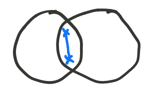
$\{x \mid x \le 0\}$ 是凸集, 是多面体, $x_0=0, x_1=-\infty$ 是单纯型.

S_+^n $n=2$ $S_+^n=\left\{\begin{bmatrix} x & y \\ y & z \end{bmatrix} \,\middle|\, x\ge 0, z\ge 0, xz \ge y^2\right\}$

交集: 若 S_1、S_2 为凸集, 则 $S_1 \cap S_2$ 为凸集

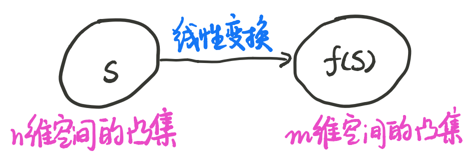


$\Downarrow$ 推广

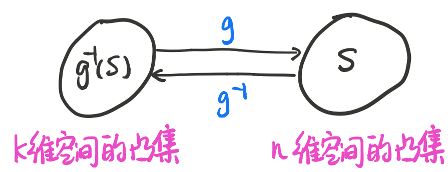
若 S_α 为凸集, $\forall \alpha\in A$, 则 $\bigcap_{\alpha\in A} S_\alpha$ 为凸集.

仿射函数: $f: \mathbb{R}^n \to \mathbb{R}^m$ 是仿射的, 当 $f(x)=Ax+b$, $A\in\mathbb{R}^{n\times m}$, $b\in\mathbb{R}^m$

若 $S\in\mathbb{R}^n$ 为凸集, $f:\mathbb{R}^n\to\mathbb{R}^m$ 仿射, 则 $f(S)=\{f(x)\mid x\in S\}$ 为凸集.



$g:\mathbb{R}^k\to\mathbb{R}^n$ 为仿射, $g^{-1}(S)=\{x\mid f(x)\in S\}$



缩放与移位是保持凸性的 $\alpha S=\{\alpha x\mid x\in S\}$.

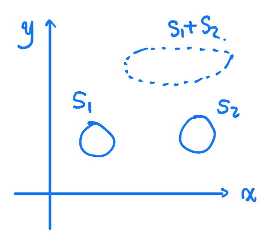
$S+a=\{x+a\mid x\in S\}$

例: 两个凸集的和是凸的.

$S_1+S_2=\{x+y\mid x\in S_1, y\in S_2\}$

$S_1\times S_2=\{(x,y)\mid x\in S_1, y\in S_2\}$

$f((x,y))=x+y$.



例: 线性矩阵不等式

$A(x)=x_1A_1+\cdots+\underbrace{x_nA_n\preceq B}$, B、A_i、$X_i\in S^m$

$(A(x)-B)\preceq 0$ 半负定

$\{x\mid A(x)\preceq B\}$ 为凸集.

证明: $\begin{cases} \text{定义仿射变换 } f(x) \triangleq B - A(x) \\ S^+_n \text{ 为凸集} \end{cases}$

想象成 1×1 的标量.

$$f^{-1}(S^n_+) = \{x \mid \underbrace{B-A(x)}_{B \geq A(x)} \geq 0\}$$

例: 椭球是球的仿射映射.

$$\mathcal{E} = \{x \mid (x-x_c)P^{-1}(x-x_c) \le 1\} \quad P \in S^n_{++}$$

$$\{u \mid \|u\|_2 \le 1\} \qquad (P^{\frac12})(P^{\frac12}) = P.$$

$$f(u) = P^{\frac12}u + x_c$$

$$\boxed{\begin{aligned} X &\triangleq P^{\frac12}u + X_c \\ u &= P^{-\frac12}(X - X_c). \end{aligned}}$$

