Differential Equations

Ordinary Differential Equations

Partial Differential Equation.

Seperable diff. Eqn

The first order ODE $y' = f(x, y)$ is said to be seperable if $f'(x, y)$ can be expressed as a product of a function of x and a function of y

$$y' = f(x) \cdot g(y)$$

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

Ex $y' = 1+y^2$

$\Rightarrow \int \frac{dy}{1+y^2} = \int dx$

$\Rightarrow \tan^{-1} y = x + c$

$\Rightarrow y = \tan(x+c)$ (c is an arbitary constant)

Ex find general sol^n.

$(x-4)y^4\,dx - x^3(y^2-3)dy = 0$

$\Rightarrow \int \frac{x-4}{x^3}dx - \int \frac{y^2-3}{y^4}dy = 0$

$\int \frac{1}{x^2}dx - 4\int \frac{dx}{x^3} - \int \frac{1}{y^2}dy + 3\int \frac{dy}{y^4} = 0$

$\Rightarrow \frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$

$\Rightarrow \boxed{\frac{2-x}{x^2} + \frac{y^2-1}{y^3} = c}$ (c is an arbitary constant)

★ $y=0$ is a sol^n which cannot be obtained from general sol^n.

Ex $2xyy' = y^2 - x^2$

$y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \left(v + x\frac{dv}{dx}\right) = \frac{v^2x^2 - x^2}{\cancel{2x}\cdot v\cancel{x}}$

$v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v}$

$\Rightarrow \int\frac{2v\,dv}{1+v^2} = -\int\frac{dx}{x}$

$=$ ~~tan~~

~~$\tan^{-1}(1+x^2) = -\tan^{-1}|x| + c$~~

$\log(1+v^2) = -\log x + c$

$\Rightarrow \log(x + v^2x) = c$

$\Rightarrow \log\left(x + \frac{y^2}{x}\right) = c$

$\Rightarrow \frac{x^2+y^2}{x} = e^c$

$\Rightarrow \underline{x^2 - xe^c + y^2 = 0}$

H.W

① $y' = \dfrac{x-\cancel{y}5}{y^2}$

② $y' = \dfrac{y-1}{x+3} \quad (x > -3)$

③ $y' = \dfrac{y\cos x}{1+2y^2}$

④ $y' = a(y(b-y))$ where $a, b > 0$

Answers.

①

$$\frac{dy}{dx} = \frac{x-\cancel{y}5}{y^2}$$

$$\cancel{dy = \frac{x}{y^2}dx - \frac{dx}{y}}$$

$$\cancel{y^2 dy = x\,dx - dx}$$

$$\int y^2 dy = \int (x-5)\, dx$$

$$\Rightarrow \frac{y^3}{3} = \frac{x^2}{2} - 5x + c$$

$$\Rightarrow y^3 = \frac{3}{2}x^2 - 15x + C$$ (c is arbitrary const.)

(2) $\frac{dy}{dx} = \frac{y-1}{x+3}$

$\Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3}$

$= \ln(y-1) = \ln(x+3) + c$

$\Rightarrow \frac{y-1}{x+3} = e^c$

$\Rightarrow y-1 = xe^c + k$

$\Rightarrow y = xe^c + \underbrace{(k+1)}_{a}$

$a = 3c+1$

$\Rightarrow \underline{y = xe^c + a}$

c is arbitrary const

(3) $\frac{dy}{dx} = \frac{y \cos x}{1+2y^2}$

$\int \frac{1+2y^2}{y} dy = \int \cos x \, dx$

$= \int \frac{1}{y} dy + 2\int y\,dy = \int \cos x \, dx$

$= \ln y + y^2 = \sin x + c$ (c is arbitrary const)

(4) ~~$\frac{dy}{dx} = ay(b-y)$~~

~~$\Rightarrow \int \frac{dy}{aby - ay^2} = \int dx$~~

(4) $\frac{dy}{dx} = ay(b-y)$

$$\Rightarrow \int \frac{dy}{y(b-y)} = \int a\,dx$$

$$\frac{1}{y(b-y)} = \frac{A}{y} + \frac{B}{b-y}$$

$$\frac{1}{y(b-y)} = \frac{A(b-y) + By}{y(b-y)}$$

$A(b-y) + By = 1$

$y=0 \Rightarrow \quad A \cdot b = 1$

$\Rightarrow A = 1/b.$

$y = b \Rightarrow \quad B \cdot b = 1$

$B = 1/b.$

$$\Rightarrow \int \frac{1}{b} \cdot \frac{1}{y}\,dy + \int \frac{1}{b} \cdot \frac{1}{b-y}\,dy = \int a\,dx$$

$$\Rightarrow \frac{1}{b} \ln y - \frac{1}{b} \ln(b-y) = ax + c$$

$$\Rightarrow \frac{1}{b} \ln\left(\frac{y}{b-y}\right) = ax + C$$

$$\Rightarrow \ln\left(\frac{y}{b-y}\right) = abx + bC$$

(C is arbitrary

Exact ODE

<u>Note</u>: If a function $u(x,y)$ has continious partial ~~differentia~~ derivatives then,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

<u>Definition</u> : A first order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact only, if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

<u>Ex</u>

find the general solution of the following differential equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

$M(x,y) = 3x^2 + 4xy$

$\frac{\partial M}{\partial y} = 4x$

$N(x,y) = 2x^2 + 2y$

$\frac{\partial N}{\partial x} = 4x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

↳ the ~~diff~~ differential is exact.

let $U(x,y) = c$ be the general solution to ①, where c is an arbitary constant

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\frac{\partial u}{\partial x} = 3x^2 + 4xy. \quad \text{and} \quad \frac{\partial u}{\partial y} = 2x^2 + 2y.$$

$$U(x,y) = x^3 + 2x^2y + \phi(y)$$

$$\frac{\partial u}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy} \quad \text{— ②}$$

$$\frac{\partial u}{\partial y} = 2x^2 + 2y$$

$$\Rightarrow \frac{d\phi(y)}{dy} = 2y$$

$$\Rightarrow \int d\,\phi(y) = \int 2y\,dy$$

$$\Rightarrow \phi(y) = y^2 + c$$ (c is arbitary constant)

