§lec 51~52.

线性等式约束的凸优化问题

$$\begin{cases} \min f(x) \\ \text{s.t. } Ax=b \end{cases} \qquad \text{KKT条件: } \begin{cases} Ax^*=b \\ \nabla f(x)+A^T v^*=0 \end{cases}$$

$$\begin{cases} \min\limits_{d} \ f(x^k+d) & \sim \quad f(x^k)+\nabla f(x^k)^T d+\frac{1}{2}d^T\nabla^2 f(x^k)d. \\ \text{s.t. } A(x^k+d)=b. & \sim \quad Ad=0 \end{cases}$$

$$d^k = \begin{bmatrix} \nabla f(x^k) & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} d^k \\ v^* \end{bmatrix} = \begin{bmatrix} -\nabla f(x^k) \\ 0 \end{bmatrix}$$

$$\alpha^k = \arg\min_{\alpha\geq 0} f(x^k+\alpha d^k)$$

$$x^{k+1} = x^k+\alpha^k d^k.$$

拉格朗日法.

$$x^{k+1} = x^k - \alpha^k(\nabla f(x^k)+A^T v^k)$$

$$v^{k+1} = v^k + \alpha^k(Ax^k-b)$$

增广拉格朗日法.

$$L_c(x,v) = f(x)+v^T(Ax-b)+\frac{c}{2}\|Ax-b\|_2^2$$

$$\begin{cases} \min f(x)+\frac{c}{2}\|Ax-b\|_2^2 \\ \text{s.t. } Ax=b \end{cases}$$

KKT条件: $\nabla_x L(x^*,v^*)=0.$ ①

$$\Rightarrow \nabla_x\{f(x^*)+(v^*)^T(Ax^*-b)\}=0.$$

$\nabla_x L_c(x^*,v^*)=0.$ ②

$$\Rightarrow \nabla_x\{f(x^*)+(v^*)^T(Ax^*-b)\}+cA^T\underbrace{(Ax^*-b)}_{0}=0.$$

①②的最优解相同.

$$x^{k+1} = x^k - \alpha^k\nabla L_c(x^k, v^k)$$

$$v^{k+1} = v^k + \alpha^k(Ax^k-b)$$

并不需要十分精确的解.

$$\left.\begin{aligned} &\Rightarrow x^{k+1} = \arg\min_x L_c(x,v^k). \\ &\Rightarrow v^{k+1} = v^k + c(Ax^{k+1}-b). \end{aligned}\right\} \text{Augmented Lagrangian Method.}$$

性质 1): 若 $v=v^*$, 则 $\forall c>0$, $x^*=\arg\min\limits_x L_c(x,v^*)$.

2) 若 $c\to+\infty$, 则 $\forall v$, $x^*=\arg\min\limits_x L_c(x,v)$.

例：$\begin{cases} \min \frac{1}{2}x_1^2+\frac{1}{2}x_2^2 \\ \text{s.t.} \quad x_1=1 \end{cases}$ $\quad x^*=\begin{pmatrix}1\\0\end{pmatrix}$

$$\Rightarrow L(x,v)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+v(x_1-1).$$

$$\begin{cases} x_1=1 \\ x_1+v=0. \\ x_2=0. \end{cases} \qquad \begin{cases} x_1^*=1 \\ v^*=-1 \\ x_2^*=0. \end{cases}$$

$$\Rightarrow L_c(x,v)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+v(x_1-1)+\frac{c}{2}(x_1-1)^2$$

$$\arg\min L_c(x,v^*)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2-(x_1-1)+\frac{c}{2}(x_1-1)^2$$

$$\begin{cases} \frac{\partial L_c(x,v^*)}{\partial x_1}=x_1-1+c(x_1-1)=0. \iff x_1=1 \\ \frac{\partial L_c(x,v^*)}{\partial x_2}=x_2=0. \end{cases}$$ 性质(1)

$$\begin{cases} x_1+v+c(x_1-1)=0 \iff x_1=\frac{c-v}{c+1} \\ x_2=0 \end{cases}$$ 性质(2)

$L_c(x,v)=\cdots$

1) $x^{k+1}=\arg\min\limits_{x} L_c(x,v^k). \iff x^{k+1}=\begin{pmatrix}\frac{c-v^k}{c+1}\\0\end{pmatrix}$

2). $v^{k+1}=v^k+c(x_1^{k+1}-1)$

$$=v^k+c\left(\frac{c-v^k}{c+1}-1\right). = v^k-\frac{c}{c+1}(v^k+1).$$

$$\begin{aligned} v^{k+1}-v^* &= v^{k+1}+1 \\ &= (v^k+1)-\frac{c}{c+1}(v^k+1) \\ &= (v^k+1)\frac{1}{c+1} \\ &= (v^k-v^*)\frac{1}{c+1} \end{aligned}$$ (线性收敛)

例：$\min\limits_{x} f(x)+g(x) \iff \begin{aligned} &\min f(x)+g(z) \\ &\text{s.t.} \quad x=z. \end{aligned}$

$$\Rightarrow L_c(x,z,v)=f(x)+g(z)+v^T(x-z)+\frac{c}{2}\|x-z\|_2^2$$

1). $\{x^{k+1},z^{k+1}\}=\arg\min\limits_{x,z} f(x)+g(z)+(v^k)^T(x-z)+\frac{c}{2}\|x-z\|_2^2$ 坐标轮换法

2) $v^{k+1}=v^k+c(x^{k+1}-z^{k+1}).$

$$(a).\ x^{k+1|t+1} = \arg\min_{x} f(x) + \frac{c}{2}\|x - z^{k+1|t} + \frac{v^k}{c}\|_2^2$$

$$z^{k+1|t}$$

$$(b).\ z^{k+1|t+1} = \arg\min_{z} g(z) + \frac{c}{2}\|z - x^{k+1|t+1} - \frac{v^k}{c}\|_2^2$$

$$(2)\ v^{k+1} = v^k + c(x^{k+1} - z^{k+1}).$$

交替方向乘子法.

例 $\min \sum_{i=1}^{n} f_i(x) \iff \min \sum_{i=1}^{n} f_i(x_i)$

s.t. $x_i = z$, $i=1,2,\cdots,n$

$$L_c = \sum_{i=1}^{n} f_i(x_i) + \sum_{i=1}^{n} v_i^T(x_i - z) + \frac{c}{2}\sum_{i=1}^{n}\|x_i - z\|_2^2$$

$$(1):\ x_i^{k+1} = \arg\min_{\{x_i\}} \sum_{i=1}^{n} f_i(x_i) + \frac{c}{2}\sum_{i=1}^{n}\|x_i - z^k + \frac{v_i^k}{c}\|_2^2$$

$$\iff x_i^{k+1} = \arg\min_{x_i} f_i(x_i) + \frac{c}{2}\|x_i - z^k + \frac{v_i^k}{c}\|_2^2$$

$$(2)\ z^{k+1} = \arg\min \frac{c}{2}\sum_{i=1}^{n}\|z - x_i^{k+1} - \frac{v_i}{c}\|_2^2.$$

$$\iff z^{k+1} = \frac{1}{n}\sum_{i=1}^{n}(x_i^{k+1} + \frac{v_i}{c})$$

$$(3)\ v_i^{k+1} = v^k + c(x_i^{k+1} - z^{k+1}),\ \forall i$$

分布式计算