y^5 - x^2y^2)\,dy = C$$

$$\cancel{-3y^6\left(\frac{3x^3}{3}\right) + 2y^3\frac{x^2}{2}} + \int = C$$

$\frac{3}{12}$ n. 4

Linear Differential equation :-

A differential equation of form $\frac{dy}{dx} + p(x)y = Q(x)$ where $p(x)$ and $Q(x)$ are function of x is called a linear D E of Ist order.

To get the solution, we write the equation as

$$\underbrace{1.dy}_{N} + [\underbrace{p(x)y - Q(x)}_{M}]\,dx = 0$$

We check that the differential eq^n is not exact $\frac{\partial M}{\partial y} = p(x) \neq 0$

$\frac{\partial N}{\partial x} = 0$

If $\mu(x)$ will be a integrating factor the D E

$\mu(x)\,dy + \underbrace{\mu(x)[p(x)y - Q(x)]}\,dx = 0$ is exact

$$\Rightarrow \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x} \Rightarrow \frac{\partial}{\partial y}[\mu(x)p(x)y - \mu(x)Q(x)]$$

$$= \frac{\partial}{\partial x}\mu(x) = \frac{d\mu}{dx}$$

$$\mu(x)p(x) = \frac{d\mu}{dx} \Rightarrow \int \frac{d\mu}{\mu} = \int p(x)\,dx + c$$

Put $c = 0$

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classmate
Date
Page

$$\ln y = \int p(x)\,dx$$

$$\boxed{y = e^{\int p(x)\,dx}}$$

Result :- The linear differential equation $\frac{dy}{dx} + p(x)\,y = Q(x)$ has the solution

$$y = e^{-\int P\,dx}\left[\int Q(x)\,e^{\int p\,dx}\,dx + C\right] \longrightarrow (*)$$

Proof :- Since I.F. $= y = e^{\int p\,dx}$ we multiply throughout the equation

$$e^{\int p\,dx}\frac{dy}{dx} + e^{\int p\,dx}\,p(x)\,y = e^{\int p\,dx}\,Q(x) \longrightarrow (**)$$

But we observe that

$$\frac{d}{dx}\left\{y\,e^{\int p\,dx}\right\} = e^{\int p\,dx}\frac{dy}{dx} + e^{\int p\,dx}\,p(x)\,y = e^{\int p\,dx}\left[\frac{dy}{dx} + p(x)\,y\right]$$

$$= e^{\int p\,dx}\,Q(x)$$

Integrating wrt x both side

$$y\,e^{\int p\,dx} = \int e^{\int p\,dx}\,Q(x)\,dx + C$$

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Date ______
Page ______

Solve

Que:- $\frac{dy}{dx} - \frac{(x+1)}{x} y = x - x^2$

$$p(x) = -\frac{(x+1)}{x}$$

$$Q(x) = x - x^2$$

$$I.F. = e^{\int p \cdot dx} = e^{\int -\frac{(x+1)}{x} \cdot dx}$$

$$= e^{\int -1 - \frac{1}{x} dx}$$

$$= e^{-x - \ln x}$$

$$= e^{-x} \cdot e^{\ln 1/x}$$

$$= \frac{1}{xe^{x}}$$

$$y \cdot \frac{1}{xe^x} = \int (x - x^2) \cdot \frac{1}{xe^x} dx + C$$

$$\frac{y}{xe^x} = \int \frac{(1-x)}{e^x} dx + C$$

$$\frac{y}{xe^x} = -e^{-x} - \int xe^{-x} dx + C$$

$$\frac{y}{xe^x} = -e^{-x} - \left[x(-e^{-x}) + \int e^{-x} dx\right] + C$$

$$\frac{y}{xe^x} = \cancel{-e^{-x}} + xe^{-x} + \cancel{e^{-x}} + C$$

$$y = x^2 + Cxe^{x}$$

Solve IVP $\frac{dy}{dx} + y\tan x = \sin 2x$ $\quad y(0)=1$

$$IF = e^{\int \tan x \, dx}$$

$$= \cancel{e}^{\cancel{\log}\sec x}$$

$$= \sec x$$

$$y \cdot \sec x = \int \sin 2x \cdot \sec x \, dx + C$$

$$y\sec x = \int 2\sin x\cancel{\cos x}\,\frac{1}{\cancel{\cos x}}\,dx + C$$

$$y\sec x = -2\cos x + C$$

~~y=~~

$$1 \cdot 1 = -2 + C$$

$$\underline{C = 3}$$

$$y\sec x = -2\cos x + 3$$

$$y = -2\cos^2 x + 3\cos x$$

Date
Page

Equation reducable to Linear form (Bernoulli eqn):-

The differential equation of the form $\frac{dy}{dx} + p(x)y = y^n Q(x)$

when $n \neq 0, 1$ is called Bernoulli eqn

If $n = 0 \rightarrow$ linear
$n = 1 \rightarrow$ Variable seperable.