So, $$U(x,y) = x^3 + 2x^2y + y^2 + c$$

Alternate solⁿ

$\Rightarrow 3x^2dx + 4xydx + 2x^2dy + 2ydy = 0.$

$\Rightarrow d(x^3) + d(y^2) + d(2x^2y) = 0$

$\Rightarrow d(x^3 + y^2 + 2x^2y) = 0.$

$\Rightarrow \underline{\underline{x^3 + y^2 + 2x^2y = c}}$

Ex

$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0.$

$M(x,y) = \cos(x+y)$

$N(x,y) = 3y^2 + 2y + \cos(x+y)$

$\frac{\partial M}{\partial y} = \frac{d\cos(x+y)}{dy} = -\sin(x+y)\cdot 1$

$= -\sin(x+y)$

$\frac{\partial N}{\partial x} = \frac{d}{dx}(3y^2 + 2y + \cos(x+y)) = -\sin(x+y)\cdot 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ d.E is exact

$$du = \cos(x+y)dx + (3y^2+2y+\cos(x+y))dy = 0$$

$$\frac{\partial u}{\partial y} = 3y^2+2y+\cos(x+y)$$

$$\frac{\partial u}{\partial x} = \cos(x+y)$$

$$u(x,y) = \int \cos(x+y)\,dx$$

$$u(x,y) = \sin(x+y) + \phi(y).$$

$$\Rightarrow \frac{d\,u(x,y)}{\partial y} = \cos(x+y) + \frac{d}{dy}\phi(y)$$

$$\frac{\partial u}{\partial y} = 3y^2+2y+\cos(x+y)$$

$$\Rightarrow \frac{d\,\phi(y)}{dy} = 3y^2+2y$$

$$\Rightarrow \phi y = \int (3y^2+2y)\,dy$$

$$\phi(y) = y^3+y^2+c$$

(c is an arbitrary constant)

$$\Rightarrow u(x,y) = y^3+y^2+\sin(x+y)+c = 0$$

Ans

Note: Integrating factor is not unique

Rule - 1

If $M(x,y)\,dx + N(x,y)\,dy = 0$ ① is such that the term $R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ $\cancel{(N)}$

only depends on x, then ① has an integrating factor $F(x)$ which is given by:

$$F(x) = e^{\int R(x)\,dx}$$

Rule - 2

If $M(x,y)\,dx + N(x,y)\,dy = 0$ ① is such that the term

$$R^{*} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on y, then ① has an integrating factor $G(y)$ which is given by:

$$G(y) = e^{\int R^{*}(y)\,dy}$$

Ex Solve the following ODE.

$$\left(e^{x+y} + ye^{y}\right)dx + \left(xe^{y} - 1\right)dy = 0, \quad y(0) = 1$$

$$M(x,y) = \left(e^{x+y} + ye^{y}\right)$$

$$N(x,y) = (xe^{y} - 1)$$

$$\frac{\partial M(x,y)}{\partial y} = e^{x+y} + e^{y} + ye^{y} = e^{x+y} + (1+y)e^{y}$$

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$$\frac{\partial N(x,y)}{\partial x} = \cancel{x}\, e^{y}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^{y} = M(x,y)$

$$\Rightarrow R^* = \frac{1}{y}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$R^* = -1$$

Now, $G(y) = e^{\int -1\,dy}$

$$= e^{-y}.$$

$$\Rightarrow du = (e^{x} + y)\,dx + (x - e^{-y})\,dy$$

$$\frac{\partial u(x,y)}{dx} = e^{x} + y$$

$$\frac{\partial u(x,y)}{dy} = x - e^{-y}$$

$$u(x,y) = \cancel{ye^{x} + \frac{y^2}{2}} \; e^{x} + xy + \phi(y)$$

$$\frac{\partial u(x,y)}{\partial y} = x + \frac{d\,\phi(y)}{dy}.$$

$$\Rightarrow \frac{d\,\phi(y)}{dy} = -e^{-y}$$

$$\Rightarrow \phi(y) = -\int e^{-y}\,dy = e^{-y} + C$$

so, $u(x,y) = e^x + xy + e^{-y} + c = 0$

Ans.

Linear ODE

A first order ODE is linear in the dependent variable y and the independent variable x if it is of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (1)}$$

Remark: An integrating factor of (1) is

$$IF = e^{\int P(x)dx}$$

Example: Solve the following ODE

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

Ans

$$P(x) = \frac{2x+1}{x}.$$

$$I.F = e^{\int (2 + 1/x)dx}$$

$$= e^{2x + \ln x}$$

$$= xe^{2x}.$$

$$\Rightarrow xe^{2x}\frac{dy}{dx} + xe^{2x}\frac{(2x+1)}{x}y = x.$$

$$\int dy = \int \frac{\cancel{1-2e^{2x}} \, x - e^{2x}}{xe^{2x}} \, \cancel{dx}$$

$$\int dy = \int \frac{(1-2e^{2x})x - e^{2x}}{x\, e^{2x}}\, dx$$

$$y = \int \frac{1-2e^{2x}}{e^{2x}}\, dx - \int \frac{dx}{x}$$

$$= \int e^{-2x} dx - 2\int dx - \int \frac{dx}{x}$$

$$y = -\frac{e^{-2x}}{2} - 2x - \ln x + c = 0$$

$$x e^{2x} \frac{dy}{dx} + e^{2x}(2x+1)y = x$$

$$x e^{2x} \cdot dy + \left(2x e^{2x} \cdot dx + e^{2x} dx\right) y = x\,dx$$

$$\boxed{y x e^{2x} = \frac{x^2}{2} + c}$$

Example

$$(x^2+1)\frac{dy}{dx} + 4xy = x.$$

$$\frac{dy}{dx} + \left(\frac{4x}{x^2+1}\right)y = \frac{x}{x^2+1}$$

$$P(x) = \frac{4x}{x^2+1}$$

$$\text{I.f} = e^{\int \frac{4x}{x^2+1}dx}$$

$$= e^{2\int \frac{dt}{t}}$$

$$= e^{\ln t^2}$$

$$= t^2 = (x^2+1)^2$$

$$\Rightarrow (x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)\cdot y = x(x^2+1)$$

$$\frac{d\left(y(x^2+1)^2\right)}{dx} = x(x^2+1)$$

$$\Rightarrow (x^2+1)^2\frac{dy}{dx} = x(x^2+1)(1-4y)$$

$$\Rightarrow \int\frac{dy}{1-4y} = \frac{1}{2}\int\frac{2x}{x^2+1}dx$$

$y(x) =$

$$= -\frac{1}{4}\ln(1-4y) = \frac{1}{4}\ln(x^2+1)^2 + C$$

(C is an arbitrary const.)

$$\Rightarrow \frac{1}{4}\ln\left[(x^2+1)^2(1-4y)\right] + C = 0$$

$y(2) = 1$

$$\Rightarrow \frac{1}{4}\ln\left[(4+1)^2(1-4)\right] + C = 0$$

Homogenous Equation

A first order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogenous if when written in the derivative form

$$\frac{dy}{dx} = f(x,y)$$

there exists a function $g()$ such that $f(x,y)$ can be expressed in the form $g(y/x)$.

