## Differentiation

$$(cu)' = cu' \qquad (c \text{ constant})$$

$$(u + v)' = u' + v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \qquad \text{(Chain rule)}$$

---

$$(x^n)' = nx^{n-1}$$

$$(e^x)' = e^x$$

$$(e^{ax})' = ae^{ax}$$

$$(a^x)' = a^x \ln a$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{\log_a e}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$(\text{arccot}\, x)' = -\frac{1}{1 + x^2}$$

## Integration

$$\int uv'\, dx = uv - \int u'v\, dx \text{ (by parts)}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n + 1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\, dx = \ln |x| + c$$

$$\int e^{ax}\, dx = \frac{1}{a} e^{ax} + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \tan x\, dx = -\ln |\cos x| + c$$

$$\int \cot x\, dx = \ln |\sin x| + c$$

$$\int \sec x\, dx = \ln |\sec x + \tan x| + c$$

$$\int \csc x\, dx = \ln |\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \text{arcsinh}\, \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arccosh}\, \frac{x}{a} + c$$

$$\int \sin^2 x\, dx = \tfrac{1}{2}x - \tfrac{1}{4} \sin 2x + c$$

$$\int \cos^2 x\, dx = \tfrac{1}{2}x + \tfrac{1}{4} \sin 2x + c$$

$$\int \tan^2 x\, dx = \tan x - x + c$$

$$\int \cot^2 x\, dx = -\cot x - x + c$$

$$\int \ln x\, dx = x \ln x - x + c$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$$

CHAPTER 1

# First-Order ODEs

Chapter 1 begins the study of ordinary differential equations (ODEs) by deriving them from physical or other problems (***modeling***), solving them by standard mathematical methods, and interpreting solutions and their graphs in terms of a given problem. The simplest ODEs to be discussed are ODEs ***of the first order*** because they involve only the first derivative of the unknown function and no higher derivatives. These unknown functions will usually be denoted by $y(x)$ or $y(t)$ when the independent variable denotes time $t$. The chapter ends with a study of the existence and uniqueness of solutions of ODEs in Sec. 1.7.

Understanding the basics of ODEs requires solving problems by hand (paper and pencil, or typing on your computer, but first without the aid of a CAS). In doing so, you will gain an important conceptual understanding and feel for the basic terms, such as ODEs, direction field, and initial value problem. If you wish, you can use your **Computer Algebra System (CAS)** for checking solutions.

**COMMENT.** ***Numerics for first-order ODEs can be studied immediately after this chapter.*** See Secs. 21.1–21.2, which are independent of other sections on numerics.

*Prerequisite:* Integral calculus.
*Sections that may be omitted in a shorter course:* 1.6, 1.7.
*References and Answers to Problems:* App. 1 Part A, and App. 2.

## 1.1 Basic Concepts. Modeling

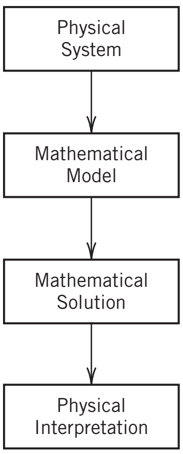


**Fig. 1.** Modeling, solving, interpreting

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical **model** of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling**.

Modeling needs experience, which we shall gain by discussing various examples and problems. (Your computer may often help you in *solving* but rarely in *setting up* models.)

Now many physical concepts, such as velocity and acceleration, are derivatives. Hence a model is very often an equation containing derivatives of an unknown function. Such a model is called a **differential equation**. Of course, we then want to find a solution (a function that satisfies the equation), explore its properties, graph it, find values of it, and interpret it in physical terms so that we can understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution, we must first define some basic concepts needed throughout this chapter.