to get the solution we divide y^n throughout

$$y^{-n} \frac{dy}{dx} + p(x)\, y^{1-n} = Q(x)$$

Substitute $y^{1-n} = z$

$$\frac{dz}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

Multiply by $(1-n)$

$$(1-n)\, y^{-n} \frac{dy}{dx} + p(x)(1-n)\, y^{1-n} = (1-n)\, Q(x)$$

$$\Rightarrow \frac{dz}{dx} + (1-n)\, p(x)\, z = (1-n)\, Q(x)$$

This is a linear DE with z and x

$$\frac{dz}{dx} + p_*(x)\, z = Q_*(x)$$ when $p_*(x) = (1-n)\, p(x)$

$Q_*(x) = (1-n)\, Q(x)$

$$z = e^{-\int P_*\,dx}\left[\int Q_*(x)\, e^{\int P_*(x)\,dx}\,dx + c\right]$$

Substitute

$z = y^{1-n}$ to get soln in y & x.

Q. Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x}\log x$

Soln:- $\quad y^{-2}\dfrac{dy}{dx} + \dfrac{1}{xy} = \dfrac{\log x}{x}$

$$y^{-1} = z$$

$$z = \frac{1}{y}$$

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$-\frac{dz}{dx} + \frac{1}{zx} = \frac{\log x}{x}$$

$$\frac{dz}{dx} - \frac{\cancel{z}}{\cancel{z}x} = -\frac{\log x}{x}$$

$$I.F = e^{-\int \frac{1}{x}dx}$$

$$= e^{\ln x^{-1}}$$

$$= \frac{1}{x}$$

classmate
Date
Page

$$z \cdot \frac{1}{x} = \int \frac{1}{x} \cdot -\frac{\log x}{x} \cdot dx + C$$

$$\cancel{\frac{z}{x} = \left[-\frac{\log x}{x} (\log x) - \int \right]}$$

$$\frac{z}{x} = \left[1 - \log x \left[\frac{-1}{x} \right] - \int \frac{-1}{x} \cdot \frac{-1}{x} \, dx \right]$$

$$\frac{z}{x} = \left[\frac{\log x}{x} + \frac{1}{x} \right] + C$$

$$z = \log x + 1 + cx$$

$$\frac{1}{xy} = \log x + 1 + cx$$

$$\cancel{y \log x + cxy + y = 1}$$

$$\cancel{1 = xy \log x + xy + cx^2 y}$$

Date

Page

Orthogonal Trajectory :- A curve which intersects each member of a family of curves at right angles is known as orthogonal trajectory.

Procedure to get orthogonal trajectory:-

Let $f(x, y, c) = 0$ — (1)

Differentiate this $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \quad — (2)$$

Eliminate the constant using (1) and (2) to get

$$g\left(x, y, \frac{dy}{dx}\right) = 0 \quad — (3)$$

Now since the slope of the curve is perpendicular to the slope of orthogonal trajectory, we get the slope of the orthogonal trajectory as

$$\frac{dy}{dx} = \frac{\partial f/\partial y}{\partial f/\partial x} \quad — (4)$$

If we solve this (4) we get a family of curve known as orthogonal trajectory.

Que:- Find a family of orthogonal trajectorys of $y = cx^2$

$$\frac{dy}{dx} = 2cx \Rightarrow \frac{dy}{dx} = 2 \cdot \frac{y}{x^{\cancel{2}}} \cdot \cancel{x} = \frac{2y}{x}$$

The slope of orthogonal trajectories is

$$\cancel{\frac{dy}{dx} = -\frac{1}{2cx}}$$

$$\cancel{\int dy = \int \frac{dx}{2cx}}$$

$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$\int 2y\,dy = \int -x\,dx$$

$$y^2 = -\frac{x^2}{2} + c$$

Que:- ① Solve $(1+y^2)\frac{dx}{\cancel{d}} = (\tan^{-1}y - x)\,dy$

$$1 = \left(\frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}\right)\frac{dy}{dx}$$