Theorem: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogenous equation, ~~ten~~ the change of variables $\boxed{y = vx}$ transforms the given ODE into a seperable form in the variables v and x.

Ex. Solve the initial - value problem.

$$\left(y + \sqrt{x^2+y^2}\right) dx - x\,dy = 0 \quad , \quad y(1) = 0$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$$

let $y = vx$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

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$$\left(x\frac{dv}{dx}+v\right)=\frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow \left(x\frac{dv}{dx}+\cancel{v}\right)=\cancel{v}+\sqrt{1+v^2}$$

$$\Rightarrow \int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}$$

$$\Rightarrow \ln\left(v+\sqrt{1+v^2}\right)=C+\ln x$$

$$\Rightarrow \ln\left(\frac{v+\sqrt{1+v^2}}{x}\right)=C$$

$$\Rightarrow \ln\left(\frac{y/x+\sqrt{\frac{x^2+y^2}{x}}}{x}\right)=C$$

$$\Rightarrow \ln\left(\frac{y+\sqrt{x^2+y^2}}{x^2}\right)=C$$

$$\ln\left(\frac{0+1}{1}\right)=C$$

$$\Rightarrow \ln 1=C$$

$$\Rightarrow C=0$$

so, $$\ln\left(\frac{y+\sqrt{x^2+y^2}}{x^2}\right)=0$$

Ans

Bernolli equation

An equation of the form,

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

is called a Bernoulli equation.

Example

Solve :

$$\frac{dy}{dx} + y = x\,y^3$$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

let $\frac{1}{y^2} = u$.

$\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{du}{dx}$

$$-\frac{1}{2}\frac{du}{dx} + u = x$$

$$\Rightarrow \frac{du}{dx} - 2u = -2x$$

I.f. $= e^{-2\int dx}$

$= e^{-2x}$

$$\Rightarrow e^{-2x}\frac{du}{dx} - 2ue^{-2x} = -2xe^{-2x}$$

$$\Rightarrow \int e^{-2x}du - 2e^{-2x}dx\,u = -\int 2xe^{-2x}dx$$

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$$u e^{-2x} = -2\int x e^{-2x} dx + c$$

Example $\cos x \, dy = y(\sin x - y)\, dx$.

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin x - y^2}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} - \tan x \cdot y = -\frac{y^2}{\cos x}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \tan x \cdot \frac{1}{y} = -\frac{1}{\cos x}$$

$$\frac{1}{y} = u.$$

$$-\frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$$

$$= \frac{du}{dx} + u \cdot \tan x = \sec x$$

$$e^{\int \tan x \, dx} = e^{\ln \sec x}$$

$$= \sec x$$

$$\sec x \frac{du}{dx} + u \sec x \tan x = \sec^2 x$$

$$\sec x \, du + u \cdot \sec x \tan x \, dx = \sec^2 x \, dx$$

$$\Rightarrow \quad u \cdot \sec x = \int \sec^2 x \, dx$$

$$= \cancel{\int \frac{1}{\cos^2 x} dx}$$

$$u \cdot \sec x = \tan x + C$$

$$\Rightarrow \quad u \cdot \sec x - \tan x = C$$

Ans

H.W

$$y' - 2\cos x \cot y + \sin^2 x \operatorname{cosec} y \cdot \cos x = 0$$

~~Step~~

~~Rule-1~~ ~~$Mdx+Ndy=0$ is homogeneous~~

Orthogonal Trajectories

Definition: let $F(x,y,c)=0$ be a given parameter(c) family of curves in the XY plane. A curve that intersects the family of curves $F(x,y,c)=0$ at right angles is called an orthogonal trajectory of the given family of curves.

Steps to get orthogonal trajectory

① from the given equation

$$F(x,y,c)=0 \quad \text{——} \quad ⓪$$

find the ~~orthogonal~~ ordinary differential eq^n.

$$\frac{dy}{dx} = f(x,y) \quad \text{——} \quad ①$$

② In the ODE ①, replace the $f(x,y)$ by its negative reciprocal, that is by

$$\frac{-1}{f(x,y)}$$

③ Now, solve $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ to get the orthogonal trajectory of ⓪

Example : Obtain the orthogonal trajectories of the family of curves $x^2+y^2=c^2$, where c is a parameter.

Sol^n.

$$x^2+y^2+c^2 \quad \text{—} ①$$

Differentiating ① w.r.t x, we get.

$$2x+2y\frac{dy}{dx}=0.$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$$

So, the ODE of the family of curves is

$$\frac{dy}{dx}=\frac{y}{x}$$

$$\Rightarrow \int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\Rightarrow \ln y=\ln x+e^k$$

$$\Rightarrow \ln y/x=e^k$$

$$\Rightarrow \frac{y}{x}=k$$

$$\Rightarrow y=kx$$, where k is a parameter.

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Example

Find the orthogonal trajectory of the family of parabola $y = cx^2$, where c is a parameter.

$$y = cx^2.$$

$$\Rightarrow c = \frac{y}{x^2}.$$

$$\frac{dy}{dx} = 2cx$$

$$\frac{dy}{dx} = \frac{2y}{x}$$

So, the ODE of orthogonal trajectory is

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{2y}.$$

$$\Rightarrow \int 2y\,dy = \int x\,dx$$

$$\Rightarrow y^2 = -\frac{x^2}{2} + k$$

$$\Rightarrow 2y^2 + x^2 = K$$ where k is a parameter.

~~Existence and Uniqueness Theorem~~

Ex 1 $\left(\frac{dy}{dx}\right)^2 + y^2 + 1 = 0$, $y(0) = 1$ $\longrightarrow$ Solution doesn't exist

Ex 2 $\frac{dy}{dx} = 2x$, $y(0) = 1$ $\longrightarrow$ has unique solution

Ex 3 $x\left(\frac{dy}{dx}\right) = y - 1$, $y(0) = 1$ $\longrightarrow$ has infinite solutions.

