lec31.

对于任意优化问题

(P). $\min f_0(x)$

$\quad$ s.t. $f_i(x)\le 0,\ i=1,\cdots,m$ $\qquad p^*$.

$\qquad\quad h_i(x)=0,\ i=1,\cdots,p.$

$$L(x,\lambda,v)=f_0(x)+\sum_{i=1}^{m}\lambda_i f_i(x)+\sum_{i=1}^{p}v_i h_i(x).$$

$$g(\lambda,v)=\inf_{x\in D} L(x,\lambda,v).$$

(D) $\max g(\lambda,v)$ $\qquad R^{m+p}$ $\qquad d^*$

$\quad$ s.t. $\lambda\ge 0$.

(1) 对偶问题一定是凸优化问题!

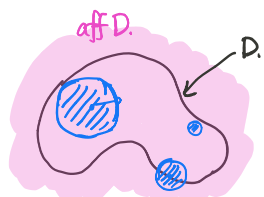
(2). $d^*\le p^*$ (weak duality, 一定成立).

$d^*=p^*$ (strong duality).

$p^*-d^*$ (duality gap).

D的 Relative Interior. (相对内部).

$$\text{Relint } D=\{x\in D \mid \exists r>0,\ B(x,r)\cap \text{aff } D\subseteq D\}.$$




把边界去掉的开集

Slater's Condition ($p^*=d^*$ 的充分条件).

若有凸问题 $\qquad \min f_0(x)$

$\qquad$ s.t. $f_i(x)\le 0,\ i=1,\cdots,m.$

$\qquad\quad Ax=b.$

其中 $f_i(x)$ 为凸, $\forall i$

当 $\exists x\in$ relint $D$, 使 $f_i(x)<0,\ i=1,\cdots,m,\ \ Ax=b$. 满足时, $p^*=d^*$.

A weaker Slater's Condition. (充分条件).

若不等式约束为仿射时, 只要可行域非空, 必有 $p^*=d^*$

$cx+d\le 0.$ $\qquad cx+d=0$ $\qquad Ax=b.$

半空间 $\qquad$ 超平面.

$$\mathrm{relint}\, D = \mathrm{relint}\{\mathrm{dom} f_0 \cap \bigcap_i \mathrm{dom} f_i\}$$
$$= \mathrm{relint}\{\mathrm{dom} f_0\}.$$

线性规划问题若可行，必有 $p^* = d^*$

例：$\min\ x^T x$　　　$p^* = d^*$.
　　s.t. $Ax = b$

$\Leftrightarrow$ (D) $\max_{v}\ -\frac{1}{4} v^T A A^T v - b^T v$

例：QCQP.

$$\begin{cases} \min\ \frac{1}{2} x^T P_0 x + q_0^T x + r_0. \\ \text{s.t. } \frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0,\ i = 1, \cdots, m. \end{cases}$$

$P_0 \in S_{++}^n$, $P_i \in S_+^n$

$$\Rightarrow L(x, \lambda) = \frac{1}{2} x^T P_0 x + q_0^T x + r_0 + \sum_{i=1}^m \left(\frac{1}{2}\lambda_i x^T P_i x + \lambda_i q_i^T x + \lambda_i r_i\right).$$
$$= \frac{1}{2} x^T \underbrace{\left(P_0 + \sum_{i=1}^m \lambda_i P_i\right)}_{P} x + \underbrace{\left(q_0 + \sum_{i=1}^m \lambda_i q_i\right)}_{Q}{}^T x + \underbrace{\left(r_0 + \sum_{i=1}^m \lambda_i r_i\right)}_{R}.$$

$$\Rightarrow g(\lambda) = \inf_x L(x, \lambda).$$
$$\underline{\underline{\lambda \ge 0}}\ -\frac{1}{2} Q^T(\lambda) P^{-1}(\lambda) Q(\lambda) + R(\lambda)$$

$$\begin{cases} \max\ -\frac{1}{2} Q^T(\lambda) P^{-1}(\lambda) Q(\lambda) + R(\lambda) \\ \text{s.t. } \lambda \ge 0. \end{cases}$$

$\exists\, x \in D = \mathbb{R}^n$; $\frac{1}{2} x^T P_i x + q_i x + r_i < 0$, $i = 1, \cdots, m$ 则 $p^* = d^*$

若 $q_i = 0$, $r_i = 0$, $\frac{1}{2} x^T P_i x < 0$. 此时 $p^* = d^* = 0$.

lec32.

几种解释 $p^* = d^*$.

$\min\ f_0(x)$　　　$x \in D$.
s.t. $f_i(x) \le 0$, $i = 1, \cdots, m$.

