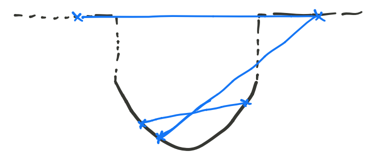
凸函数的扩展.

$f: \mathbb{R}^n \to \mathbb{R}$ 为凸函数, $\mathrm{dom} f = C \subseteq \mathbb{R}^n$

$$\tilde{f} = \begin{cases} f(x) & x \in \mathrm{dom} f \\ +\infty & x \notin \mathrm{dom} f \end{cases} \qquad \boxed{\begin{array}{l} \tilde{f}: \mathbb{R}^n \to \mathbb{R} \\ \mathrm{dom} \tilde{f} = \mathbb{R}^n \end{array}}$$

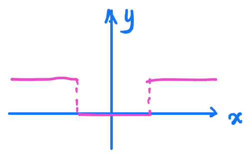


例: 示性函数是凸函数.

凸集 $C \subseteq \mathbb{R}^n$, $f_C(x) = \begin{cases} \text{无定义} & x \notin C \\ 0 & x \in C \end{cases}$

$$I_C(x) = \begin{cases} \infty & x \notin C \\ 0 & x \in C \end{cases}$$

$$J_C(x) = \begin{cases} 1 & x \notin C \\ 0 & x \in C \end{cases}$$

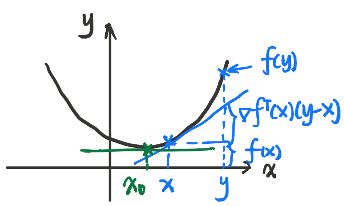


一阶条件

设 $f: \mathbb{R}^n \to \mathbb{R}$ 可微, 即梯度 ∇f 在 $\mathrm{dom} f$ 上均存在, 则 f 为凸 等价于:

① $\mathrm{dom} f$ 为凸集

② $f(y) \geq f(x) + \nabla f^T(x)(y-x)$, $\forall x, y \in \mathrm{dom} f$.



$\exists x_0 \quad \nabla f(x_0) = 0 \quad \forall y.$

$f(y) \geq f(x_0) + \nabla f(x_0)(y-x) = f(x_0)$

x_0是最小值点.

证明一阶条件

考虑一维情况: $f: \mathbb{R} \to \mathbb{R}$ 为凸 $\Leftrightarrow$ $\mathrm{dom} f$ 为凸 且 $f(y) \geq f(x) + f'(x)(y-x)$.

证明: $(\Rightarrow)$

f 为凸函数. $x, y \in \mathrm{dom} f$ 为凸集.

$\forall t \quad 0 < t \leq 1 \quad x + t(y-x) \in \mathrm{dom} f.$

$f(x + t(y-x)) \leq (1-t) f(x) + t f(y)$

$t f(y) \geq t f(x) + f(x + t(y-x)) - f(x)$

$$f(y) \geq f(x) + \frac{f(x + t(y-x)) - f(x)}{t}$$

$$\lim_{t \to 0_+} \Rightarrow \quad f(y) \geq f(x) + f'(y)(y-x).$$

$(\Leftarrow)$ 设 $\forall x \neq y$, $x, y \in \mathrm{dom}\, f$

$0 \le \theta \le 1$. 构造 $z = \theta x + (1-\theta) y \in \mathrm{dom}\, f$.

$$f(x) \ge f(z) + f'(z)(x - z).$$

$$f(y) \ge f(z) + f'(z)(y - z)$$

$$\underbrace{\theta f(x) + (1-\theta) f(y) \ge f(z)}_{\text{凸函数.}} + f'(z)\underbrace{\big(\theta(x-z) + (1-\theta)(y-z)\big)}_{\theta x + (1-\theta) y - z = 0.}$$