Ans 2

$$\frac{dy}{dx} = 2x$$

$$\int dy = \int 2x\,dx$$

$$\Rightarrow y = x^2 + c$$

$y(0) = 1$ $\Rightarrow 1 = 0 + c$

$\Rightarrow c = 1$

So, $y = x^2 + 1$

Ans 3

$$x\left(\frac{dy}{dx}\right) = (y-1)$$

$$\Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x}$$

$$\Rightarrow \ln(y-1) = \ln x + e^c$$

$$\Rightarrow \ln\left(\frac{y-1}{x}\right) = e^c$$

$$\Rightarrow \frac{y-1}{x} = c$$

$$\Rightarrow y = cx + 1$$

$y(0) = 1$

Existence and Uniqueness theorem $\left(\frac{dy}{dx} = f(x,y),\ y(x_0)=y_0\right)$

Existance thm: Suppose that $f(x,y)$ is continious in some region

$$R^* = \{ (x,y) : |x-x_0| \le a,\ |y-y_0| \le b \}$$

$a, b > 0$.

Since f is continious in the closed and bounded domain, f is necessary bounded in R, that is there exists a $k>0$ such that $|f(x,y)| \le k \ \forall (x,y) \in R^*$

Then, the initial value function (1) has at least one solution $y = y(x)$ defined in the interval $|x-x_0| \le \alpha$, where:

$$\alpha = \min\{a, b/k\}$$

Uniqueness theorem:

Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continious functions in R^*.

That is. (a) $|f(x,y)| \le k$ and (b) $\left|\frac{\partial f}{\partial y}\right| \le L \ \forall (x,y) \in R$

Then the initial value function (1) has at must one solution $y = y(x)$, defined in $|x-x_0| \le \alpha$, where

$$\alpha = \min\{a, b/k\}$$

Note: The condition in (b) can be substituted by the following condition. Lipochitz condition

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|, \quad L>0$$

$$\forall (x,y) \in R^*,$$

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Example

$$\frac{dy}{dx} = xy - \sin y \quad , \quad y(0) = 2$$

$x_0 = 0, \ y_0 = 2$

$f(x,y) = xy - \sin y$

$\frac{\partial f}{\partial y} = x - \cos y$ → both continious.

So, at one solⁿ.

Example

$$\frac{dy}{dx} = y^{1/3} + x \quad , \quad y(1) = 0$$

$f(x,y) = y^{1/3} + x$

$$\frac{\partial f}{\partial y} = \frac{1}{3}(y)^{-2/3}$$

$$= \frac{1}{3(y)^{2/3}} \longrightarrow \text{not contd.}$$

Laplace Transformation

Let $f(x)$ be a function for $t \geq 0$. Then, the Laplace transformation of $f(t)$ is defined by

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

Note: The domain of $f(t)$ is the set of non-negative real numbers. But the domain of the transformed function $F(s)$ depends on the function $f(\cdot)$. Actually, it varries from function to function.

Inverse Laplace Transformation

If $\mathcal{L}(f(t)) = F(s)$

then $f(t)$ is known as the inverse Laplace Transformation of $F(s)$. $f(t) = \mathcal{L}^{-1}(F(s))$

Ex-1 let $f(t) = 1$; $t \geq 0$

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \frac{1}{s}, \text{ provided } s > 0.$$

So, $\mathcal{L}^{-1}(F(\frac{1}{s})) = 1$ also $\mathcal{L}^{-1}\left(\frac{K}{s}\right) = K$

Ex Let $f(t) = e^{at}$, $t \geq 0$

Find the Laplace transformation of $f(s)$

$\rightarrow$ $F(s) = \mathcal{L}(e^{at})$

$$= \int_0^\infty e^{at} \cdot e^{-st} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \frac{1}{s-a}, \text{ provided } s-a > 0$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

Laplace Transformation is linear

$f(t)$, $g(t)$

$$\mathcal{L}[a f(t) + b g(t)] = a \mathcal{L} f(t) + b \mathcal{L} g(t)$$

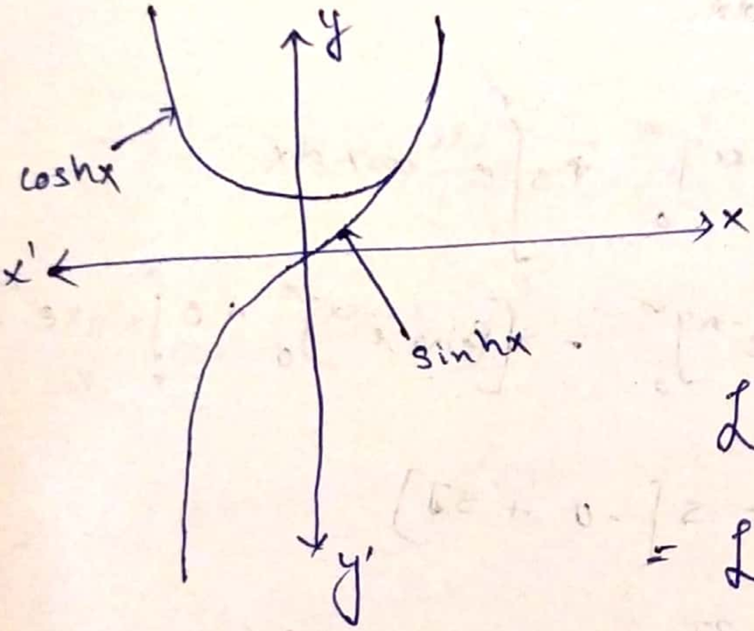
Ex Obtain the Laplace transformation of $\sinh x$ and $\cosh x$

hyperbolic sin function

hyperbolic cos function

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



$\mathcal{L}(\sinh x)$

$= \mathcal{L}\left(\frac{e^{x} - e^{-x}}{2}\right)$

$= \frac{1}{2}\mathcal{L}(e^{x}) - \frac{1}{2}\mathcal{L}(e^{-x})$

$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$

$= \frac{1}{s^2-1}$

$\mathcal{L}(\cosh x)$

$= \mathcal{L}\left(\frac{e^{x} + e^{-x}}{2}\right)$

$= \frac{1}{2}\mathcal{L}(e^{x}) + \frac{1}{2}\mathcal{L}(e^{-x})$

$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right]$

$= \frac{s}{s^2-1}$

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Ex Obtain Laplace Transformation of sinx & cosx.