② Solve $\frac{dy}{dx} + \frac{y}{x} = \frac{\sin x}{x}$, $y(0) = 0$

③ Solve $x \log x \frac{dy}{dx} + y = 2 \log x$

④ $\sin y \, dy/dx = \cos x \,(2\cos y - \sin^2 x)$

Date

Page

Solution:- 2 ~~$(1+y^2)\cdot d$~~

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\sin x}{x}, \quad y(0) = 0$$

$$I.F = e^{\int \frac{1}{x} \cdot dx}$$

$$= e^{\int \ln x} = x$$

$$y \cdot x = \int \frac{\sin x}{x} \cdot x + C$$

$$xy = -\cos x + C$$

$$\boxed{c = 1}$$

$$\boxed{xy + \cos x = 1}$$

Solution:- 1 $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1} y}$$

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$$x \cdot e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2}\,dy + c$$

Put $\tan^{-1}y = t$

$\frac{1}{1+y^2}\,dy = dt$

$$xe^t = \int e^t\, t\, dt + c$$

$$xe^t = t.e^t - \int e^t dt + c$$

$$xe^t = te^t - e^t + c$$

$$xe^{\tan^{-1}y} = (\tan^{-1}y - 1)\cancel{e^{\tan^{-1}y}} + c$$

$$x = (\tan^{-1}y - 1) + ce^{-\tan^{-1}y}$$

Solution-3

$$x\log x \frac{dy}{dx} + y = 2\log x$$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

$$I.F = e^{\int \frac{1}{x\log x}dx}$$

Put $\log x = t$

$\frac{1}{x}dx = dt$

$$= e^{\int \frac{1}{t}dt}$$

$$= e^{\ln t} = \log x$$

$$y \cdot \log x = \int \frac{2}{x} . \log x + c$$

$$\cancel{y\log x = 2\log x[\log x] - \int \frac{2}{x}\log x + c}$$

$$y_{\log x} = (\log x)^2 + c$$

(4) $\sin y \frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$

$\sin y \frac{dy}{dx} - 2\cos y \cos x = -\sin^2 x \cos x$

$\cancel{\frac{dy}{dx}}$ $\qquad -\cos y = t$

$+\sin y \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + 2t\cos x = -\sin^2 x \cos x$

$I.F = e^{\int 2\cos x \cdot dx}$

$\Rightarrow I.F. = e^{+2\sin x}$

$t.e^{2\sin x} = \int -\sin^2 x \cos x . e^{2\sin x}\, dx + c$

$\sin x = t$

$\cos x\, dx = dt$

$te^{2t} = \int -t^2 e^{2t}\, dt + c$

$te^{2t} = -t^2 . \frac{e^{2t}}{2} - \int (-\cancel{2}t)\frac{e^{2t}}{\cancel{2}}\, dt + c$

$te^{2t} = -t^2 \frac{e^{2t}}{2} + \int t e^{2t}\, dt + c$

$te^{2t} = -t^2 \frac{e^{2t}}{2} + t . \frac{e^{2t}}{2} - \int 1 . \frac{e^{2t}}{2} . dt + c$

classmate

Date

Page

Linear Differential equation of second order :-

A differential equation of the form $y'' + p(x)y' + q(x)y = r(x)$ is called as linear differential equation of 2^{nd} order. Here $p(x)$, $q(x)$ and $r(x)$ are function of x only. It is linear in y, y', y''. If $r(x) = 0$ we call the equation as homogenous otherwise non homogenous.

eg. $(1-x^2)y'' - 2xy' + 6y = 0$

A solution of Linear 2^{nd} diff. equation is a function $y(x)$ such that y', y'' should exist ~~or~~ if we substitute in the eqn we should get the identity.

Homogenous linear second order $(r(x)=0)$:-

Result :- (Fundamental theorem of homogenous linear equation).

For a homogenous Li. differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad — (1)$$

any linear combination of two solution on an open interval I is again a solution. In particular the sum and constant multiple of soln are also a sol.

let y_1 and y_2 are two soln.

classmate

Date

Page

let $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are arbitrary constants.

$$y' = c_1 y_1' + c_2 y_2'$$

$$y'' = c_1 y'' + c_2 y_2''$$

Substituting

$$\left(y_1'' + p(x) y_1' + q(x)\, y_1\right) + \left(y_2'' + p(x) y_2' + q(x)\, y_2\right) = 0$$

Fundamental soln :-

A second order linear homogenous DE has general ~~eqn~~ solution is of the form $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are arbitrary constants this y_1 and y_2 are linearly independent and form a fundamental system.

Basis $\rightarrow \{y_1, y_2\} \rightarrow$ fundamental system.

I.V.P. for second order :- A second differential equation with the intial condition

$$y'' + p(x) y' + q(x) y = 0$$

$$y(x_0) = \alpha$$

$$y'(x_0) = \beta$$

Particular Solution:- Any solution that can be obtained by assigning particular value to c_1, c_2 in general soln is called particular solution.