几何解释　$\min\ f_0(x)$
　　　　　s.t. $f_1(x) \le 0$.

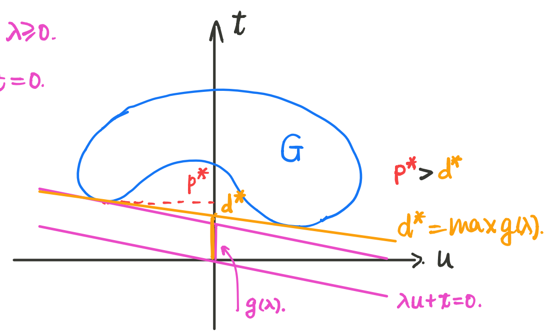
$G = \{ (f_1(x), f_0(x)) \mid x \in D \}$.

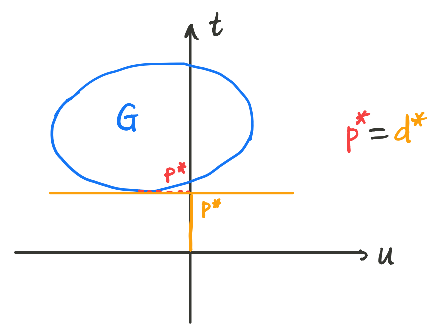
$p^* = \inf \{ t \mid (u,t) \in G, u \le 0 \}$

$g(\lambda) = \inf \{ \underbrace{\lambda \cdot u + t}_{L(x,\lambda)} \mid (u,t) \in G \}$.

$g(\lambda)$ $\lambda \ge 0$.

$\lambda u + t = 0$.




固定λ，将 $\lambda u + t$ 往上平移直到与G相接触 (inf).

此时，截距为 $g(\lambda)$.

鞍点的解释. Saddle Point.

$L(x,\lambda)$.

$$\inf_{x\in D} \sup_{\lambda \ge 0} L(x,\lambda) = \underbrace{\sup_{\lambda \ge 0} \overbrace{\boxed{\inf_{x\in D} L(x,\lambda)}}^{g(\lambda)}}_{d^* = \max g(\lambda)}$$

$(x^*, \lambda^*)$.

$$d^* = \sup_{\lambda \ge 0} \inf_{x \in D} L(x,\lambda)$$

$$p^* = \inf_{x\in D} \sup_{\lambda \ge 0} L(x,\lambda).$$

min $f_0(x)$.

s.t. $f_i(x) \le 0, i = 1, \cdots, m$.

$$\sup_{\lambda \ge 0} f_0(x) + \sum_{i=0}^{m} \lambda_i f_i(x). = \begin{cases} f_0(x) + 0 & \forall i \ f_i(x) \le 0 \\ +\infty & \exists i \ f_i(x) > 0 \end{cases}$$

$\inf_{x\in D} \sup_{\lambda \ge 0} L(x,\lambda)$.

多目标优化解释.

$\min \{ f_0(x), f_1(x), \cdots, f_m(x) \}$.

$$\{\lambda_i\} \Rightarrow \min \underbrace{f_0(x) + \sum_i \lambda_i f_i(x)}_{L(x,\lambda)}.$$

min $f_0(x)$.

s.t. $f_i(x) \le 0, i = 1, \cdots, m$

给定 $\lambda$，$\min\limits_{x\in D} L(x,\lambda) \longrightarrow \tilde{x}$

$\max\limits_{\lambda\ge 0} g(\lambda) \longrightarrow \tilde{\lambda}$

$\min\limits_{x\in D} L(x,\tilde{\lambda}) \rightarrow x^* \leftarrow \{\lambda_i\} \Rightarrow \min f_0(x) + \sum\limits_i \tilde{\lambda}_i f_i(x)$.

经济学的解释

$$\begin{cases} \min\ f_0(x) \\ \text{s.t. } f_i(x)\le 0,\ i=1,\cdots,m. \end{cases}$$

计划经济（原材料不能动，可能有余，可能不够）.

$x$：产品数量　$-f_0(x)$：利润　$f_0(x)$：损失　$f_i(x)$：原材料的约束.

设原材料可交易，$\lambda_i \ge 0$.

$$\underbrace{\min_x\ f_0(x) + \sum_{i=1}^m \lambda_i f_i(x)}_{} = g(\lambda)$$

市场经济（原材料可交易）.

$(x^*, \lambda^*)$.

最大损失 $\max\ g(\lambda)$ s.t. $x\ge 0$. $\Rightarrow d^*$

$d^* \le p^*$. 市场经济 $\ge$ 计划经济.

$d^* = p^*$.