$\mathcal{L}(\sin x)$

$$I = \int_0^\infty \sin x \, e^{-sx} \, dx$$

$$\Rightarrow I = \left[-\cos x \cdot e^{-sx}\right]_0^\infty \mp s\int_0^\infty e^{-sx} \cos x \, dx$$

$$I = \left[-\cos x \cdot e^{-sx}\right]_0^\infty - s\left[\left[\sin x \, e^{-sx}\right]_0^\infty + s\int_0^\infty \sin x \, e^{-sx} dx\right]$$

$$I = 1 - s\left[-0 + sI\right]$$

$$\Rightarrow I = 1 - s^2 I$$

$$\Rightarrow I = \frac{1}{1+s^2}$$

$\mathcal{L}(\cos x)$

$$I = \int_0^\infty \cos x \, e^{-sx} dx$$

$$I = \left[\sin x \, e^{-sx}\right]_0^\infty + s\int_0^\infty e^{-sx} \sin x \, dx$$

$$I = 0 + s\left[\left[-\cos x \, e^{-sx}\right]_0^\infty - s\int_0^\infty e^{-sx} \cos x \, dx\right]$$

$I = s[1 - sI]$

$$I = \frac{s}{1+s^2}$$

$$\mathcal{L}(e^{ix}) = \frac{1}{s-i} = \frac{s+i}{s^2+1}$$

$$\mathcal{L}(\cos x + i\sin x) = \frac{s+i}{s^2+1}$$

$$\Rightarrow \mathcal{L}(\cos x) + i\,\mathcal{L}(\sin x) = \frac{s}{s^2+1} + \frac{i}{s^2+1}.$$

$$\Rightarrow \mathcal{L}(\cos x) = \frac{s}{s^2+1}$$

$$\mathcal{L}(\sin x) = \frac{1}{s^2+1}$$

Ex Obtain

(i) $\mathcal{L}(3+e^{6t})^2$ &

(ii) $\mathcal{L}(3+e^{6t})$

The homogenous LODE with constant coefficients

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where $a_0, a_1, \ldots a_{n-1}, a_n$ are real constant

→ let $y = e^{mx}$ be a solⁿ of (1), $m \in R$.

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2y}{dx^2} = m^2 e^{mx}, \quad \ldots \quad \frac{d^ny}{dx^n} = m^n e^{mx}$$ — (1)

$$e^{mx}\left[a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n\right] = 0$$

$$\underbrace{a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0}$$ — (ii)

↑

auxilary equation or
Characterisic eqⁿ of (1)

Case 1: Roots are real and distinct

(Th) If (2) has n distinct real roots, say $m_1, m_2, \ldots m_n$, then the general solⁿ of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$ where

$c_1, c_2 \ldots c_n$ are real roots.

Case II : Repeated real roots.

Result Suppose the roots of the auxilary eq^n are double real root m an $(n-1)$ distinct real roots $m_1, m_2, m_3, \ldots, m_{n-1}$

Then the G.S of (1) is

$$y = (c_1 + c_2 x)e^{mx} + c_3 e^{m_1 x} + c_4 e^{m_2 x} + \cdots + c_n e^{m_{n-2} x}$$

where $c_1, c_2, c_3, \ldots, c_n$ are real constant.

$$\left(\frac{dy}{dx} - y\right)^2 = 0$$

$$(m-1)^2 = 0$$

$\Rightarrow m = 1, 1$

Result Suppose for (2), we have real root m occurring k times and the remaining roots are distinct real numbers, say, $m_{k+1}, m_{k+2}, \ldots m_n$ then the G.S of (1) is

$$y = (c_1 + c_2 x + \cdots + c_k x^{k-1})e^{mx} + c_{k+1} e^{m_{k+1} x} \cdots + c_n e^{m_n x}$$

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Case 3 : Conjugate Complex roots.

Result : If $a+ib$ and $a-ib$ are each k fold roots of the auxiliary equation, then the corresponding general sol^n.

$$y = e^{ax}\left[\left(c_1 + c_2 x + c_3 x^3 + \dots + c_k x^{k-1}\right)\sin bx + \left(c_{k+1} + c_{k+2}x + c_{k+3}x^2 + \dots + c_{2k}x^{k-1}\right)\cos bx\right]$$

Questions

① $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0$

$e^{m}\left[m^2 - 3m + 2\right] = 0$

$\Rightarrow \left[m^2 - 2m - m + 2\right] = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 2$ or 1

(2) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

$m^3 - 4m^2 + m + 6 = 0$

$m^3 + m^2 - 5m^2 - 5m + 6m + 6 = 0$

$m^2(m+1) - 5m(m+1) + 6(m+1) = 0$

$(m+1)[m^2 - 5m + 6] = 0$

$(m+1)(m-2)(m-3) = 0$

$m = -1, 2, 3$

(3) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$.

(4) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

Ans (3)

$m^3 - 4m^2 - 3m + 18 = 0$

(-2)

$m^3 + 2m^2 - 6m^2 - 12m + 9m + 18 = 0$

$\Rightarrow (m+2)[m^2 - 6m + 9]$

$\Rightarrow (m+2)(m+3)^2 = 0$

$\Rightarrow m = -2, 3, 3$

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(4) $m^2 - 6m + 25 = 0$

$m = 3 \pm 4i$

H.W

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 14\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 25y = 0$$

Let us consider the following equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \quad a_0(x) \neq 0$$

or

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{—} (1)$$

where $P(x) = \frac{a_1(x)}{a_0(x)}$ and $Q(x) = \frac{a_2(x)}{a_0(x)}$

Assumption : It is given that $y_1(x)$ is a ~~solution~~ nontrivial solution to (1)

let us take another ~~substitution~~ solution of (1) of the form

$y_2(x) = y_1(x)\, v(x)$, where $v_1(x)$ is a real valued function.

$$\cancel{\frac{d^2y_2}{dx^2}} \quad \frac{dy_2}{dx} = \frac{dy_1}{dx}v + y_1\frac{dv}{dx}$$

$$\frac{d^2y_2}{dx^2} = \frac{d^2y_1}{dx^2}u + 2\frac{dy_1}{dx}\frac{dy}{dx} + y_1\frac{d^2u}{dx^2}$$

