Optimization / Mathematical Programming.

(min)   minimize $f_0(x)$
(s.t)   subject to $f_i(x) \le b_i$ $i = 1, \cdots, m$
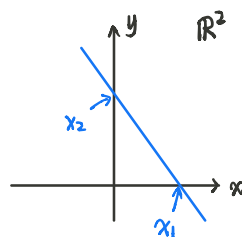$$x = [x_1, \cdots, x_n]^T$$

凸规划 / 非凸规划

Chapter 2   Convex Sets

仿射集 Affine sets

$x_1 \ne x_2 \in \mathbb{R}^n, \theta \in \mathbb{R} \Rightarrow y = \theta x_1 + (1-\theta) x_2$ 直线
$$= x_2 + \theta(x_1 - x_2)$$

$x_1 \ne x_2 \in \mathbb{R}^n, \theta \in [0,1] \Rightarrow y = \theta x_1 + (1-\theta) x_2$ 线段

仿射集: 一个集合 C 是仿射集. 若 $\forall x_1, x_2 \in C$, 则连接 x_1 与 x_2 的直线也在集合内.

直线、$\mathbb{R}^2$ 是仿射集; 线段、闭合图形不是仿射集

设 $x_1, \cdots, x_k \in C$, $\theta_1, \cdots, \theta_k \in \mathbb{R}$, $\theta_1 + \cdots + \theta_k = 1$
（K个点）

仿射组合: $\theta_1 x_1 + \cdots + \theta_k x_k$ (一个点)

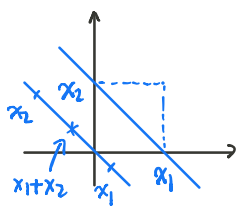$$\theta_1 x_1 + \cdots + \theta_k x_k \in C \Longleftrightarrow C \text{ 是仿射集}$$

有仿射集 C, $x_1, x_2, x_3 \in C$. $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, $\theta_1 + \theta_2 + \theta_3 = 1$

$$\frac{\theta_1}{\theta_1 + \theta_2} x_1 + \frac{\theta_2}{\theta_1 + \theta_2} x_2 \in C$$

$$(\theta_1 + \theta_2)\left(\frac{\theta_1}{\theta_1 + \theta_2} x_1 + \frac{\theta_2}{\theta_1 + \theta_2} x_2\right) + (1 - \theta_1 - \theta_2) x_3 \in C.$$

$$= \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$$

$x_1, x_2 \in C$, C 是仿射集, 则 $\theta x_1 + (1-\theta) x_2 \in C$

仿射集 $\alpha x_1 + \beta x_2 \in C$ $(\alpha, \beta \in \mathbb{R})$

$V = C - x_0 = \langle x - x_0 \mid x \in C \rangle$ $\forall x_0 \in C$

与 C 相关的子空间.   对仿射集 C 的平移

证明: $\forall v_1, v_2 \in V$, $\forall \alpha, \beta \in \mathbb{R} \overset{?}{\Rightarrow} \alpha v_1 + \beta v_2 \in V$

$\quad\quad \alpha v_1 + \beta v_2 + x_0 \in C$

$\quad\quad \Leftarrow \underline{\alpha} (\underbrace{v_1 + x_0}_{\in C}) + \underline{\beta} (\underbrace{v_2 + x_0}_{\in C}) + \underline{(1-\alpha-\beta)} \underbrace{x_0}_{\in C} \in C$

例: 线性方程组的解集是仿射集

$\quad C = \{x \mid Ax = b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

$\quad \forall x_1, x_2 \in C$, 则 $Ax_1 = b$, $Ax_2 = b$

$\quad \theta \in \mathbb{R}$, $\theta x_1 + (1-\theta) x_2 \overset{?}{\in} C$

$\quad A(\theta x_1 + (1-\theta) x_2) = b$?

