lec 29.

Chapter 5 Duality

$$\min f_0(x)$$
$$\text{s.t. } f_i(x) \le 0,\ i=1,\cdots,m$$
$$h_i(x) = 0,\ i=1,\cdots,p.$$

$x \in \mathbb{R}^n$, $D = \bigcap_{i=0}^{m} \mathrm{dom} f_i \cap \bigcap_{i=1}^{p} \mathrm{dom}\, h_i$ $\qquad$ p^* optimal value.

Lagrangian function / Lagrangian 拉格朗日函数.

$$L(x,\lambda,v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} v_i h_i(x).$$

Lagrange Dual Function / Dual Function 对偶函数.

$$g(\lambda,v) = \inf_{x\in D} L(x,\lambda,v).$$

Lagrange Multiplier / Multipler λ, v.

性质: 1. 对偶函数为凹函数.

$\sup_{x\in D} L(x,\lambda,v)$ 凸. $\cup$ $\rightarrow$ $\inf_{x\in D} L(x,\lambda,v)$ 凹. $\cap$

2. $\forall \lambda \ge 0, \forall v, g(\lambda,v) \le p^*$.

证明: 设x^*是原问题最优解, 则必可行.

则 $f_i(x^*) \le 0$, $h_i(x^*) = 0$.

当 $\forall \lambda \ge 0, \forall v$, 有 $\sum_{i=1}^{m} \lambda_i f_i(x^*) + \sum_{i=1}^{p} v_i h_i(x^*) \le 0$

$$L(x^*,\lambda,v) = \underbrace{f_0(x^*)}_{p^*} + \sum_{i=1}^{m} \lambda_i f_i(x^*) + \sum_{i=1}^{p} v_i h_i(x^*) \le p^*$$

$$g(\lambda,v) \le p^*$$

例: $\min\ x^T x$

s.t. $Ax = b$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$.

$$\Rightarrow L(x,v) = x^T x + v^T (Ax - b)$$

$$\Rightarrow g(v) = \inf_{x\in D} L(x,v) = \inf_{x\in D} x^T x + v^T A x - v^T b.$$

对x偏导 $2x + A^T v = 0$. $x = -\frac{A^T v}{2}$ 代回去求最优值.

$$\frac{1}{4}(v^T A A^T v) - \frac{1}{2}(v^T A A^T v) - v^T b = -\frac{1}{4} v^T \underbrace{AA^T}_{半正定} v - b^T v \quad 凹.$$

例: $\min c^T x$

s.t. $Ax \geq b$, $x \geq 0$ ⋮ $Ax - b = 0$, $-x \leq 0$

$$\Rightarrow L(x,\lambda,v) = c^T x - \lambda^T x + v^T(Ax-b)$$
$$= -b^T v + (c + A^T v - \lambda)^T x$$

$$\Rightarrow g(\lambda, v) = \inf_{x\in D} L(x,\lambda,v) = \begin{cases} -b^T v & c^T + A^T v - \lambda = 0. \\ +\infty & \text{otherwise.} \end{cases}$$

例: $\min \ x^T W x.$

s.t. $\underline{x_i = \pm 1, \ i=1,\cdots,m.}$

$\Updownarrow$

$x_i^2 - 1 = 0.$ (二次约束)

$$\Rightarrow L(x,v) = x^T W x + \sum_{i=1}^{m} v_i (x_i^2 - 1)$$
$$= x^T(W + \mathrm{diag}\{v\})x - 1^T \cdot v$$

$$\Rightarrow g(v) = \inf_{x\in D} x^T(W + \mathrm{diag}\{v\})x - 1^T \cdot v.$$
$$= \begin{cases} -1^T \cdot v & W + \mathrm{diag}\{v\} \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

函数的共轭.

f^* 是 $f: \mathbb{R}^n \to \mathbb{R}$ 的共轭, 若 $f^*(y) = \sup_{x\in \mathrm{dom} f} (y^T x - f(x)).$