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$$\frac{dw}{dx} + \frac{2y_1'(x) + P(x)\,y_1(x)}{y_1(x)}\, w = 0$$

$$\Rightarrow \frac{dw}{w} = -\frac{2y_1'(x) + P(x)y_1(x)}{y_1(x)}\, dx$$

$$\Rightarrow \int \frac{dw}{w} = -2\int \frac{y_1'(x)}{y_1(x)}\, dx - \int P(x)\, dx$$

$$\Rightarrow \ln w = -2 \ln y_1(x) - \int P(x)\, dx + c$$

$$\Rightarrow \ln\left(w\, y_1^2(x)\right) = -\int P(x)\, dx + c$$

$$\Rightarrow w = \frac{1}{y_1^2(x)}\, e^{-\int P(x)dx + c}$$

without loss of generality $c=0$ can be taken

$$w = \frac{1}{y_1^2(x)}\, e^{-\int P(x)dx}$$

$$\frac{du}{dx} = w$$

$$\Rightarrow \boxed{u(x) = \int \frac{1}{y_1^2(x)}\, e^{-\int P(x)dx}\, dx}$$

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$$\begin{vmatrix} y_1(x) & v(x)\, y_1(x) \\ y_1'(x) & v(x)\, y_1'(x) + v'(x)\, y_1(x) \end{vmatrix}$$

$$\cdot\ y_1^2 \frac{dv}{dx} = e^{-\int P(x)dx} \neq 0$$

<u>Ex</u> verify that $y = e^{2x}$ is a solution of

$$(2x+1)\frac{d^2y}{dx^2} - 4(x+1)\frac{dy}{dx} + 4y = 0$$

find the general sol^n.

$$\int \frac{1}{e^{4x}}\, e^{\int \frac{-4x+1}{2x+1} dx}\, dx$$

Laplace Transformation (Contd.)

Ex

① $\mathcal{L}(3+e^{6t})^2$

$\mathcal{L}(3^2 + e^{12t} + 6e^{6t})$

$= \mathcal{L}(9) + \mathcal{L}(e^{12t}) + 6\mathcal{L}e^{6t}$

$= 9\int_0^\infty e^{-st}\,dt + \int_0^\infty e^{12t}\cdot e^{-st}\,dt + 6\int_0^\infty e^{6t}\cdot e^{-st}\,dt$

$= \frac{9}{s} + \int_0^\infty e^{-(s-12)t}\,dt + 6\int_0^\infty e^{-(s-6)t}\,dt$

$= \frac{9}{s} + \frac{1}{s-12} + \frac{6}{s-6}$; $\underline{s > 12}$

$\mathcal{L}$

Ex Obtain

$\mathcal{L}(t^a)$

$$I_a = \int_0^\infty t^a e^{-st}\, dt$$

$$I_a = \cancel{\left[\frac{t^{a+1} e^{-st}}{a+1}\right]_0^\infty + s \int_0^\infty e^{-st} \frac{t^{a+1}}{a+1}}$$

$\cancel{I_a}$ $\cancel{I}$

Ex $\mathcal{L}^{-1}\left[\dfrac{3s+5}{(s-5)(s+2)}\right]$

$$\frac{3s+5}{(s-5)(s+2)} = \frac{A}{s-5} + \frac{B}{s+2}$$

$3s+5 = A(s+2) + B(s-5)$

$s = 5.$

$A = \dfrac{20}{7}$, $B = \dfrac{1}{7}$

$\Rightarrow \mathcal{L}^{-1}\left[\dfrac{A}{s-5} + \dfrac{B}{s+2}\right]$ $\qquad A = \dfrac{20}{7}$, $B = \dfrac{1}{7}$

$= A\, \mathcal{L}^{-1}\left(\dfrac{1}{s-5}\right) + B\, \mathcal{L}^{-1}\left(\dfrac{1}{s+2}\right)$

$= \dfrac{20}{7} e^{5t} + \dfrac{1}{7} e^{-2t}$ Ans

Ex Obtain

$$\mathcal{L}^{-1}\left[\frac{3s+4}{(s-2)(s^2+7)}\right]$$

$$\frac{3s+4}{(s-2)(s^2+7)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+7}$$

$$3s+4 = A(s^2+7) + (Bs+C)(s-2)$$

$s=2$

$10 = A \times 11$

$\Rightarrow A = \frac{10}{11}$

$s=0$

$4 = 7A - 2C$

$2C = \frac{70}{11} - 4$

$2C = \frac{26}{11}$ 13

$C = 13/11$

$s=1$

$7 = 8A - \left(B + \frac{13}{11}\right)$

$B = \frac{80}{11} - \frac{13}{11} - \frac{77}{11}$

$B = \frac{-10}{11}$

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$$

H.W.

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

First Shifting Theorem

Statement: If a function $f(t)$ has Laplace transformation $F(s)$, then the Laplace transformation of $e^{at} f(t)$ is $F(s-a)$.

Mathematically,

If $\mathcal{L}(f(t)) = F(s)$,

then $\mathcal{L}(e^{at} f(t)) = F(s-a)$

Again, if $\mathcal{L}^{-1}(F(s)) = f(t)$, then

$$\mathcal{L}^{-1}(F(s-a)) = e^{at} \mathcal{L}^{-1}(F(s))$$

$\mathcal{L}(e^{at} f(t))$

$$= \int_0^\infty e^{at} f(t) e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= F(s-a)$$

Ex → Find the Laplace transformation of.

$f(t) = \cosh(at) \cdot \cos bt$

$$= \frac{e^{at} + e^{-at}}{2} \cdot \cos bt$$

$$= \frac{1}{2}\left[e^{at}\cos bt + e^{-at}\cos bt\right]$$

$$\mathcal{L}(f(t)) = \frac{1}{2}\left[\mathcal{L}[e^{at}\cos bt] + \mathcal{L}[e^{-at}\cos bt]\right]$$

If $= \cos bt$.

$$\mathcal{L}(f(t)) = \frac{s}{s^2+b^2} = F(s)$$

If ~~f(t) = cos bt~~

~~$\mathcal{L}(f(t))$~~ $\quad \mathcal{L}(e^{at} f(t)) = F(s-a)$

$$= \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}(e^{-at} f(t)) = \frac{s+a}{(s+a)^2+b^2}$$

$$\mathcal{L}(f(t)) = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$$

Change of scale Property

Theorem If $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$\rightarrow$ $$\mathcal{L}(f(at)) = \int_0^\infty e^{-st} f(at)\, dt$$

let $at = u$

$dt = \frac{du}{a}$

$$= \int_0^\infty e^{-\frac{su}{a}} f(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)u} f(u)\, du$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

Ex If $\mathcal{L}(f(t)) = \frac{e^{-1/s}}{s} = F(s)$

Find $\mathcal{L}(e^{-t} f(3t))$

$$\int_0^\infty e^{-t} f(3t)\, e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty e^{-(s+1)t} f(3t)\, dt$$

$3t = u \quad \Rightarrow t = \frac{u}{3}$.