Method of Solution:- (Reduction of order)

$$y'' + p(x)y' + q(x)y = 0$$

In this case we will guess one of the soln.

Let y_1 be a known soln to this (2)

Since the second solution is L.I.D to this y_1.

$$y_1 \neq K y_2 \quad \text{or} \quad \frac{y_1}{y_2} \neq \text{constant}$$

Let $y_2 = u(x) y_1$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

$$u'' y_1 + u'(2y_1' + p y_1) + u(y_1'' + p y_1' + q y_1) = 0$$

$$u'' y_1 + 2u' y_1' + u y_1'' + p(u' y_1 + u y_1')$$
$$+ q(u(x) y_1) = 0$$

$$u'' y_1 + u'(2y' + p y_1) + u\cancel{(p y_1' + q y_1 + y_1'')} = 0$$

0

$$\Rightarrow u''y_1 + u'(2y_1' + py_1) = 0$$

Let $u' = U$

$$U'y_1 + U(2y_1' + py_1) = 0$$

$$U' + U\left(\frac{2y_1'}{y_1} + p\right) = 0$$

This is linear OD in U and x,

$$\frac{U'}{U} = -\left(\frac{2y_1'}{y_1} + p\right)$$

Integrating both sides

$$\Rightarrow \ln|U| = -2\ln|y_1| - \int p\,dx$$

$$\boxed{U = \frac{1}{y_1^2} e^{-\int p\,dx}}$$

$$U = u' \Rightarrow u = \int U.dx \Rightarrow \int \frac{1}{y_1^2} e^{-\int p\,dx}$$

$$y_2 = u y_1$$

Basis $= \{y_1, y_2\}$

Solve the equation :-

$x^2y'' - xy' + y = 0$ by assuming $y_1 = x$.

Let $y_2 = u x$

$$u = \int \frac{1}{x^2} e^{-\int -x\,dx} \cdot dx$$

$$u = \int \frac{1}{x^2} \cdot e^{x^2/2} \cdot dx$$

put $x^2 = t$

$2x\,dx = dt$

$$p = \frac{-x}{x^2} = -\frac{1}{x}$$

$$u = \int \frac{1}{x^2} e^{\int \frac{1}{x} \cdot dx} \cdot dx$$

$$u = \int \frac{1}{\cancel{x^2}} \cdot \cancel{x}\,dx$$

$$u = \ln x$$

$$y_2 = x \ln x$$

1. Second order homogenous linear differential eq^n

$$y'' + p(x)y' + q(x)y = 0 \quad -(1)$$

2. The general sol^n of this is of the form $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are arbitrary constants. The set $\{y_1, y_2\}$ is called basis of sol^n or fundamental system then y_1 and y_2 are L.I.

3. Method of sol for this differential, we discussed the method of Reduction for sol^n.

Given a sol^n y_1 we chose $y_2 = 4y_1$. Finally we get

$$4 = \int U\,dx, \text{ where } U = \frac{1}{y_1^2} e^{-\int P \cdot dx}$$

* If the equation is not in the form (1) we can not use this formula directly, however we can substitute some new variable so that the second order diff eq^n is reduced to first order and can be solved.

classmate

Date

Page

Que:-1 $x^2y'' - 5xy' + 9y = 0$, $y_1 = x^2$,

Que:-2 $y'' = y'$, $y_1 = e^x$

Que:-3 $yy'' = 2y'^2$

$$y' = z \qquad z' = \frac{dz}{dy}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z'z$$

$$yzz' = 2z^2 \Rightarrow y\frac{dz}{dy} = 2z$$

$$\Rightarrow \frac{1}{2}\frac{dz}{z} = \frac{dy}{y}$$

$$\Rightarrow 2\ln y = \ln z + \ln C$$

$$z = y^2c_1$$

$$\Rightarrow \frac{dy}{dx} = y^2c_1$$

$$\int y^{-2}dy = \int c_1\,dx$$

$$\Rightarrow -\frac{1}{y} = c_1x + c_2$$

Date
Page

Solve :-

① $\frac{dy}{dx} + y \tan x = y^2 \sec^3 x$

$y^{-2} \frac{dy}{dx} + y^{-1} \tan x = \sec^3 x$

$\frac{1}{y} = z$

$+\frac{1}{y^2} \frac{dy}{dx} = -\frac{dz}{dx}$

$-\frac{dz}{dx} + \tan x \, z = \sec^3 x$

$\frac{dz}{dx} - (\tan x) z = -\sec^3 x$

$$\text{I.F.} = e^{\int P \cdot dx}$$

$$= e^{-\int \tan x \, dx}$$

$$= e^{+\log_e \cos x}$$

$$= \cos x$$

$$z \cdot \cos x = \int \cos x \cdot \left(-\frac{1}{\cos^3 x}\right) \cdot dx + C$$