$\quad = \theta A x_1 + (1-\theta) A x_2 = b$

$V = \{x - x_0 \mid x \in C\}$ $\forall x_0 \in C$
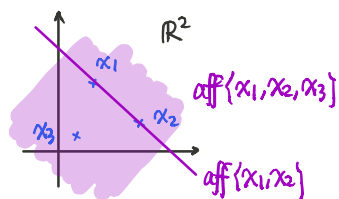
$\quad = \{x - x_0 \mid Ax = b\}$ $Ax_0 = b$

$\quad = \{x - x_0 \mid A(x - x_0) = 0\}$

$\quad = \{y \mid Ay = 0\}$ ↰ 平移

$C = \{x \mid Ax = b\}$

任意集合 C, 构造尽可能小的仿射集

仿射包: $\text{aff } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid \forall x_1, \cdots, x_k \in C , \theta_1 + \cdots + \theta_k = 1\}$

仿射集一定是凸集.

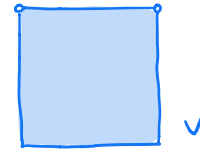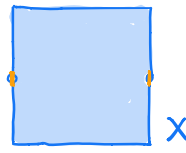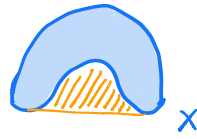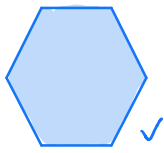凸集 (Convex Set): 一个集合 C 是凸集, 当任意两点之间的线段仍然在 C 内.

C 为凸集 $\Longleftrightarrow \forall x_1, x_2 \in C$, $\forall \theta$, $\theta \in [0,1]$, $\theta x_1 + (1-\theta) x_2 \in C$.

凸组合: $\theta_1 x_1 + \cdots + \theta_k x_k$, $\theta_1, \cdots, \theta_k \in \mathbb{R}$, $\theta_1 + \cdots + \theta_k = 1$, $\underline{\theta_1, \cdots, \theta_k \in [0,1]}$

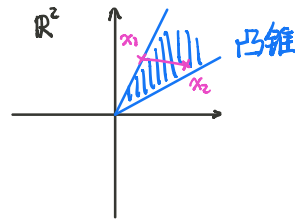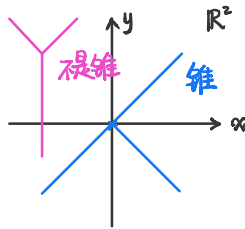C 为凸集 $\Longleftrightarrow$ 任意元素凸组合 $\in C$

凸包: $C \in \mathbb{R}^n$

$\quad \text{Conv } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid \forall x_1, \cdots, x_k \in C , \forall \theta_1, \cdots, \theta_k \in [0,1] , \theta_1 + \cdots + \theta_k = 1\}$
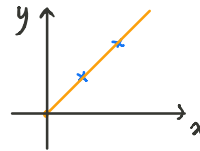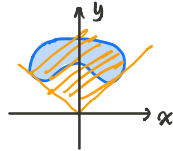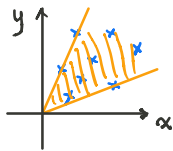
锥 Cone   凸锥 Convex Cone.

C是锥 $\iff \forall x \in C$, $\theta \geq 0$, 有 $\theta x \in C$.

C是凸锥 $\iff \forall x_1, x_2 \in C$, $\underline{\theta_1, \theta_2 \geq 0}$, 有 $\theta_1 x_1 + \theta_2 x_2 \in C$

凸锥组合: $\theta_1 x_1 + \cdots + \theta_k x_k$ $\theta_1, \cdots, \theta_k \geq 0$

凸锥包: $x_1, \cdots, x_k \in C$ $\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C, \theta_1, \cdots, \theta_k \geq 0\}$

仿射组合 $\forall \theta_1, \cdots, \theta_k$ $\theta_1 + \cdots + \theta_k = 1$

凸组合 $\forall \theta_1, \cdots, \theta_k$ $\theta_1 + \cdots + \theta_k = 1$, $\theta_1, \cdots, \theta_k \in [0,1]$

凸锥组合 $\forall \theta_1, \cdots, \theta_k$ $\theta_1, \cdots, \theta_k \geq 0$

$C = \{x\}$ $\theta_1 x + \theta_2 x = x$ 仿射集 $\emptyset$ 也是仿射集

$\emptyset$ 是仿射集、凸集、凸锥.