$dt = \frac{du}{3}$

$$\int_0^\infty e^{-\frac{(s+1)u}{3}} f(u) \frac{du}{3}$$

$$\Rightarrow \frac{1}{3} \int_0^\infty e^{-\frac{(s+1)}{3} u} f(u)\, du$$

$$\Rightarrow \cancel{\frac{1}{3}} \frac{e^{-\frac{3}{s+1}}}{\frac{s+1}{\cancel{3}}}$$

$$= \frac{e^{-3/s+1}}{s+1}$$

Ex Find the inverse Laplace transformation of $F(s) = \frac{3s-137}{s^2+2s+401}$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{3s-137}{s^2+2s+401}\right)$$

~~3s-137~~

$$= \mathcal{L}^{-1}\left[\frac{3s-137}{s^2+2s+1+400}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{3s-140+3}{(s+1)^2+(20)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{3(s+1)}{(s+1)^2+(20)^2}\right] - 7\,\mathcal{L}^{-1}\left[\frac{20}{(s+1)^2+(20)^2}\right]$$

Ex $\mathcal{L}^{-1}\left[\frac{64}{81s^4-256}\right]$

$$= \mathcal{L}^{-1}\left[\frac{64}{(3s)^4-(4)^4}\right]$$

$$= \mathcal{L}^{-1}\left(\frac{64}{((3s)^2-4^2)((3s)^2+4^2)}\right)$$

$$\Rightarrow 2\mathcal{L}^{-1}\left[\frac{((3s)^2+4^2)-((3s)^2-4^2)}{((3s)^2-4^2)((3s)^2+4^2)}\right]$$

$$= 2\left[\mathcal{L}^{-1}\left[\frac{1}{(3s)^2-4^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(3s)^2+4^2}\right]\right]$$

$$= \frac{2}{12}\left[\mathcal{L}^{-1}\frac{(3s+4)-(3s-4)}{(3s+4)(3s-4)}\right] - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{\frac{4}{3}}{(s)^2+\left(\frac{4}{3}\right)^2}\right)$$

$$= \frac{1}{2}\left[\mathcal{L}^{-1}\left(\frac{1}{3s-4}\right) - \mathcal{L}^{-1}\left(\frac{1}{3s+4}\right)\right] - \frac{3}{4}\frac{1}{6}\sin^{-1}\left(\frac{4s}{3}\right)$$

$$= \frac{1}{6}\left[\sinh(4t) - \sin\frac{4t}{3}\right]$$

Ex: Find.

$$\mathcal{L}^{-1}\left[\frac{3s+1}{(s+1)^4}\right]$$

$$= e^{-t}\,\mathcal{L}^{-1}\left[\frac{3(s-1)+1}{s^4}\right]$$

$$= e^{-t}\left[3\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s^4}\right)\right]$$

$$= \mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^{a+1}}\right) = \frac{1}{\Gamma(a+1)}t^a$$

Piecewise Continious Function.

A function is called piecewise continious on ~~of~~ an interval if the interval can be broken into finite no. of sub-intervals on which the function is continious on each open subintervals (without its end point) and has finite limit at the end points of every subinterval.

Ex-1

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ t+1, & 2 < t \le 3 \end{cases}$$

Ex-2

$$f(t) = \begin{cases} \frac{1}{2-t}, & 0 \le t < 2 \\ t+1, & 2 \le t \le 3 \end{cases}$$

Sufficient conditions for existance of Laplace Transformation

Theorem

Hypothesis :

(i) $f(t)$ is piecewise continuous on $[0, A]$ for each $A > 0$

(ii) $|f(t)| \le Ke^{at}$ for $k > 0$ and $a \in R$

Conclusion :

$\mathcal{L}(f(t))$ exists for $s > 0$

Ex

Check weather Laplace transformation

$$f(t) = \begin{cases} 0, & 0 < t \leq 1 \\ t, & 1 < t \leq 4 \\ 0, & t > 4 \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$= \cancel{\int_0^1 e^{-st} f(t)\,dt} + \int_1^4 e^{-st} f(t)\,dt + \cancel{\int_4^\infty e^{-st} f(t)\,dt}$$

$$= \int_1^4 e^{-st} \cdot t \cdot dt.$$

$$= \left[\frac{e^{-st}}{s}\right]_1^4 + \frac{1}{s}\int_1^4 e^{-st}\,dt$$

$$= \left[\frac{e^{-s} - e^{-4s}}{s}\right] + \frac{1}{s^2}\left[e^{-st}\right]_1^4$$

$$\Rightarrow \frac{(e^{-s} - e^{-4s})}{s} + \frac{(e^{-s} - e^{-4s})}{s^2}$$

$$= (e^{-s} - e^{-4s})\left[\frac{1}{s} + \frac{1}{s^2}\right]$$

$$= (e^{-s} - e^{-4s})\frac{(s+1)}{s^2}$$

Laplace Transformation of the derivative

Theorem-1

Hypothesis:

- f is continious, f' is picewise continious on $[0, A]$ for every $A > 0$
- $|f(t)| \leq ke^{at}$ for $k, a \geq 0$

Conclusion:

$\mathcal{L}(f')$ exists and

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0) \quad , s > 0$$

Theorem-2

- $f, \ldots, f^{(n-1)}$ continious, f^{n} is picewise contd. on $[0, A]$ for every $A > 0$
- $|f(t)|, \ldots, |f^{(n-1)}(t)| \leq ke^{at}$ for $k, a \geq 0$.

Conclusion:

$\mathcal{L}(f^{n})$ exists and

$$\mathcal{L}(f^{(n)}) = s^{n}\mathcal{L}(f) - s^{n-1} f(0) \ldots\ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Ex Using Laplace Transformation solve the following IVP

$$y'' - 2y' - 8y = 0. \quad y(0) = 3$$
$$y'(0) = 6$$

Ans Taking Laplace transformation bothside,

$$\mathcal{L}(y'') - 2\mathcal{L}(y') - 8\mathcal{L}(y) = 0$$

$$\Rightarrow s^2 \mathcal{L}(y) - s\, y(0) - y'(0)$$
$$-2\left[s\mathcal{L}(y) - y(0) \right] - 8\mathcal{L}(y) = 0$$

$$\Rightarrow s^2 \mathcal{L}(y) - 3s - \cancel{6} - 2\{ s\mathcal{L}(y) + \cancel{6} - 8\mathcal{L}(y) = 0$$