$$z \cos x = \int -\sec^2 x \, dx + C$$

$$z\cos x = -\tan x + c$$

$$\cos x = -y\tan x + cy$$

$$\boxed{\cos x + y\tan x = cy}$$

② $$y - \frac{dy}{dx}\cos x = y^2\cos x(1-\sin x)$$

$$y\sec x - \frac{dy}{dx} = y^2(1-\sin x)$$

$$\frac{dy}{dx} - y\sec x = y^2(\sin x - 1)$$

$$y^{-2}\frac{dy}{dx} - \frac{1}{y}\sec x = \sin x - 1$$

Put $-\frac{1}{y} = z$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + z\sec x = \sin x - 1$$

$$I.F = e^{\int \sec x \cdot dx}$$

$$= \sec x + \tan x$$

$$z(\sec x + \tan x) = \int (\sec x + \tan x)(\sin x - 1)\cdot dx$$

$$\Rightarrow z(\sec x + \tan x) = \int \frac{\sin^2 x - 1}{\cos x}\cdot dx$$

$$= (\sec x + \tan x) = \int -\cos x \, dx + C$$

$$= \left(\frac{1+\sin x}{\cos x} \right) = -\sin x + C$$

$$-\frac{1}{y} \left(\frac{1+\sin x}{\cos x} \right) = -\sin x + C$$

$$\boxed{\frac{1+\sin x}{y\cos x} = \sin x + C_1}$$

③ $$x \log x \left(1 - \frac{dy}{dx} \right) = y$$

$$-x \log x \frac{dy}{dx} + x \log x = y$$

$$\frac{dy}{dx} - 1 = \frac{-y}{x \log x}$$

$$\frac{dy}{dx} + \frac{y}{x \log x} = 1$$

$$I.F = e^{\int \frac{1}{x \log x} \cdot dx}$$

Put $\log x = t$

$\frac{1}{x} dx = dt$

$$= e^{\int \frac{dt}{t}}$$

$$= e^{\ln t} = \log x$$

$$y \log x = \int 1 . \log x + c$$

$$y \log x = x \log x - x + c$$

(4) $\tan x \, dy + \tan y \, dx + \sec y \cos(\sin x)\, dx = 0$

$$\tan x \frac{dy}{dx} + \tan y + \sec y \cos(\sin x) = 0$$

Second order homogenous linear with constant coefficients :-

A differential eqn of the form $y'' + ay' + by = 0$ where a and b are constants is known as second order homogenous linear with constant coefficients.

To solve the differential eqn :-

Say $\Rightarrow y = e^{\lambda x}$ (λ is unknown). Find y' and y''

$$y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 e^{\lambda x} + a(\lambda e^{\lambda x}) + b e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

$\because e^{\lambda x}$ can not be zero.

This eq^n in λ is called characteristics eq^n or auxilliary eq^n.

This is a quadratic eq^n in λ and we get two values.

$$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

$$\lambda_1 = -a + \sqrt{a^2-4b} \qquad \lambda_2 = \frac{-a - \sqrt{a^2-4b}}{2}$$

There are 3 cases depending on sign of a^2-4b

Case-I $\quad a^2-4b>0$

(Two real distinct roots)

Let the two real distinct roots be

$$\lambda_1 = \frac{-a+\sqrt{a^2-4b}}{2}$$

$$\lambda_2 = \frac{-a-\sqrt{a^2-4b}}{2}$$

The two linearly independent sol^n are

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

A basis is $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$

General sol^n is $\quad y = c_1 y_1 + c_2 y_2$.

classmate
Date
Page

1. $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$

$$y = e^{\lambda x}$$

To get auxiliary eq^n in λ put $y = 1$,
$\frac{dy}{dx} = y' = \lambda$, $\frac{d^2y}{dx^2} = y'' = \lambda^2$

$$\lambda^2 + \lambda - 2 = 0$$
$$\lambda^2 + 2\lambda - \lambda - 2 = 0$$
$$\lambda(\lambda+2) - 1(\lambda+2) = 0$$
$$\lambda = 1 \ \& \ -2$$

$y_1 = e^x$ $\quad$ $y_2 = e^{-2x}$
Basis = $\{e^x, e^{-2x}\}$

$$y = c_1 e^x + c_2 e^{-2x}$$
where c_1 and c_2 are arbitrary constants

its given that
$$y(0) = 4$$

$$4 = c_1 + c_2$$

$$y' = c_1 e^x - 2c_2 e^{-2x}$$ $\boxed{y = e^x + 3e^{-2x}}$

$$-5 = \cancel{c_1} - 2c_2$$
$$\underline{\overset{-}{4} = \overset{-}{\cancel{c_1}} + \overset{-}{c_2}}$$
$$-9 = -3c_2 \quad \boxed{c_2 = 3}$$

$\boxed{c_1 = 1}$

Case-II $a^2 - 4b = 0$. (Real double root).