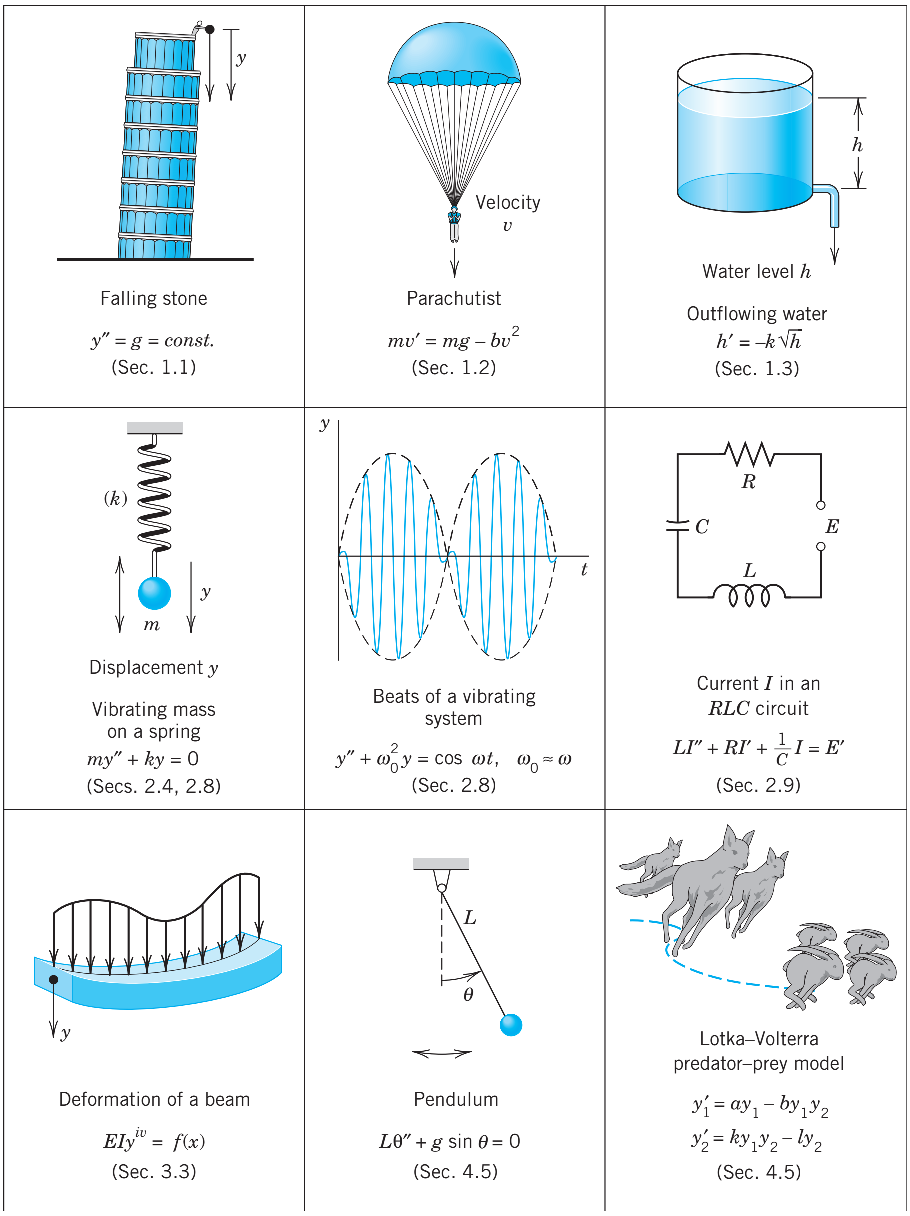


**Fig. 2.** Some applications of differential equations

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time $t$). The equation may also contain $y$ itself, known functions of $x$ (or $t$), and constants. For example,

$$y' = \cos x \tag{1}$$

$$y'' + 9y = e^{-2x} \tag{2}$$

$$y'y''' - \tfrac{3}{2}y'^2 = 0 \tag{3}$$

are ordinary differential equations (ODEs). Here, as in calculus, $y'$ denotes $dy/dx$, $y'' = d^2y/dx^2$, etc. The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function $u$ of two variables $x$ and $y$ is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDEs have important engineering applications, but they are more complicated than ODEs; they will be considered in Chap. 12.

An ODE is said to be of **order** $n$ if the $n$th derivative of the unknown function $y$ is the highest derivative of $y$ in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs**. Such equations contain only the first derivative $y'$ and may contain $y$ and any given functions of $x$. Hence we can write them as

$$F(x, y, y') = 0 \tag{4}$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form*, in contrast to the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

## Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if $y$ and $y'$ are replaced with $h$ and $h'$, respectively. The curve (the graph) of $h$ is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints $a$ and $b$ are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

**EXAMPLE 1 Verification of Solution**

Verify that $y = c/x$ ($c$ an arbitrary constant) is a solution of the ODE $xy' = -y$ for all $x \neq 0$. Indeed, differentiate $y = c/x$ to get $y' = -c/x^2$. Multiply this by $x$, obtaining $xy' = -c/x$; thus, $xy' = -y$, the given ODE. ■