$$\Rightarrow (s^2 - 2s - 8)\mathcal{L}(y) = 3s$$

$$\Rightarrow \mathcal{L}(y) = \frac{3s}{s^2 - 2s - 8}$$

$$= \frac{3s}{(s-4)(s+2)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{3s}{(s-4)(s+2)}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{(s-4) + 2(s+2)}{(s-4)(s+2)}\right]$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-4}\right)$$

$$y(t) = e^{-2t} + 2e^{4t}$$

Ex Solve

$$y'' - y = t \qquad y(0) = 1$$
$$y'(0) = 1$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(t)$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \frac{s+1}{s^2}$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) - \mathcal{L}(y) = \frac{s+1}{s^2}$$

$$(s^2-1)\mathcal{L}(y) - (s+1) = \frac{s+1}{s^2}$$

$$(s^2-1)\mathcal{L}(y) = \frac{(s+1)(1+s^2)}{s^2}$$

$$\mathcal{L}(y) = \frac{\cancel{(s+1)}(1+s^2)}{s^2(\cancel{1-s^2})(1-s)}$$

$$\mathcal{L}(y) = \frac{1+s^2}{s^2(1-s)}$$

Ex-1. Obtain the Laplace transformation of

$f(t) = t\cos(at)$

Ex 2 given $\mathcal{L}(\sin\sqrt{t}) = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$,

obtain the Laplace transformation of

$f(t) = \dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Ex 3 solve

$y''' - 3y'' + 3y' - y = t^2 e^t$

with $y(0)=1$, $y'(0)=0$, $y''(0)=-2$

Ex 4 solve

$y'' + 6y' + 5y = \delta(t)$, $y(0)=1$

$y'(0)=0$

where $\delta(t)$ is the Dirac-delta function.

Answers.

1)

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot t\cos(at)\, dt$$

$f(t) = t e^{iat} \rightarrow$ $\mathcal{L}(t \cdot (co$

$$\rightarrow \mathcal{L}(f(t)) = \int_0^\infty t \cdot e^{-(s-ia)t}\, dt$$

$$= \left[-\frac{t \cdot e^{-(s-ia)t}}{(s-ia)} \right]_0^\infty + \frac{1}{(s-ia)} \int_0^\infty e^{-(s-ia)t}\, dt$$

$$= 0 + \frac{1}{(s-ia)^2}$$

$$= \frac{1}{(s-ia)^2}$$

$$= \frac{1}{(s^2-a^2)-i2sa}$$

$$= \frac{(s^2-a^2)+i2sa}{(s^2-a^2)^2+4s^2a^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2} + i\frac{2sa}{(s^2+a^2)^2}$$

$$\Rightarrow \mathcal{L}(t\cos(at)) = \cancel{\frac{2sa}{(s^2+a^2)^2}} \underline{\underline{\frac{s^2-a^2}{(s^2+a^2)^2}}}$$

(2) $\mathcal{L}(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$

$$f(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$\mathcal{L}(\sin\sqrt{t}) = \int_0^\infty e^{-st}\sin\sqrt{t}\,dt$$

$$= \cancel{\left(-\frac{\sin\sqrt{t}\,e^{-st}}{s}\right]_0^\infty}^{\,0} + \int_0^\infty \frac{\cos\sqrt{t}}{2\sqrt{t}} \cdot \frac{e^{-st}}{s}\,dt$$

$$\frac{\sqrt{\pi}}{\cancel{2s}^{3/2}} e^{-1/4s} = \frac{1}{\cancel{2s}}\int_0^\infty \frac{\cos\sqrt{t}}{\sqrt{t}} e^{-st}\,dt$$

$$\mathcal{L}\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$$

(3) $y''' - 3y'' + 3y' - y = t^2 e^t$

$$\mathcal{L}(y''') - 3\mathcal{L}(y'') + 3\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(t^2 e^t)$$

$$s^3\mathcal{L}(y) - s^2 y(0) - sy'(0) - y''(0)$$

$$-3\left[s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right]$$

$$+3\left[s\mathcal{L}(y) - y(0)\right] - \mathcal{L}(y) = \mathcal{L}(t^2e^t)$$

$$\Rightarrow \mathcal{L}(y)\left[s^3 - 3s^2 + 3s - 1\right] - \left[s^2 - 3s + 3\right]y(0) - s\cancel{y'(0)}^{0} - y''(0) = \mathcal{L}(t^2 e^t)$$

$$\Rightarrow \mathcal{L}(y)(s-1)^3 - (s^2 - 3s - 3) + 2 = \mathcal{L}(t^2 e^t)$$

$$\Rightarrow \mathcal{L}(y)(s-1)^3 - s^2 + 3s + 5 = \mathcal{L}(t^2 e^t)$$

$$\Rightarrow \mathcal{L}(y) = \frac{\mathcal{L}(t^2.e^t) + (s^2 - 3s - 5)}{(s-1)^3}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left(\frac{\frac{2}{(s-1)^3} + (s^2 - 3s + 1)}{(s-1)^3}\right) = \mathcal{L}^{-1}\left(\frac{2}{(s-1)^6}\right) + \mathcal{L}^{-1}$$

(K) Dirac - Delta Function

A function $\delta(t)$ satisfying the following properties,

(i) $\delta(t) = \infty$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$,

(ii) $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

is called dirac-delta function.

~~$\delta(t) = \infty$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$~~

$$\mathcal{L}(\delta(t)) = \int_0^{\infty} e^{-st}\, \delta(t)\, dt.$$

$$= \int_0^{\infty} 1 \cdot \delta(t)\, dt$$

$$= \int_0^{\infty} \delta(t)\, dt$$

$$= 1$$

(4) $\mathcal{L}(y'') + 6\mathcal{L}(y') + 5\mathcal{L}(y) = \mathcal{L}(\delta(t))$

$\Rightarrow s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 6[s\mathcal{L}(y) - y(0)] + 5\mathcal{L}(y) = 1$

$\Rightarrow \mathcal{L}(y)(s^2 + 6s + 5) - (s+6)\,y(0) - \cancelto{0}{y'(0)} = 1$

$\Rightarrow \mathcal{L}(y)\,(s+5)(s+1) - (s+6) = 1$

$\Rightarrow \mathcal{L}(y) = \dfrac{s+7}{(s+5)(s+1)} \Rightarrow y = \mathcal{L}^{-1}\left(\dfrac{(s+7)}{(s+5)(s+1)}\right)$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{(s+5)+2}{(s+5)(s+1)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 2\,\mathcal{L}^{-1}\left(\frac{1}{(s+5)(s+1)}\right)$$

$$= e^{-1} + 2\left[\quad\right]$$