In this case, $\lambda_1 = \lambda_2 = -\frac{a}{2}$

One of the roots is $y_1 = e^{-\frac{a x}{2}}$

The other solution can be obtained by considering $y_2 = u y_1$ (use method of reduction of order)

$$y_2 = u e^{-\frac{a}{2}x}$$

$$u = \int \frac{1}{y_1^2} e^{-\int P \cdot dx} dx$$

$$u = \int \frac{1}{e^{-ax}} e^{-\int a \cdot dx}$$

$$= \int \frac{1}{e^{-ax}} \cdot e^{-ax} dx$$

$$= x$$

Hence, $y_2 = x e^{-\frac{a}{2}x}$

Basis $= \{ e^{-a/2 x}, x e^{-\frac{a x}{2}} \}$

General soln is $y = (c_1 + c_2 x) e^{-\frac{a}{2}x}$

Solve:- $y'' + 8y' + 16y = 0$

$$\lambda^2 + 8\lambda + 16 = 0$$

$$(\lambda + 4)^2 = 0$$

$$\lambda = -4$$

Hence

$y_1 = e^{-4x}, y_2 = x e^{-4x}$ Basis $= \{e^{-4x}, xe^{-4x}\}$

$$y = (c_1 + c_2 x) e^{-4x}$$

where c_1 and c_2 are arbitary constants.

$y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0$ $y(0) = 3$

$\lambda = 2$ $y'(0) = 1.$

Hence,

$y_1 = e^{2x}, y_2 = x e^{2x}$

Basis $= \{e^{2x}, xe^{2x}\}$

$y = (c_1 + c_2 x) e^{2x}$ → General Solution

$\boxed{3 = (c_1)}$ $\boxed{y = (3 - 5x) e^{2x}}$

$y' = 2c_1 e^{2x} + c_2 [x(2e^{2x}) + e^{2x}]$

$1 = 2c_1 + c_2$ $\boxed{c_2 = -5}$

classmate

Date
Page

Case - III $a^2 - 4b < 0$ (Complex conjugate roots).

The two roots of the auxilliary eqns are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} + i\frac{\sqrt{4b - a^2}}{2} = \alpha + i\beta$$

$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} - i\frac{\sqrt{4b - a^2}}{2} = \alpha - i\beta$$

$e^{ix} = \cos x + i \sin x$
$e^{-ix} = \cos x - i \sin x$

real part $= \cos x$
= add and divide by 2.
Imaginary part $= \sin x$
= subtract and divide by $2i$.

$$e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$e^{\lambda_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x}(\cos\beta x - i \sin\beta x)$$

$$y_1 = e^{\alpha x} \cos\beta x$$

$$y_2 = e^{\alpha x} \sin\beta x$$

Basis $= \{e^{\alpha x} \cos\beta x, e^{\alpha x} \sin\beta x\}$.

General soln is

$$y = e^{\alpha x}[A \cos\beta x + B \sin\beta x]$$

Where A & B are arbitrary constants.

Solve:- $y'' + y = 0$

$\lambda^2 + 1 = 0$

$\lambda = i, -i$

$\alpha = 0, \ \beta = 1$

$y_1 = \cos x$
$y_2 = \sin x$

Summary of $y'' + ay' + by = 0$. a & b constants

Case	Roots	Basis	General soln
I	λ_1 and λ_2 real & distinct		

Case - I	Roots	Basis	General Solution
I	λ_1 and λ_2 distinct and real $\lambda_1 = \frac{-a+\sqrt{a^2-4b}}{2}$ $\lambda_2 = \frac{-a-\sqrt{a^2-4b}}{2}$	$\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ where c_1 and c_2 are arbitrary constants.
II	$\lambda_1 = \lambda_2 = -\frac{a}{2}$	$\{e^{\lambda_1 x}, xe^{\lambda_1 x}\}$	$y_1 = (c_1 + c_2 x)e^{\lambda_1 x}$ where c_1 and c_2 are arbitrary
III	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$	$\{e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x\}$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$ where A and B arbitrary constar

Euler - Cauchy equation :-

A differential eqⁿ of the form $x^2y'' + axy' + by = 0$ is known as Euler - Cauchy eqⁿ, here a & b are constants.