**EXAMPLE 2 Solution by Calculus. Solution Curves**

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where $c$ is an arbitrary constant. This is a ***family of solutions***. Each value of $c$, for instance, 2.75 or 0 or $-8$, gives one of these curves. Figure 3 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$. ■

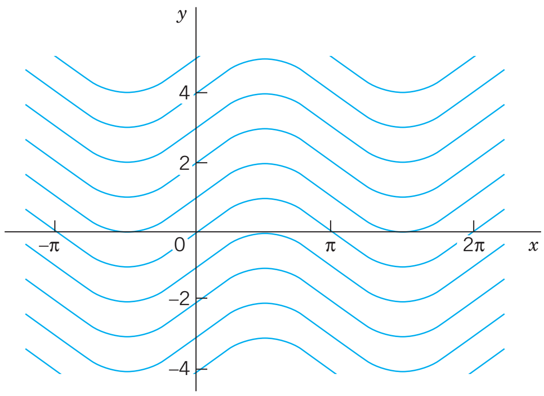


Fig. 3. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

**EXAMPLE 3 (A) Exponential Growth. (B) Exponential Decay**

From calculus we know that $y = ce^{0.2t}$ has the derivative

$$y' = \frac{dy}{dt} = 0.2e^{0.2t} = 0.2y.$$

Hence $y$ is a solution of $y' = 0.2y$ (Fig. 4A). This ODE is of the form $y' = ky$. With positive-constant $k$ it can model exponential growth, for instance, of colonies of bacteria or populations of animals. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as **Malthus's law**.[1] We shall say more about this topic in Sec. 1.5.

**(B)** Similarly, $y' = -0.2$ (with a minus on the right) has the solution $y = ce^{-0.2t}$, (Fig. 4B) modeling **exponential decay**, as, for instance, of a radioactive substance (see Example 5). ■

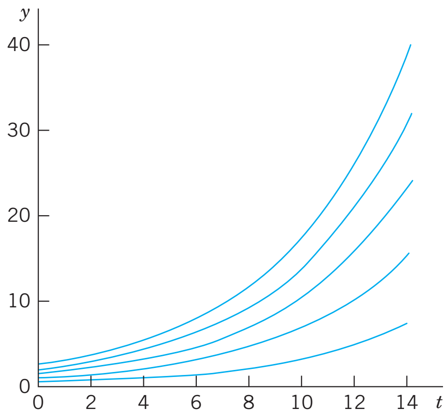


Fig. 4A. Solutions of $y' = 0.2y$ in Example 3 **(exponential growth)**

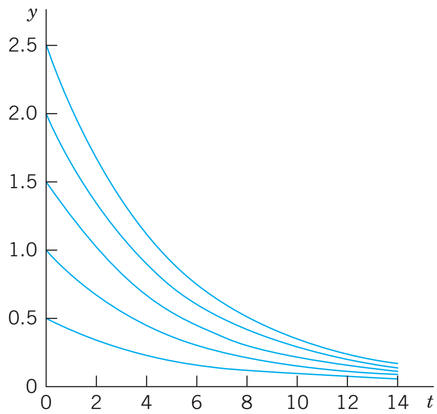


Fig. 4B. Solutions of $y' = -0.2y$ in Example 3 **(exponential decay)**

[1] Named after the English pioneer in classic economics, THOMAS ROBERT MALTHUS (1766–1834).

We see that each ODE in these examples has a solution that contains an arbitrary constant $c$. Such a solution containing an arbitrary constant $c$ is called a **general solution** of the ODE.

(We shall see that $c$ is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.)

We shall develop methods that will give general solutions *uniquely* (perhaps except for notation). Hence we shall say ***the*** general solution of a given ODE (instead of ***a*** general solution).

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant $c$. If we choose a specific $c$ (e.g., $c = 6.45$ or 0 or $-2.01$) we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

In most cases, general solutions exist, and every solution not containing an arbitrary constant is obtained as a particular solution by assigning a suitable value to $c$. Exceptions to these rules occur but are of minor interest in applications; see Prob. 16 in Problem Set 1.1.

## Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values $x_0$ and $y_0$, that is used to determine a value of the arbitrary constant $c$. Geometrically this condition means that the solution curve should pass through the point $(x_0, y_0)$ in the $xy$-plane. An ODE, together with an initial condition, is called an **initial value problem**. Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{5}$$

**EXAMPLE 4** **Initial Value Problem**

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7.$$

***Solution.*** The general solution is $y(x) = ce^{3x}$; see Example 3. From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

## More on Modeling

The general importance of modeling to the engineer and physicist was emphasized at the beginning of this section. We shall now consider a basic physical problem that will show the details of the typical steps of modeling. Step 1: the transition from the physical situation (the physical system) to its mathematical formulation (its mathematical model); Step 2: the solution by a mathematical method; and Step 3: the physical interpretation of the result. This may be the easiest way to obtain a first idea of the nature and purpose of differential equations and their applications. Realize at the outset that your ***computer*** (your ***CAS***) may perhaps give you a hand in Step 2, but Steps 1 and 3 are basically your work.

And Step 2 requires a solid knowledge and good understanding of solution methods available to you—*you* have to choose the method for your work by hand or by the computer. Keep this in mind, and always check computer results for errors (which may arise, for instance, from false inputs).

### EXAMPLE 5 Radioactivity. Exponential Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

*Physical Information.* Experiments show that at each instant a radioactive substance decomposes—and is thus decaying in time—proportional to the amount of substance present.

***Step 1. Setting up a mathematical model of the physical process.*** Denote by $y(t)$ the amount of substance still present at any time $t$. By the physical law, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. This gives the ***first-order ODE***

$$\frac{dy}{dt} = -ky \tag{6}$$

where the constant $k$ is positive, so that, because of the minus, we do get decay (as in [B] of Example 3). The value of $k$ is known from experiments for various radioactive substances (e.g., $k = 1.4 \cdot 10^{-11}\ \text{sec}^{-1}$, approximately, for radium ${}^{226}_{88}\text{Ra}$).

Now the given initial amount is 0.5 g, and we can call the corresponding instant $t = 0$. Then we have the ***initial condition*** $y(0) = 0.5$. This is the instant at which our observation of the process begins. It motivates the term *initial condition* (which, however, is also used when the independent variable is not time or when we choose a $t$ other than $t = 0$). Hence the mathematical model of the physical process is the ***initial value problem***

$$\frac{dy}{dt} = -ky, \qquad y(0) = 0.5. \tag{7}$$

***Step 2. Mathematical solution.*** As in (B) of Example 3 we conclude that the ODE (6) models exponential decay and has the general solution (with arbitrary constant $c$ but definite given $k$)

$$y(t) = ce^{-kt}. \tag{8}$$

We now determine $c$ by using the initial condition. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing our process is (cf. Fig. 5)

$$y(t) = 0.5e^{-kt} \qquad (k > 0). \tag{9}$$

***Always check your result***—it may involve human or computer errors! Verify by differentiation (chain rule!) that your solution (9) satisfies (7) as well as $y(0) = 0.5$:

$$\frac{dy}{dt} = -0.5ke^{-kt} = -k \cdot 0.5e^{-kt} = -ky, \qquad y(0) = 0.5e^{0} = 0.5.$$

***Step 3. Interpretation of result.*** Formula (9) gives the amount of radioactive substance at time $t$. It starts from the correct initial amount and decreases with time because $k$ is positive. The limit of $y$ as $t \to \infty$ is zero. ∎

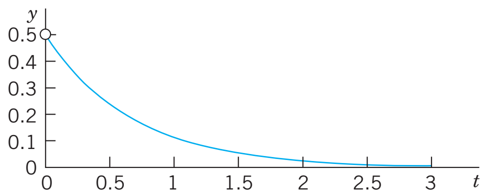


**Fig. 5.** Radioactivity (Exponential decay, $y = 0.5e^{-kt}$, with $k = 1.5$ as an example)

# PROBLEM SET 1.1

## 1–8 CALCULUS

Solve the ODE by integration or by remembering a differentiation formula.

1. $y' + 2\sin 2\pi x = 0$
2. $y' + xe^{-x^2/2} = 0$
3. $y' = y$
4. $y' = -1.5y$
5. $y' = 4e^{-x}\cos x$
6. $y'' = -y$
7. $y' = \cosh 5.13x$
8. $y''' = e^{-0.2x}$

## 9–15 VERIFICATION. INITIAL VALUE PROBLEM (IVP)

**(a)** Verify that $y$ is a solution of the ODE. **(b)** Determine from $y$ the particular solution of the IVP. **(c)** Graph the solution of the IVP.

