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Matrices Using Python

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Problem Statement:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

SOLUTION:

Given:

Equation of circle is

$$x^2 + y^2 = 16 \tag{1}$$

Equation of Parabola is

$$y^2 = 6x \tag{2}$$

From (2) we can say that Parabola is concave towards positive xaxis.

From equation (1) radius of circle is,

$$r = 4 \tag{3}$$

To Find

To find the intersection points and area of unshaded region shown in figure

STEP-1

The given circle and parabola can be expressed as conics with parameters,
For circle,

$$\mathbf{V}_1 = \mathbf{I} \tag{4}$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$\mathbf{u_1} = 0 \tag{6}$$

$$f_1 = -16 \tag{7}$$

For Parabola,

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

$$\mathbf{u_2} = -\begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{9}$$

$$f_2 = 0 \tag{10}$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu\mathbf{V}_2\right)\mathbf{x} + 2\left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top \mathbf{x} \tag{11}$$

$$+ \left(f_1 + \mu f_2\right) = 0 \tag{12}$$

$$\mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \tag{13}$$

$$\mathbf{u}_1 + \mu\mathbf{u}_2 = -\begin{pmatrix} 3\mu \\ 0 \end{pmatrix} \tag{14}$$

$$f_1 + \mu f_2 = -16 \tag{15}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ \left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{16}$$

And,

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| = 0 \tag{17}$$

Substituting equation (13),(14) and (15) in equation (16) We get,

$$\implies \begin{vmatrix} 1 & 0 & -3\mu \\ 0 & 1+\mu & 0 \\ -3\mu & 0 & -16 \end{vmatrix} = 0 \tag{18}$$

Solving the above equation we get,

$$9\mu^3 + 9\mu^2 + 16\mu + 16 = 0 \tag{19}$$

gives,

$$\mu = -1 \tag{20}$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{21}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{22}$$

$$f = -16, \tag{23}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{24}$$

with the conic section

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{25}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{26}$$

$$q = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \tag{27}$$

Thus, the desired pair of straight lines are

$$\begin{pmatrix} 2 \\ -8 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \end{pmatrix} y \tag{28}$$

$$\tag{29}$$

upon substituting from The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \tag{30}$$

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \tag{31}$$

$$with the conic section we have, \tag{32}$$

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \tag{33}$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \Big(-\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \Big) \tag{34}$$

On substituting

$$\mathbf{q} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \tag{35}$$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{36}$$

With the given circle

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{37}$$

$$\mathbf{u} = -\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{38}$$

$$f = -16 \tag{39}$$

The value of q ,

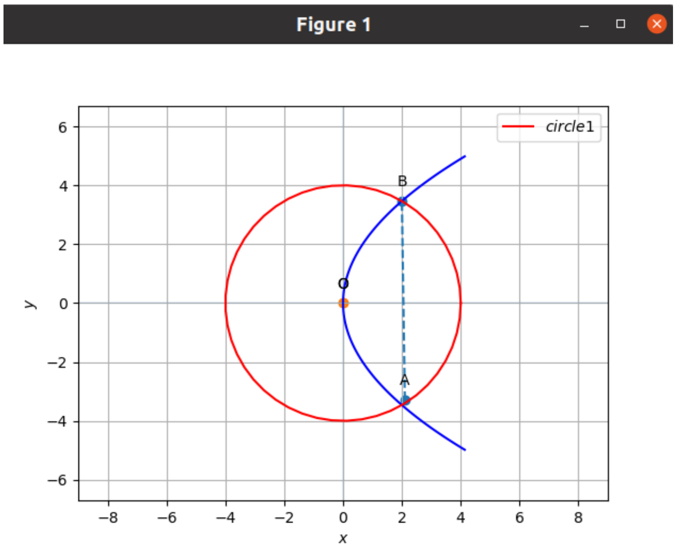
$$q = 2, -8 \tag{40}$$

The points of intersection with Parabola along circle are

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \tag{41}$$

$$\mathbf{B} = \begin{pmatrix} 2 \\ -3.46 \end{pmatrix} \tag{42}$$

Result



From the figure,

Total area of circle exterior to the parabola is given by,

A=Area of circle-2[Area(OAD)+Area(DAC)]

$$B = \int_0^2 y\,dx + \int_2^4 y\,dx \tag{43}$$

$$B = \int_0^2 f(x)\,dx + \int_2^4 g(x)\,dx \tag{44}$$

Where g(x) is area of circle and f(x) is the area of parabola around the points

$$B = \int_0^2 \sqrt{16x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx \tag{45}$$

$$B = \frac{4}{3}(4\pi + \sqrt{3}) \tag{46}$$

$$\tag{47}$$

Area of circle

$$C = \pi(r^2)\,dx \tag{48}$$

$$C = \pi(4^2)\,dx = 16\pi \tag{49}$$

$$A = C - B \tag{50}$$

Area A is,

$$A = 16\pi - \frac{4}{3}(4\pi + \sqrt{3})\,m^2 \tag{51}$$

$$A = 31.200\,square units \tag{52}$$

Construction

Points	coordinates
A	$\begin{pmatrix} 2 \\ 3.46 \end{pmatrix}$
B	$\begin{pmatrix} 2 \\ -3.46 \end{pmatrix}$

*Verify the above proofs in the following code.

`https://github.com/Susi9121/FWC/tree/main/matrix/line`