Let a solⁿ of the differential equation $y = x^m \neq 0$ where $m \in R$.

Now, substitute the differential eqⁿ $y' = m x^{m-1}$, $y'' = m(m-2)x^{m-2}$.

$$x^2 m(m-1)x^{m-2} + ax\, m x^{m-1} + bx^m = 0$$

$$x^m[m(m-1) + am + b] = 0$$

Since $x^m \neq 0$.

$$m^2 + m(a-1) + b = 0$$

This is called auxilliary eqⁿ/ characteristics eqⁿ in m. So it will have two roots

$$m_1 = \frac{-(a-1) + \sqrt{(a-1)^2 - 4b}}{2}$$

$$m_2 = \frac{-(a-1) - \sqrt{(a-1)^2 - 4b}}{2}$$

Depending on value of D we have 3 cases:

Case : 1 $(a-1)^2 - 4b > 0$ (Two real and distinct roots)

Say the roots $m_1 = ?$, $m_2 = ?$

In this we can get soln $y_1 = x^{m_1}$, $y_2 = x^{m_2}$,

Basis $= \{x^{m_1}, x^{m_2}\}$,

General soln :

$y = c_1 x^{m_1} + c_2 x^{m_2}$ where c_1 & c_2 are arbitrary coefficien

$x^2 y'' - 2.5 x y' - 2y = 0$

Put $y = x^m$

$m(m-1) - 2.5m - 2 = 0$

$m^2 - 3.5m - 2 = 0$

$m_1 = 4$, $m_2 = -0.5$

The two linearly independent solutions are $y_1 = x^4$ & $y_2 = x^{-0.5}$

Basis : $\{x^4, x^{-0.5}\}$

$y = c_1 x^4 + c_2 x^{-1/2}$ → General soln

where c_1 & c_2 are arbitrary constent

Case-II $(a-1)^2 - 4b = 0$ (Real double root)

let $m_1 = m_2 = \frac{1-a}{2}$

we get one of the soln say $y_1 = x^{\frac{1-a}{2}}$. The other linearly independent order is

$$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0$$

$$y = \int \frac{1}{x^{1-a}} \cdot e^{-\int \frac{a}{x} \cdot dx} \cdot dx$$

$$= \int \frac{1}{x^{1-a}} e^{\cancel{a} \ln x^{-a}} \cdot dx$$

$$= \int \frac{1}{x^{1-a}} \cdot x^{-a} \cdot dx$$

$$= \int \frac{1}{x^{1-a+a}} dx$$

$$= \ln x$$

Here, $y_2 = x^{\frac{1-a}{2}} \ln x$

The two linearly indepent solns are $y_1 = x^{1-a/2}$ & $y_2 = \ln x \, [x^{1-a/2}]$

Basis $= \{ \cdot x^{\frac{1-9}{2}}, \ln x\,(x)^{\frac{1-9}{2}} \}$

$y = x^{\frac{1-9}{2}} [c_1 + c_2 \ln x]$

where c_1 & c_2 are arbitrary constants.

1. $x^2 y'' - 3xy' + 4y = 0$

Auxilliary eqn.

$m^2 + m(a-1) + b = 0$

$m^2 - 4m + 4 = 0$

$\boxed{m = 2}$

$y_1 = x^2$

Say, $y_2 = u y_1$

$u = \ln x$

$y_2 = (\ln x)(x^2)$

Basis $= \{ \ln x, x^2 \ln x \}$

General soln $\Rightarrow y = x^2 (c_1 + c_2 \ln x)$

Case-III :— $(a-1)^2 - 4b < 0$ (Complex roots).

Let complex conjugate roots are $m_1 = \mu \pm i\nu$

$$= \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

$$m_1 = \mu + i\nu \quad \boxed{\mu = 1 - a/2}$$
$$m_2 = \mu - i\nu \quad \& \quad \nu = \frac{\sqrt{4b - (a-1)^2}}{2}$$

$$\left. \begin{aligned} y_1^* &= x^{\mu + i\nu} \\ y_2^* &= x^{\mu - i\nu} \end{aligned} \right\} \quad x^{\mu+i\nu} = x^{\mu}\left[\cos(\nu \ln x) + i \sin(\nu \ln x)\right]$$

$$x^{\mu - i\nu} = x^{\mu} x^{-i\nu} = x^{\mu}\left[\cos(\nu \ln x) - i \sin(\nu \ln x)\right]$$