9. $y' + 4y = 1.4, \quad y = ce^{-4x} + 0.35, \quad y(0) = 2$
10. $y' + 5xy = 0, \quad y = ce^{-2.5x^2}, \quad y(0) = \pi$
11. $y' = y + e^x, \quad y = (x + c)e^x, \quad y(0) = \frac{1}{2}$
12. $yy' = 4x, \quad y^2 - 4x^2 = c \ (y > 0), \quad y(1) = 4$
13. $y' = y - y^2, \quad y = \dfrac{1}{1 + ce^{-x}}, \quad y(0) = 0.25$
14. $y' \tan x = 2y - 8, \quad y = c\sin^2 x + 4, \quad y(\frac{1}{2}\pi) = 0$
15. Find two constant solutions of the ODE in Prob. 13 by inspection.
16. **Singular solution.** An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of this kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

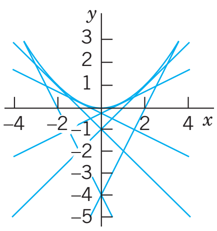


**Fig. 6.** Particular solutions and singular solution in Problem 16

## 17–20 MODELING, APPLICATIONS

These problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. **Half-life.** The *half-life* measures exponential decay. It is the time in which half of the given amount of radioactive substance will disappear. What is the half-life of $^{226}_{88}\mathrm{Ra}$ (in years) in Example 5?
18. **Half-life.** Radium $^{224}_{88}\mathrm{Ra}$ has a half-life of about 3.6 days.
    **(a)** Given 1 gram, how much will still be present after 1 day?
    **(b)** After 1 year?
19. **Free fall.** In dropping a stone or an iron ball, air resistance is practically negligible. Experiments show that the acceleration of the motion is constant (equal to $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$, called the **acceleration of gravity**). Model this as an ODE for $y(t)$, the distance fallen as a function of time $t$. If the motion starts at time $t = 0$ from rest (i.e., with velocity $v = y' = 0$), show that you obtain the familiar law of free fall
$$y = \tfrac{1}{2}gt^2.$$
20. **Exponential decay. Subsonic flight.** The efficiency of the engines of subsonic airplanes depends on air pressure and is usually maximum near 35,000 ft. Find the air pressure $y(x)$ at this height. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure. At 18,000 ft it is half its value $y_0 = y(0)$ at sea level. *Hint.* Remember from calculus that if $y = e^{kx}$, then $y' = ke^{kx} = ky$. Can you see without calculation that the answer should be close to $y_0/4$?

# 1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point $(x_0, y_0)$ must have, at that point, the slope $y'(x_0)$ equal to the value of $f$ at that point; that is,

$$y'(x_0) = f(x_0, y_0).$$

Using this fact, we can develop graphic or numeric methods for obtaining approximate solutions of ODEs (1). This will lead to a better conceptual understanding of an ODE (1). Moreover, such methods are of practical importance since many ODEs have complicated solution formulas or no solution formulas at all, whereby numeric methods are needed.

**Graphic Method of Direction Fields. Practical Example Illustrated in Fig. 7.** We can show directions of solution curves of a given ODE (1) by drawing short straight-line segments (lineal elements) in the $xy$-plane. This gives a **direction field** (or *slope field*) into which you can then fit (approximate) solution curves. This may reveal typical properties of the whole family of solutions.

Figure 7 shows a direction field for the ODE

$$y' = y + x \tag{2}$$

obtained by a CAS (Computer Algebra System) and some approximate solution curves fitted in.

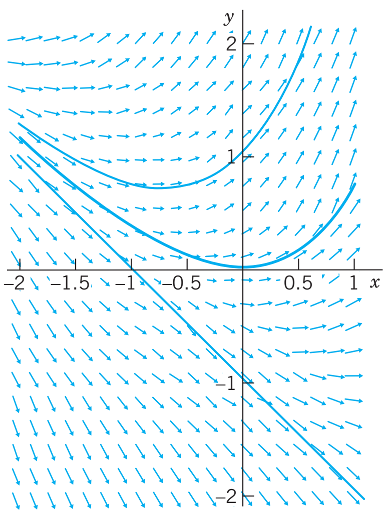


Fig. 7. Direction field of $y' = y + x$, with three approximate solution curves passing through (0, 1), (0, 0), (0, −1), respectively