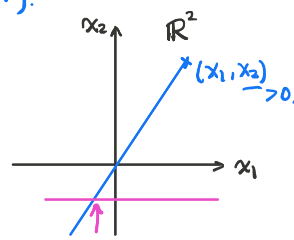
$$\{f(u) \mid \|u\|_2 \le 1\} = \{P^{\frac12}u + x_c \mid \|u\|_2\le1\} = \{x \mid \|P^{-\frac12}(x-x_c)\|_2 \le 1\}$$

$$= \{x \mid (x-x_c)^T P^{-1}(x-x_c) \le 1\}.$$

透视函数 perspective function

$$P: \mathbb{R}^{n+1} \to \mathbb{R}^n \qquad \mathrm{dom}\, P = \mathbb{R}^n \times \mathbb{R}_{++}$$

$$P(z,t) = \frac{z}{t} \quad z \in \mathbb{R}^n,\ t \in \mathbb{R}_{++}$$



$$(-\frac{x_1}{x_2}, -1) = (-P(x_1,x_2), -1)$$

例: 考虑 $\mathbb{R}^{n+1}$ 内线段、$x = (\underbrace{\tilde{x}}_{\in\mathbb{R}^n}, \underbrace{x_{n+1}}_{\in \mathbb{R}_{++}})$ $y = (\underbrace{\tilde y}_{\in\mathbb{R}^n}, \underbrace{y_{n+1}}_{\in\mathbb{R}_{++}})$

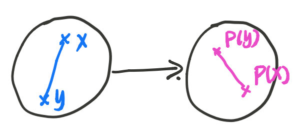
$\theta\geq0$, 线段为 $\theta x+(1-\theta)y$

证明 线段 $\xrightarrow{P}$ 线段. $x \xrightarrow{P} P(x)$ $y \xrightarrow{P} P(y)$.

$$\theta x + (1-\theta)y \xrightarrow{P} P(\theta x+(1-\theta)y)$$

$$P(\theta x+(1-\theta)y) = \frac{\theta\tilde x + (1-\theta)\tilde y}{\theta x_{n+1}+(1-\theta)y_{n+1}} = \underbrace{\frac{\theta x_{n+1}}{\theta x_{n+1}+(1-\theta)y_{n+1}}}_{\mu\in[0,1]}\cdot\underbrace{\frac{\tilde x}{x_{n+1}}}_{P(x)} + \underbrace{\frac{(1-\theta)y_{n+1}}{\theta x_{n+1}+(1-\theta)y_{n+1}}}_{1-\mu}\cdot\underbrace{\frac{\tilde y}{y_{n+1}}}_{P(y)}$$

$\theta \to \mu$ 是一一映射



例: 任意凸集的反透视映射仍是凸集.

$$P^{-1}(C) = \{(x,t)\in\mathbb{R}^{n+1} \mid \frac{x}{t}\in C,\ t>0\}$$

考虑 $(x,t)\in P^{-1}(C)$ $(y,s)\in P^{-1}(C)$, $0\le\theta\le1$.

$$\frac{\theta x+(1-\theta)y}{\theta t+(1-\theta)s} = \overset{\mu}{\boxed{\frac{\theta t}{\theta t+(1-\theta)s}}}\cdot\overset{\in C}{\boxed{\frac{x}{t}}} + \overset{1-\mu}{\boxed{\left(1-\frac{\theta t}{\theta t+(1-\theta)s}\right)}}\overset{\in C}{\boxed{\frac{y}{s}}} \in C.$$

线性分数函数.

$g: \mathbb{R}^n \to \mathbb{R}^{m+1}$ 为仿射映射. , $g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$ $A \in \mathbb{R}^{m\times n}$ $b \in \mathbb{R}^m$ $c \in \mathbb{R}^n$ $d \in \mathbb{R}$

$p: \mathbb{R}^{m+1} \to \mathbb{R}^m$ 为透视变换

$f: \mathbb{R}^n \to \mathbb{R}^m \triangleq p \circ g$ (复合函数). 线性分数函数 $P(g(c))$

$$f(x) = \frac{Ax+b}{c^T x + d}, \quad \operatorname{dom} f = \{x \mid c^T x + d > 0\}$$

例: 两个随机变量的联合概率 $\longrightarrow$ 条件概率

u, v $\{1, \cdots, n\}$ $\{1, \cdots, m\}$

$p_{ij} = P(u=i, v=j)$ 联合概率

$f_{ij} = P(u=i \mid v=j)$ 条件概率

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}$$

$$(p_{1j} + \cdots + p_{nj})\,(0, \cdots, 1, \underset{(i)}{0}, \cdots 0).$$