The two linearly independent solⁿ (Real) are

$$y_1 = x^{\mu} \cos(\nu \ln x)$$
$$y_2 = x^{\mu} \sin(\nu \ln x)$$

$$\text{Basis} = \{ x^{\mu} \cos(\nu \ln x), \ x^{\mu} \sin(\nu \ln x) \}$$

General solⁿ is $y = c_1 x^{\mu} \cos(\nu \ln x) + c_2 x^{\mu} \sin(\nu \ln x)$

Solve! $x^2y'' + 7xy' + 13y = 0$

Auxilliary eq^n

$$m^2 + m(7-1) + 13 = 0$$

$$m^2 + 6m + 13 = 0$$

$$m_1 = -3+2i \quad , \quad m_2 = -3-2i$$

The two linearly independent solns are:

$$y_1 = x^{-3}\cos(2\ln x) \qquad y_2 = x^{-3}\sin(2\ln x)$$

Basis: $\{x^{-3}\cos(2\ln x) , x^{-3}\sin(2\ln x)\}$

General soln:

$$y = c_1 x^{-3}\cos(2\ln x) + c_2 x^{-3}\sin(2\ln x)$$

where c_1 & c_2 are arbitrary constants

Que:- $x^2D^2 - 0.2xD + 0.36y = 0$

$$m^2 + (-0.2-1)m + 0.36 = 0$$

$$m - 1.2m + 0.36 = 0$$

classmate

Date
Page

Existence and uniqueness Results

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad -①$$

$$y(x_0) = K_0 \,, \quad y'(x_0) = K_1 \quad -③$$

Theorem: 1

If $p(x)$ and $q(x)$ are continous function in some open interval I and $x_0 \in I$, the the IVP (1 & 2) has a unique soln $y(x)$ in I.

L.D $\longrightarrow c_1 y_1 + c_2 y_2 \Rightarrow$ at least $c_1 \& c_2 \neq 0$
L.I $\longrightarrow c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 = c_2 = 0$.

Wronskian of y_1 and $y_2 = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= y_1 y_2' - y_2 y_1'$$

Theorem: 2

Suppose that ① has continuous coefficients $p(x)$ and $q(x)$ ~~are~~ on some open interval I then the two soln y_1 and y_2 of ① on I are linearly dependent on I iff the $W(y_1, y_2)$ is 0 at some $x_0 \in I$. Furthermore of $W = 0$, for $x = x_0$, then $W = 0$, on I, Hence

Hence if there is one $x_1 \in I$, at which $W \neq 0$ then y_1 and y_2 are linearly dependent on I.

<u>Proof</u> :— (a). $y'' + p(x) y' + q(x) y = 0$ — (1)

Let y_1 and y_2 are two sol'ns such that they are linearly dependent.
Say $y_2 = k y_1$ (where k is a constant)

$$W(y_1, y_2) = W(y_1, ky_1)$$

$$= \begin{vmatrix} y_1 & ky_1 \\ y_1' & ky_1' \end{vmatrix}$$

$$= 0$$

(b) Let the $W(y_1, y_2)$ is zero
We need to prove that y_1 & y_2 are linear dependent.
Consider the following linear system of eq'ns

$$k_1 y_1(x_0) + k_2 y_2(x_0) = 0$$
$$k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0$$

the determinant of the coefficient matrix is equal to the y_1 and y_2 which is zero. ∵ we have a non trivial sol'n. So at least one of k_1, k_2 are non-zer

$$y(x) = k_1 y_1 + k_2 y_2$$

It also satisfies initial condition

$$y(x_0) = 0 \ \& \ y'(x_0) = 0$$

$$y^* = 0 \Rightarrow y = y^* = 0$$

$$k_1 y_1 + k_2 y_2 = 0$$

$\Rightarrow y_1$ and y_2 are linearly dependent.

(2) Let $W = 0$ for some $x_0 \in I$.
$\therefore y_1$ and y_2 are linearly dependent.

$$\Rightarrow W \equiv 0$$

Given any point $x_1 \in I$, $W(y_1, y_2) \neq 0$
$\Rightarrow y_1$ and y_2 are L.I. otherwise
y_1, y_2 will be linearly dependent.

$p(x)$ are $q(x)$ are continous on interval (I)
then eqn (1) has a general solution.

Theorem:-4

Suppose that (1) has constant coefficients $p(x)$ and $q(x)$ on some open interval I, then every soln $y = Y(x)$ of (1) on I is of the form $Y(x) = c_1 y_1 + c_2 y_2$ where y_1 & y_2 are L.I & c_1 and c_2 are suitable constants