例: $\min f_0(x)$

s.t. $x = 0.$

$$\Rightarrow L(x,v) = f(x) + v^T x, \quad \mathrm{dom} L = \mathrm{dom} f \times \mathbb{R}^n$$

$$\Rightarrow g(v) = \inf_{x\in \mathrm{dom} f} (f(x) + v^T x).$$
$$= -\sup_{x\in \mathrm{dom} f} (-v^T x - f(x))$$
$$= -f^*(-v).$$

例: $\min f_0(x)$

$\quad$ s.t. $Ax \le b$

$\qquad\quad Cx = d.$

$$\Rightarrow L(x,\lambda,v) = f_0(x) + \lambda^T(Ax-b) + v^T(Cx-d)$$

$$= f_0(x) + (\lambda^T A + v^T C)x - \lambda^T b - v^T d.$$

$$\Rightarrow g(\lambda,v) = \inf_{x\in \mathrm{dom} f} L(x,\lambda,v).$$

$$= -f_0^*(-(\lambda^T A + v^T C)^T) - \lambda^T b - v^T d.$$

(D) $\begin{cases} \max & g(\lambda,v) \\ \text{s.t.} & \lambda \ge 0 \end{cases}$ 最优值 d^* (Dual problem).

(P) $\begin{cases} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0, \ i=1,\cdots,m \\ & h_i(x) = 0, \ i=1,\cdots,p. \end{cases}$

(1) $d^* \le p^*.$

(2). λ^*, v^*: optimal lagrange multiplier.

例: $\boxed{\begin{aligned} &\min \ c^T x \\ &\text{s.t.} \ Ax=b \\ &\qquad x \ge 0 \end{aligned}}$ $\xrightarrow{x\text{替换成}\lambda.}$ $\boxed{\begin{aligned} P_1: &\min \ c^T\lambda. \\ &\text{s.t.} \ A\lambda = b \\ &\qquad \lambda \ge 0. \end{aligned}}$

① P_1的对偶问题的对偶问题是自身

② 对偶问题一定是凸问题.

③ 对偶的对偶不一定等于自身.

$$g(\lambda,v) = \begin{cases} -b^T v & A^T v + \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

$$(D) \Rightarrow \begin{aligned} &\max \ g(\lambda,v) \\ &\text{s.t} \ \lambda \ge 0. \end{aligned} \Leftrightarrow \begin{aligned} &\max \ -b^T v \\ &\text{s.t} \ \lambda \ge 0 \\ &\quad A^T v - \lambda + c = 0 \end{aligned} \Bigg\} \rightarrow \begin{aligned} &\min \ b^T v \\ &A^T v + C \ge 0. \end{aligned} \Leftrightarrow \boxed{\begin{aligned} D_1: &\min \ b^T x \\ &\text{s.t.} \ A^T x + C \ge 0 \end{aligned}}$$

例: $\boxed{\begin{aligned} &\min \ c^T x \\ &\text{s.t.} \ Ax \le b. \end{aligned}}$ $\Leftrightarrow$ $\boxed{\begin{aligned} P_2: &\min \ c^T x \\ &\text{s.t.} \ -Ax+b \ge 0. \end{aligned}}$

$$\Rightarrow L(x,\lambda) = c^T x + \lambda^T(Ax-b)$$

$$= (c^T + A^T\lambda)^T x - b^T\lambda.$$ (拉格朗日函数)

$$\Rightarrow g(\lambda) = \inf_x L(x,\lambda) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$ (对偶函数)

$$D \Rightarrow \begin{aligned} &\max \ -b^T\lambda. \\ &\text{s.t.} \ A^T\lambda + c = 0 \\ &\qquad \lambda \ge 0. \end{aligned}$$ (对偶问题) $\Leftrightarrow$ $\boxed{\begin{aligned} D_2: &\min \ b^T\lambda. \\ &\text{s.t.} \ -A^T\lambda = c \\ &\qquad \lambda \ge 0. \end{aligned}}$